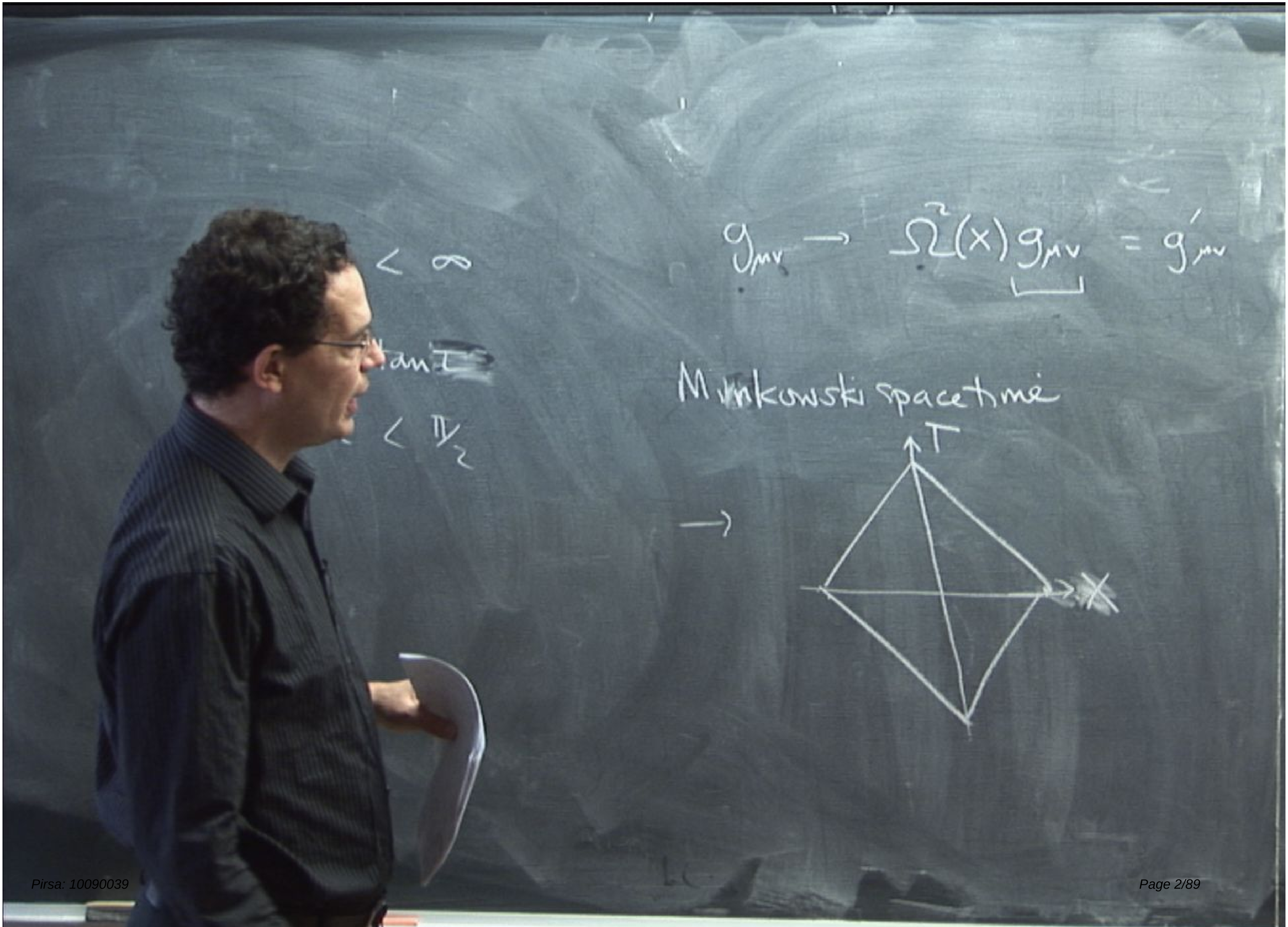
Title: Relativity (PHYS 604) - Lecture 14

Date: Sep 30, 2010 10:30 AM

URL: http://pirsa.org/10090039

Abstract:



< ∞
tan
< π/2
gμν → Ω²(x) gμν = g′μν
Minkowski spacetime
T
X

$$-\infty < t < \infty$$

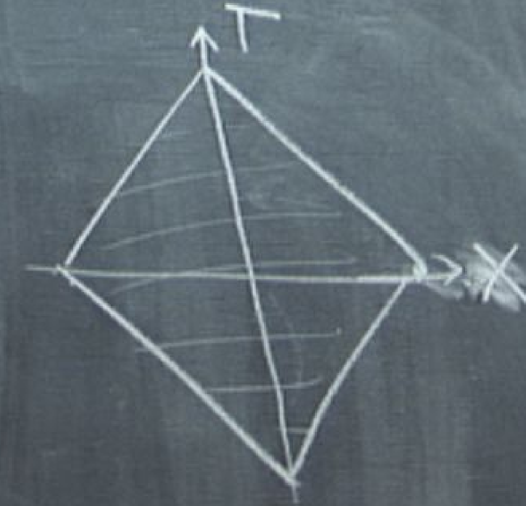
$$t = \tan \tau$$

$$-\pi/2 < \tau < \pi/2$$

$$g_{\mu\nu} \longrightarrow \Omega^2(x)\,\underbrace{g_{\mu\nu}} = g'_{\mu\nu}$$

Minkowski spacetime

$\longrightarrow$



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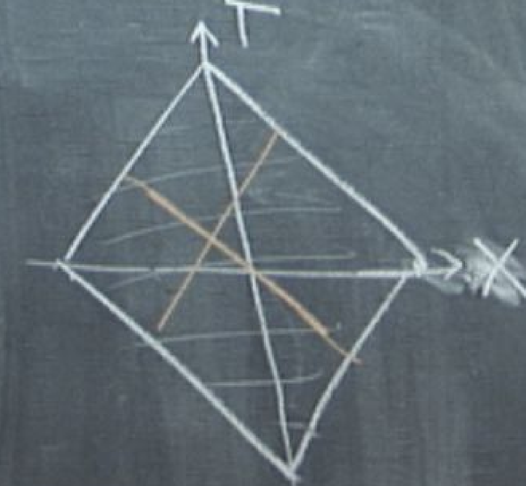

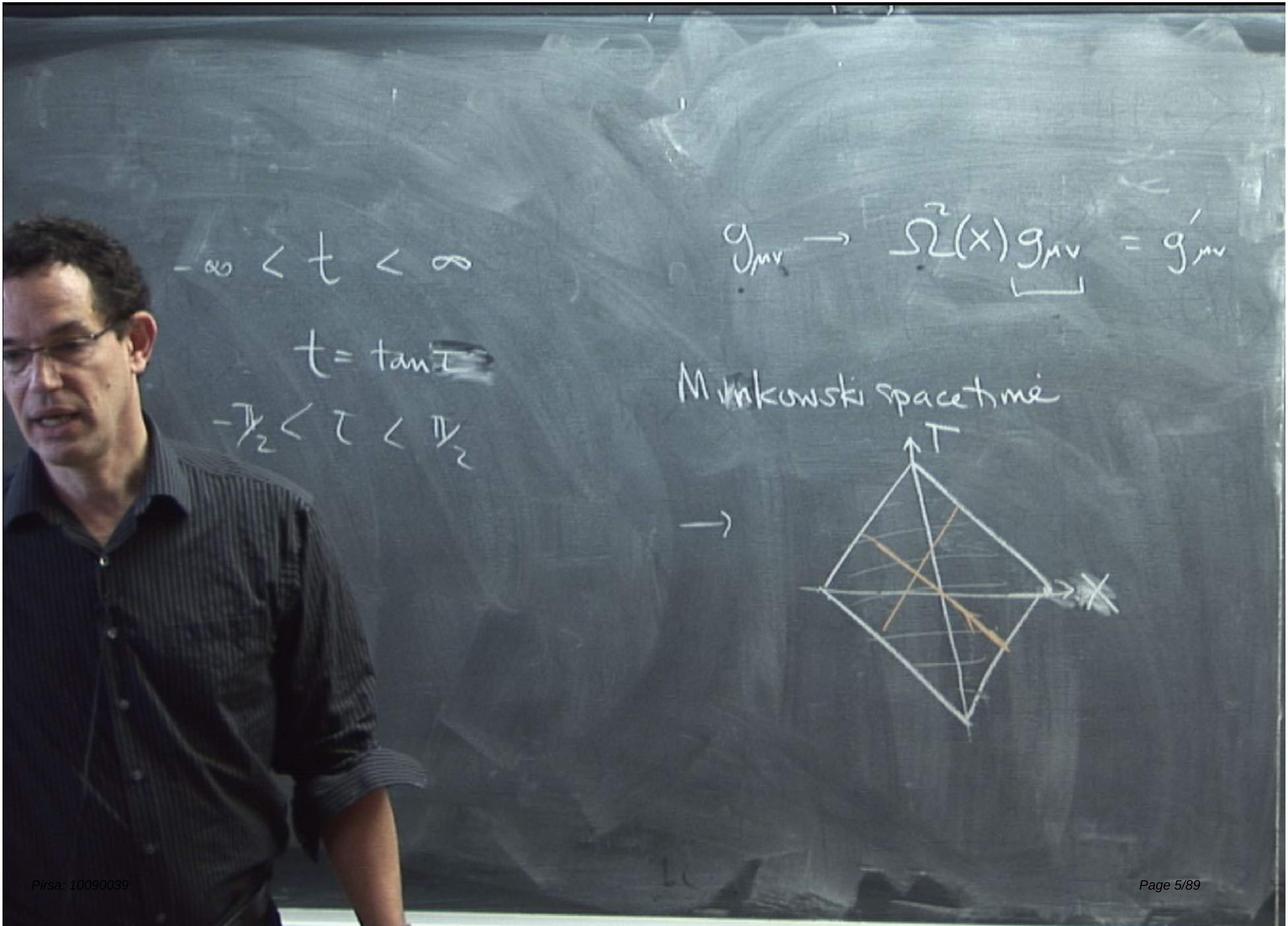
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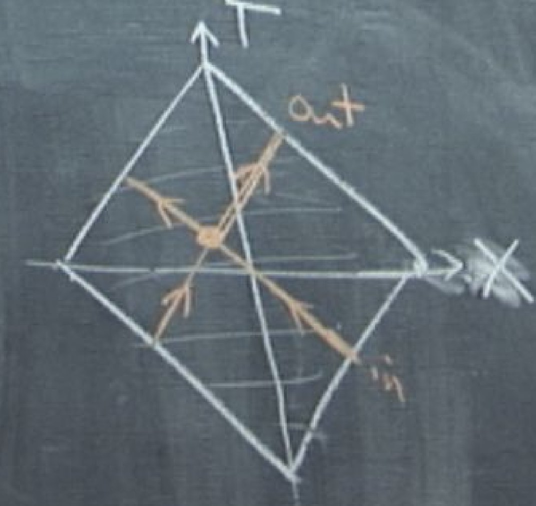

