Title: Entanglement and topological order in cluster states

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Abstract: We study a Hamiltonian system describing a three-spin 1/2 cluster like interaction competing with an Ising-like exchange. We show that a cluster state, the ground state of the Hamiltonian in the absence of the Ising term, is provided by a hidden order of topological nature. In the presence of the cluster and Ising couplings, a continuous quantum phase transition occurs in the system, directly connecting a local broken symmetry phase to a cluster phase with the hidden order. At the critical point the Hamiltonian is self-dual. We analyze the geometric entanglement and demonstrate that it can capture the transition, as a single parameter.

Entanglement and topological order in cluster states

Luigi Amico

MATIS - CNR-INFM & DMFCI, Università di Catania.

***Ma**terials and **T**echnologies*
*for **I**nformation and communication **S**ciences*

- **Collaborations:**

W. Son (Singapore)
R. Fazio (Pisa-Singapore)
V. Vedral (Oxford-Singapore)

S. Pascazio (Bari)
P. Facchi (Bari)
P. Smacchia (Bari)

Outline

- General ideas.
- Cluster-Ising model Hamiltonian.
- Exact solution and duality.
- Quantum phase transition and geometric entanglement

Entanglement measures

Review: Horodecki's family quant-ph/0702225, Rev. Mod. Phys. 2009.

(Bennet, Bernstein, Shumacher, Popescu, PRA1996.)

General aim:

Quantify Entanglement in many body systems.

Possible questions:

- Correlation Vs Entanglement of the STATES.
  - Entanglement and Critical phenomena ?
- Entanglement as a resource (q-compuation....)

(Review: Amico, Fazio, Osterloh, Vedral RMP 2008)

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[redacted]

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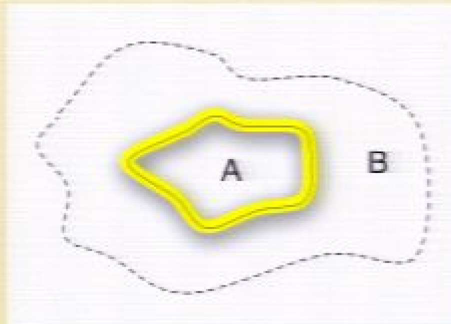
Correlations are information of one system about another



Entanglement: $\Psi \neq \Psi_A \Psi_B$

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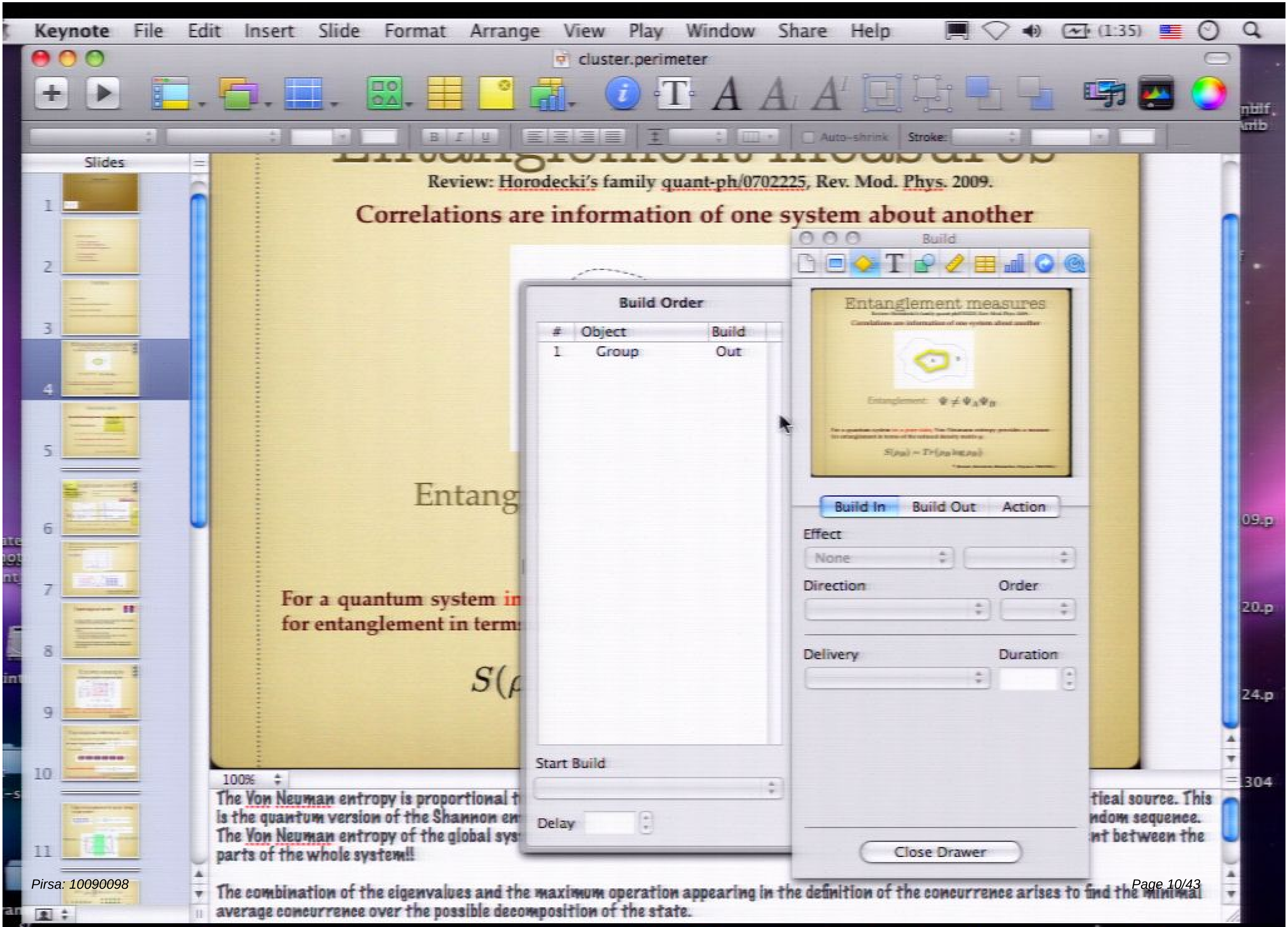
The combination of the eigenvalues and the maximum operation appearing in the definition of the concurrence arises to find the minimal average concurrence over the possible decomposition of the state.



