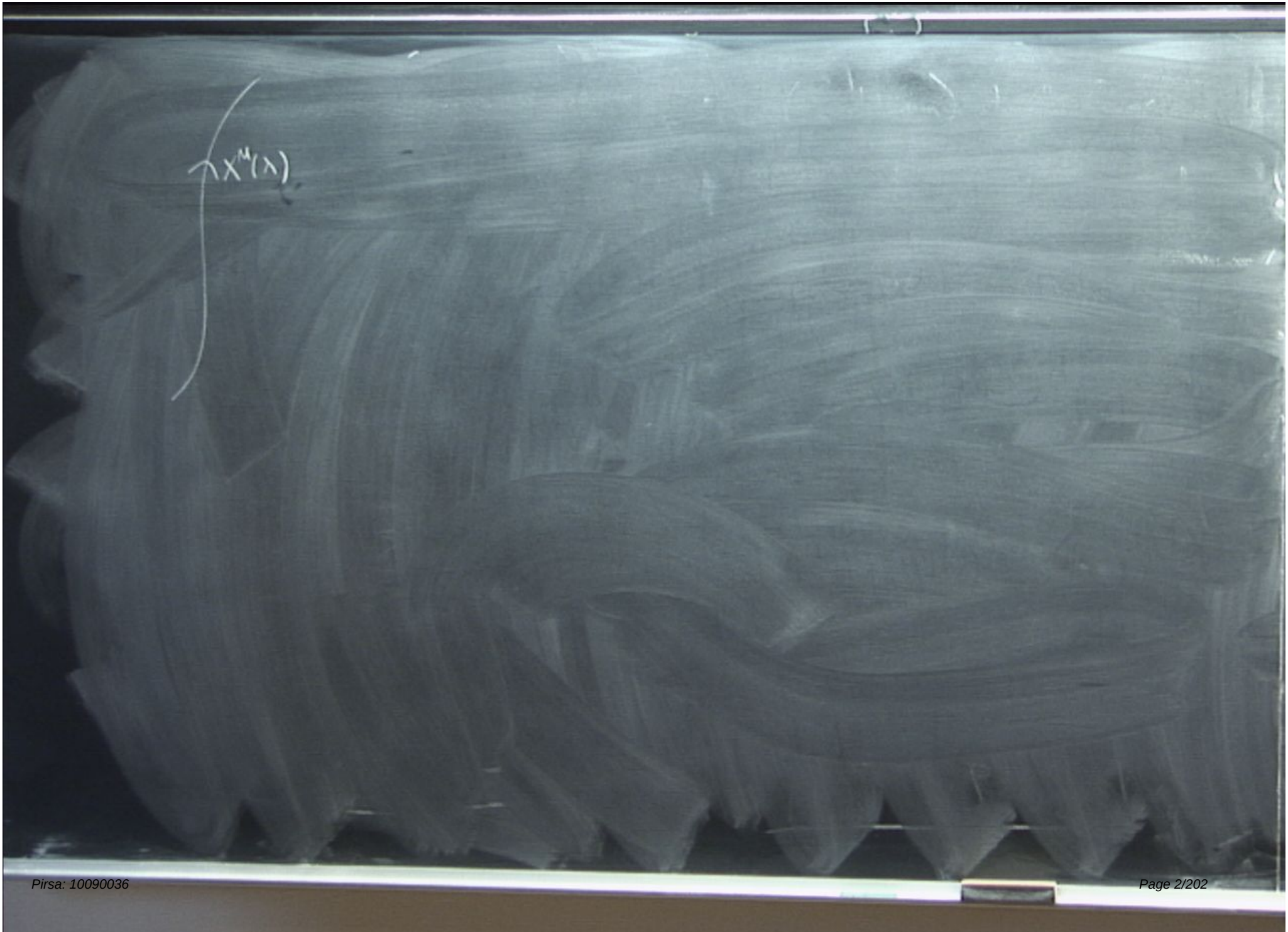
Title: Relativity (PHYS 604) - Lecture 11

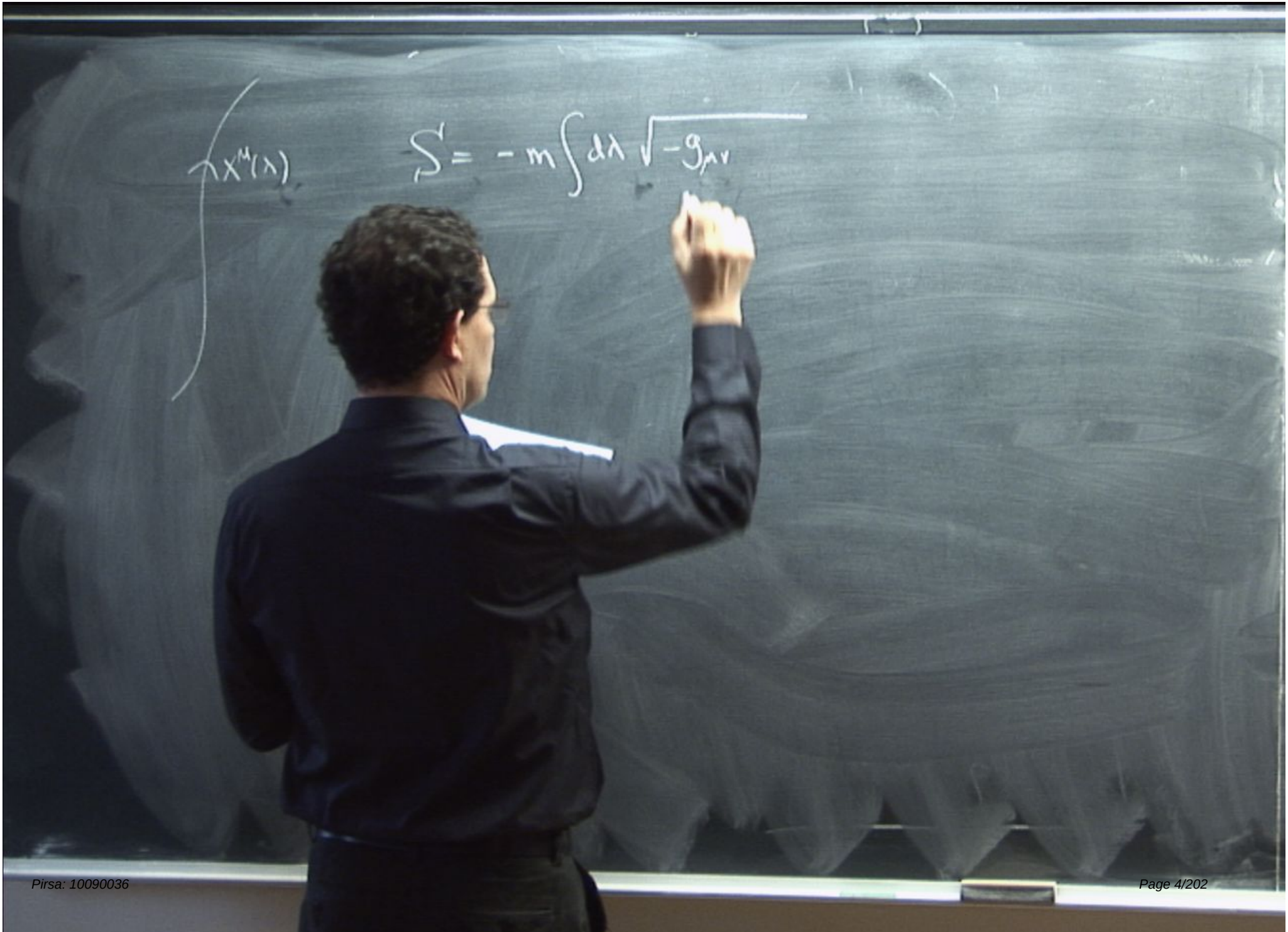
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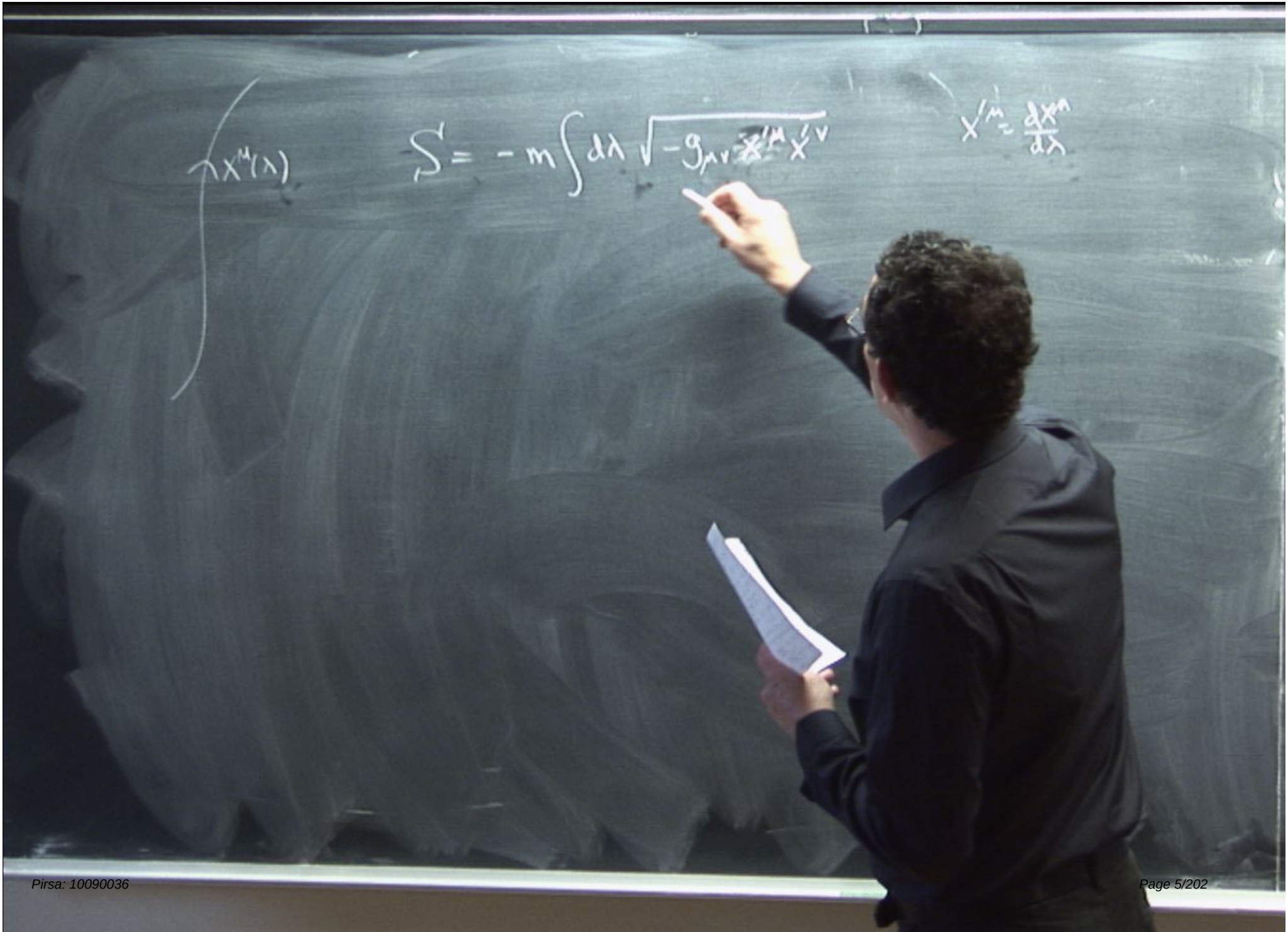
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Abstract:

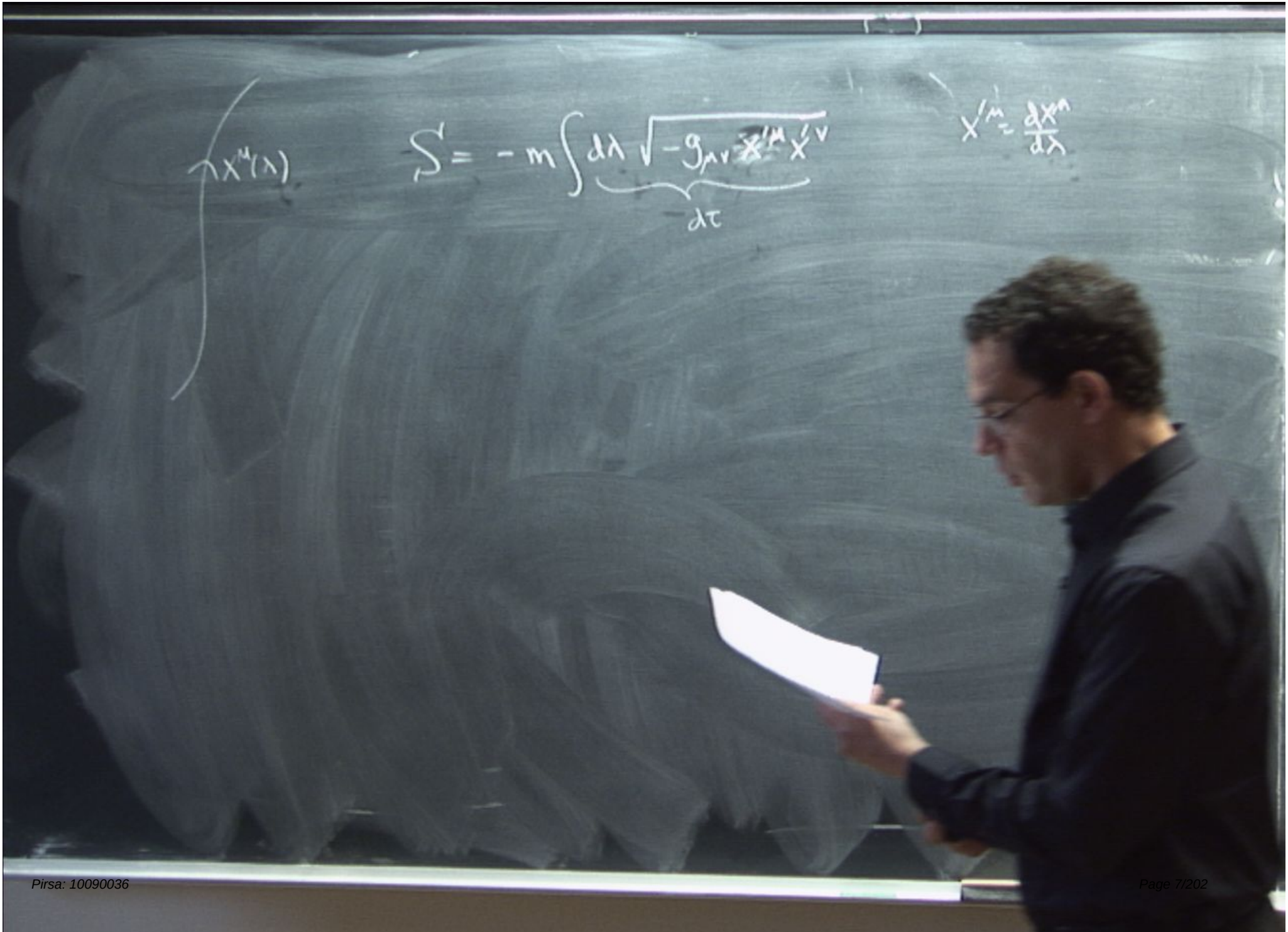

$X^M(\lambda)$

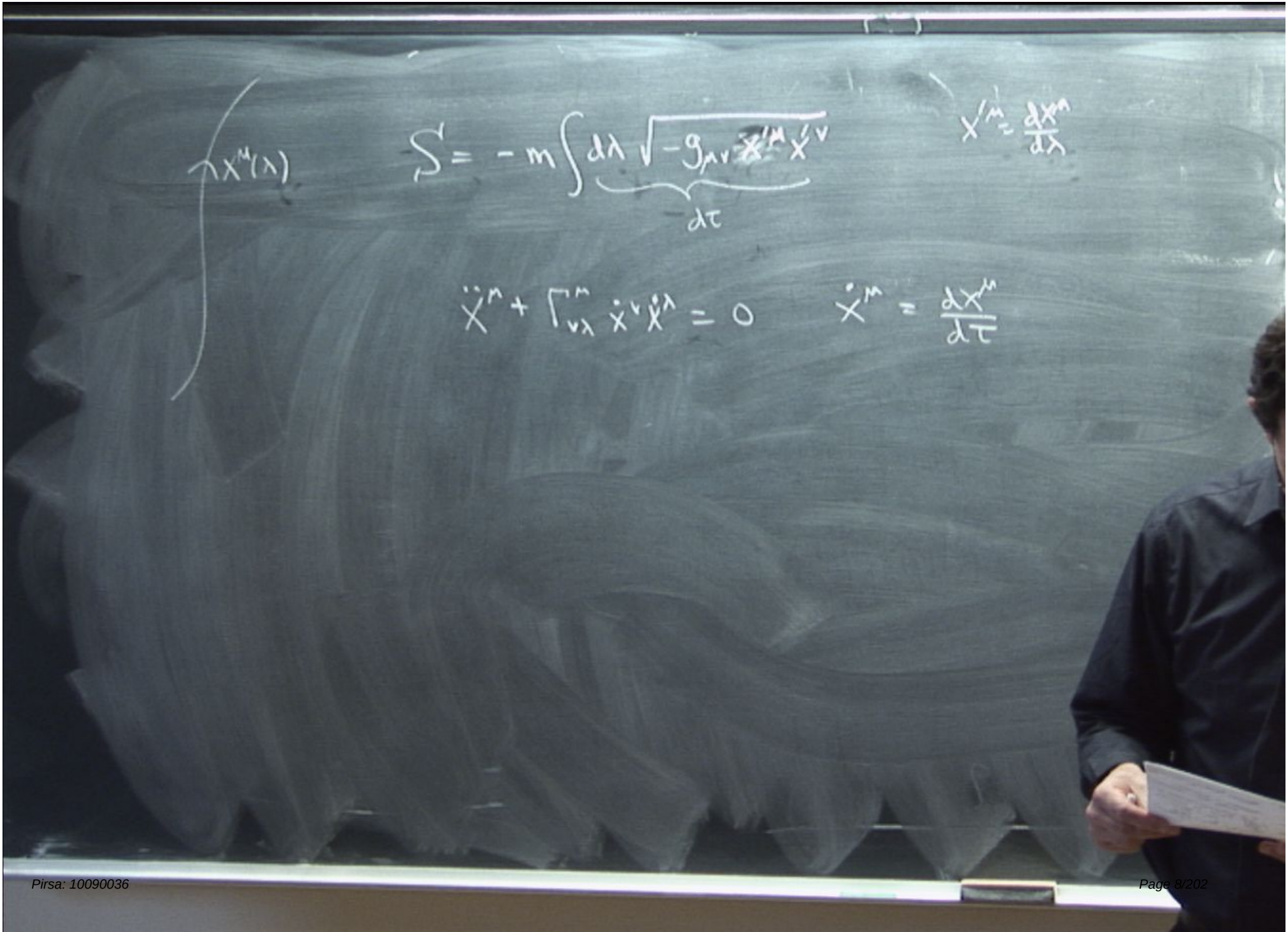
$\int X^M(\lambda)$ $\quad S = -m\int$


$\int X^{\mu}(\lambda)$
$S = -m \int d\lambda \sqrt{-g_{\mu\nu}}$


$\int x^\mu(\lambda)$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$x'^m = \frac{dx^m}{d\lambda}$

$$\int_{x^\mu(\lambda)} \qquad S = -m \int \underbrace{d\lambda \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}_{d\tau} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