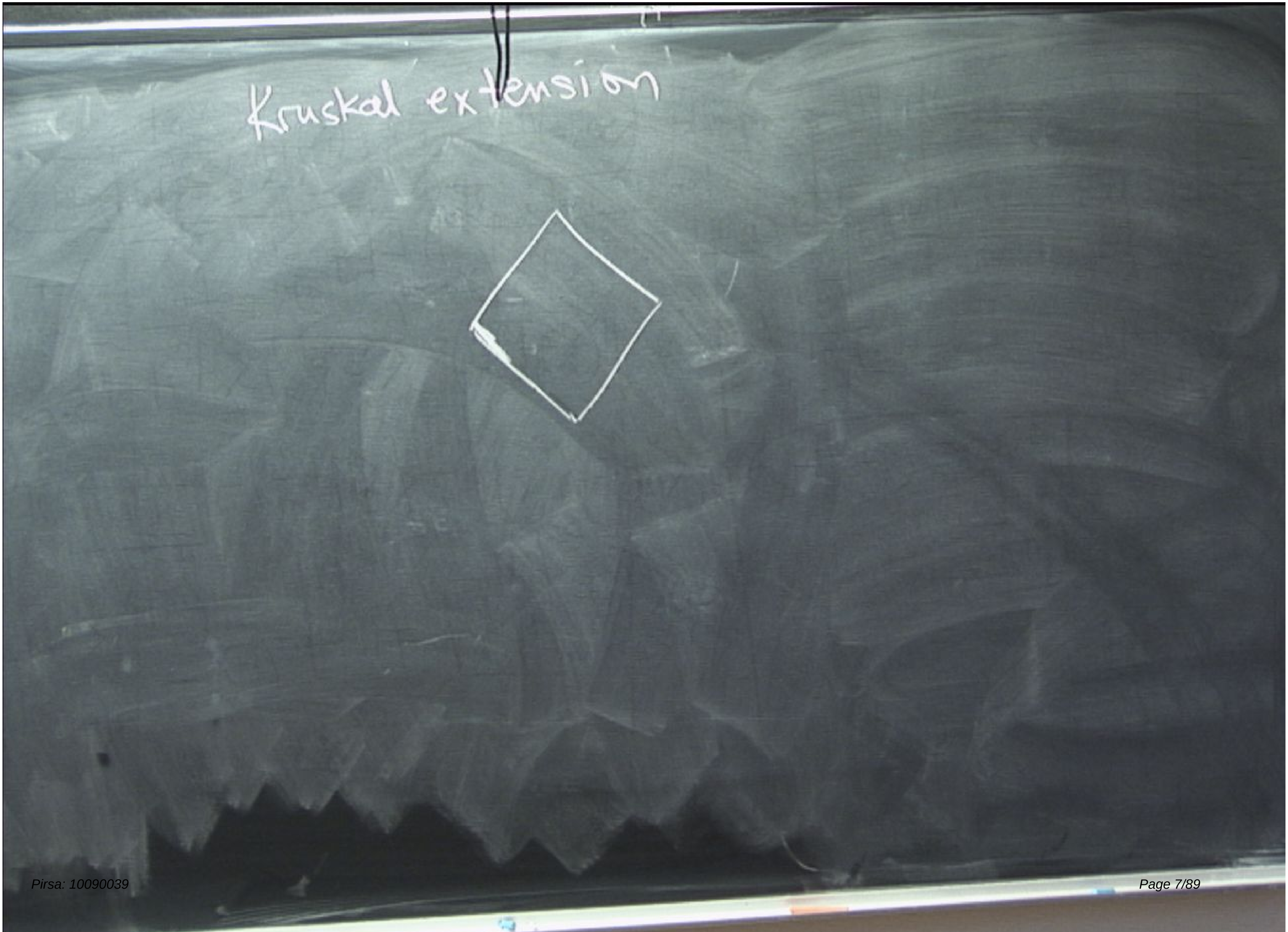
Kruskal extension

$$-\infty < t < \infty$$

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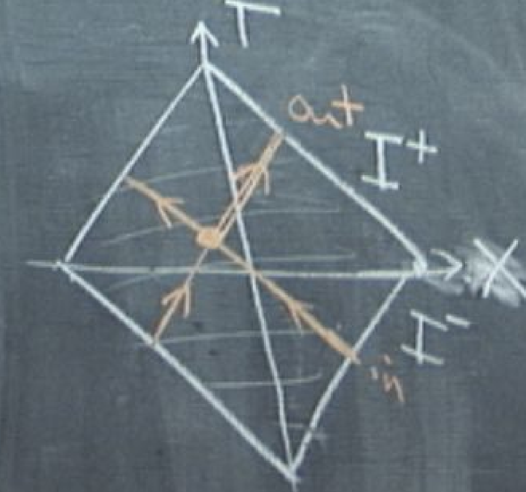
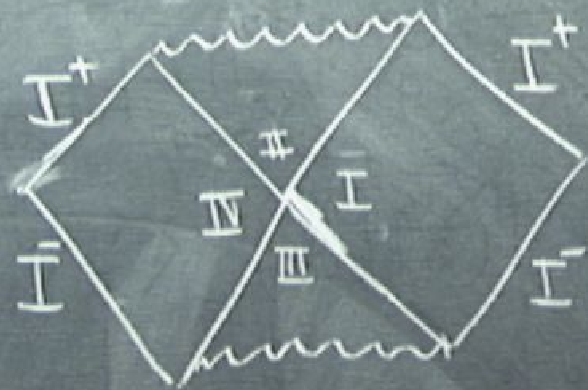
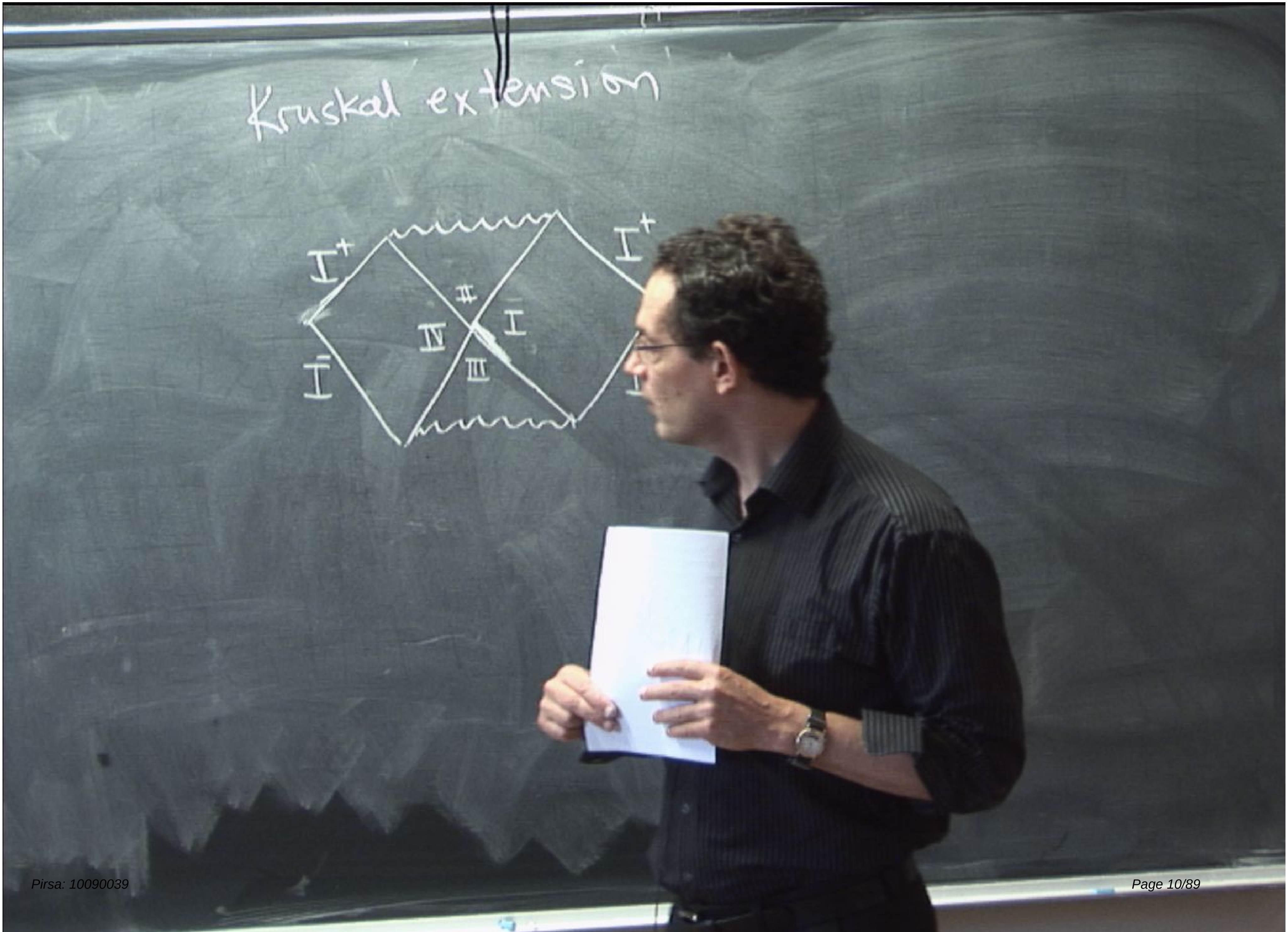
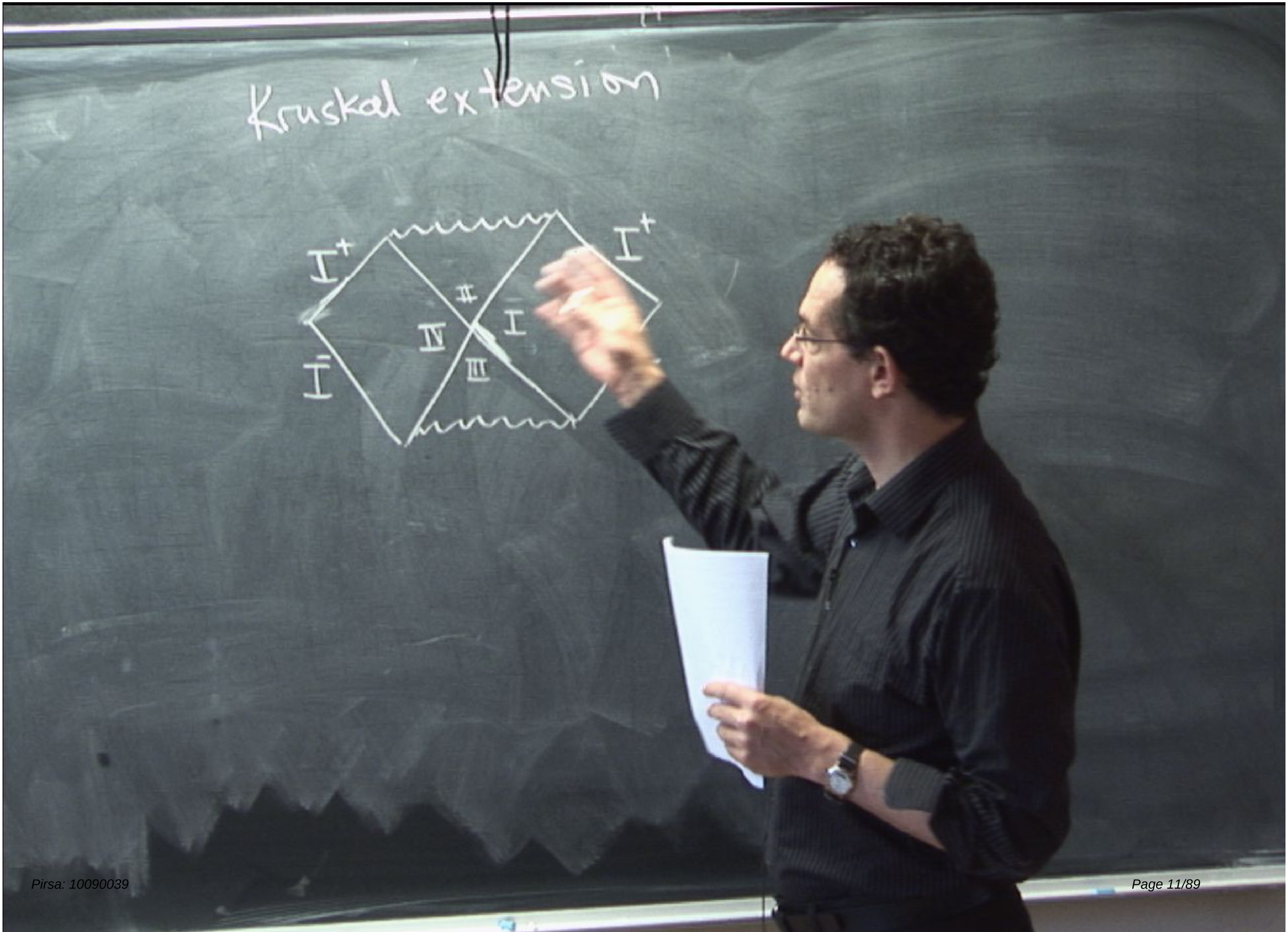


Kruskal extension




Kruskal extension
$\mathcal{I}^+$
$\mathcal{I}^+$
II
I
IV
III
$\mathcal{I}^-$


Kruskal extension
$\mathcal{I}^+$
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II
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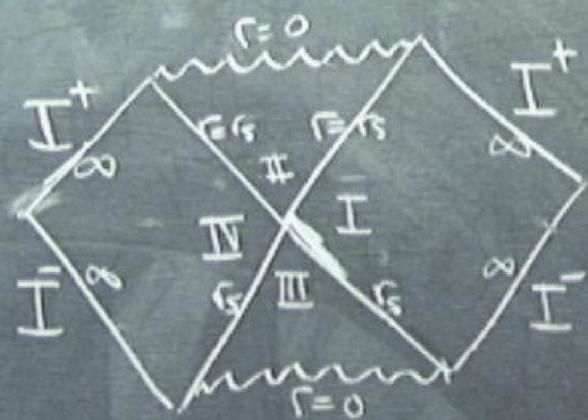
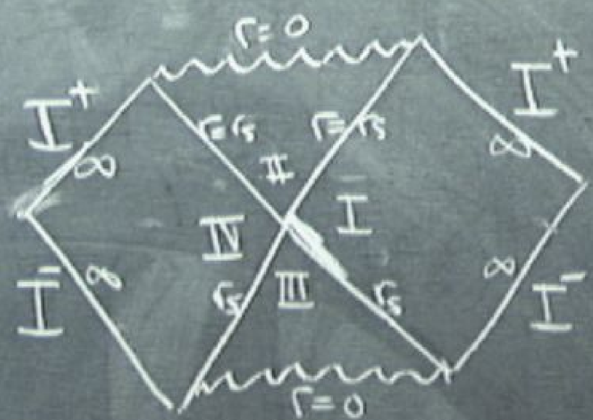
Kruskal extension



Kruskal extension



Kerr metric (1963)

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- black hole with angular momentum.

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- stationary but not static.

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$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2 c^2}\right)dt^2 - \frac{2GMar}{c^2\rho^2}\left(dt\,d\phi + d\phi\,dt\right)$$

Kerr metric (1963)

- black hole with angular momentum. $g_{\mu\nu} dx^\mu dx^\nu$
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$$+ \frac{\rho^2}{\Delta} r^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin\theta\right] d\phi^2$$

where $\Delta = r^2 - \frac{2GMr}{c^2} + a^2$

$$\rho^2 = r^2 + a^2$$

Interpretation: black hole of mass M,
angular momentum $J=$

dt

ϕ^2

Interpretation: black hole of mass M,
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(Boyer -Lindquist coordinates)

Interpretation: black hole of mass M,
angular momentum
$J = Mac$

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solves $R_{\mu\nu} = 0$.

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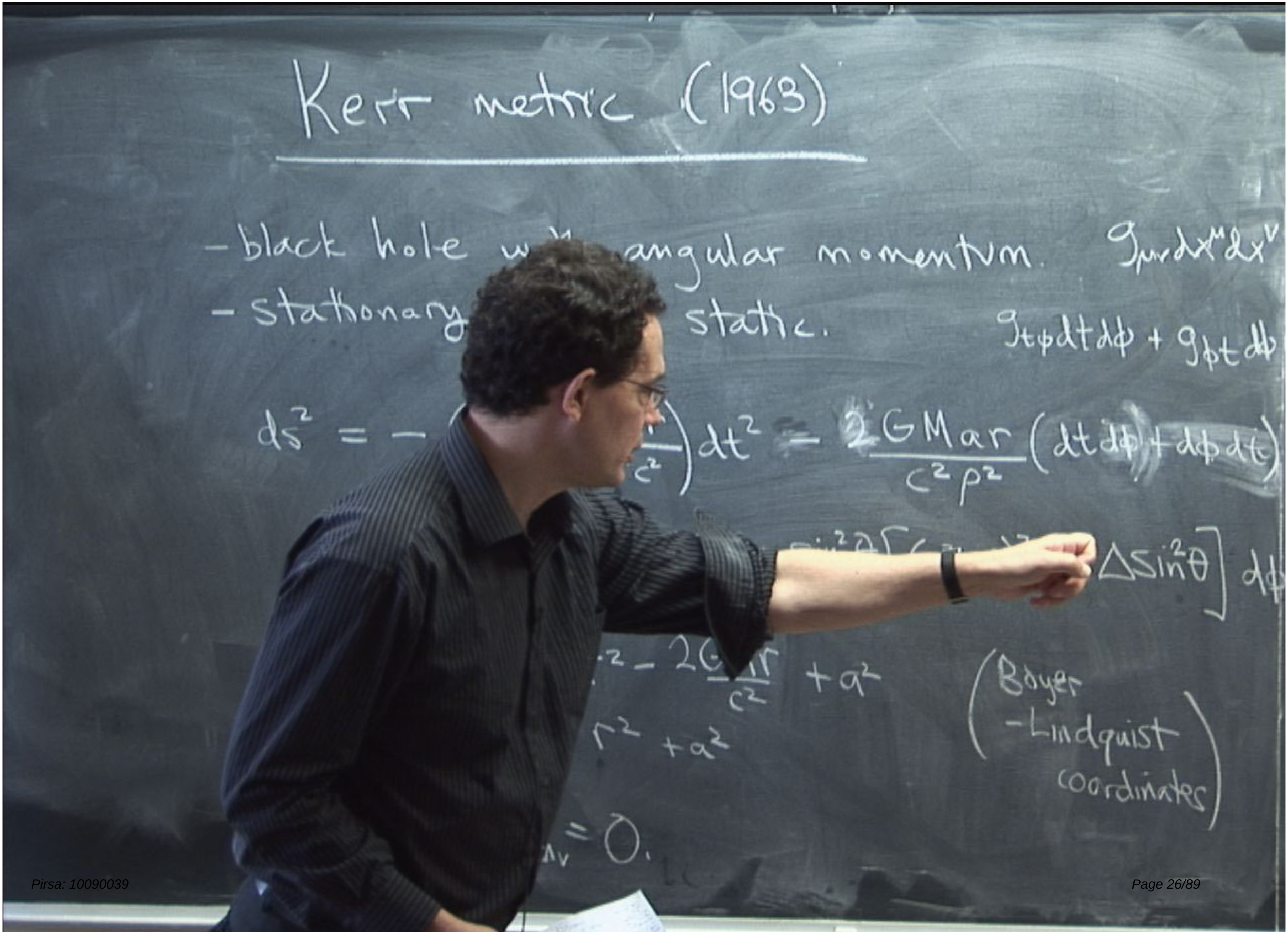
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angular momentum.
$g_{\mu\nu}dx^\mu dx^\nu$
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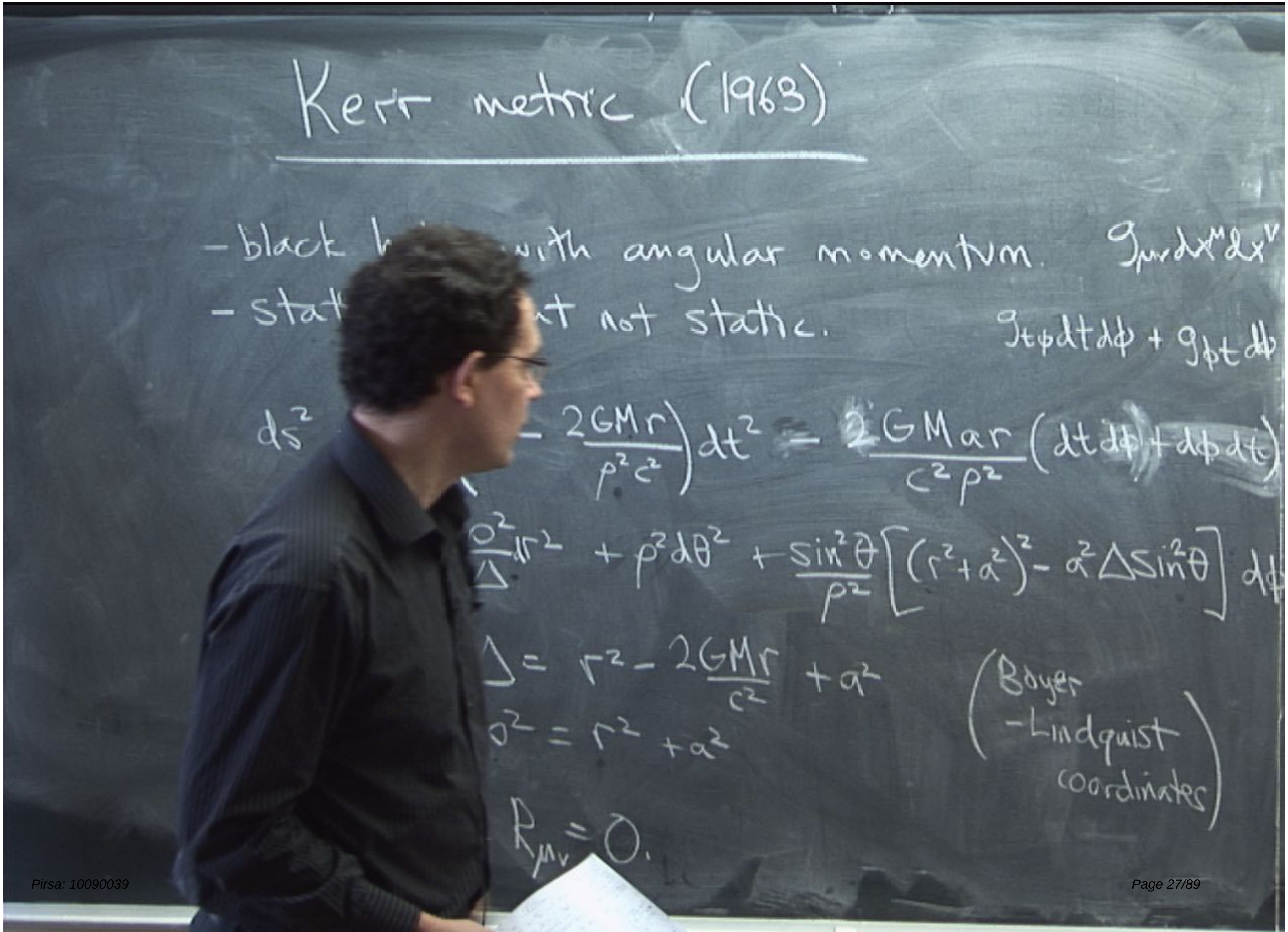
$g_{t\phi}dt\,d\phi + g_{\phi t}\,d\phi$
$ds^2 = -$
$)dt^2 - \frac{2GMar}{c^2\rho^2}(dt\,d\phi + d\phi\,dt)$
$\Delta\sin^2\theta]\,d\phi$
$+a^2$
$r^2 + a^2$
(Boyer-Lindquist coordinates)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$g_{t\phi}dt\,d\phi + g_{\phi t}d\phi$...
ds^2 ... $\frac{2GMr}{\rho^2c^2}\Big)dt^2 - \frac{2GMar}{c^2\rho^2}(dt\,d\phi + d\phi\,dt)$
... $\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi$
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interpretation: black hole of mass M,
angular momentum
$J = Mac$