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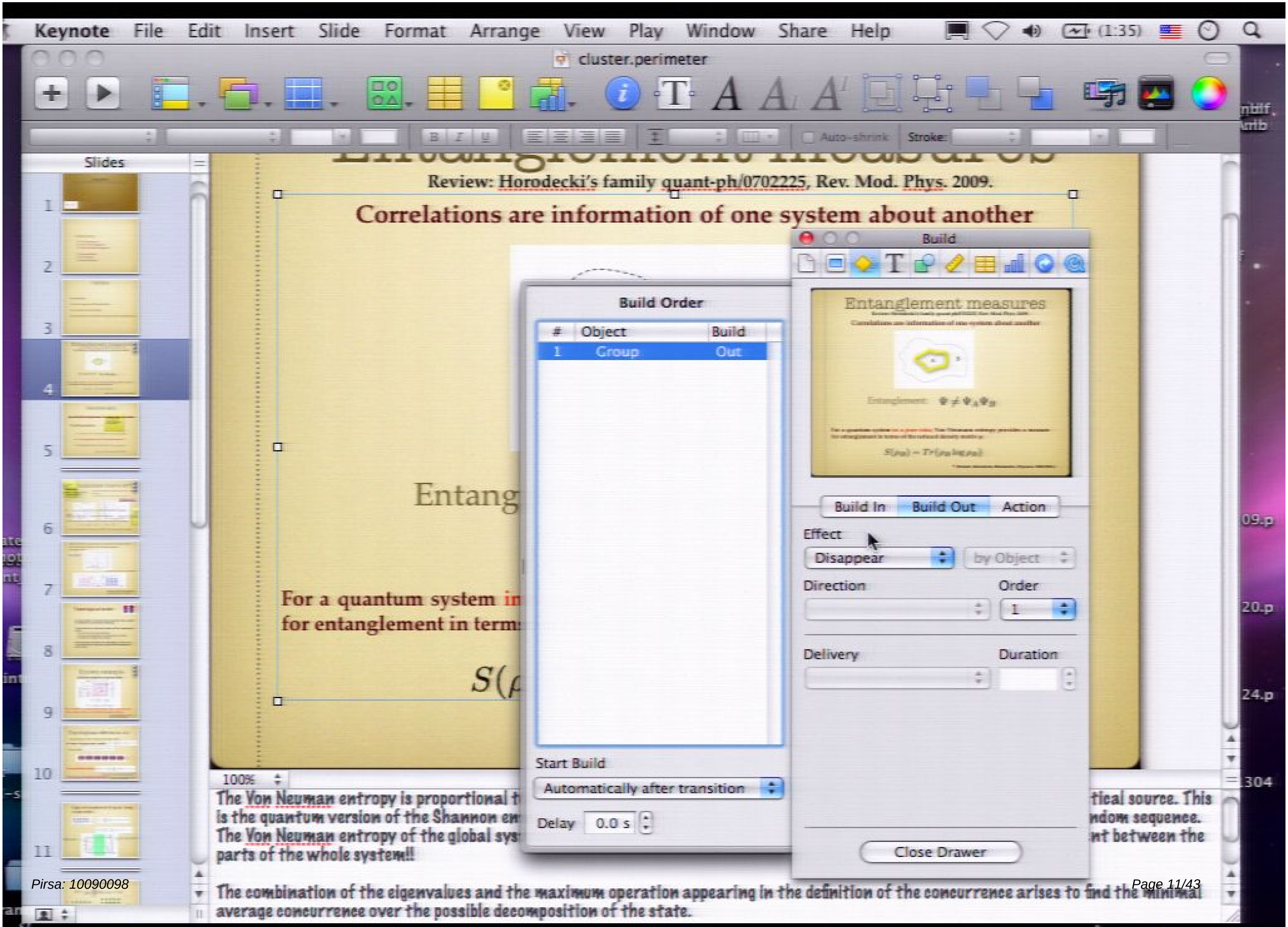
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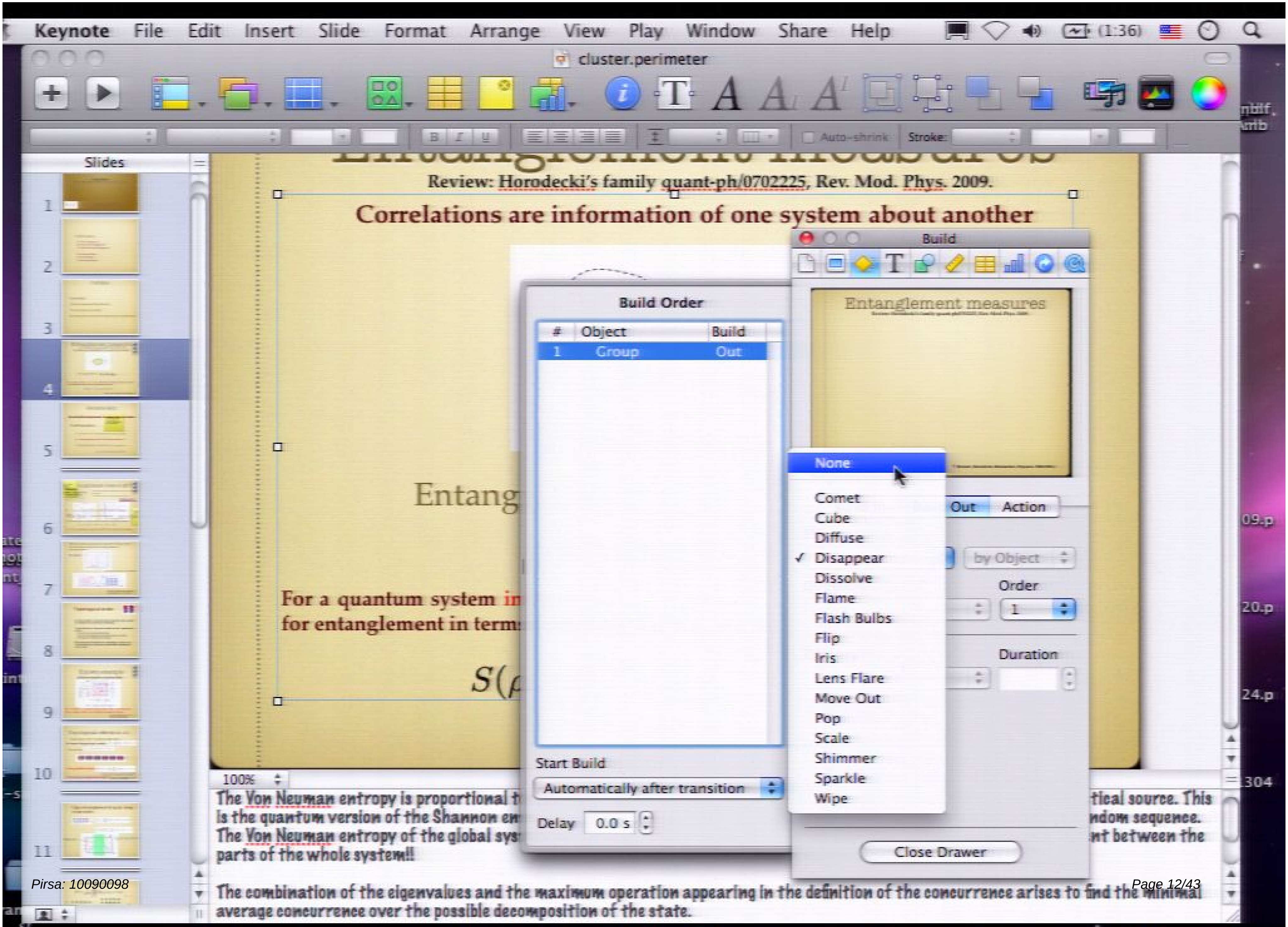
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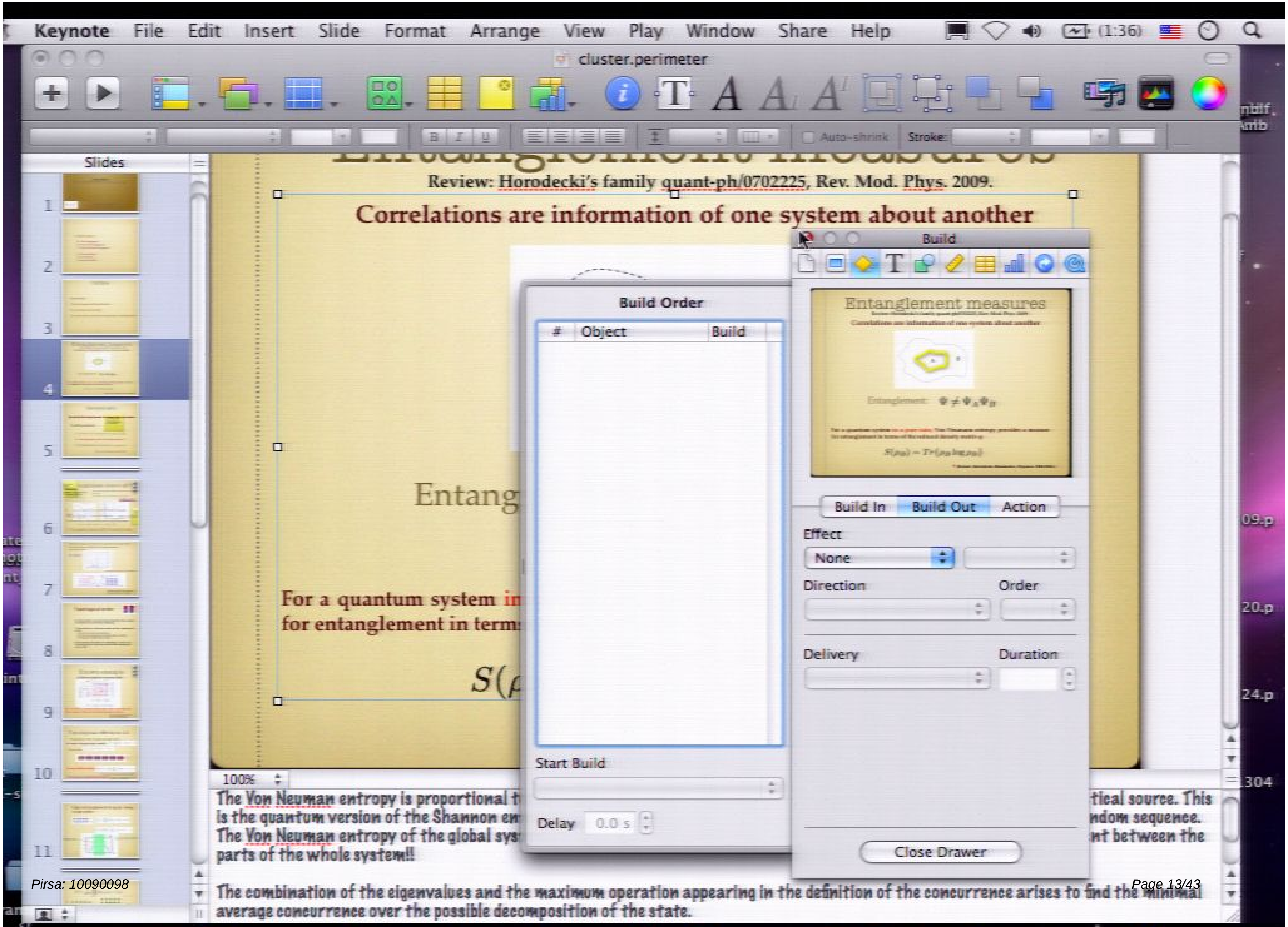
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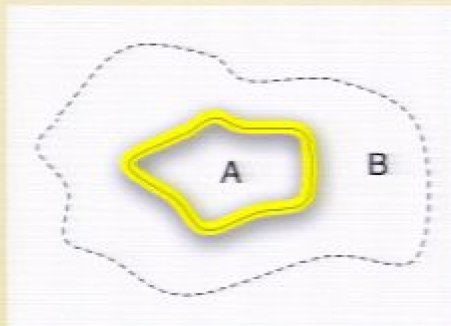