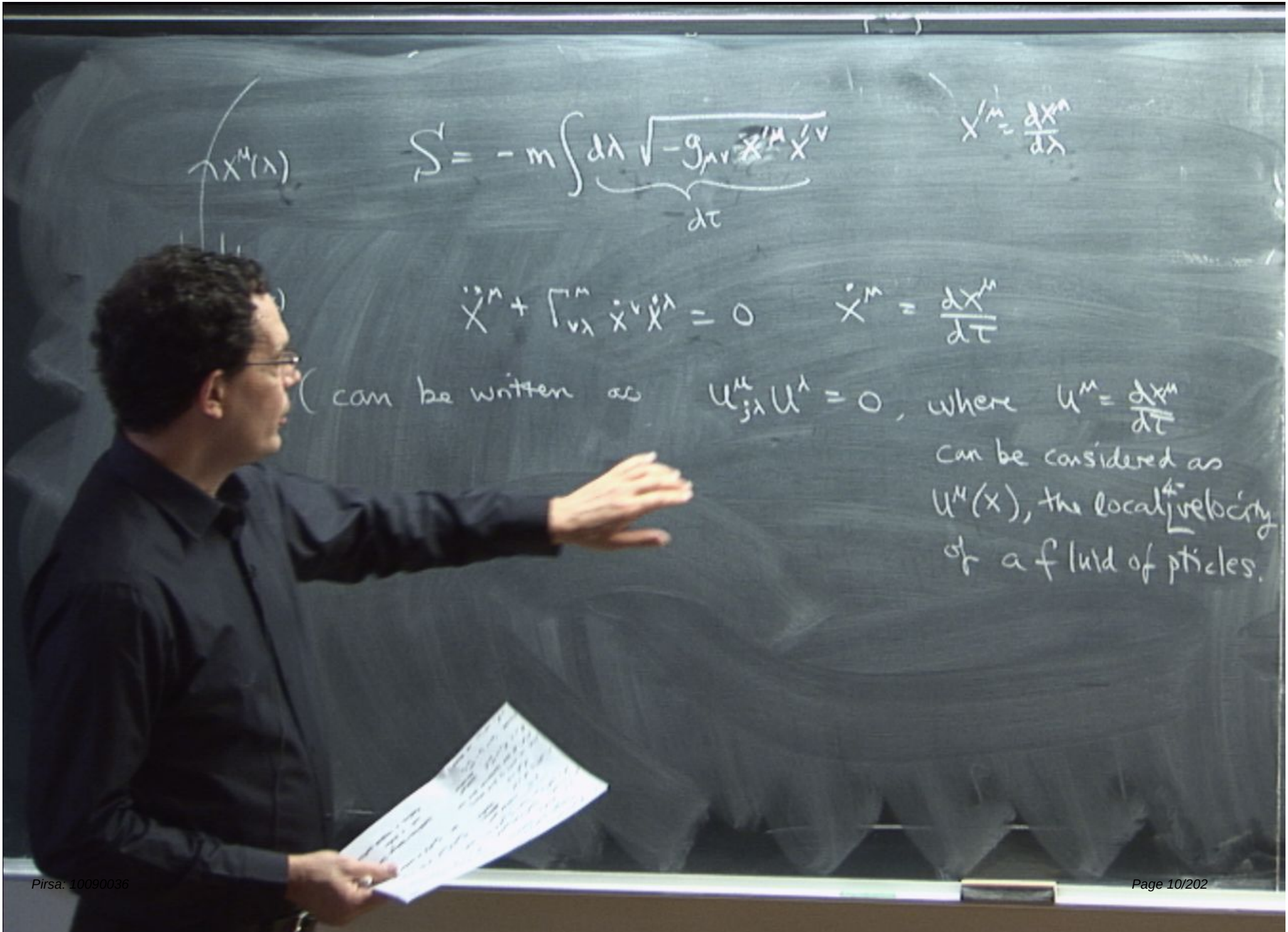

$\int_{x^\mu(\lambda)}$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$d\tau$
$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$

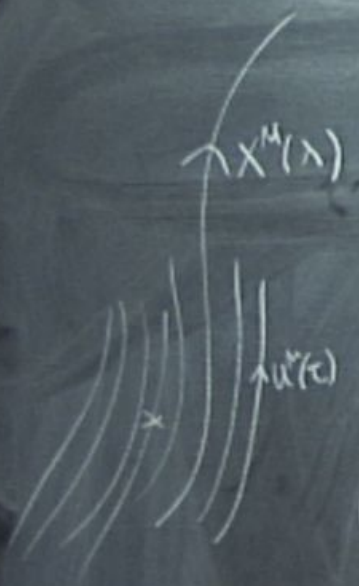

$\int_{x^\mu(\lambda)}$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$d\tau$
$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$
$\ddot{x}^\lambda + \Gamma^\lambda_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
$\dot{x}^\mu = \frac{dx^\mu}{d\tau}$

$$\int_{x^\mu(\lambda)} \qquad S = -m \int \underbrace{d\lambda \sqrt{-g_{\mu\nu} x'^\mu x'^\nu}}_{d\tau} \qquad x'^\mu = \frac{dx^\mu}{d\lambda}$$

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0 \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

(can be written as $u^\mu_{;\lambda} u^\lambda = 0$, where $u^\mu = \frac{dx^\mu}{d\tau}$


$x^\mu(\lambda)$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
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can be considered as
$u^\mu(x)$, the local 4-velocity
of a fluid of ptricles.



$$S = -m\int d\lambda \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{d\tau} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

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$x^\mu(\lambda)$

$u^\mu(x)$

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

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can be considered as $u^\mu(x)$, the local 4-velocity of a fluid of ptcles.

$x^\mu(\lambda)$

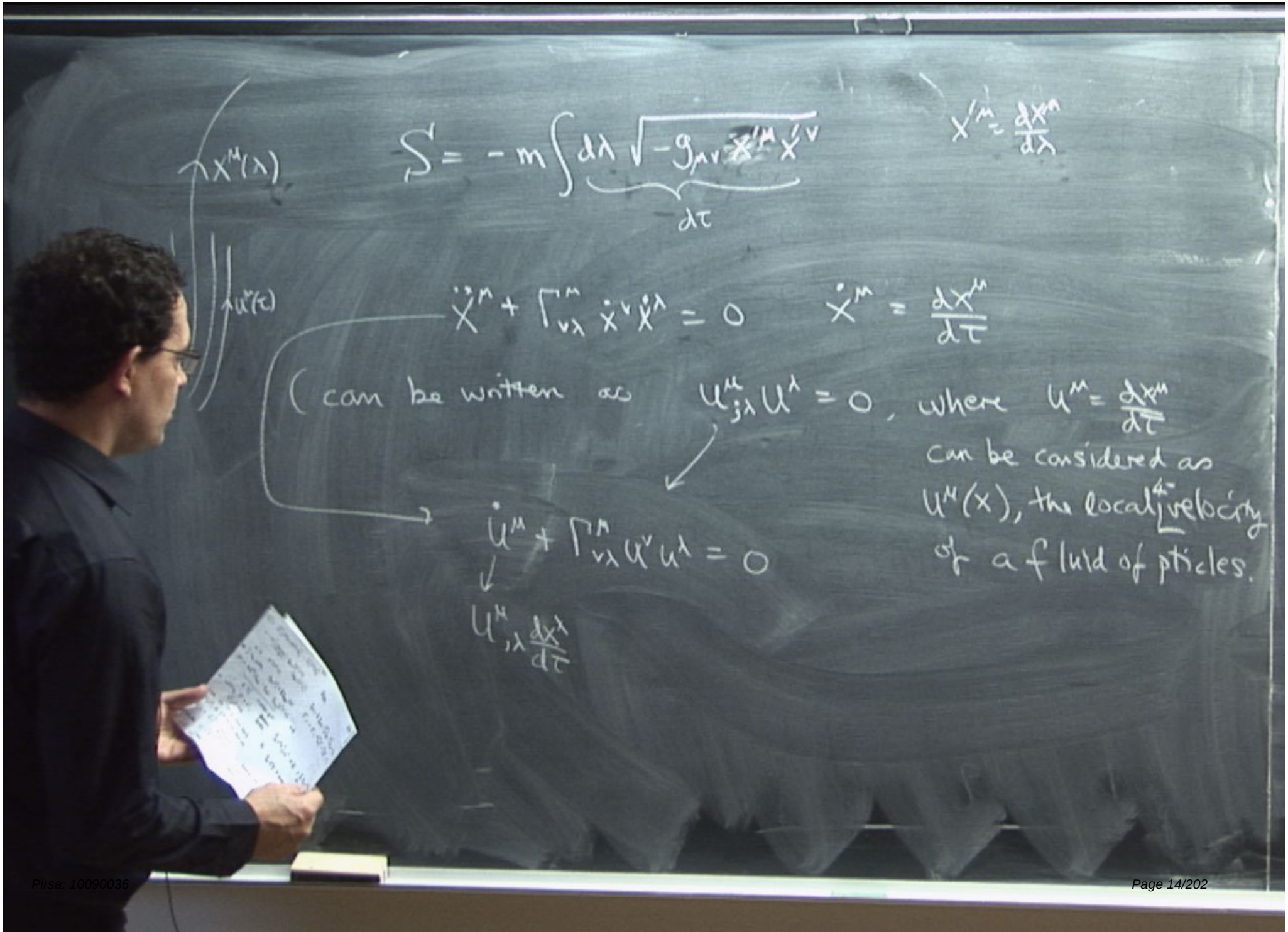
$$S = -m\int d\lambda \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{d\tau} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