Take some limits

M fixed, $a \to 0 \Rightarrow$ Schwarzschild

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$$x = (r^2+a^2)^{1/2} \sin\theta \cos\phi$$
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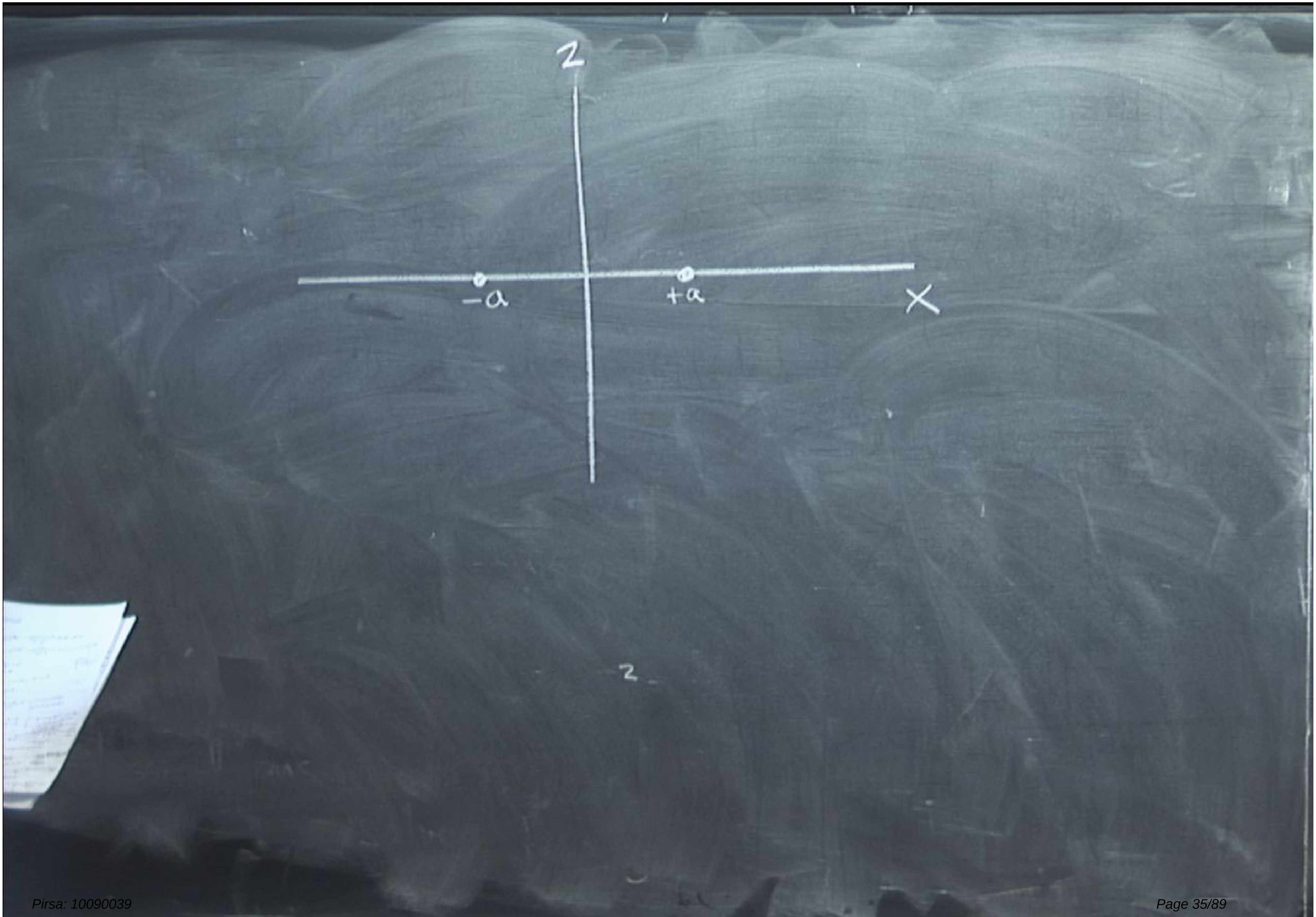
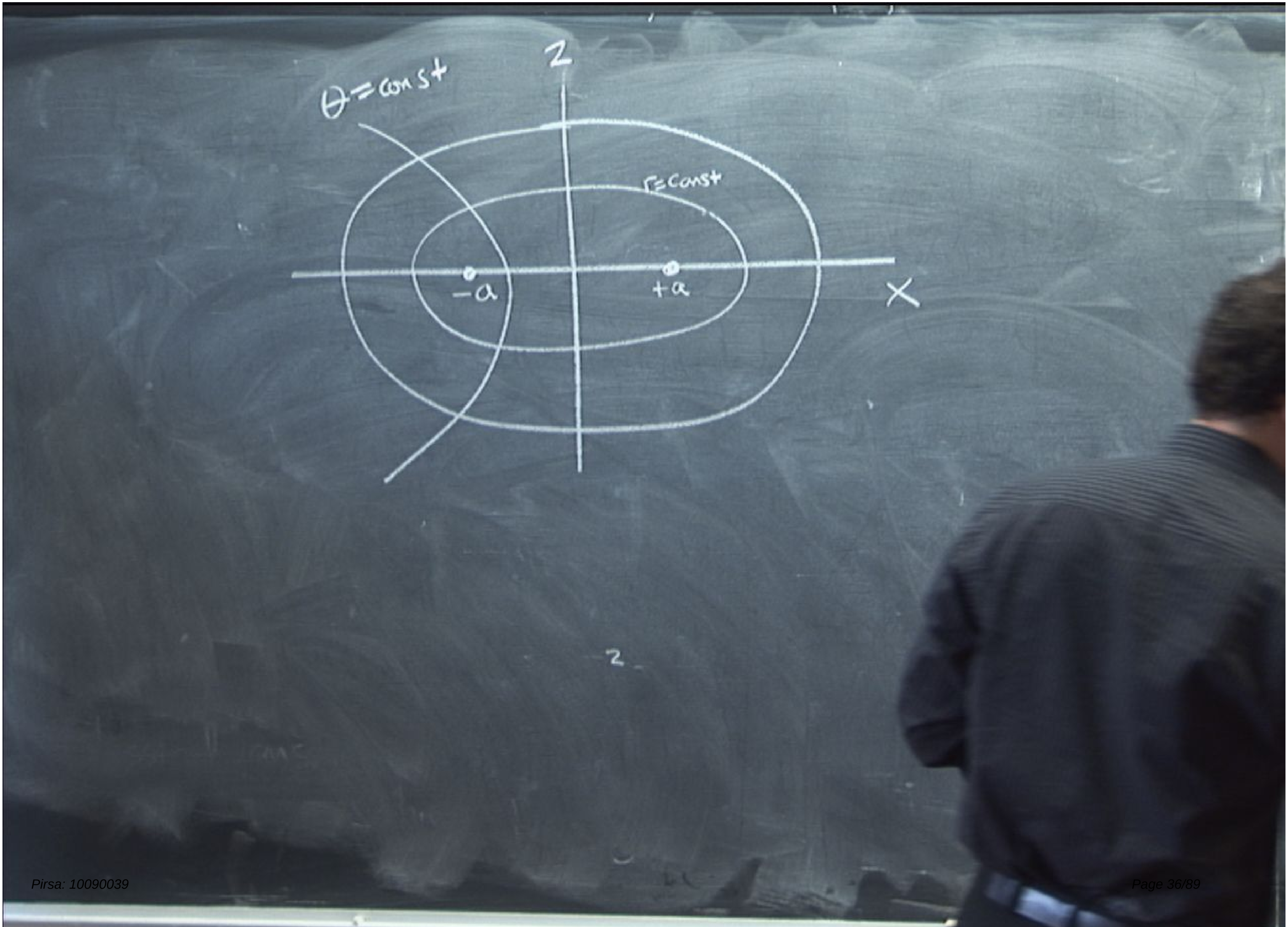
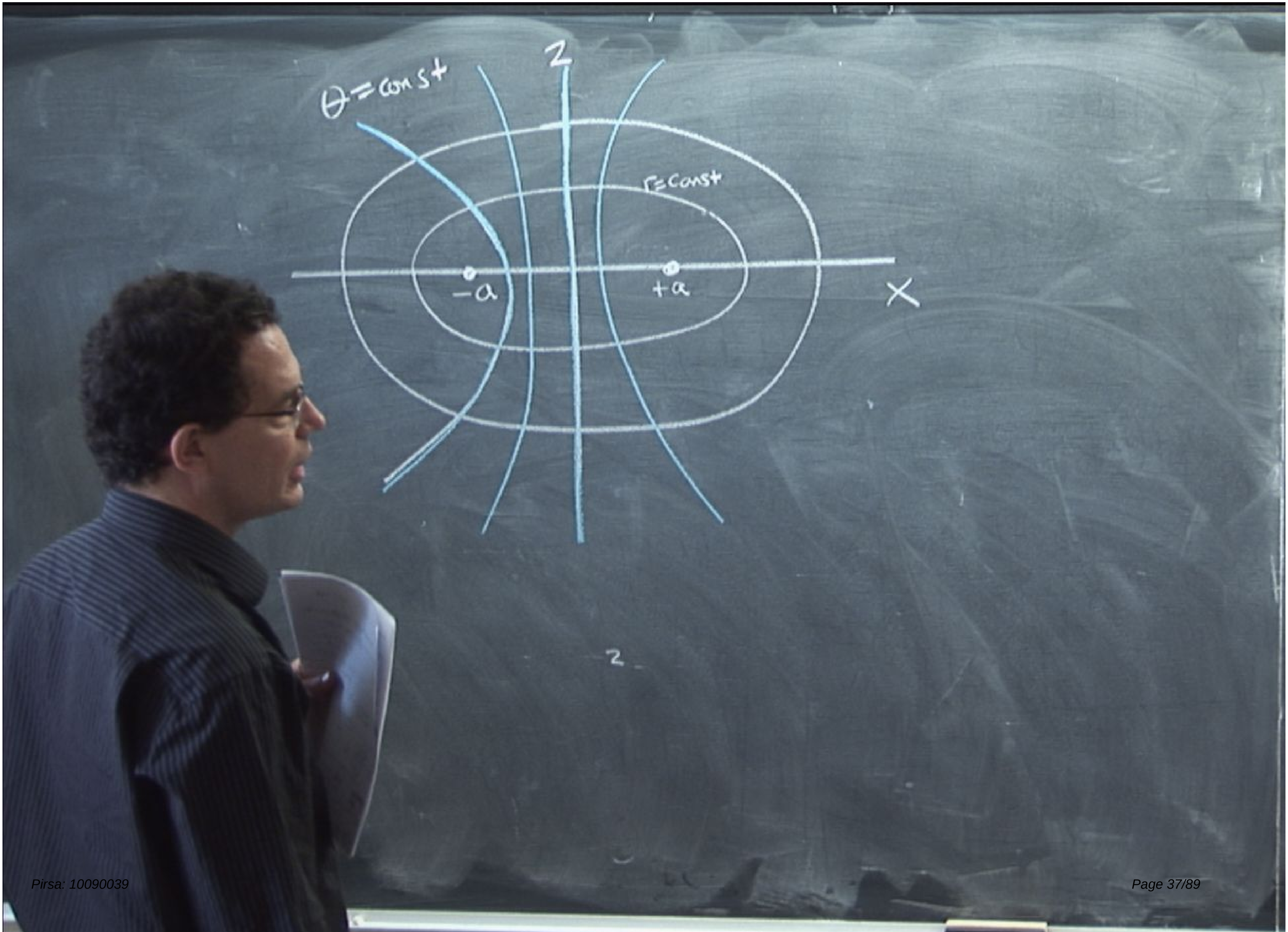
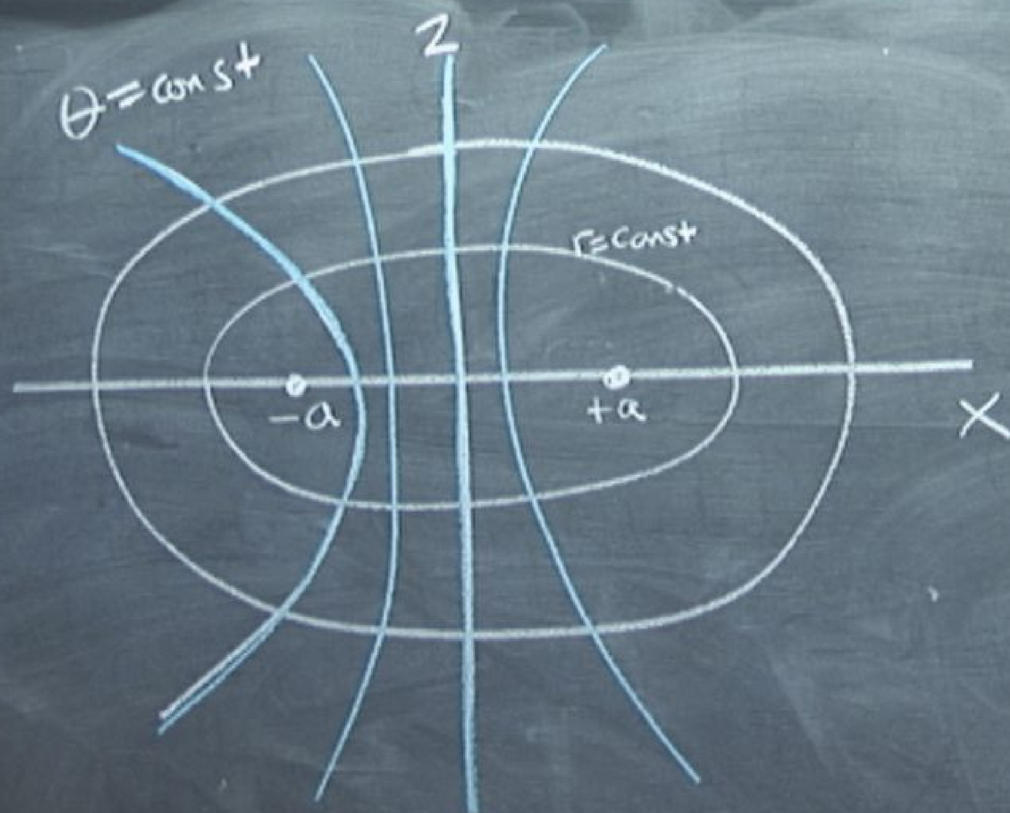
Ellipsoidal polar coordinates


z
−a
+a
x



z
θ=const
r=const
−a
+a
x


θ=const
z
r=const
$-a$
$+a$
x
z



2

Interpretation: black hole of mass M, angular momentum $J = Mac$

Take some limits

M fixed, $a \to 0$ $\Rightarrow$ Schwarzschild

a fixed, $M \to 0$

$$-dt^2 + \frac{(r^2+a^2\cos^2\theta)}{(r^2+a^2)}\,dr^2 + (r^2+a^2\cos^2\theta)^2\,d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2$$