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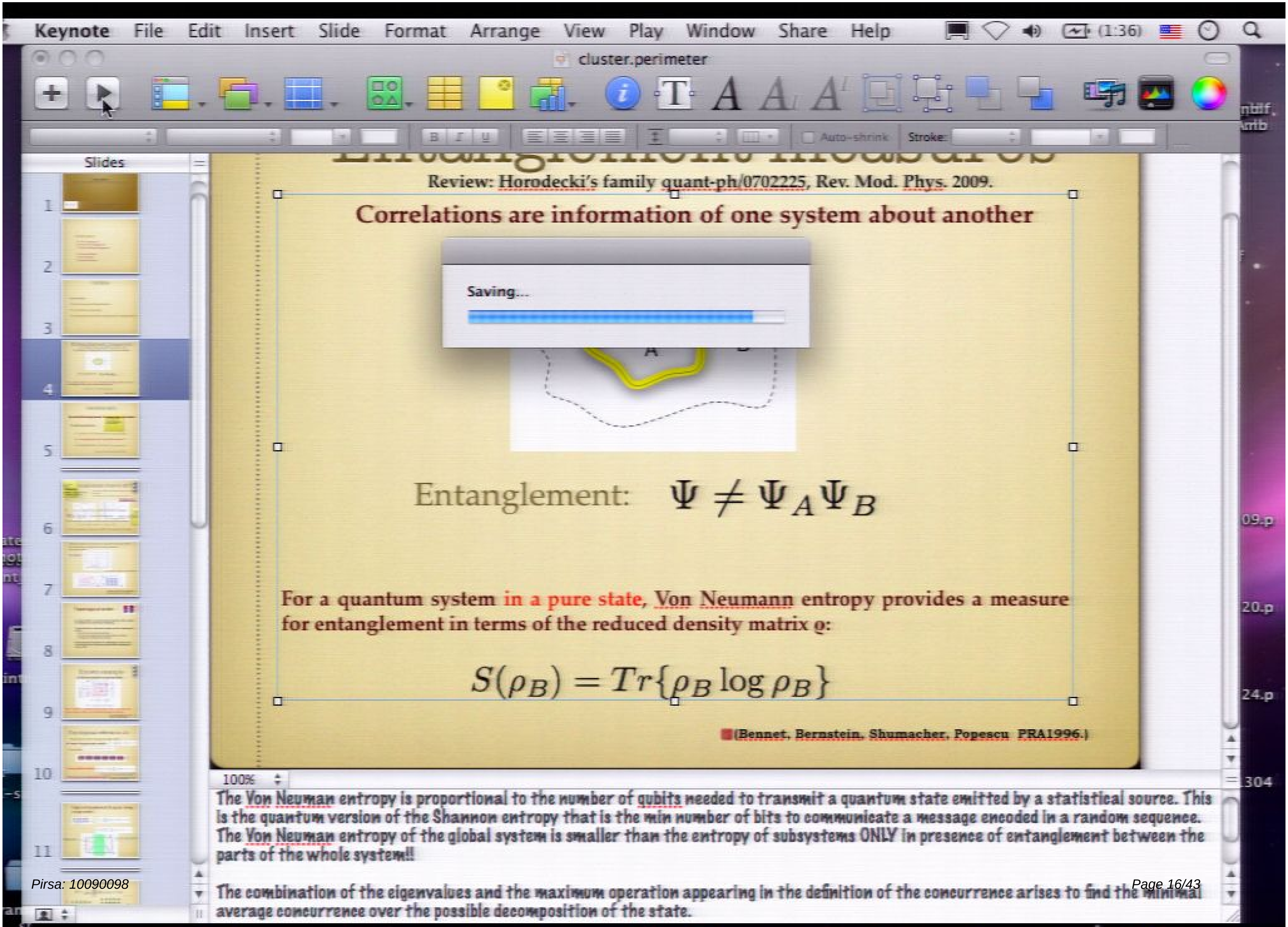
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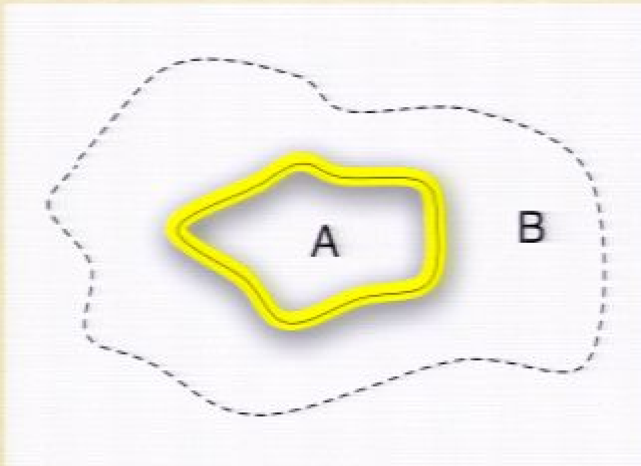

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(Review: Amico, Fazio, Osterloh, Vedral RMP 2008)

Entanglement close to QPT

letters to nature

Scaling of entanglement close to a quantum phase transition

A. Osterloh*†, Luigi Amico*†, G. Falci*† & Rosario Fazio†‡

* Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), viale A. Doria 6, 95125 Catania, Italy
† NEST-INFM, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
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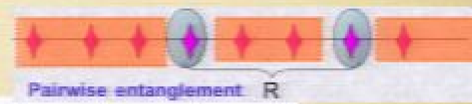
Critical change of the bipartite entanglement (Concurrence) at the quantum critical point.

At criticality: In general NO long range Concurrence.



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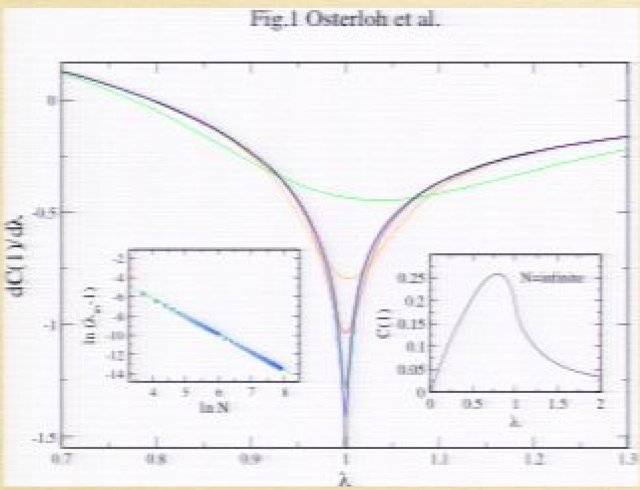

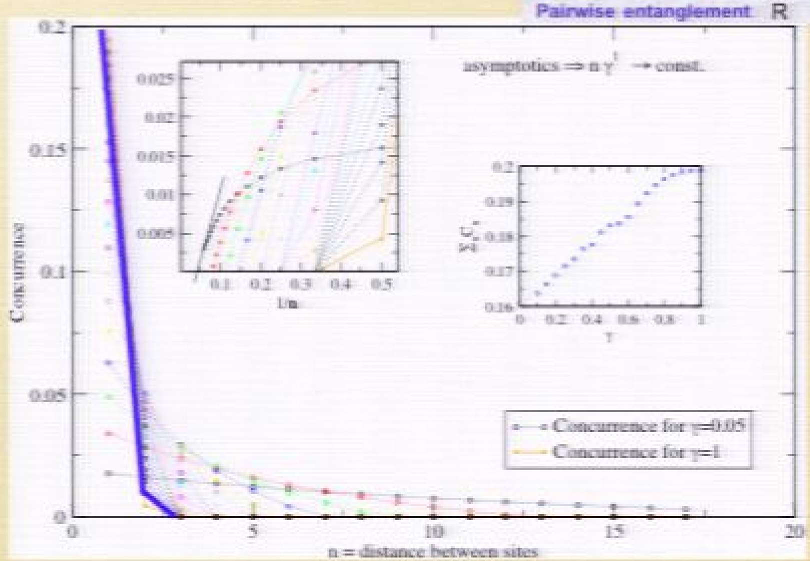

$$H(\gamma,0,h/J)=\sum_i (1+\gamma)S_i^x S_{i+1}^x+(1-\gamma)S_i^y S_{i+1}^y-\frac{2h}{J}S_i^z$$

- Clear quantitative picture for QPT with symmetry breaking.