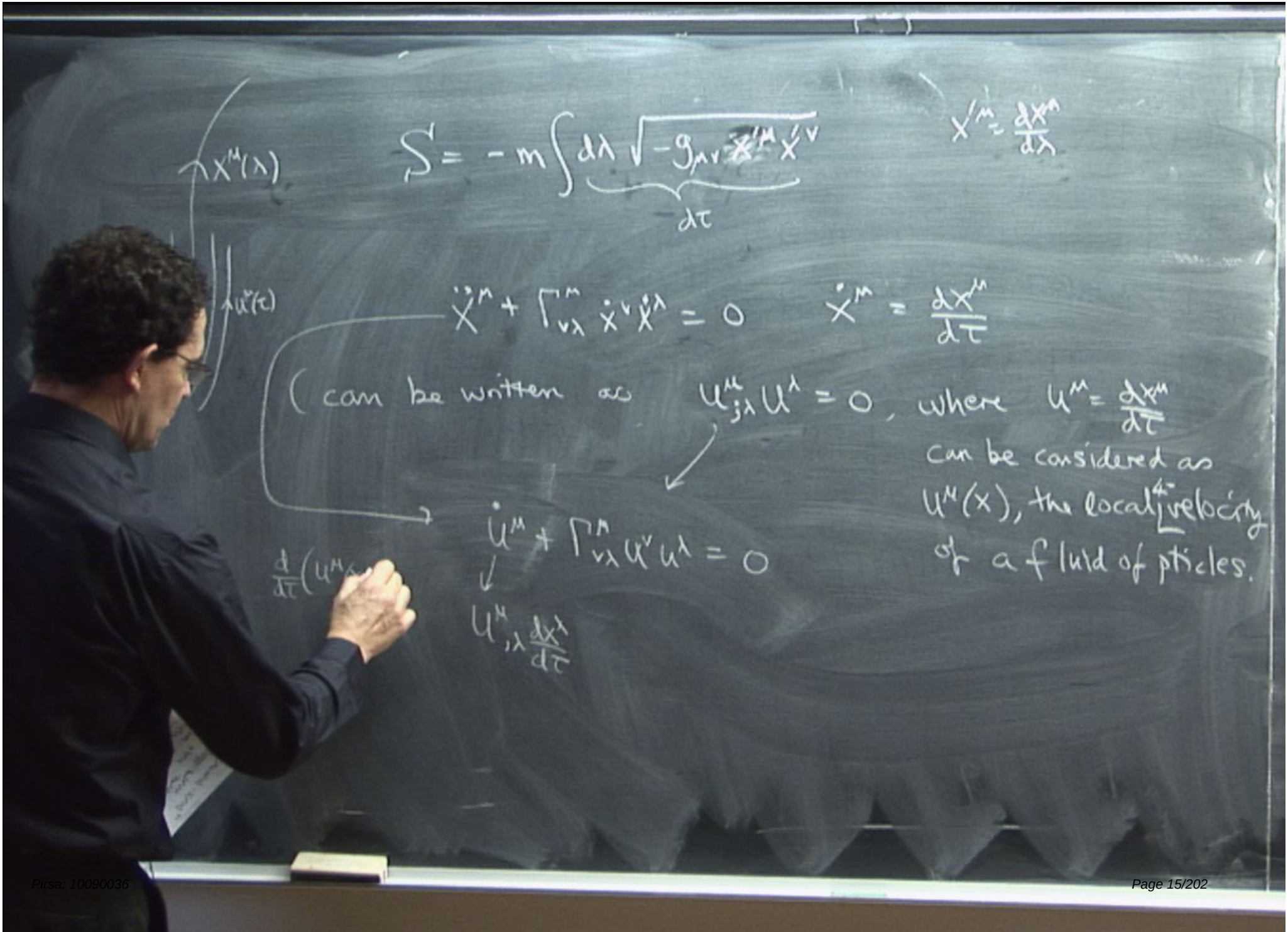
$u^\mu(\tau)$

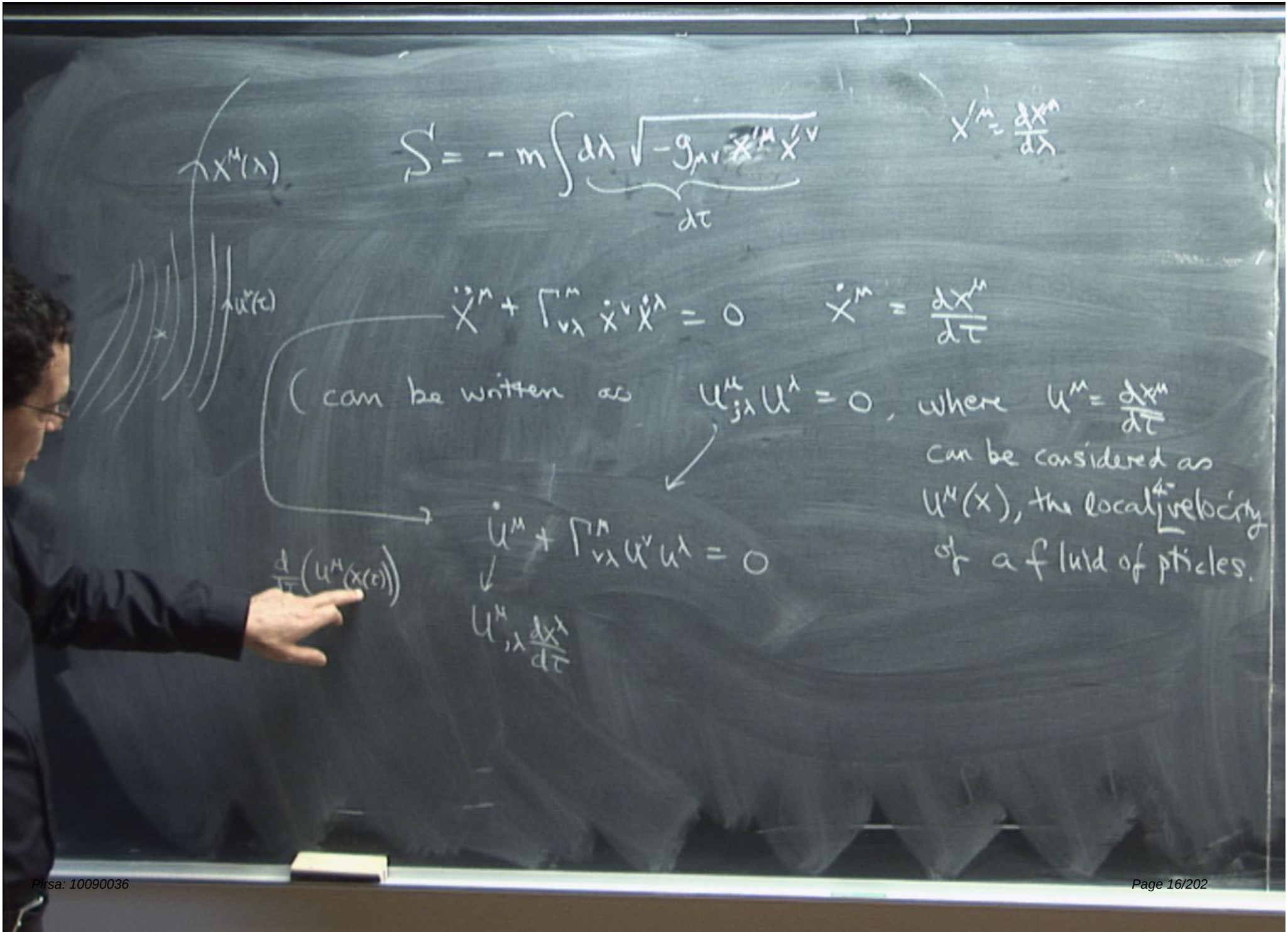
$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

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$$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$$


$x^\mu(\lambda)$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$d\tau$
$x'^\mu = \frac{dx^\mu}{d\lambda}$
$u^\mu(\tau)$
$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
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$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$
$u^\mu{}_{,\lambda}\frac{dx^\lambda}{d\tau}$


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$\frac{d}{d\tau}(u^\mu(x(\tau)))$
$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$
$u^\mu_{,\lambda}\frac{dx^\lambda}{d\tau}$

$x^\mu(\lambda)$

$$S = -m\int d\lambda \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{d\tau}$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$u^\mu(\tau)$