– Minkowski in coordinates

$$\left.\begin{aligned} x &= (r^2+a^2)^{1/2}\sin\theta\cos\phi \\ y &= (r^2+a^2)^{1/2}\sin\theta\sin\phi \\ z &= (r^2+a^2)\cos\theta \end{aligned}\right)$$

Ellipsoidal polar coord

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$J = Mac$

Take some limits

M fixed, $a \to 0 \implies$ Schwarzschild

a fixed, $M \to 0$

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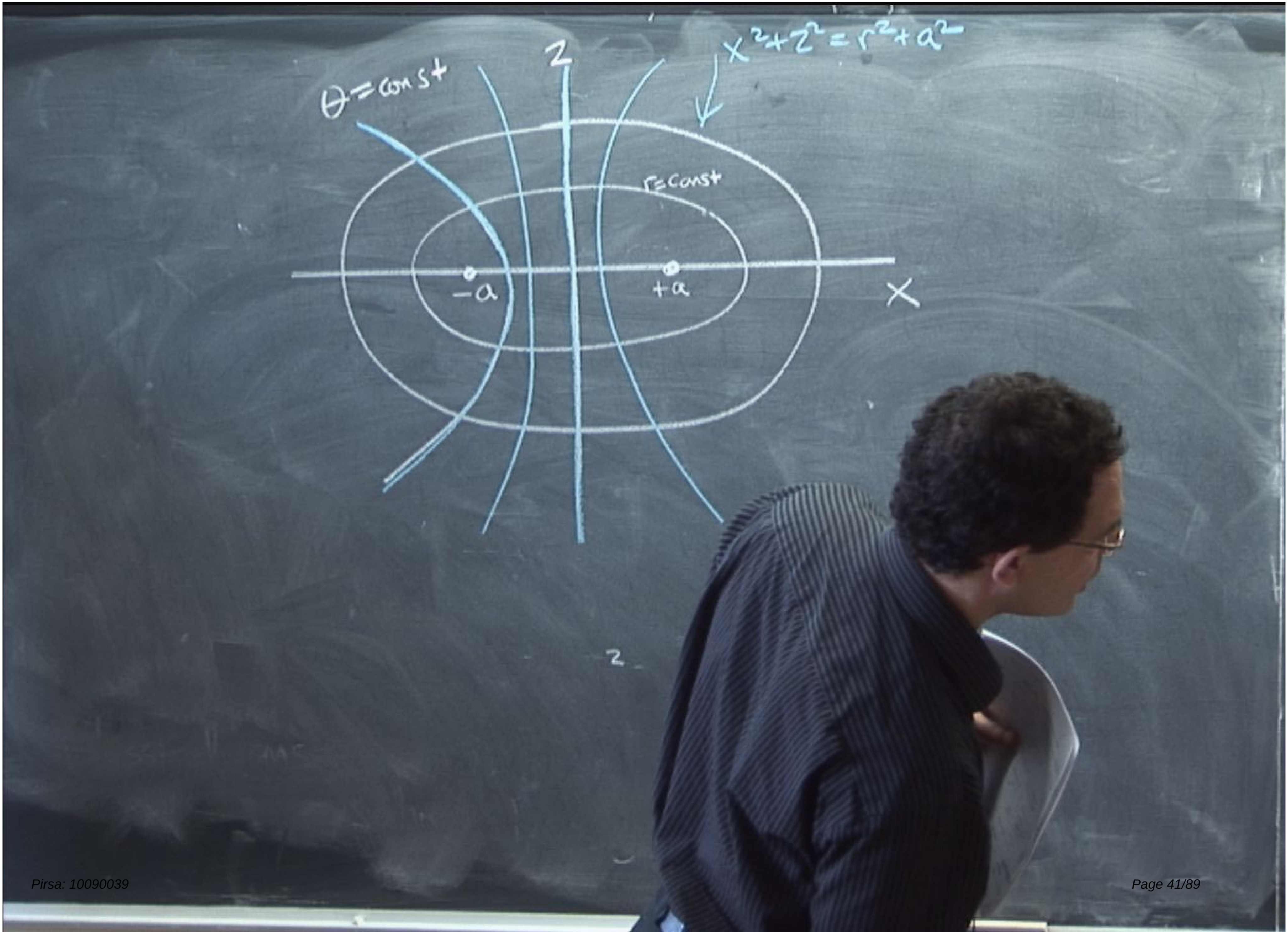

$x^2+z^2=r^2+a^2$
z
θ=const
r=const
$-a$
$+a$
x

Interpretation: black hole of mass M, angular momentum $J = Mac$

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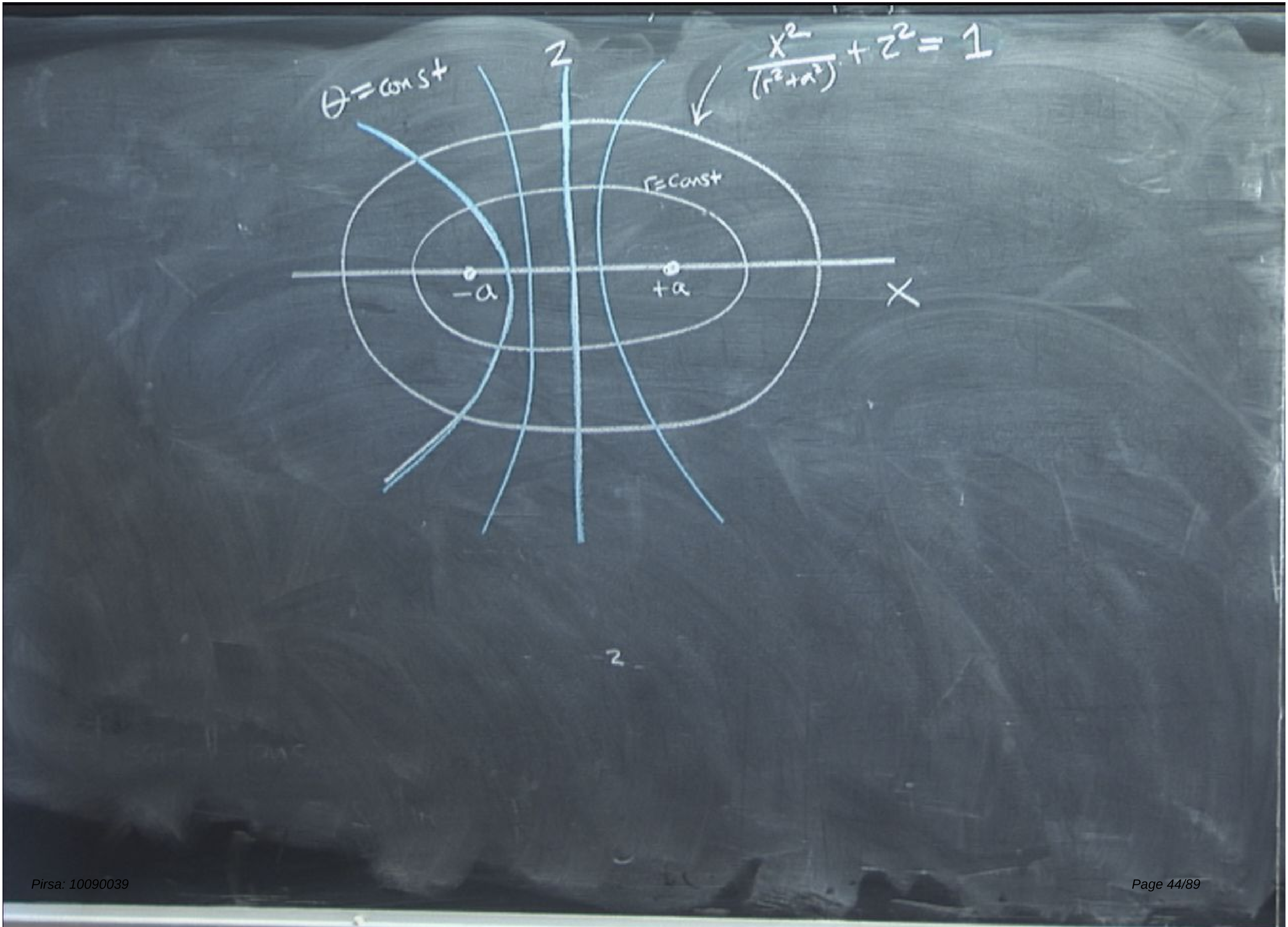
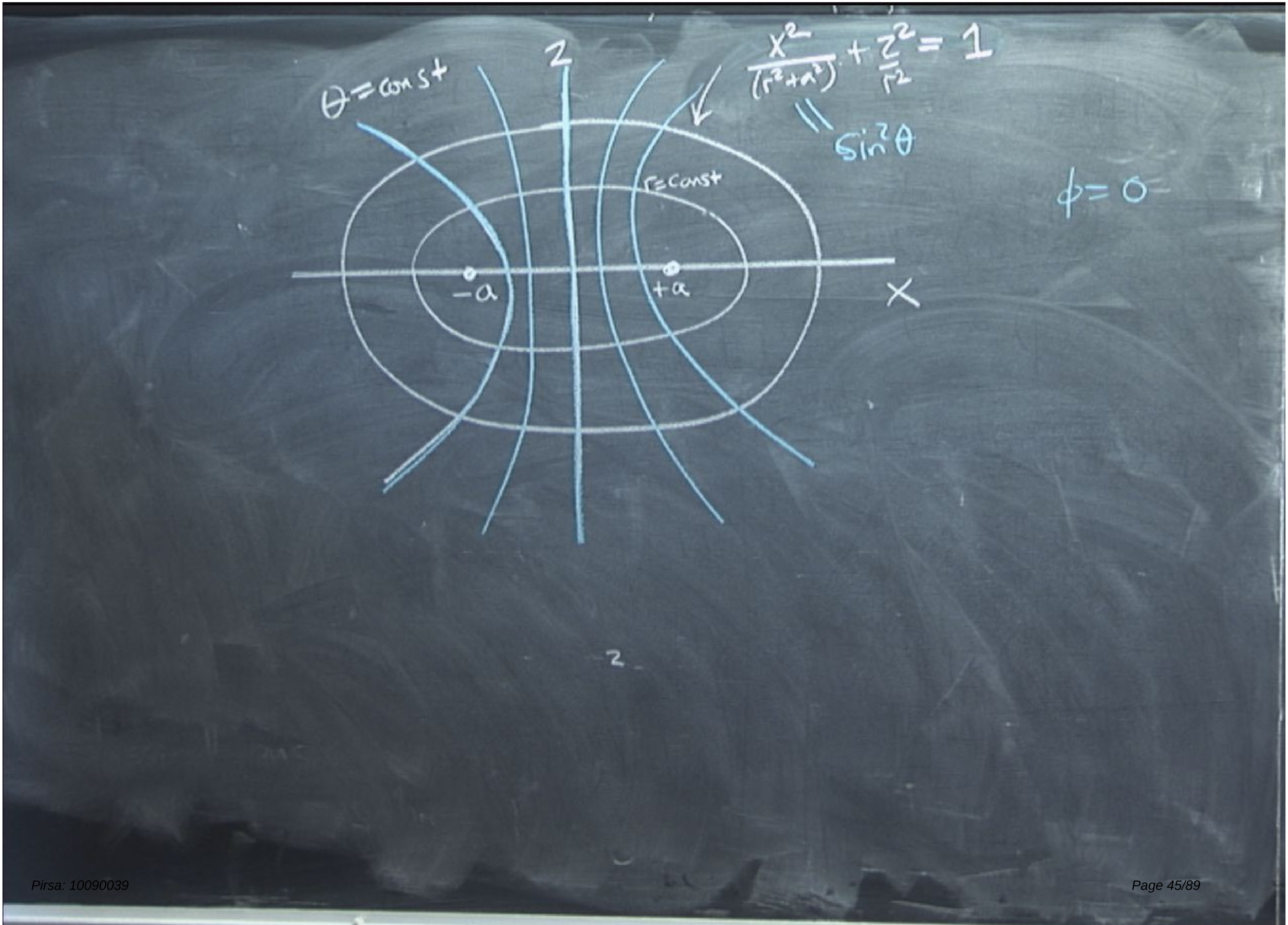
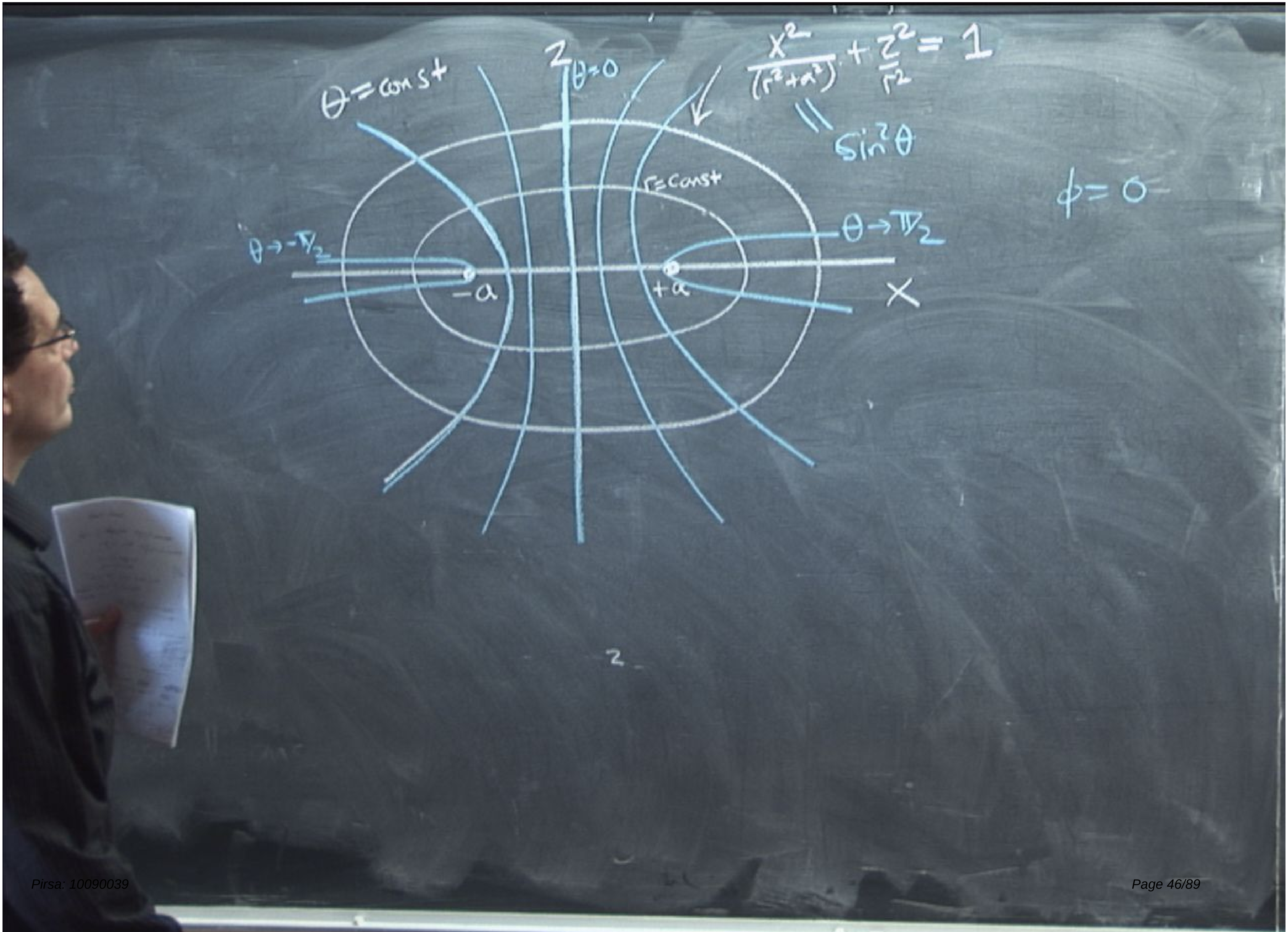
Ellipsoidal polar coordinates