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$$H = H_0 + \sum_l \lambda_l A_l$$

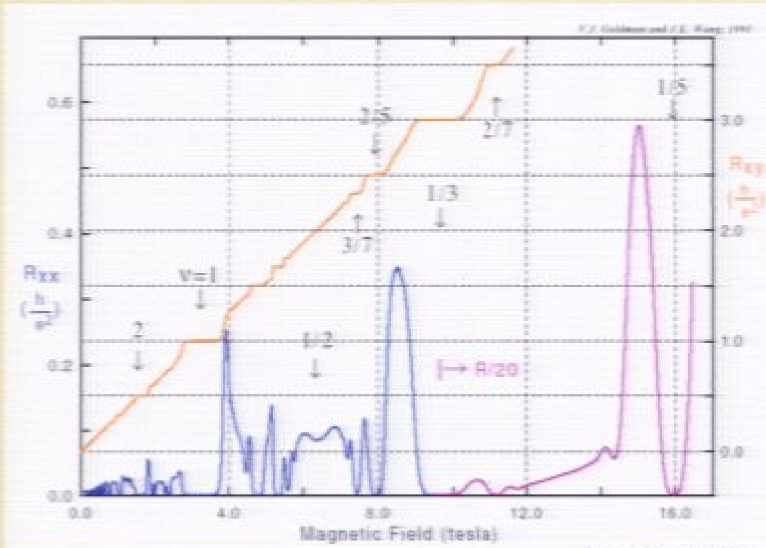
Measure of entanglement: $M(\langle A_l \rangle) = M\left(\frac{\partial E_{gs}}{\partial \lambda_l}\right)$

Wu, Sarandy, Lidar Phys. Rev. Lett. 2004; Wu, Sarandy, Lidar, Sham 2005. Yang 2005; Gu, Tian, Lin 2005.

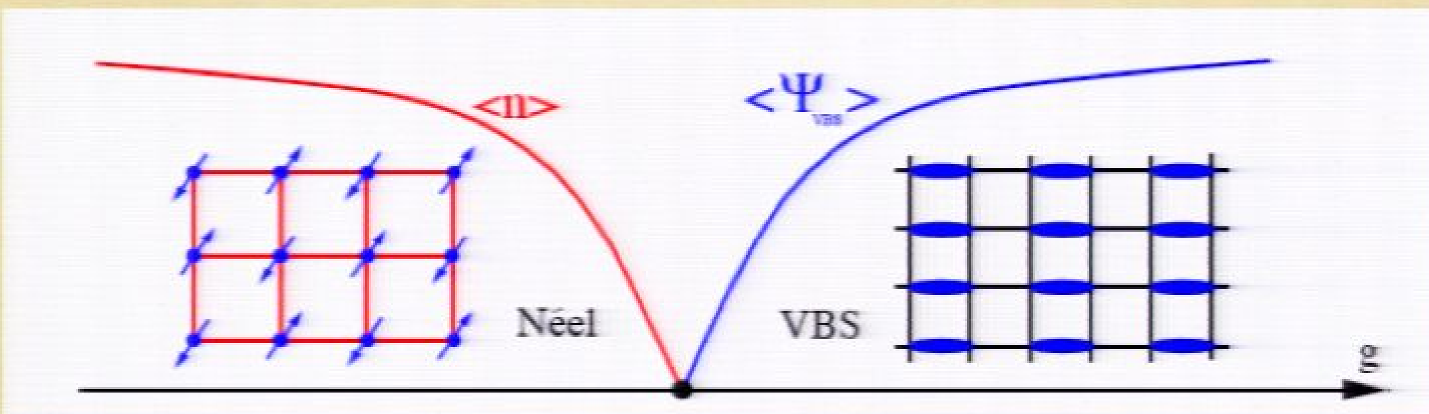
- Clear quantitative picture for QPT with symmetry breaking.

Beyond Landau mechanism of symmetry breaking: More subtle transitions without local order parameter - topological order.

Examples:

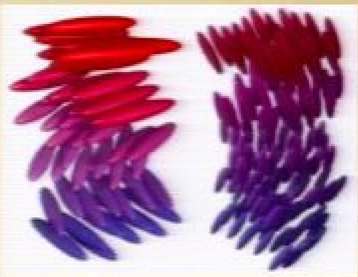


H. L. Stormer, D. C. Tsui, and A. C. Gossard, RMP 1999.



T. Senthil, et al, Science 303, 1490 (2004);
S. Sachdev, Nat. Phys. 4, 173 (2008)....

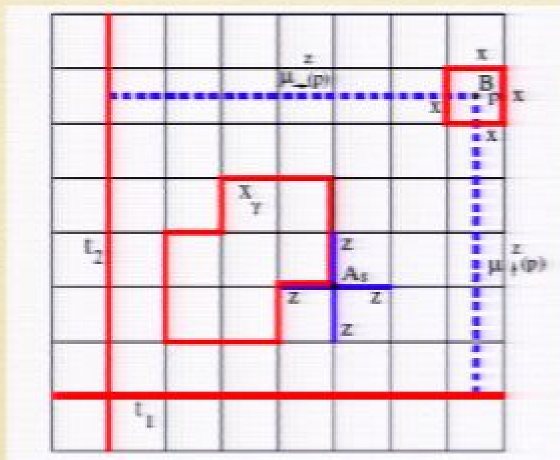
Topological order



- A phase which cannot be described by the Landau framework of symmetry breaking.
- Three different characterization of the topological order.
  - Insensitivity to local perturbation.
  - Ground state degeneracy to the boundary condition.
  - Topological entanglement entropy.
- Relationship between the topological order and fault tolerance. Symmetry protected topological order (1d).

Known example

2d-Kitaev model in a trasverse field



$$H = -J\sum_s A_s - J\sum_p B_p$$

with $A_s = \prod_{i\in s}\sigma_i^x$, and $B_p = \prod_{i\in p}\sigma_i^z$

QPT between a spin-polarized phase and a topologically ordered phase with non vanishing topological entropy.

Kitaev 2003-2004; Hamma, Lidar 2008....
Castelnovo and Chamon 2008;

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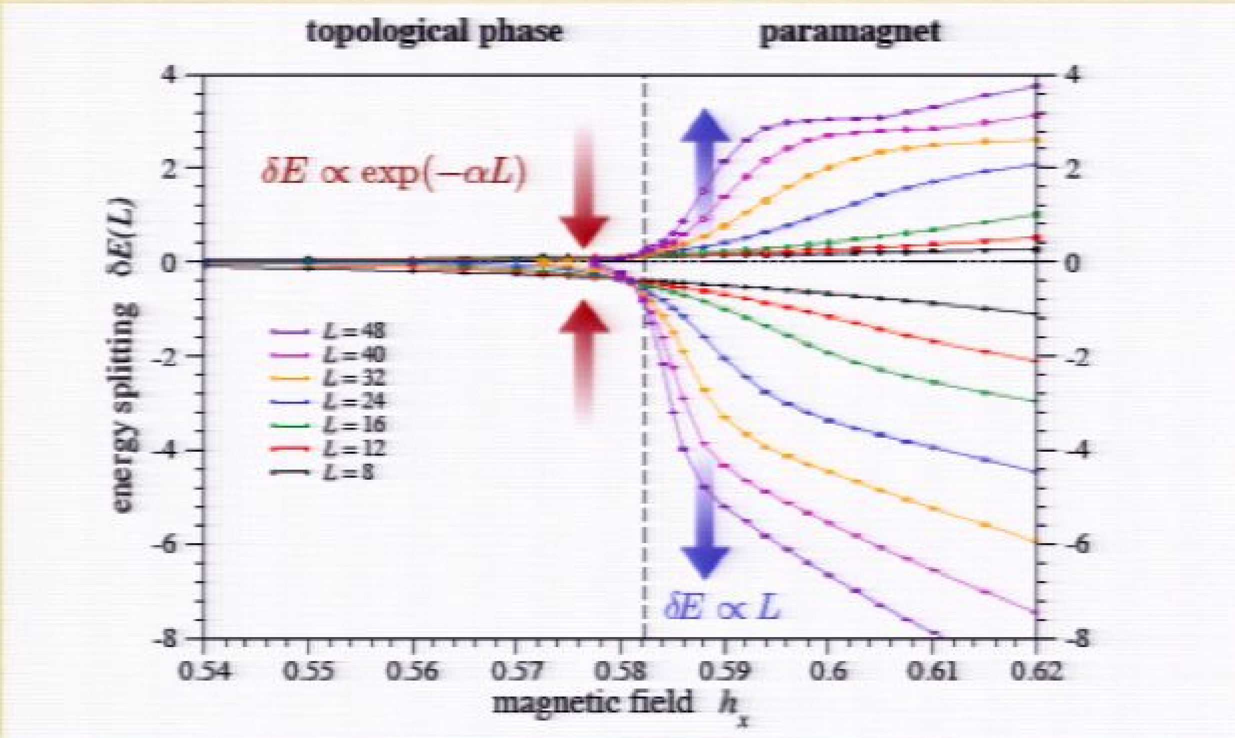