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$$\frac{d}{d\tau}\left(u^\mu(x(\tau))\right)$$

$$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$$

$$u^\mu_{,\lambda}\frac{dx^\lambda}{d\tau} = u^\mu_{,\lambda}u^\lambda$$

$x^\mu(\lambda)$

$$S = -m\int d\lambda \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{d\tau}$$

$$x'^\mu = \frac{dx^\mu}{d\lambda}$$

$u^\nu(\tau)$

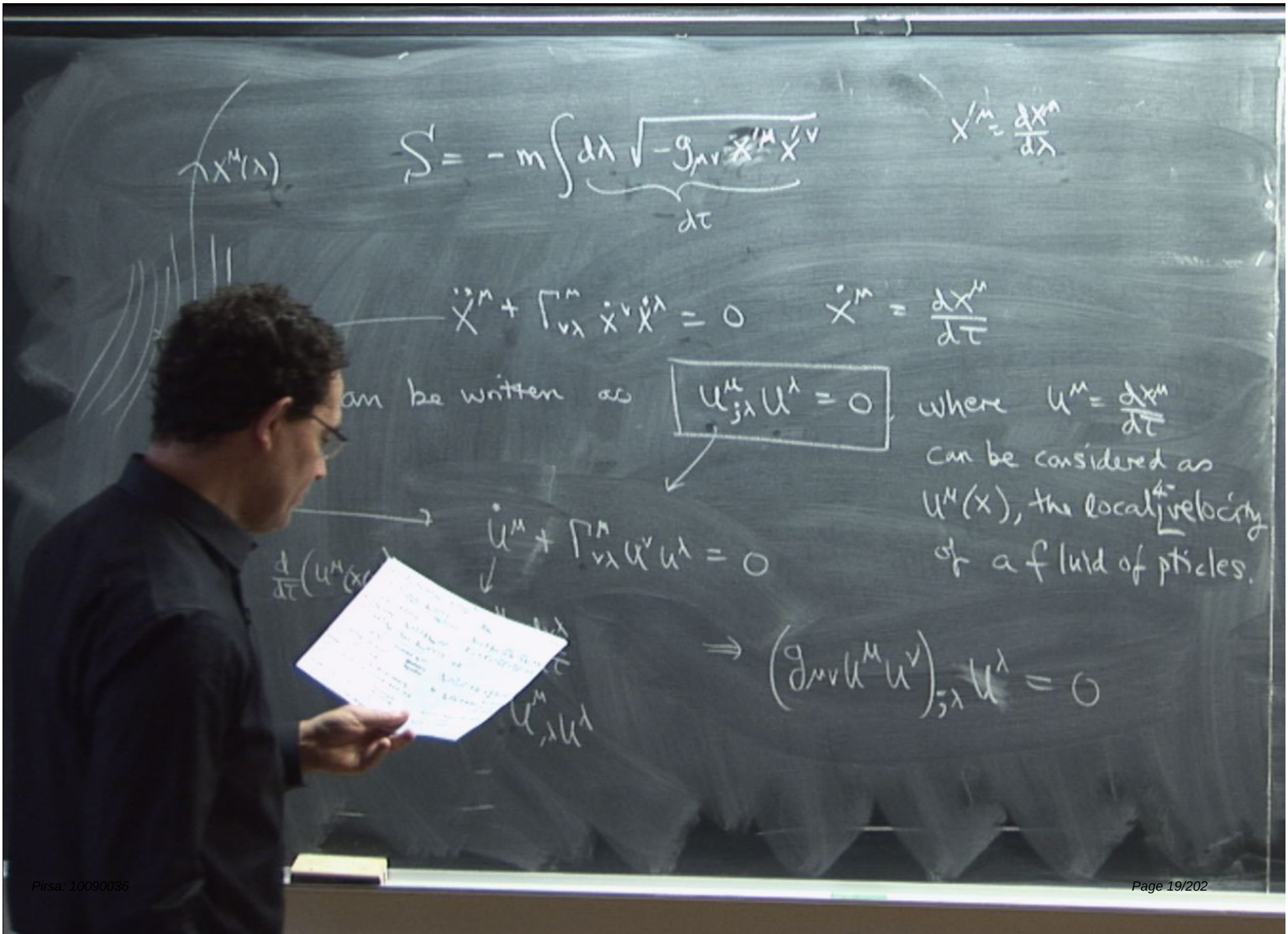
$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

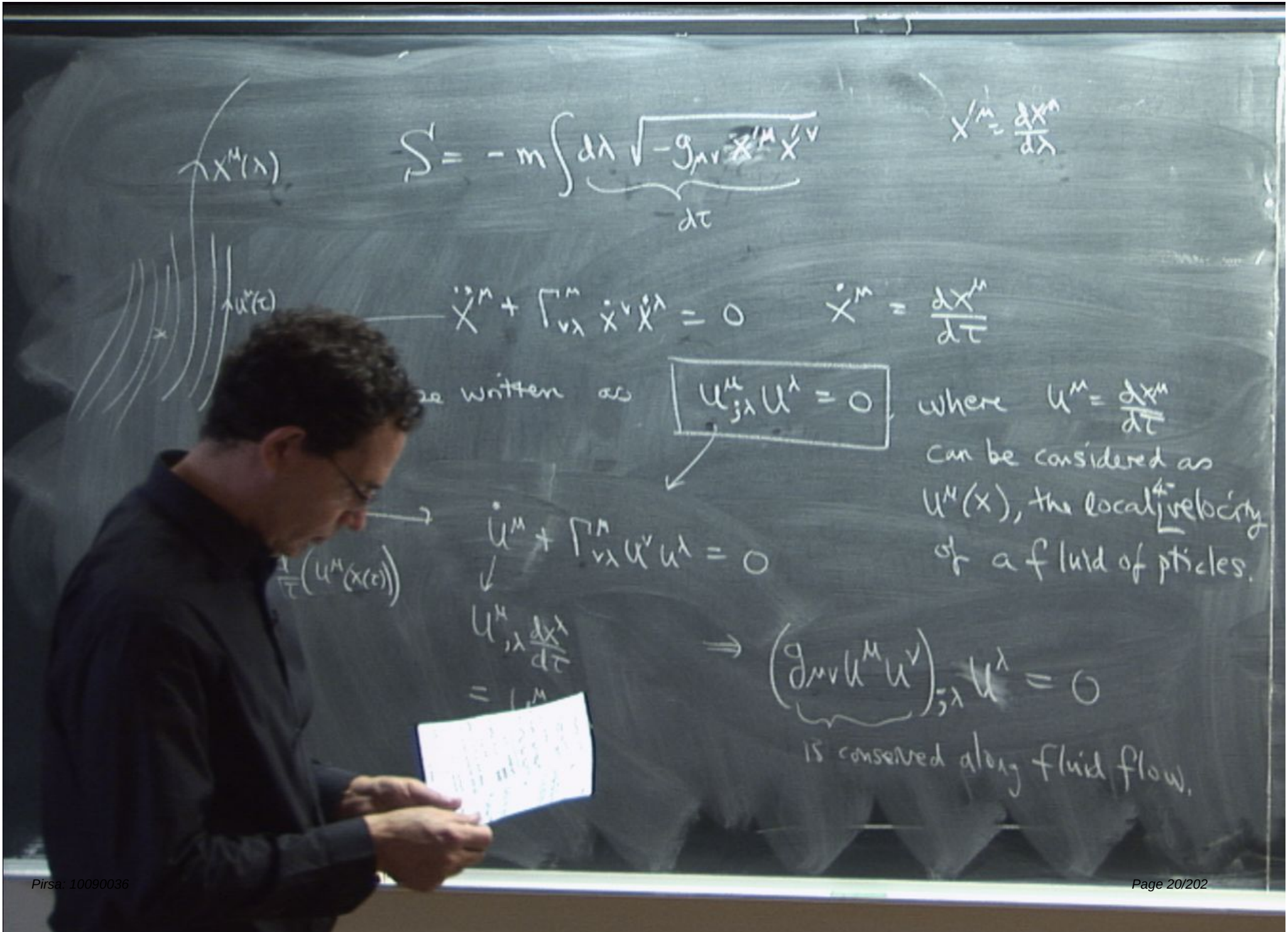
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$x^\mu(\lambda)$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$d\tau$
$x'^\mu = \frac{dx^\mu}{d\lambda}$
$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
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can be written as
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$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$
$\Rightarrow \left(g_{\mu\nu}u^\mu u^\nu\right)_{;\lambda}u^\lambda = 0$


$x^\mu(\lambda)$
$S = -m\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$
$d\tau$
$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$
$u^\mu(\tau)$
$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
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written as
$u^\mu_{;\lambda} u^\lambda = 0$
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$u^\mu(x)$, the local 4-velocity
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$\frac{1}{\tau}(u^\mu(x(\tau)))$
$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0$
$u^\mu_{,\lambda}\frac{dx^\lambda}{d\tau}$
$\Rightarrow \left(g_{\mu\nu}u^\mu u^\nu\right)_{;\lambda} u^\lambda = 0$
is conserved along fluid flow.

$x^\mu(\lambda)$

$$S = -m\int d\lambda \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{d\tau} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$u^\mu(\tau)$