θ = const
z
$\frac{x^2}{(r^2+a^2)} + z^2 = 1$
r = const
−a
+a
x

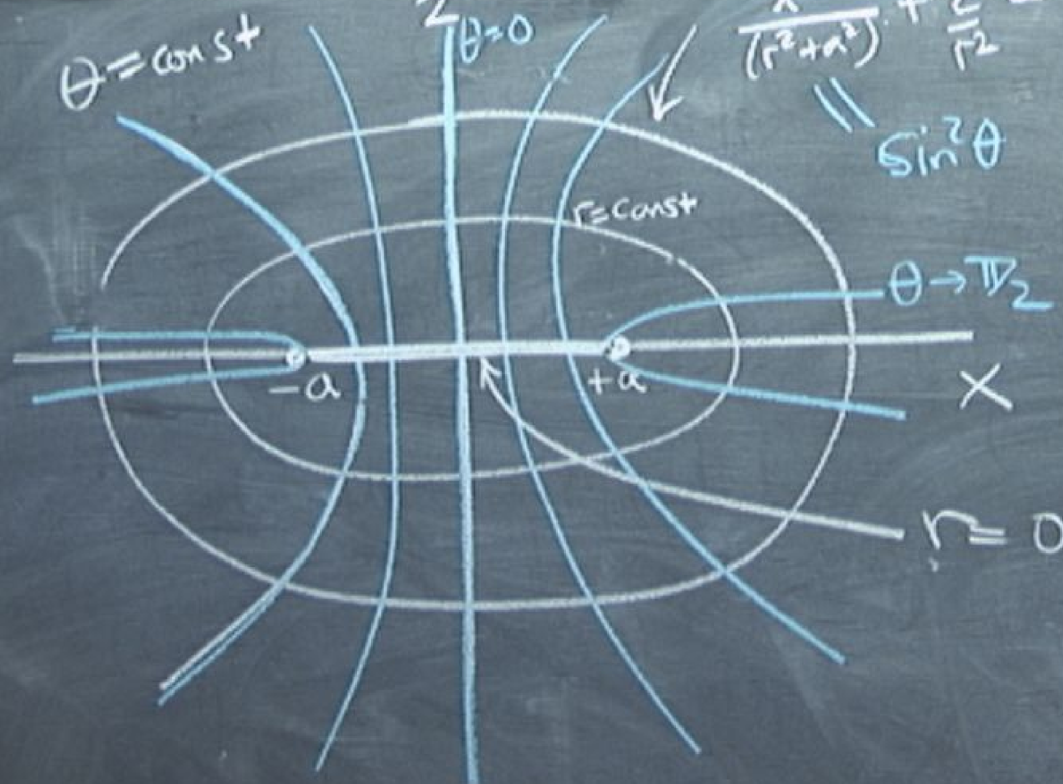

$\frac{x^2}{(r^2+a^2)} + \frac{z^2}{r^2} = 1$
$\sin^2\theta$
$\theta = \text{const}$
z
$r = \text{const}$
$-a$
$+a$
x
$\phi = 0$
2


$\frac{x^2}{(r^2+a^2)} + \frac{z^2}{r^2} = 1$
$\sin^2\theta$
$\theta = \text{const}$
z
$\theta = 0$
$r = \text{const}$
$\phi = 0$
$\theta \to \pi/2$
$\theta \to -\pi/2$
$-a$
$+a$
x

$$\frac{x^2}{(r^2+a^2)} + \frac{z^2}{r^2} = 1$$

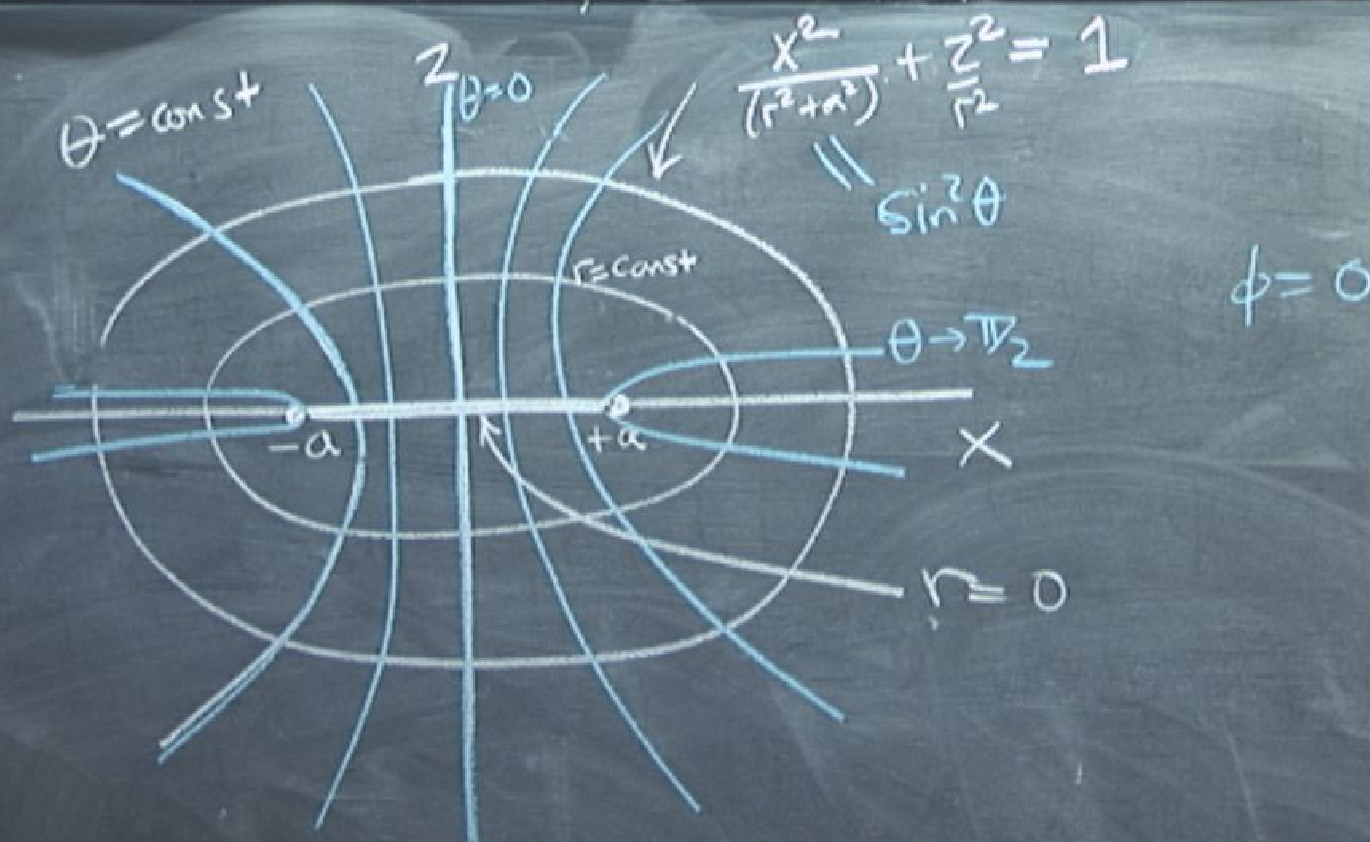
$$\frac{x^2}{(r^2+a^2)} = \sin^2\theta$$

$\phi = 0$



$$-\left(1-\frac{2GMr}{\rho^2c^2}\right)dt^2 - 2\frac{GMar\sin^2\theta}{c^2\rho^2}\left(dt\,d\phi+d\phi\,dt\right)$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$



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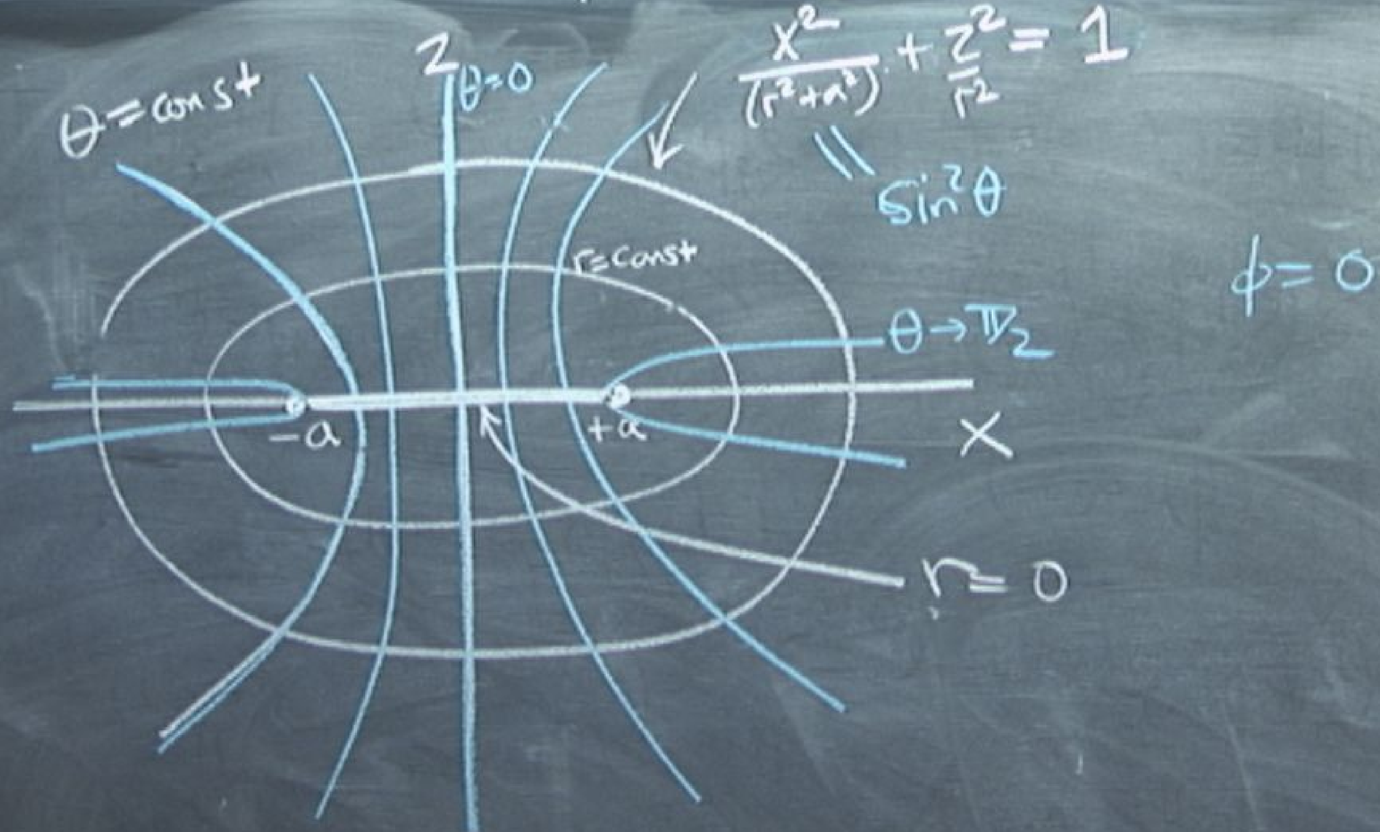
$$\Delta = r^2 - \frac{2GMr}{c^2} + a^2 \;; \quad \rho^2 = r^2 + a^2\cos^2\theta$$



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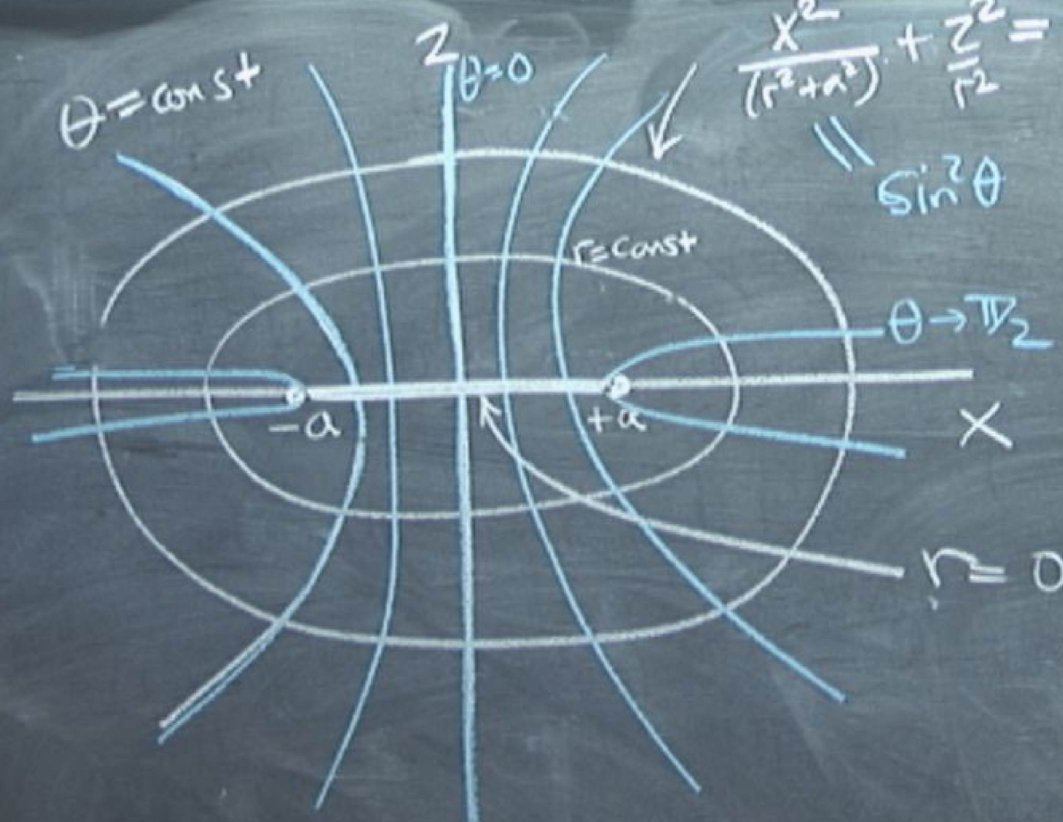