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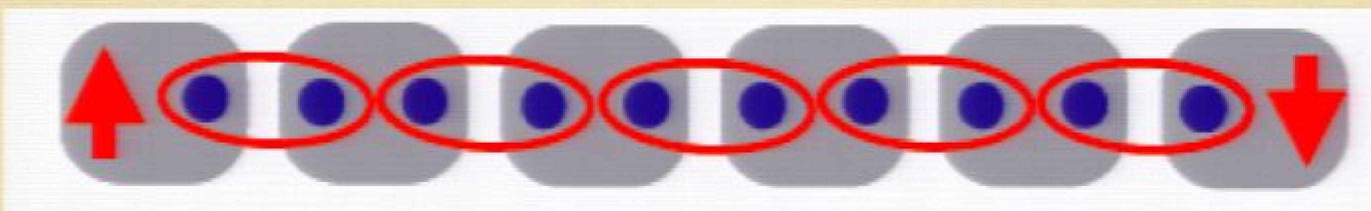
Topological effects in 1d

- **Topological order: localized edge states.**

◆**Example: 1d-gapped spin models:** $$H^{\mathrm{AKLT}} := \sum_{k=0}^{N} \left(\vec{S}_k \vec{S}_{k+1} + \frac{1}{3} (\vec{S}_k \vec{S}_{k+1})^2 + \frac{2}{3} \right)$$

Ground state: $$|V\rangle = \left(\otimes_{k=1}^{N} A_{\bar{k}k}\right) |I\rangle_{\bar{0}1} |I\rangle_{\bar{1}2} \cdots |I\rangle_{\bar{N}N+1}.$$

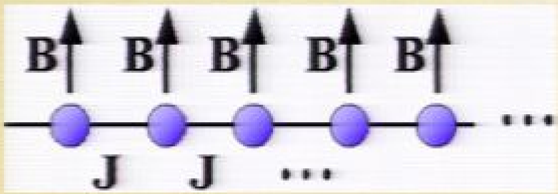


String (hidden) order: $$\mathcal{O}_1^a(n,n') = -\langle S_n^a (\exp \sum_{j=n+1}^{n'-1} i\pi S_j^a) S_{n'}^a \rangle, \quad a = (x,y,z).$$

M. den Nijs and K. Rommelse: Phys. Rev. B **40** (1989) 4709[APS].
T. Kennedy and H. Tasaki: Phys. Rev. B **45** (1992) 304[APS].

Diverging entanglement lenght! **F. Verstraete, M.A. Martin-Delgado, J.I. Cirac**

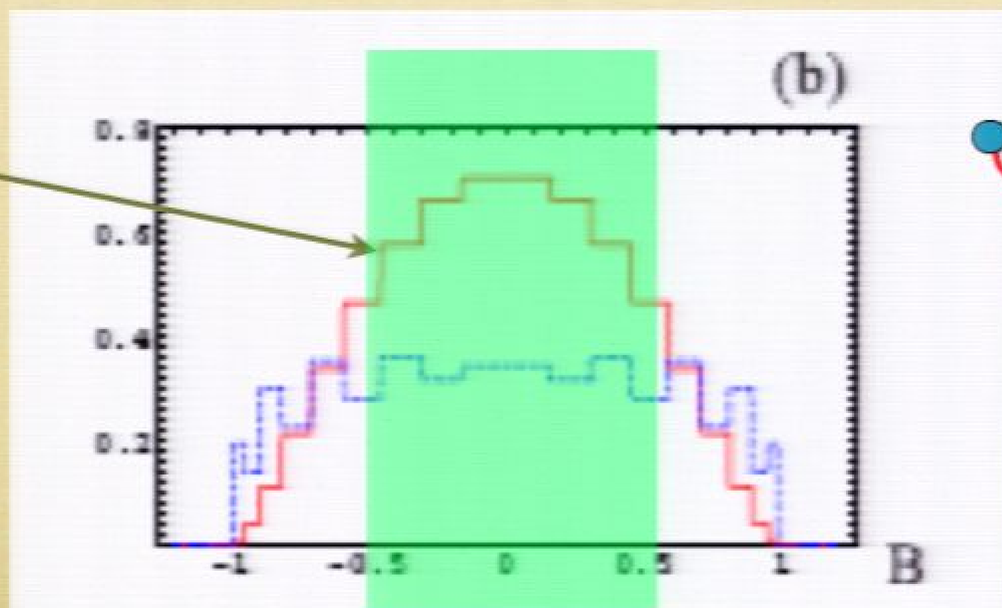
Edge entanglement & quasi-long range order

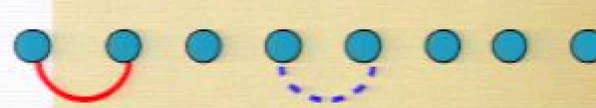


$$H = -\Big[\sum_{i=1}^{N} \frac{J}{2}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + B\sigma_i^z\Big]$$

$$|\varphi_g^k\rangle = \left(\frac{2}{N+1}\right)^{\frac{k}{2}} \sum_{l_1<l_2<\cdots<l_k} C_{l_1 l_2 \cdots l_k} |l_1, l_2, \cdots, l_k\rangle$$

Edge spins





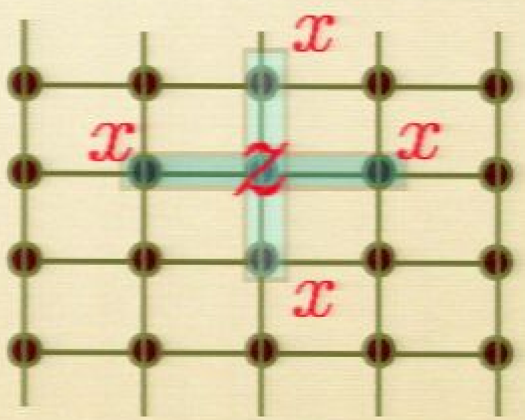
Cluster states

Briegel and Raussendorf 2001

$$|C\rangle = \frac{1}{2^{N/2}} \prod_{j=1}^{N} \left(\sigma^z_{j+1} | \downarrow\rangle_j + | \uparrow\rangle_j \right)$$

$$\sigma^z_{N+1} = 1$$

1d



2d

$$K_l = \sigma^z_l \prod_{j=n.n.(l)} \sigma^x_j$$

$$|C\rangle = K_l |C\rangle \quad \forall l$$

Gottesman 1997

- **Cluster states can be constructed as the ground state of the Hamiltonian.**

$$H_C = -\sum K_l$$

Example N=4 (PBC)

$$H_C = -\sum_{l=1}^{N} K_l$$

$E_4 = +4$ $\quad \sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z|C_4\rangle$

$E_3 = +2$ $\quad \sigma_k^z\sigma_j^z\sigma_i^z|C_4\rangle$

$E_2 = 0$ $\quad \sigma_j^z\sigma_i^z|C_4\rangle$

$E_1 = -2$ $\quad \sigma_i^z|C_4\rangle$

$E_0 = -4$ $\quad |C_4\rangle$

$$|C_4\rangle = |\uparrow + \uparrow +\rangle + |\uparrow - \uparrow -\rangle + |\downarrow - \downarrow +\rangle + |\downarrow + \downarrow -\rangle$$

$$\sigma^z|\uparrow/\downarrow\rangle = \pm|\uparrow/\downarrow\rangle\,, \quad \sigma^x|+/-\rangle = \pm|+/-\rangle$$

- **Bipartite order**
- **Polarization along z controls the σ^x configuarion.**

Correlations and entanglement

- Any two-spin-correlation is vanishing.
- Any two-spin-entanglement is vanishing.