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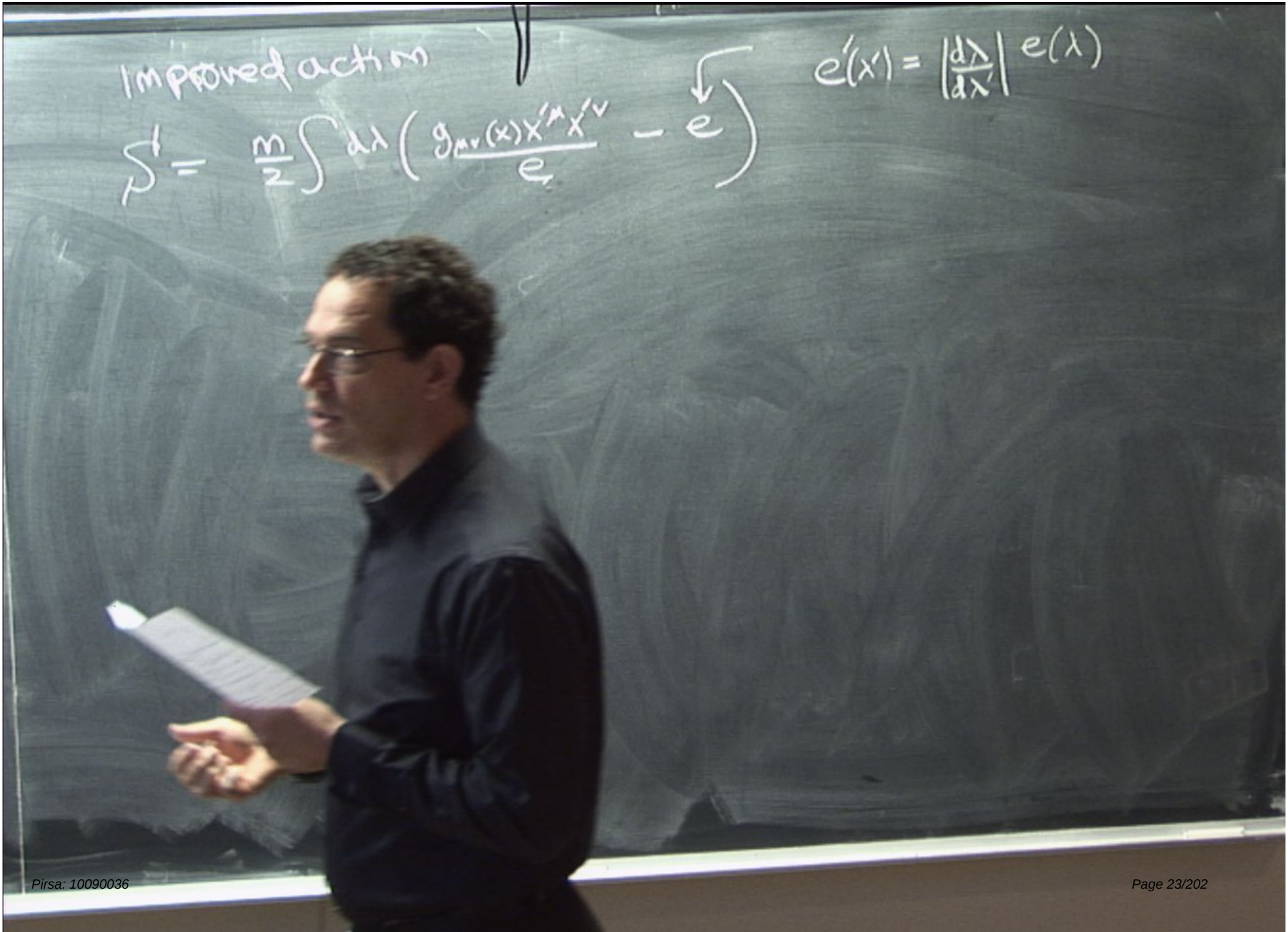
$\frac{d}{d\tau}\left(u^\mu(x(\tau))\right)$

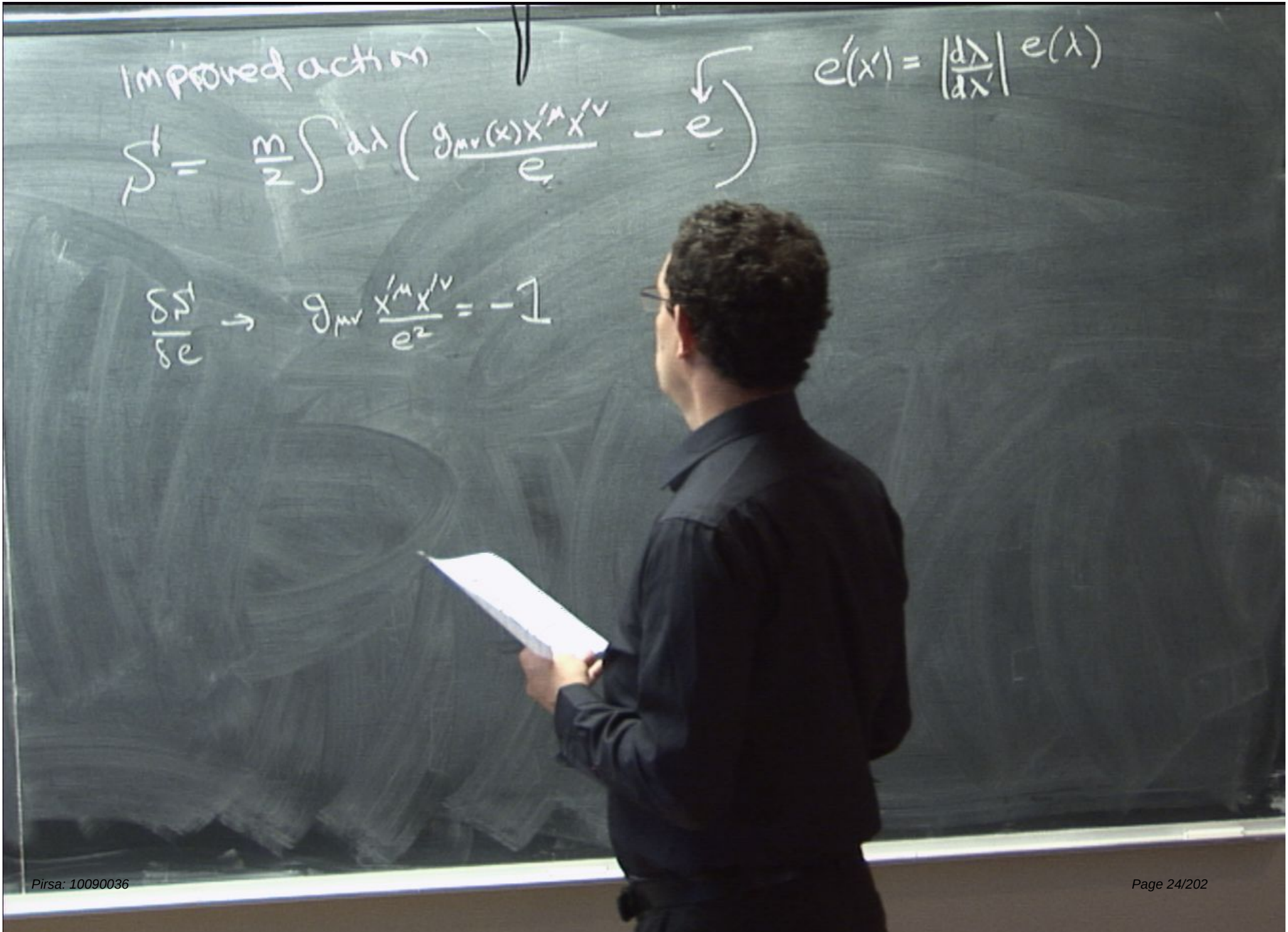
$$\dot{u}^\mu + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$$

$$u^\mu_{,\lambda}\frac{dx^\lambda}{d\tau} = u^\mu_{,\lambda}u^\lambda \Rightarrow \underbrace{\left(g_{\mu\nu}u^\mu u^\nu\right)}_{\text{is conserved along fluid flow.}}{}_{;\lambda}u^\lambda = 0$$

Improved action

$$S' = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}(x) x'^{\mu} x'^{\nu}}{e} - e \right)$$


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Improved action

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$$S' = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

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$\Rightarrow +1$ tachyons

0 massless ptcles.

$$\frac{\delta S'}{\delta e} \rightarrow g_{\mu\nu}\frac{\dot{x}^\mu \dot{x}^\nu}{e^2} = -1 \Rightarrow \underline{d\tau = e\,d\lambda}$$

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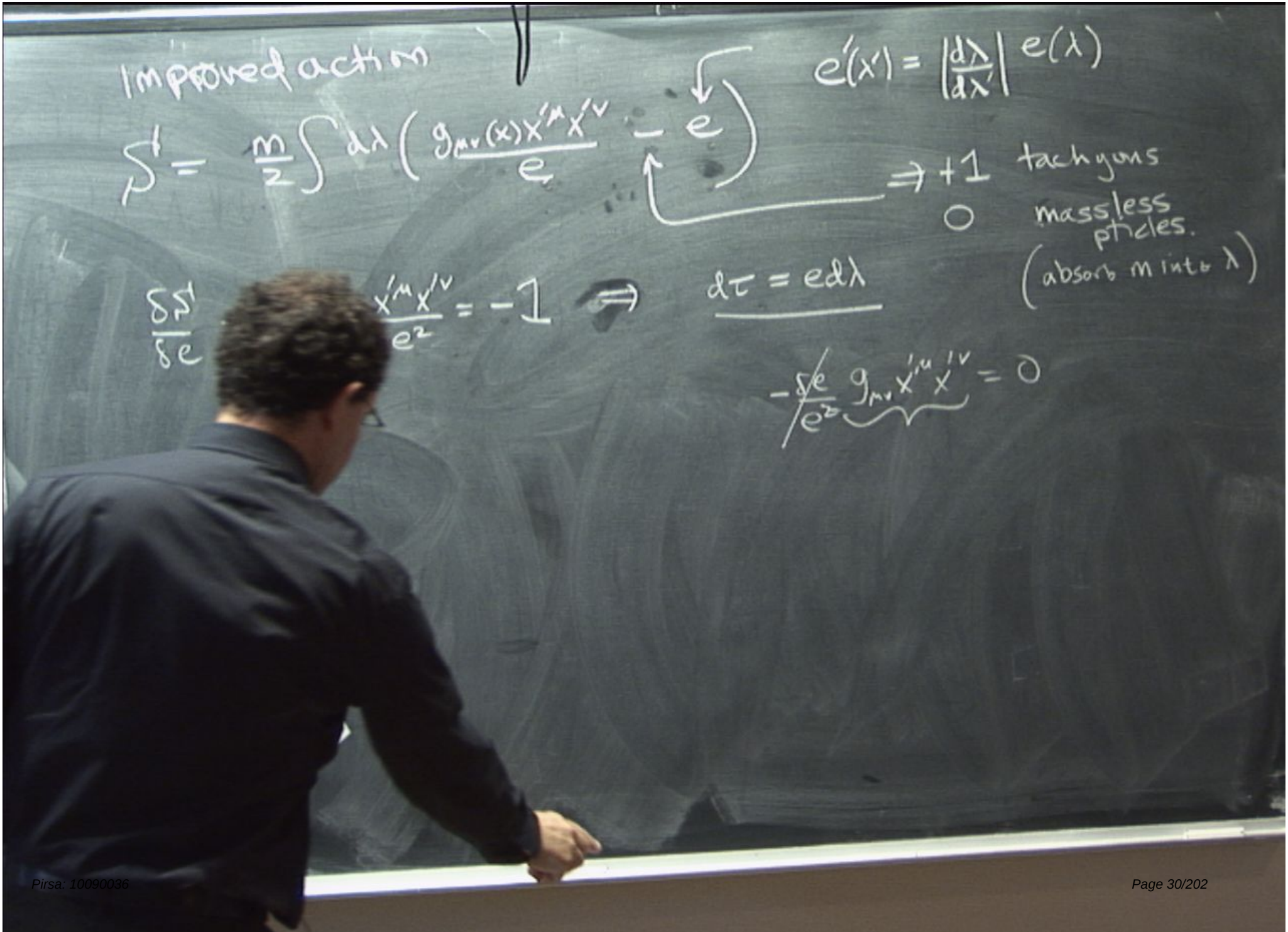
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$$\frac{\delta S'}{\delta e} \rightarrow g_{\mu\nu}\frac{x'^{\mu}x'^{\nu}}{e^2} = -1 \Rightarrow \underline{d\tau = e\,d\lambda}$$