$$\frac{x^2}{(r^2+a^2)} + \frac{z^2}{r^2} = 1$$

$$\frac{x^2}{(r^2+a^2)} = \sin^2\theta$$

$\phi = 0$

$$-\left(1-\frac{2GMr}{\rho^2c^2}\right)dt^2 - 2\frac{GMar\sin^2\theta}{c^2\rho^2}\left(dt\,d\phi+d\phi\,dt\right)$$

$$+\frac{\rho^2}{\Delta}dr^2+\rho^2d\theta^2+\frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2-a^2\Delta\sin^2\theta\right]d\phi^2$$

$$\Delta = r^2-\frac{2GMr}{c^2}+a^2\;;\quad \rho^2 = r^2+a^2\cos^2\theta$$

Event horizon: boundary of a region from which light cannot escape to infinity

- null surface

$r = $ const surface

Event horizon: boundary of a region from which light cannot escape to infinity

– null surface

$r = \text{const}$ surface

Event horizon: boundary of a region from which light cannot escape to infinity

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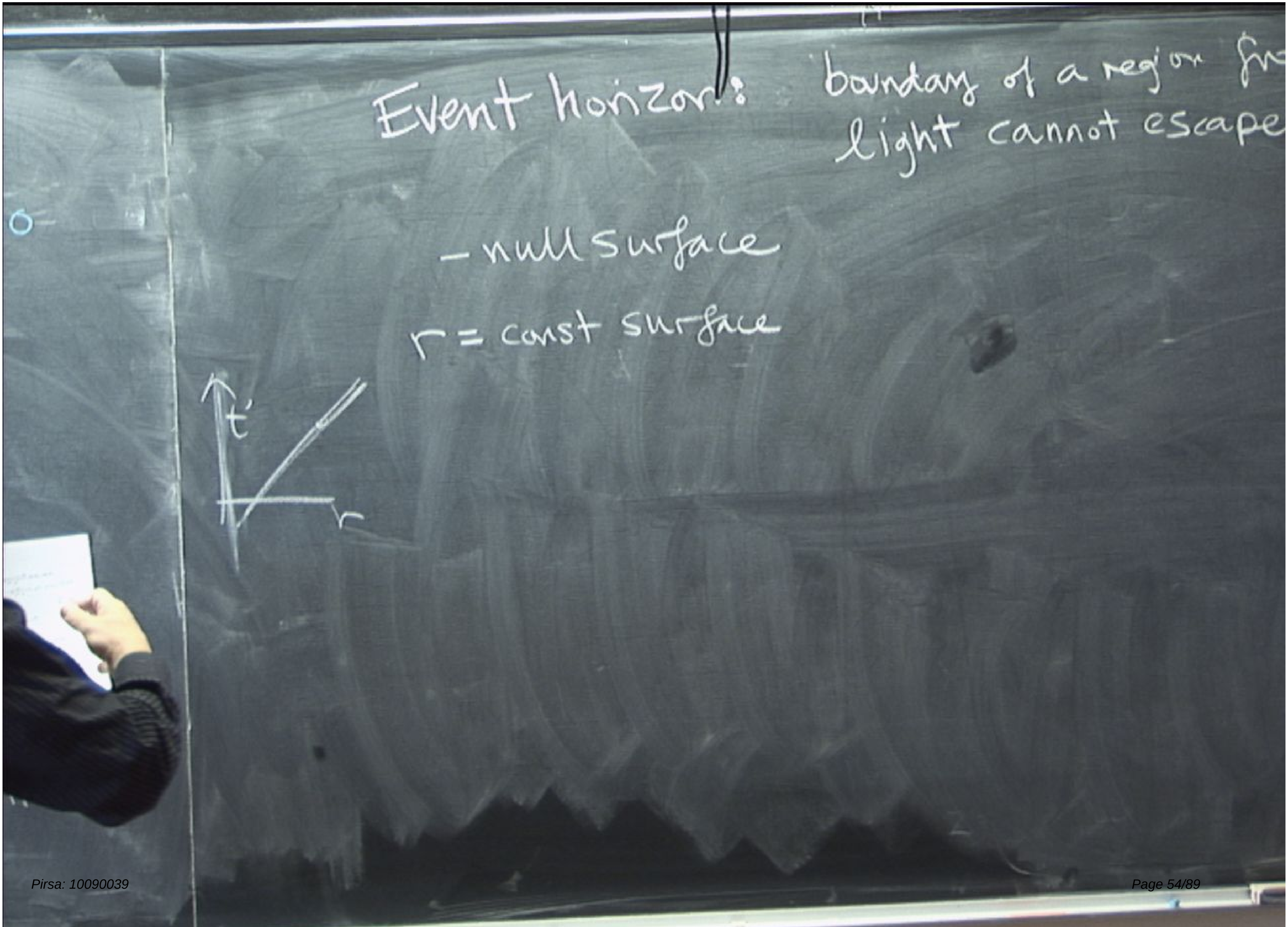
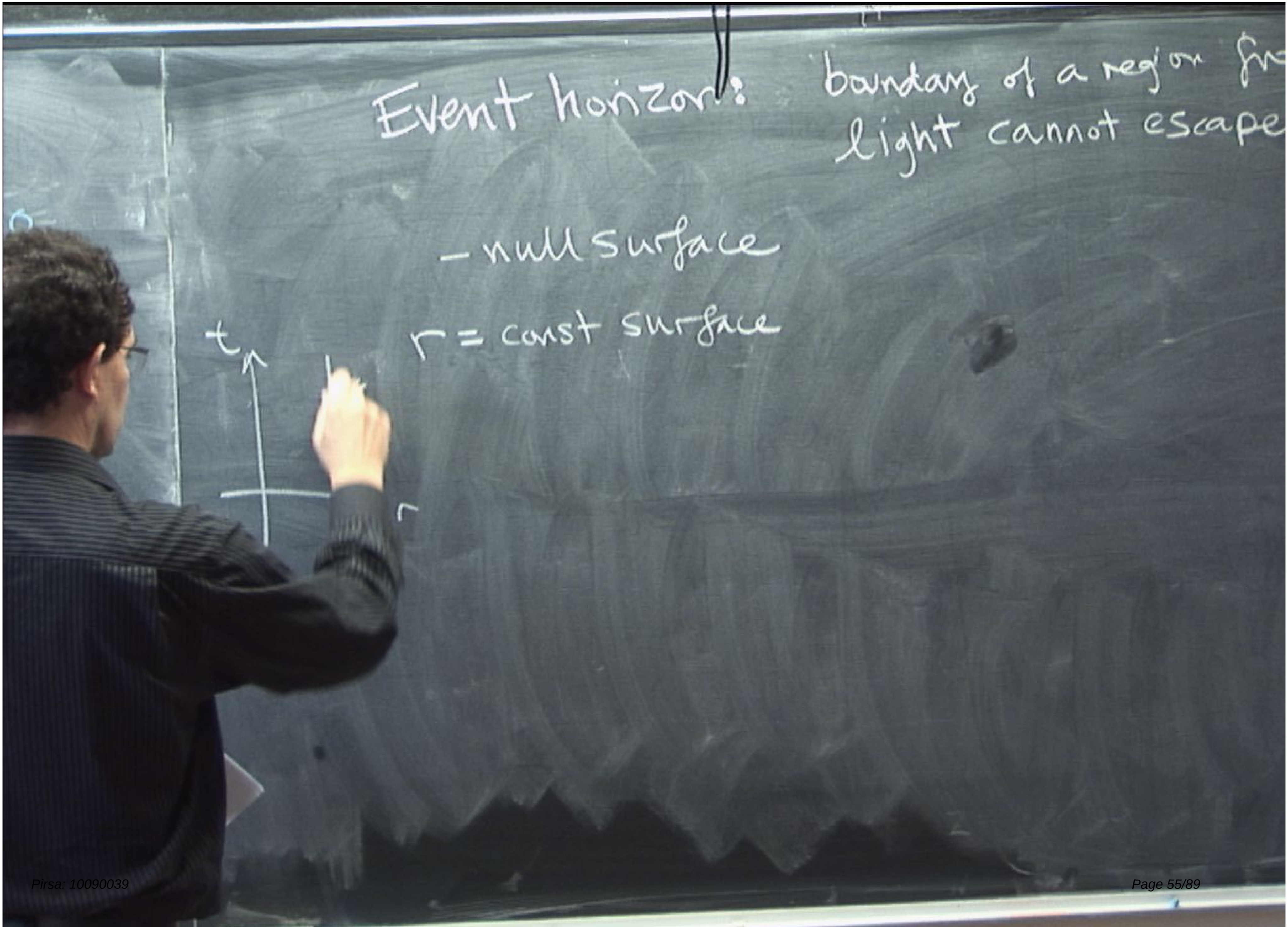
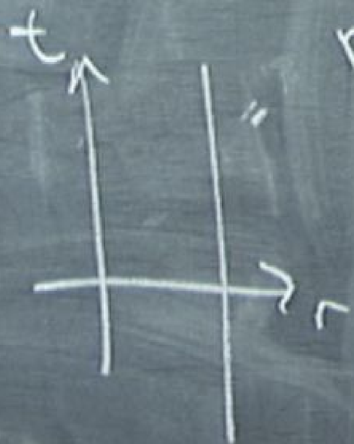

Event horizon: boundary of a region from
light cannot escape
- null surface
r = const surface
t'
r


Event horizon:
boundary of a region
light cannot escape
– null surface
r = const surface
t
r

Event horizon: boundary of a region fr[om which] light cannot escape

- null surface

$r =$ const surface



Event horizon: boundary of a region from which light cannot escape

- null surface

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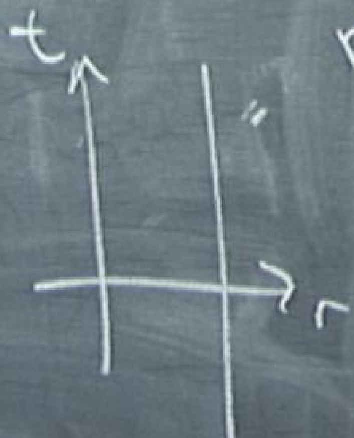

Event horizon: boundary of a region fr
light cannot escape

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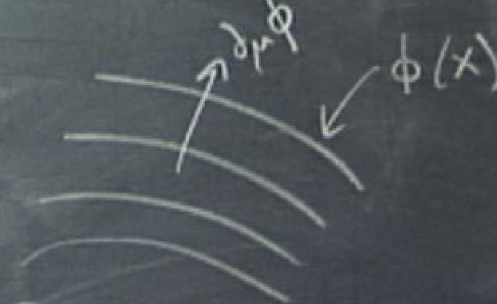
$\partial_\mu r =$ normal vector to $r = \text{constan}$

Event horizon: boundary of a region from which light cannot escape to infinity



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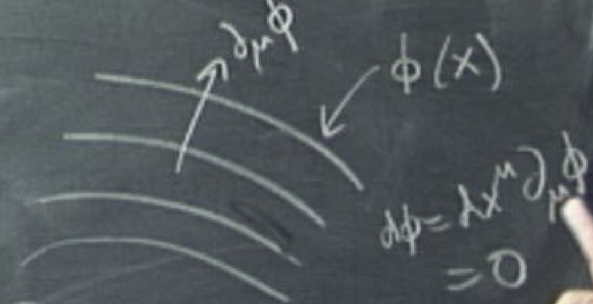
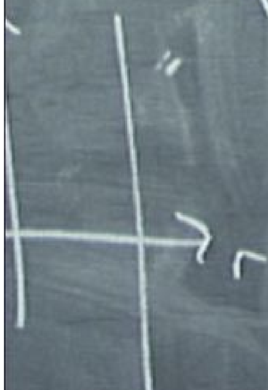


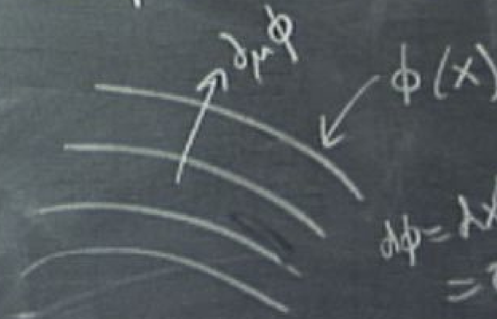
- null surface

$r = \text{const surface} = \text{null}$?



$\partial_\mu r$ = normal vector to $r = \text{constant}$

Event horizon: boundary of a region from which light cannot escape to infinity



$d\phi = dx^\mu \partial_\mu \phi = 0$

- null surface

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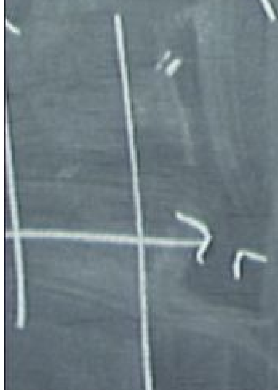

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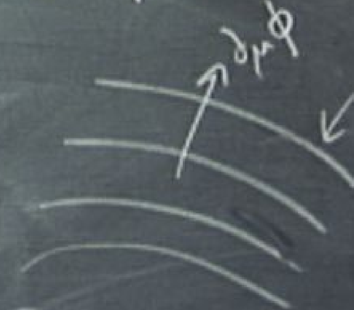

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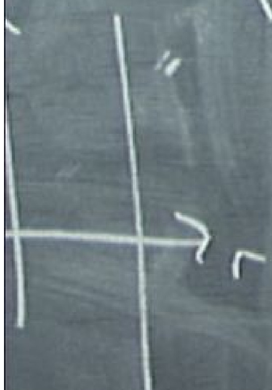

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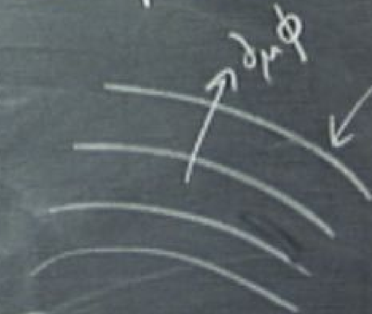