* Global entanglement (ex.: one-tangle saturates) = N/2.
* String correlations (open chain):

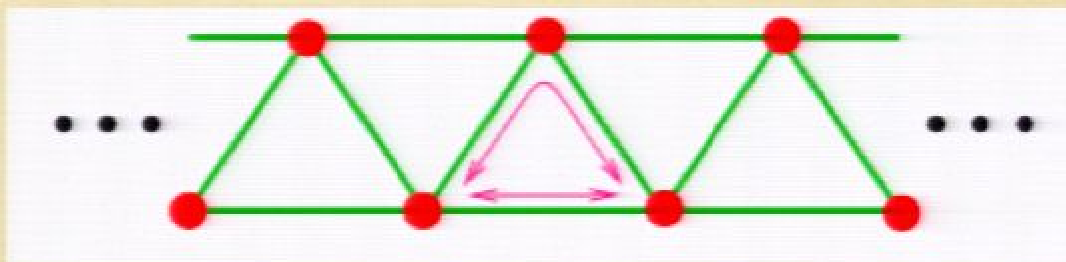
$$\mathcal{O}_z = (-)^{N-2}\langle \sigma_1^y \prod_j^{N-1} \sigma_j^z \sigma_N^y \rangle = 1$$

Skrovseth and Bartlett 2009; Doherty and Bartlett 2009

Quench of cluster states by local interactions: Cluster-Ising Hamiltonian

$$H = \sum_j \sigma^x_{j-1}\sigma^z_j\sigma^x_{j+1} - \lambda\sigma^y_j\sigma^y_{j+1} - \vec{B}\cdot\vec{\sigma}_j$$

- The model can be realized with two bosonic species in a triangular optical lattice.



Pachos and Plenio 2004

C. Becker, P. Soltan-Panahi, J. Kronjäger, S. Dörscher, K. Bongs, K. Sengstock 2009

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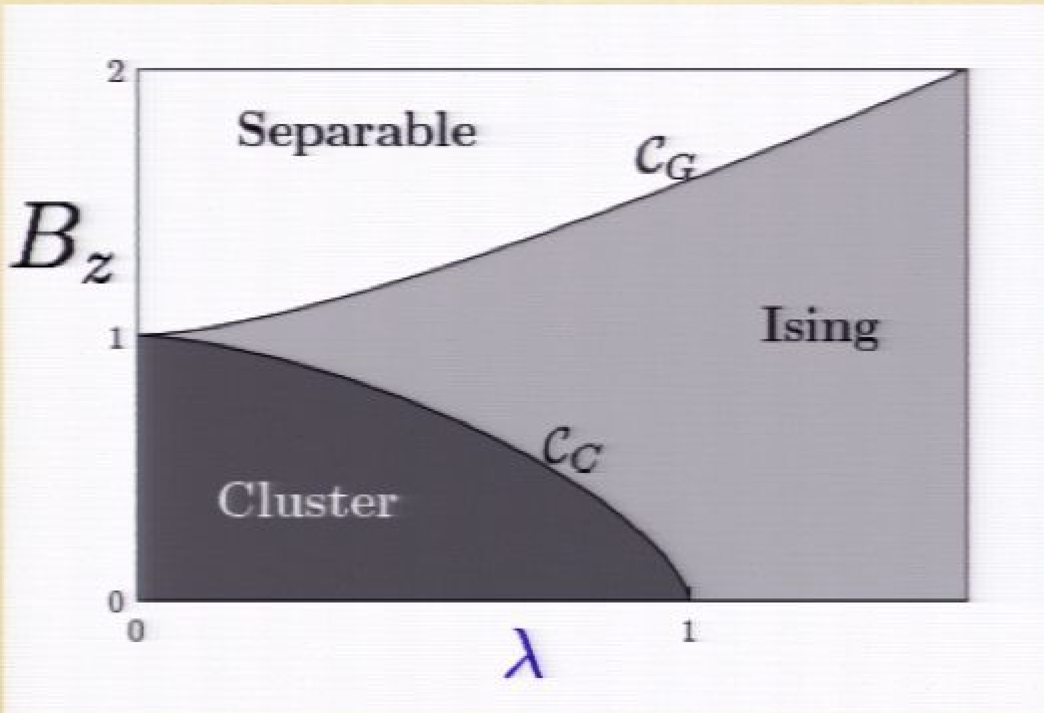

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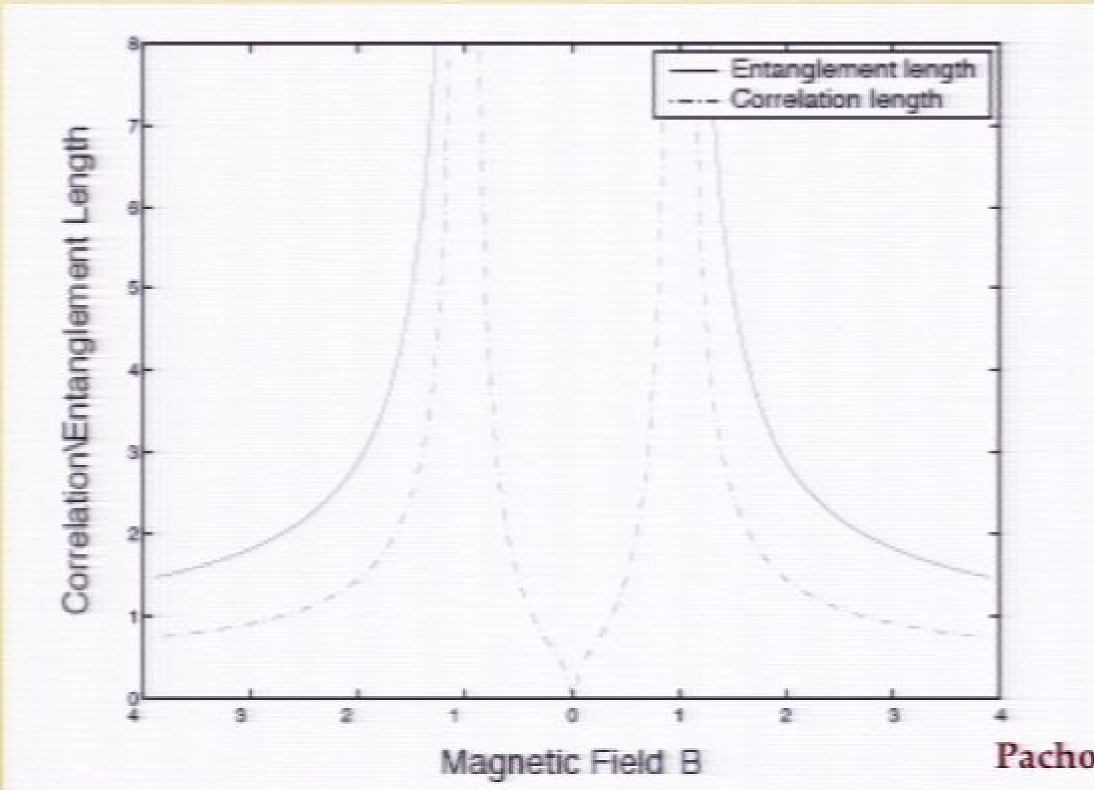
$$\lambda = 0$$



Pachos and Plenio 2004

Exact solution, duality and quantum phase transitions $(\vec{B}=0)$

- Free fermions.

$$H(\lambda)=\sum_l (c_{l-1}^\dagger - c_{l-1})(c_{l+1}^\dagger + c_{l+1}) + \lambda (c_l^\dagger + c_l)(c_{l+1}^\dagger - c_{l+1}) = \sum_k \Lambda_k (\gamma_k^\dagger \gamma_k - 1/2)$$

The dispersion: $\Lambda_k = \sqrt{(\lambda^2+1) - 2\lambda\cos(6\pi k/N)}$

- **Continuous QPT with z=ν=1 at λ=1.**

- Duality.