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Improved action
$S' = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}(x)x'^{\mu}x'^{\nu}}{e} - e \right)$
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0 massless ptcles.
(absorb m into λ)
$\frac{\delta S'}{\delta e}$
$\frac{x'^{\mu}x'^{\nu}}{e^2} = -1 \Rightarrow$
$d\tau = e\,d\lambda$
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Action is easier in this form: can set $e=1$ (to fix $d\lambda = d\tau$) directly.

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then vary w.r.t. X^μ to get geodesic equation (Ex)

Improved action

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$\Rightarrow +1$ tachyons

0 massless ptcles.

$\frac{\delta S'}{\delta e} \rightarrow g_{\mu\nu}\frac{x'^{\mu}x'^{\nu}}{e^2} = -1 \Rightarrow$ $d\tau = e\,d\lambda$ (absorb m into λ)

$-\frac{\delta e}{e^2} g_{\mu\nu} x'^{\mu} x'^{\nu} = 0$

Action is easier in this form: L can for a massive pkc

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article

Improved action

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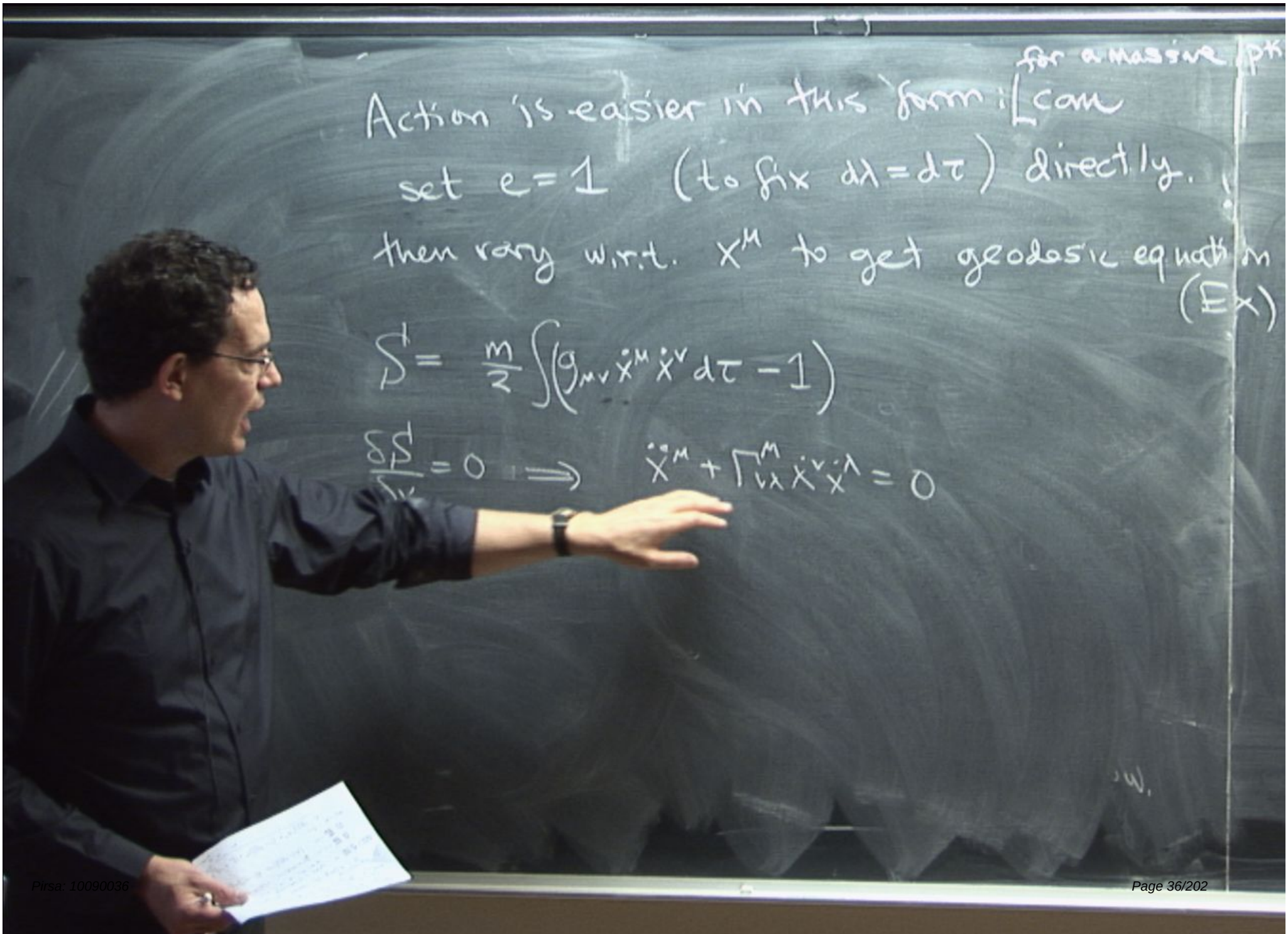
$\Rightarrow +1$ tachyons

0 massless particles.

$$\frac{\delta S'}{\delta e} \rightarrow g_{\mu\nu}\frac{x'^{\mu}x'^{\nu}}{e^2} = -1 \Rightarrow \underline{d\tau = e\,d\lambda}$$

(absorb m into λ)

$$-\frac{\delta e}{e^2} g_{\mu\nu} x'^{\mu} x'^{\nu} = 0$$


for a massive p*
Action is easier in this form: L = cam
set e = 1 (to fix dλ = dτ) directly.
then vary w.r.t. x^μ to get geodesic equation
(Ex)
$S = \frac{m}{2}\int (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau - 1)$
$\frac{\delta S}{\delta x^\mu} = 0 \Rightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$

Action is easier in this form: ∫ L dτ for a massive pt

set $e=1$ (to fix $d\lambda = d\tau$) directly.

then vary w.r.t. x^μ to get geodesic equation (Ex)

$e=1$ $\quad S = \frac{m}{2}\int (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau - 1)$

$$\frac{\delta S}{\delta x} = 0 \Longrightarrow \quad \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0$$