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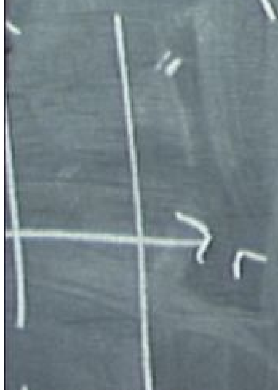
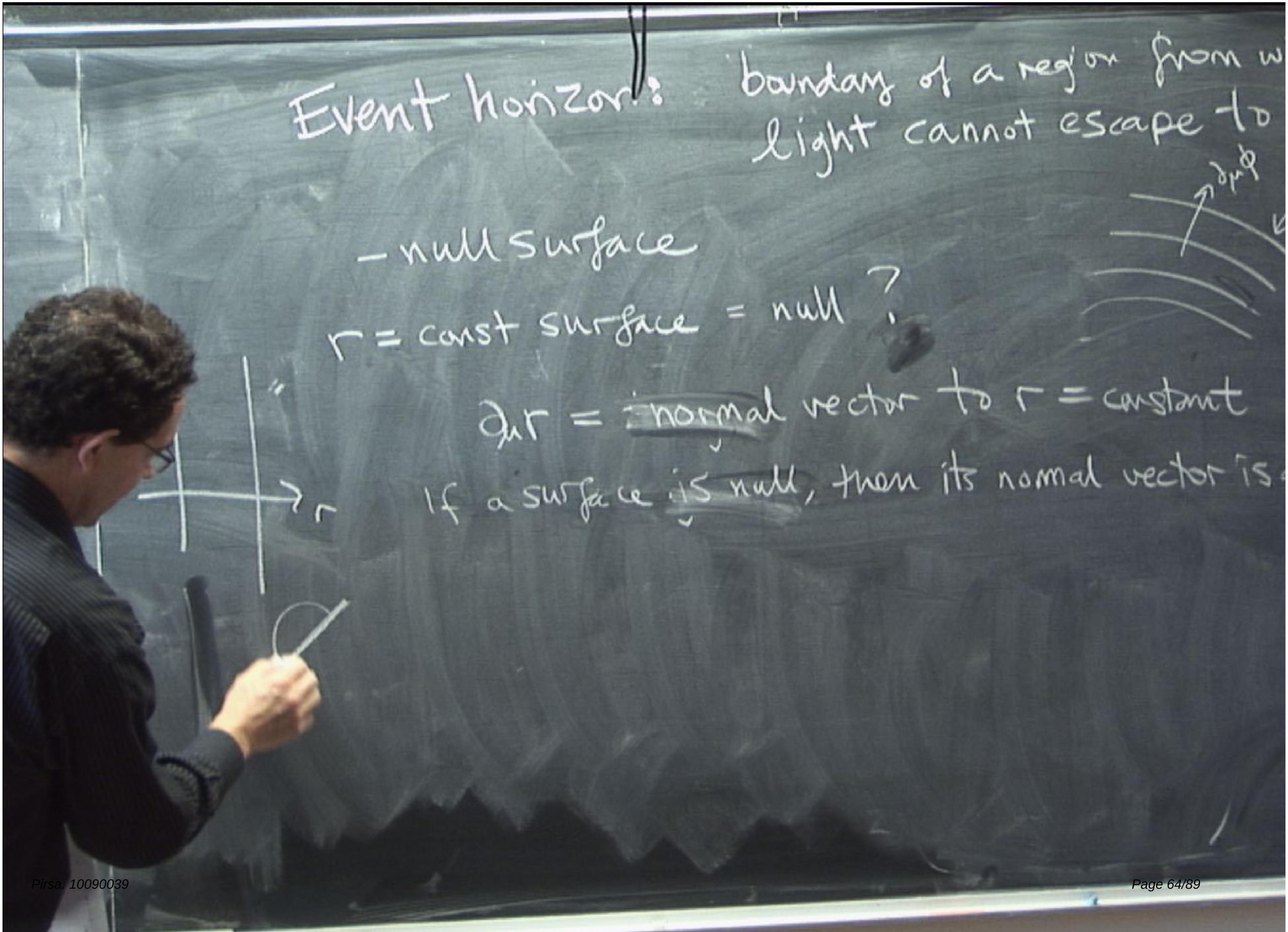
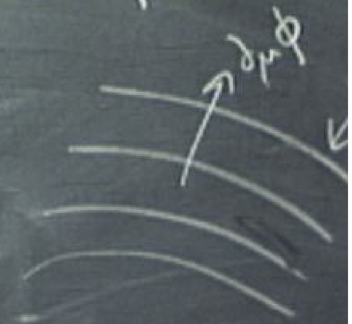


$\partial_\mu r =$ normal vector to $r =$ constant

If a surface is null, then its normal vector is also null

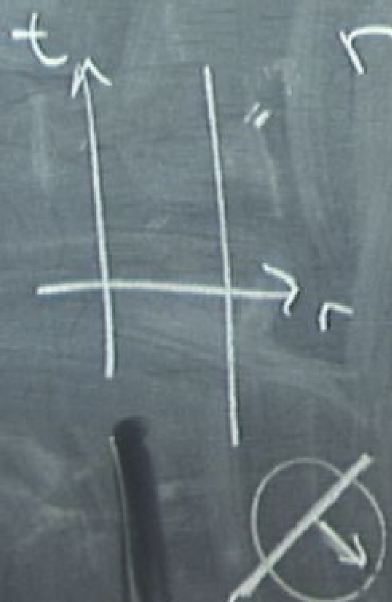

Event horizon: boundary of a region from w
light cannot escape to
$\partial_\mu \phi$
- null surface
r = const surface = null ?
$\partial_\mu r$ = normal vector to r = constant
If a surface is null, then its normal vector is
r

Event horizon: boundary of a region from w
light cannot escape to



– null surface

r = const surface = null ?



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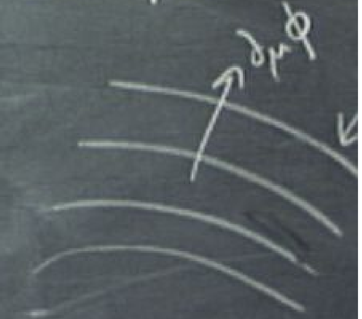
If a surface is null, then its normal vector is

Event horizon: boundary of a region from w
light cannot escape to



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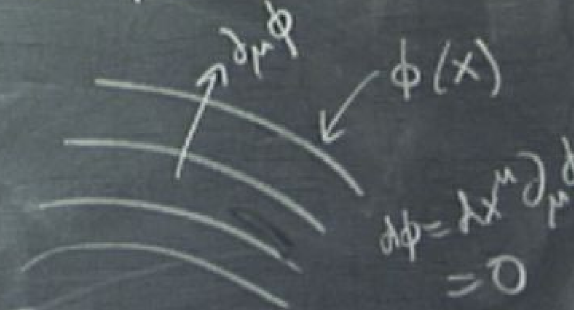

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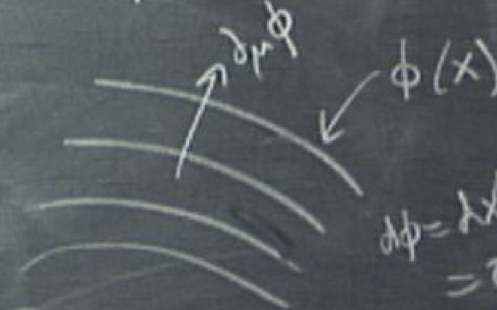
If a surface is null, then its normal vector is also null

$$g^{\mu\nu} \partial_\mu r \, \partial_\nu r = 0$$

Event horizon: boundary of a region from which light cannot escape to infinity



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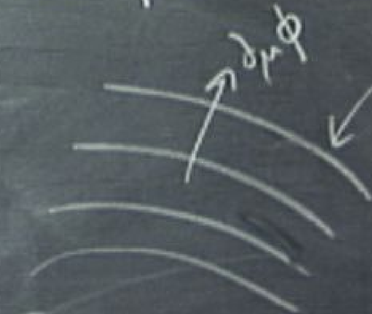

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$$g^{\mu\nu}\partial_\mu r\, \partial_\nu r = 0$$

$\partial_\mu r = (0,1,0,0)$ $\qquad = g^{rr}$

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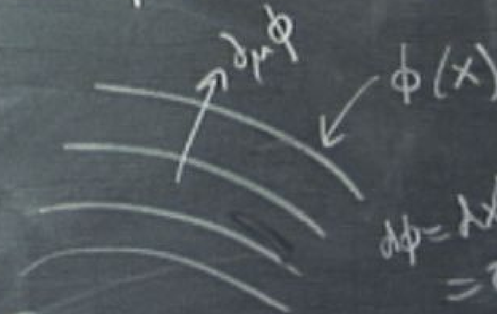

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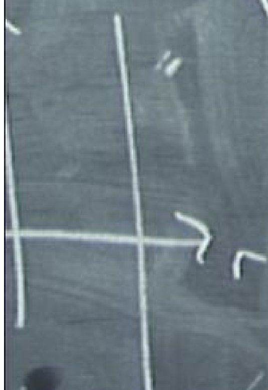


$\partial_\mu r =$ normal vector to $r =$ constant

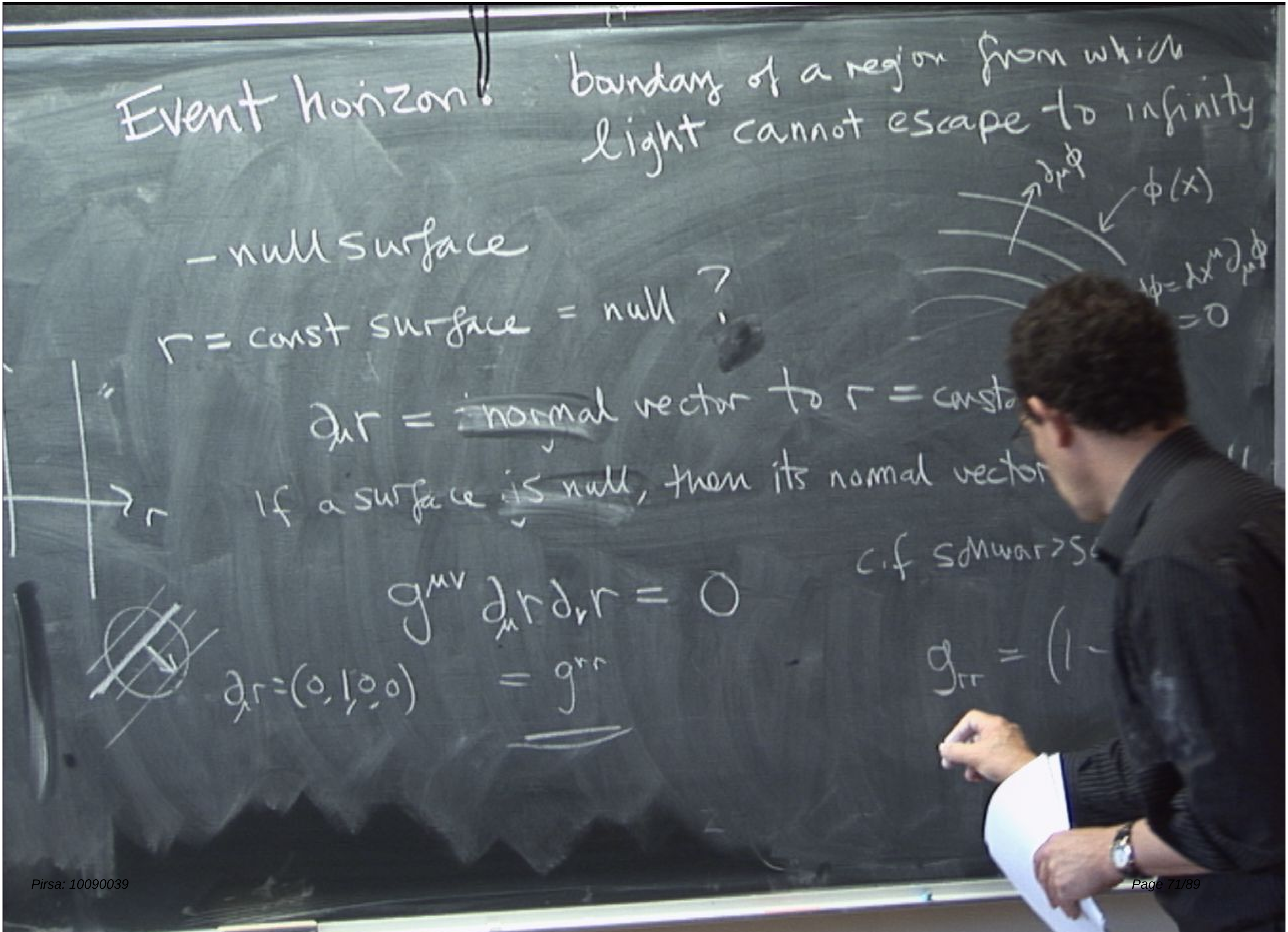
If a surface is null, then its normal vector is also null

c.f. Schwarzschild

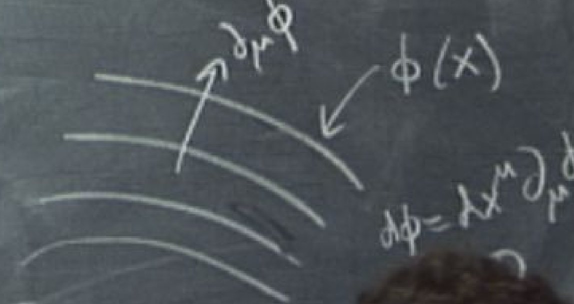
$$g^{\mu\nu} \partial_\mu r \partial_\nu r = 0$$

$\partial_\lambda r = (0, 1, 0, 0)$ $\quad = \underline{\underline{g^{rr}}}$

$g_{rr} = (\quad\quad)^{-1}$


Event horizon: boundary of a region from which light cannot escape to infinity
- null surface
r = const surface = null ?
∂_μ r = normal vector to r = const
If a surface is null, then its normal vector
g^μν ∂_μ r ∂_ν r = 0
∂_μ r = (0, 1, 0, 0)
= g^rr
c.f. schwarzs
g_rr = (1 -
∂_μ φ
φ (x)

Event horizon: boundary of a region from which light cannot escape to infinity



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$g^{rr} =$

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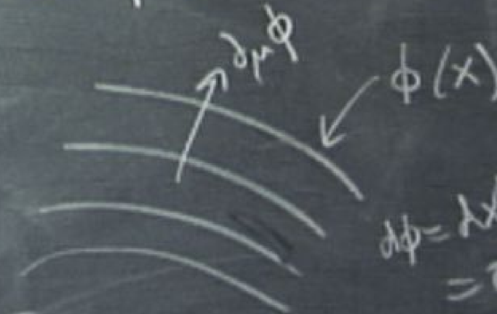

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c.f. schwarzschild

$$g_{rr} = \left(1 - 2\frac{GM}{rc^2}\right)^{-1}$$

$$g^{rr} = \left(1 - \frac{2GM}{rc^2}\right).$$

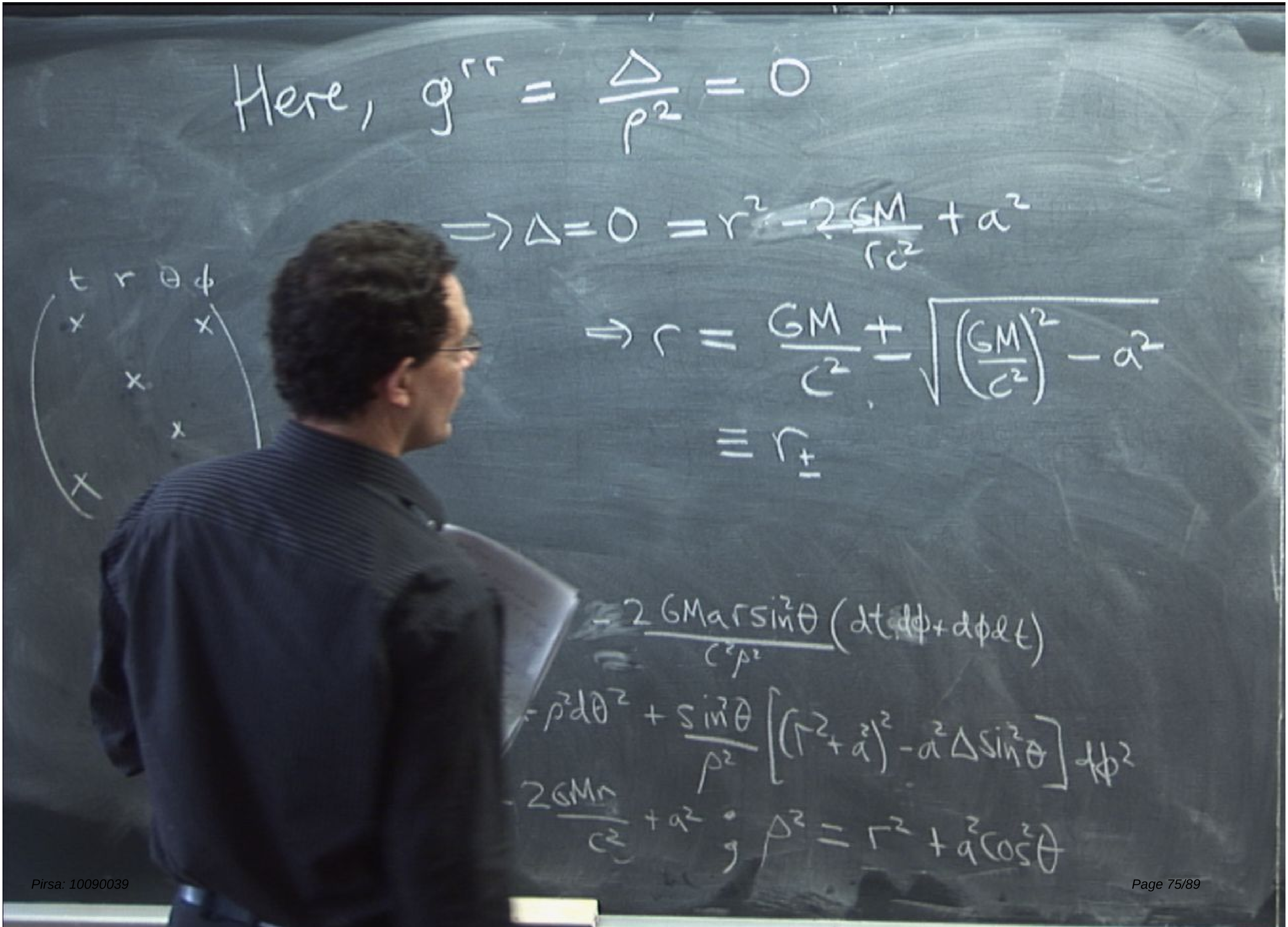
Here, $g^{rr} =$

$$
\begin{array}{c}
\begin{matrix} t & r & \theta & \phi \end{matrix} \\
\begin{pmatrix} x & & & x \\ & x & & \\ & & x & \\ x & & & x \end{pmatrix}
\end{array}
$$

$$
-\left(1-\frac{2GMr}{\rho^2c^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{c^2\rho^2}\left(dt\,d\phi + d\phi\,dt\right)
$$

$$
+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2
$$

$$
\Delta = r^2 - \frac{2GMr}{c^2} + a^2 \; ; \; \rho^2 = r^2 + a^2\cos^2\theta
$$