$$\mu_j^z = \sigma_j^x \sigma_{j+1}^x\,, \quad \mu_j^x = \prod_{l=1}^{j} \sigma_j^z$$

$$H(\lambda) \to H(\lambda)_{dual} = \sum_j \mu_j^y \mu_{j+1}^y - \lambda \mu_{j-1}^x \mu_j^z \mu_{j+1}^x - B.T.$$

$$N \to \infty : \lambda H(\lambda^{-1}) = H(\lambda)_{dual}$$

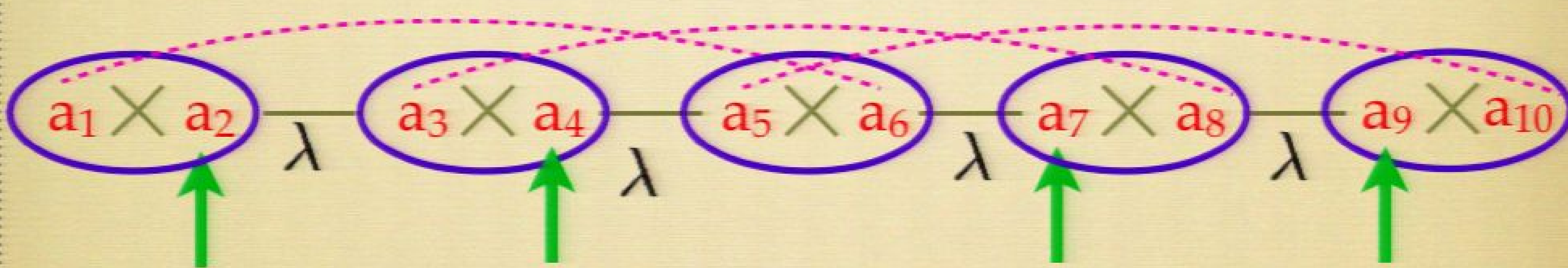
- **QPT at self dual point.**

Pachos and Plenio 2004;
Doherty and Bartlett 2009;

Majorana fermions

$$H(\lambda) = \sum_{l=2}^{N-1} a_{2j-3}a_{2j+2} + \lambda \sum_{l=1}^{N-1} a_{2j}a_{2j+1}$$

$$a_{2j-1} = \frac{c_j - c_j^\dagger}{i}, \quad a_{2j} = c_j + c_j^\dagger$$



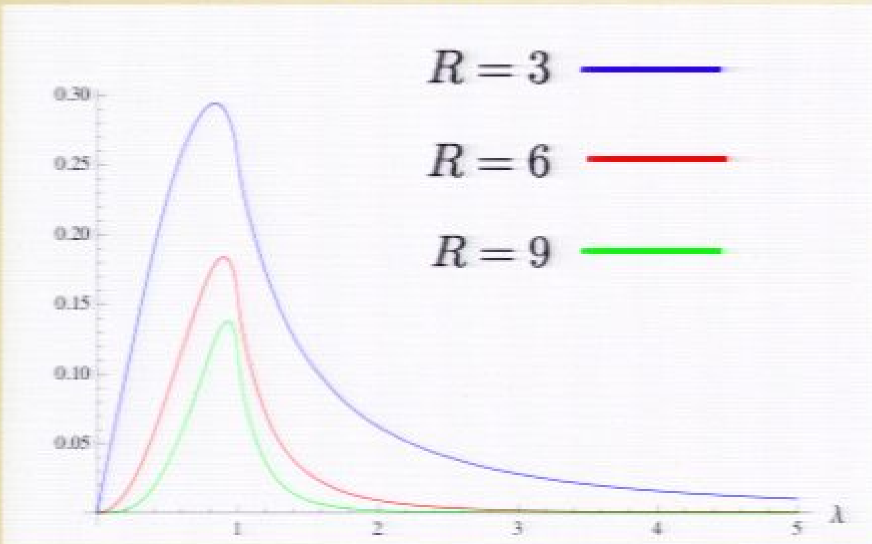
Kiatev Laumann 2009; Wen 2004

λ=0 (Cluster state): four-fold degeneracy.

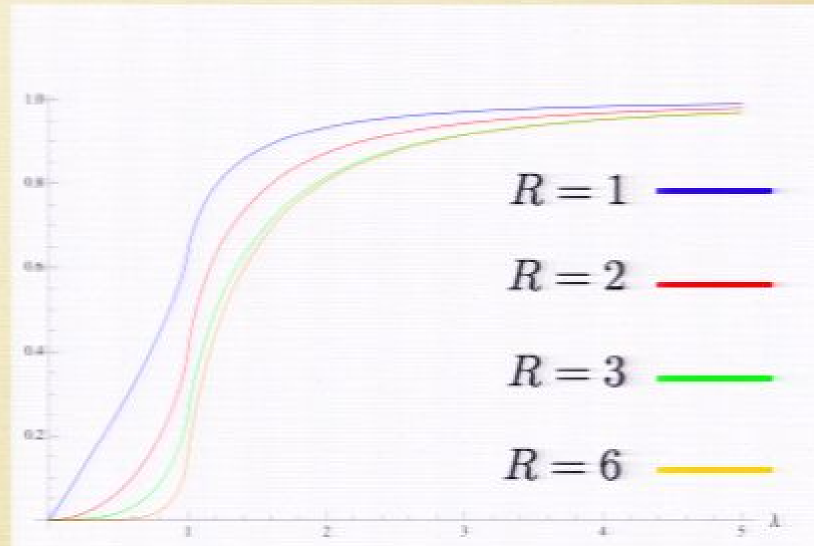
λ large (Ising): two fold degeneracy.

Correlation functions

$\langle \sigma_0^x \sigma_R^x \rangle$



$\langle \sigma_0^y \sigma_R^y \rangle$



$$\langle \sigma^z \rangle = \langle \sigma_0^z \sigma_R^z \rangle = 0 \,, \quad \forall R$$

Szego theorem:

$$\sqrt{\lim_{R\to\infty} \langle \sigma_0^y \sigma_R^y \rangle} = m_{stag} = (1 - \lambda^{-2})^{-3/8}$$

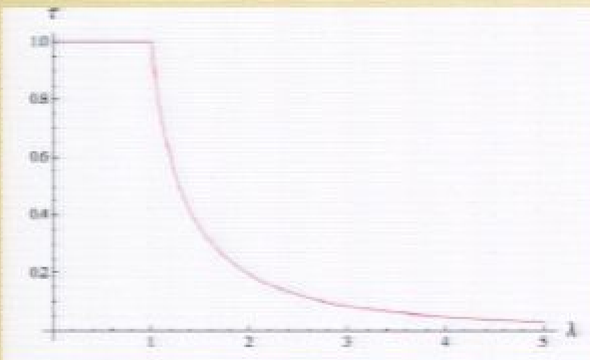
$$\beta = \frac{3}{8}$$

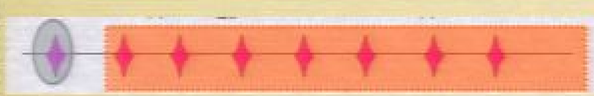
Entanglement

- Any two-spin entanglement is vanishing (thermal ground state).

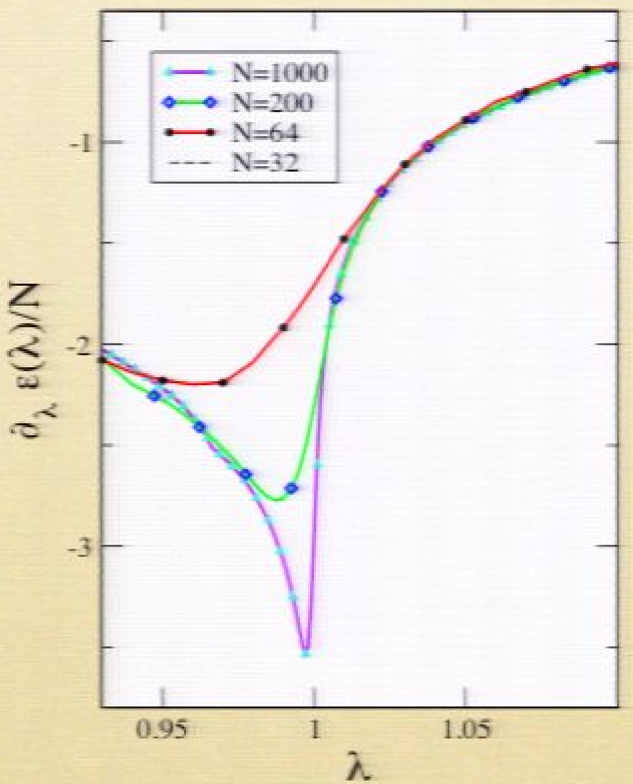
$$\varepsilon(|\psi\rangle) = -\log_2\left[\max_{\theta}|\langle S(\theta)|\psi\rangle|\right]$$

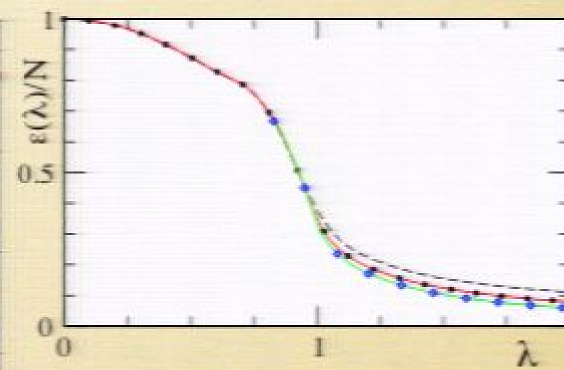
- Global entanglement.

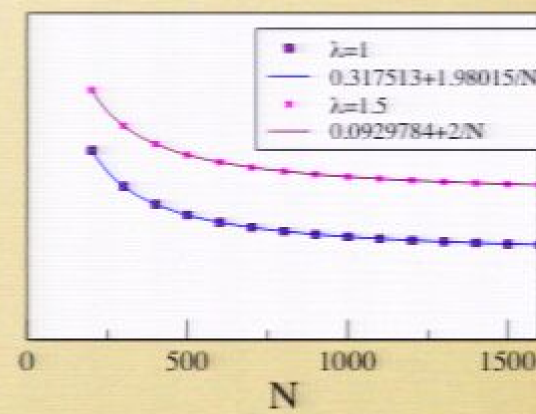




$$\tau = 4\det\rho = \frac{1}{4}\left(1 - m^2_{stag}\right)$$







$$|S(\theta_j,\phi_j)\rangle = \prod_{j=1}^{N}[\cos(\theta_j) + e^{-\phi_j}\sin(\theta_j)\sigma_j^x]|\uparrow\rangle^{\otimes N}$$

Son, Amico, Fazio, Pascazio, Vedral 2010

2-D ?

$$H_{MF} = \underbrace{- \sum_i \sigma_i^z \prod_{j(i)} \sigma_j^x}_{H_0} + \underbrace{z\lambda\psi \sum_i \sigma_i^y}_{V}$$

$\psi = \langle \sigma_i^y \rangle$

take the order parameter real and space independent

The self-consistent equation reads

$$1 = z\lambda \sum_j \int_{-\infty}^{+\infty} d\tau \, \langle T_\tau \, \sigma_j^y(\tau) \, \sigma_\ell^y(0) \rangle \qquad \sigma_j^y(\tau) = e^{\tau H_0} \sigma_j^y(0) e^{-\tau H_0}$$

$$\lambda_c = 1$$

Independent of dimensionality: multispin interaction

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **70**, 060302(R) (2004)

Valence-bond states for quantum computation

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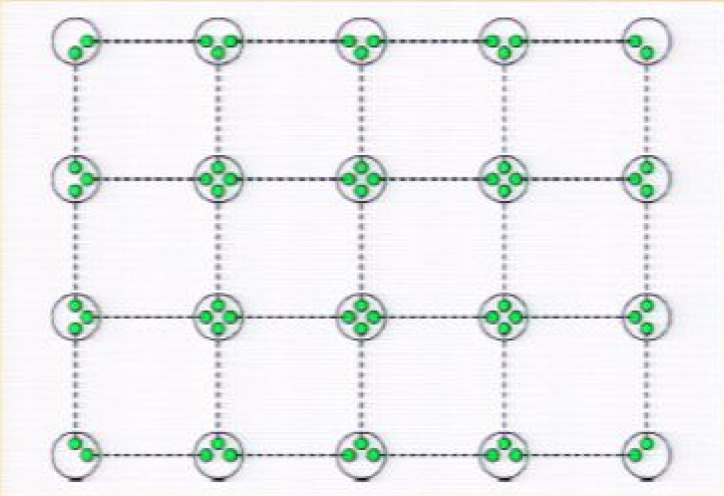
(Received 14 April 2004; published 14 December 2004)

We propose a way of universal quantum computation by doing joint measurements on distributed singlets. We show how these joint measurements become local measurements when the singlets are interpreted as the virtual components of a large valence-bond state. This proves the equivalence of the cluster-state-based quantum computational model and the teleportation-based model, and we discuss several features and possible extensions. We show that all stabilizer states have a very simple interpretation in terms of valence-bond solids, which allows to understand their entanglement properties in a transparent way.

DOI: 10.1103/PhysRevA.70.060302 PACS number(s): 03.67.Mn, 75.10.Pq, 03.65.Ud

$$|C\rangle \Leftrightarrow$$



2-d: antiferro-vbs quantum phase transitions (?)

Conclusions

- Hidden order in cluster states is quenched by the Ising interaction through a continuous quantum phase transition to antiferromagnetic phase.
- Entanglement can be used to describe the transition, beyond the Landau symmetry breaking mechanism.

[redacted]

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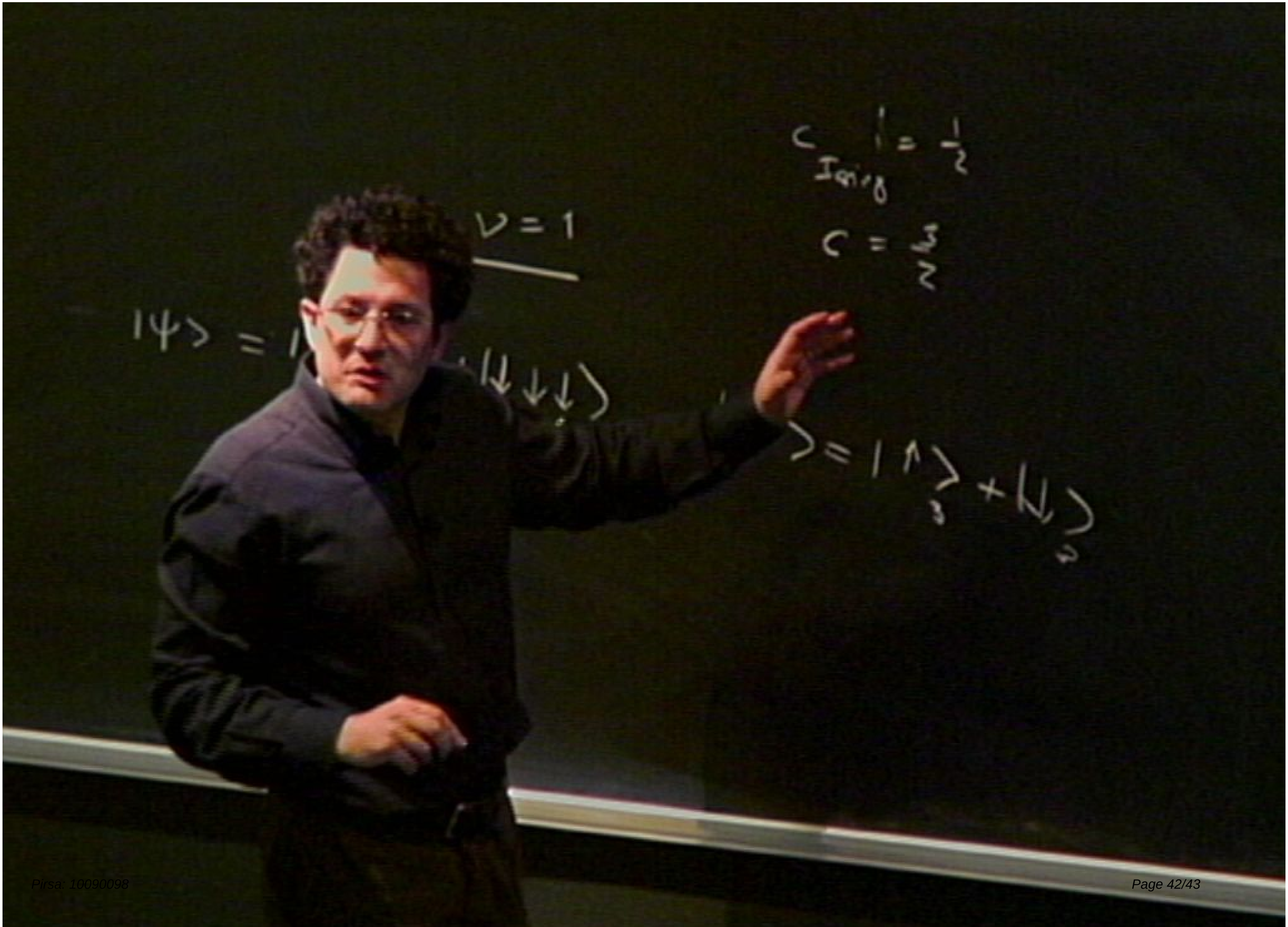
c Ising = 1/2
ν = 1
c = 3/2

Conclusions

- Hidden order in cluster states is quenched by the Ising interaction through a continuous quantum phase transition to antiferromagnetic phase.
- Entanglement can be used to describe the transition, beyond the Landau symmetry breaking mechanism.