Here, $g^{rr} = \frac{\Delta}{\rho^2} = 0$
$\Rightarrow \Delta = 0 = r^2 - 2\frac{GM}{rc^2} + a^2$
t r θ φ
$\Rightarrow r = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}$
$\equiv r_\pm$
$-2\frac{GMar\sin^2\theta}{c^2\rho^2}(dt\,d\phi + d\phi\,dt)$
$\rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$
$-\frac{2GMr}{c^2} + a^2$; $\rho^2 = r^2 + a^2\cos^2\theta$

$$\begin{array}{cccc} t & r & \theta & \phi \end{array}$$
$$\begin{pmatrix} \times & & & \times \\ & \times & & \\ & & \times & \\ \times & & & \times \end{pmatrix}$$

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$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

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$$\begin{pmatrix} \times & & & \times \\ & \times & & \\ & & \times & \\ \times & & & \times \end{pmatrix}$$

$$-\left(1 - \frac{2GMr}{\rho^2 c^2}\right)dt^2 - 2\frac{GMar\sin^2\theta}{c^2\rho^2}(dt\,d\phi + d\phi\,dt)$$

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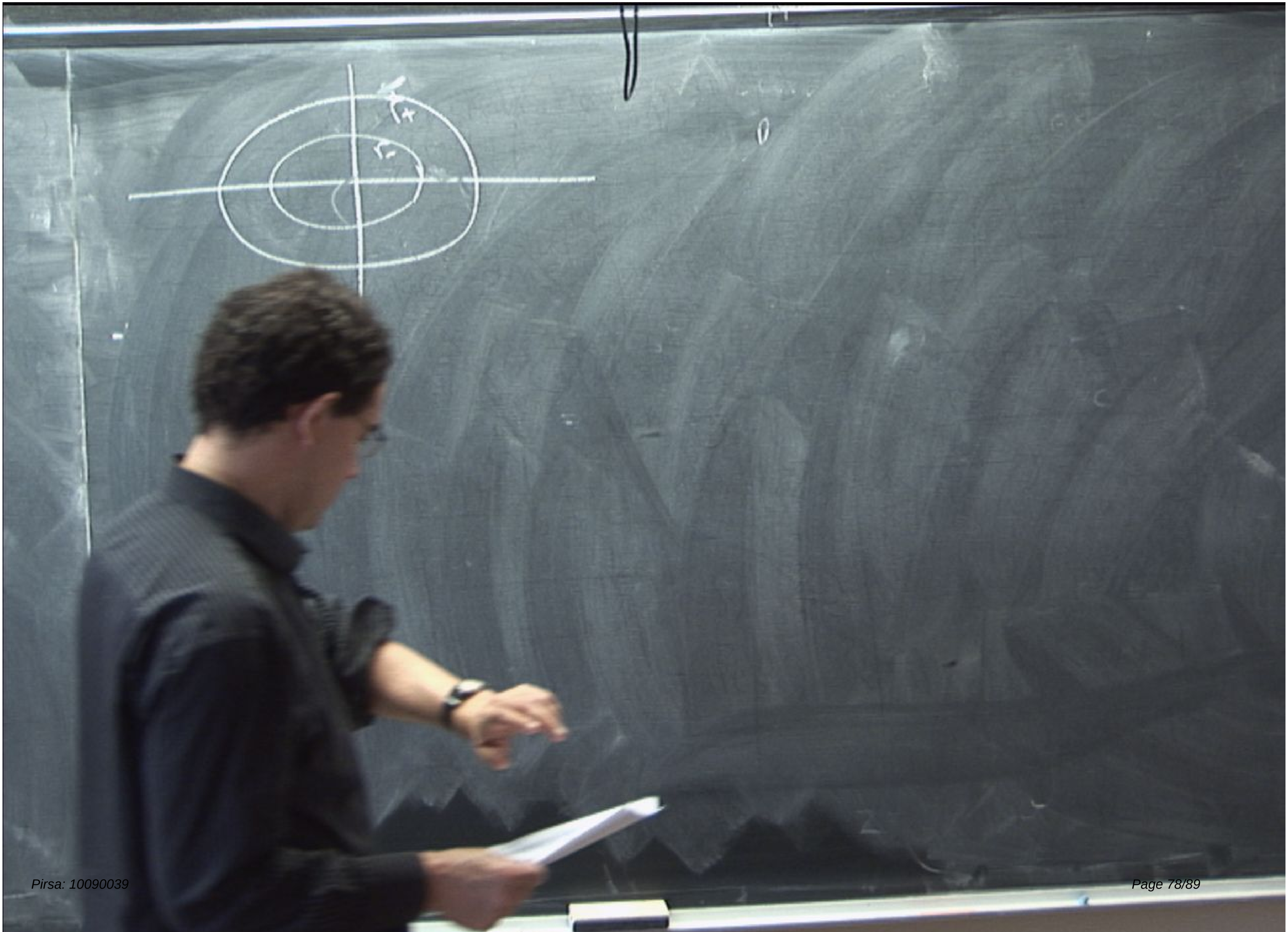
r_+
r_-


r+
r−

$$R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \sim \frac{1}{\rho^6}$$

$$R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \sim \frac{1}{\rho^6}$$

Singularity at $\rho = 0$.

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

$$R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \sim \frac{1}{\rho^6}$$

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$$\rho^2 = r^2 + a^2 \cos^2\theta$$
$$= 0 \quad \text{iff} \quad r = 0 \text{ and } \theta = \pi/2.$$

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Singularity is a ring

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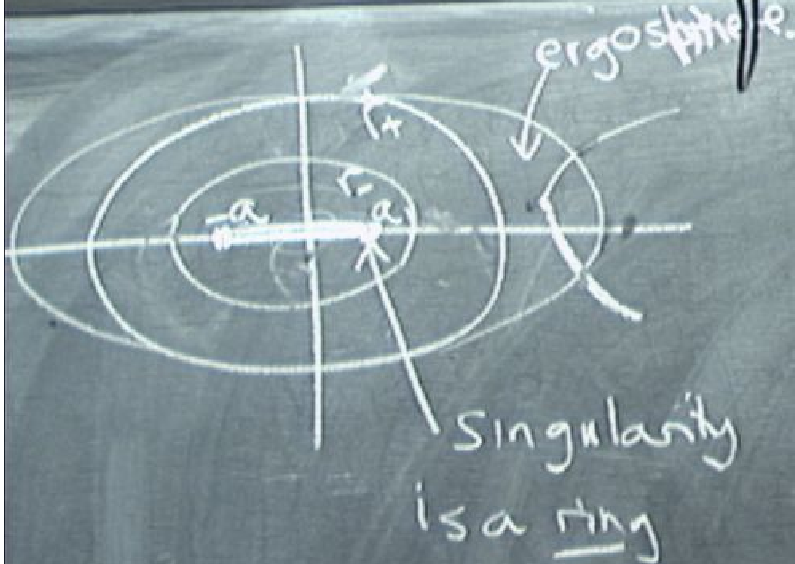

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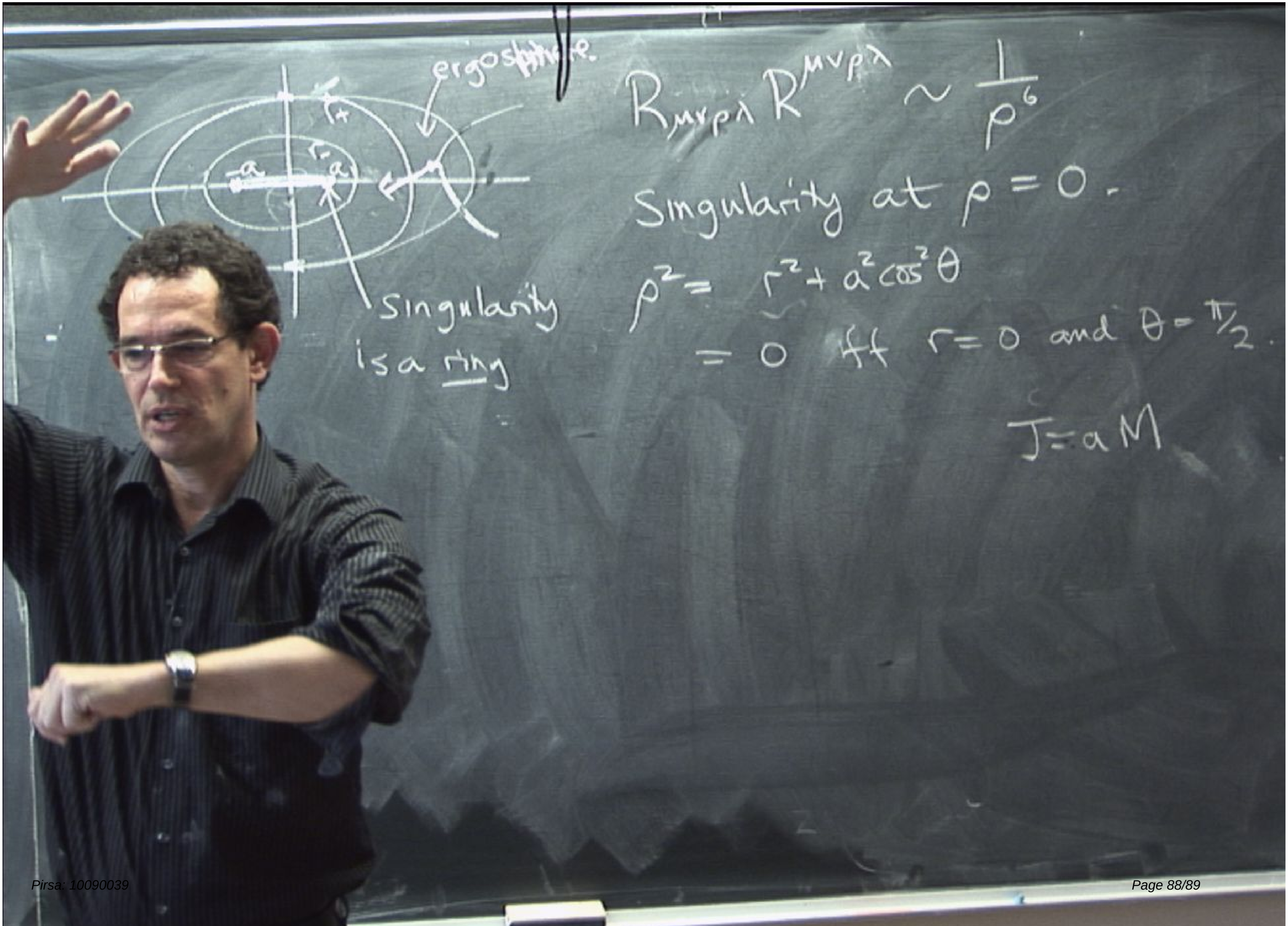

ergosphere.
r_+
r_-
$-a$
a
Singularity is a ring
$R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \sim \frac{1}{\rho^6}$
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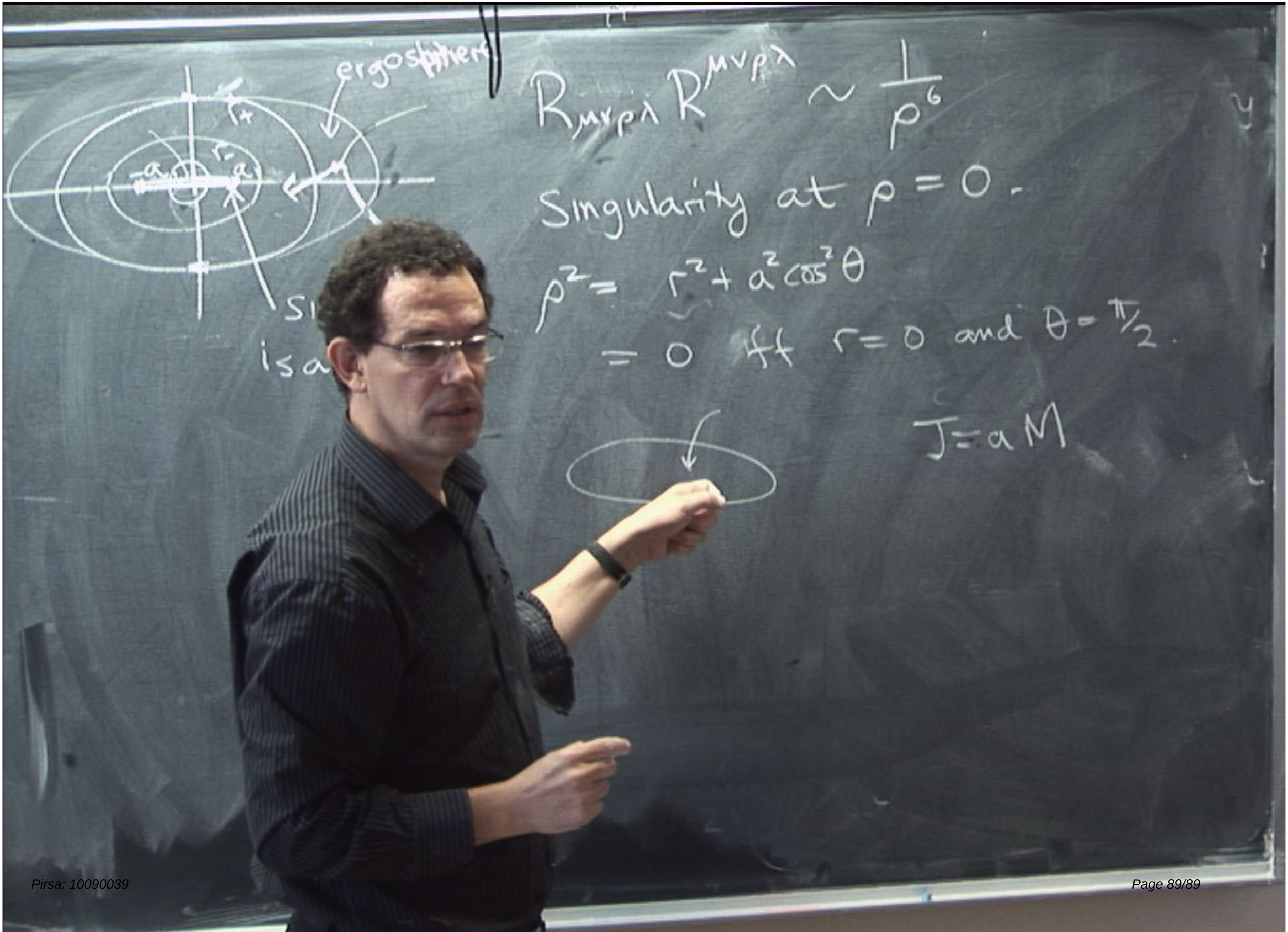
$J = aM$


ergosphere
r_+
r_-
$-a$
a
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