Action is easier in this form: ∫ can for a massive pt

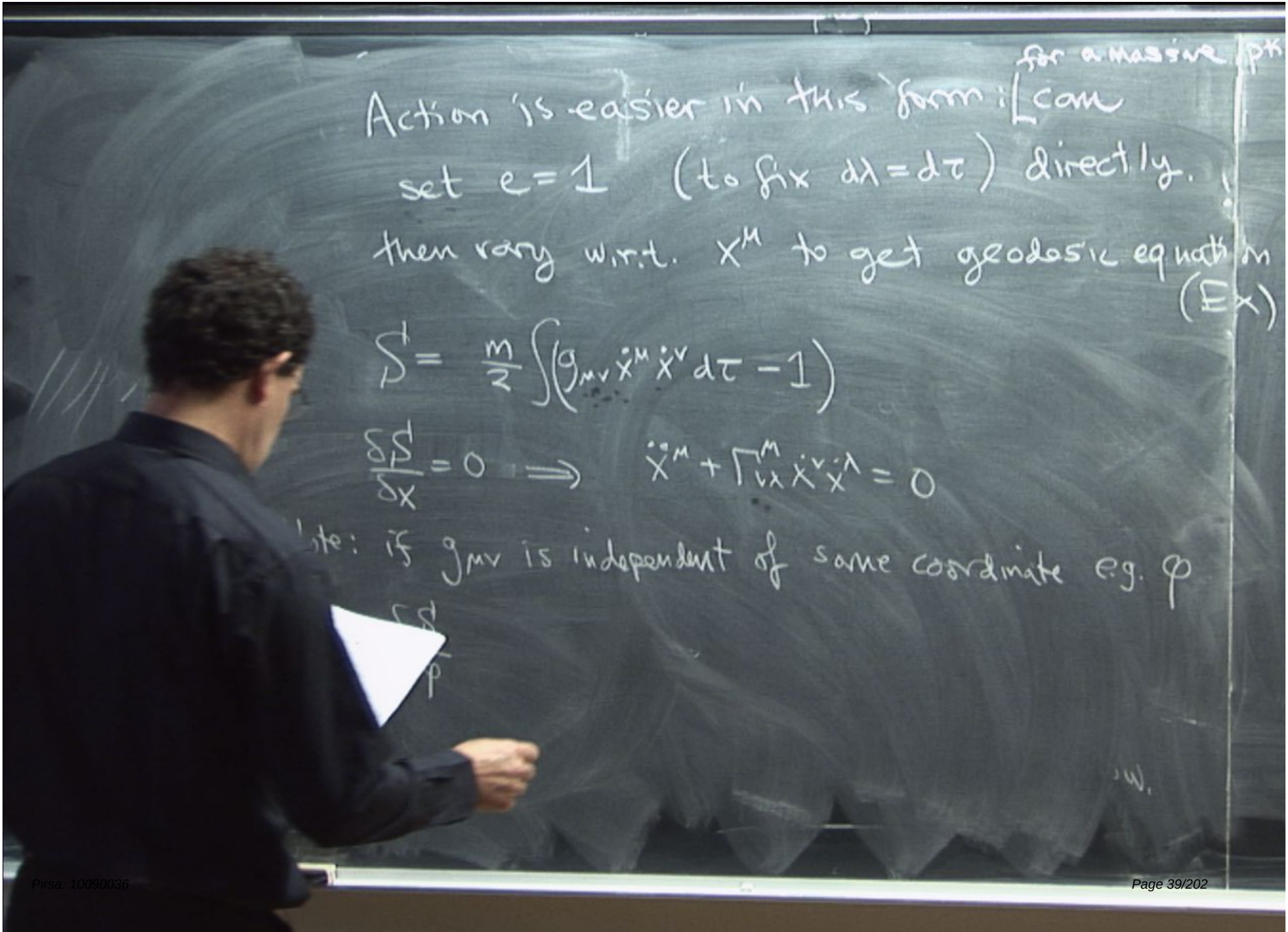
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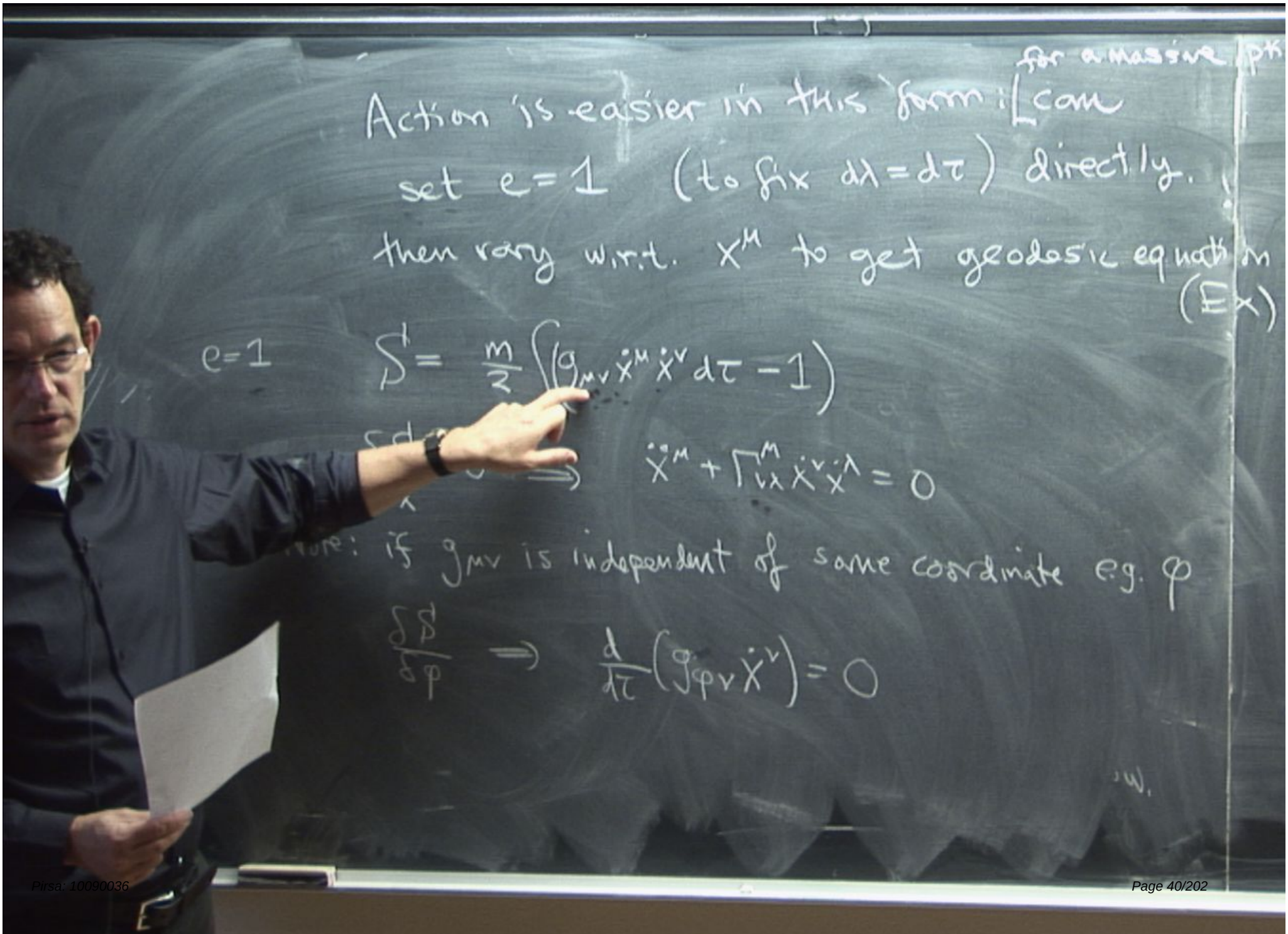
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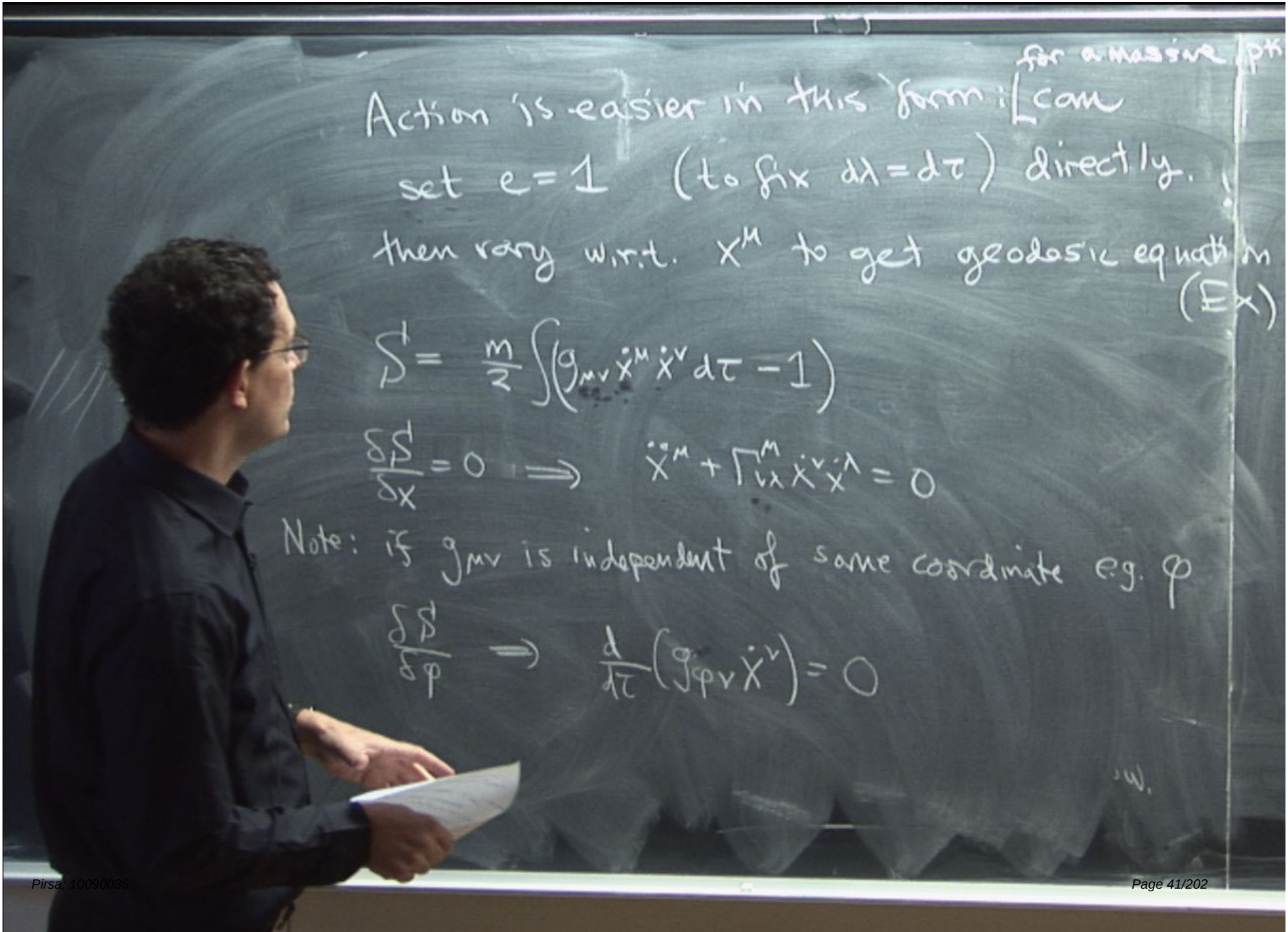
$e=1$ $\qquad S = \frac{m}{2}\int (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau - 1)$

$$\frac{\delta S}{\delta x} = 0 \implies \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

Note: if $g_{\mu\nu}$ is independent of some coordinate e.g. φ


for a massive pt
Action is easier in this form: Lcan
set e=1 (to fix dλ=dτ) directly.
then vary w.r.t. x^μ to get geodesic equation
(Ex)
$S = \frac{m}{2}\int(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu d\tau - 1)$
$\frac{\delta S}{\delta x} = 0 \Rightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
Note: if $g_{\mu\nu}$ is independent of some coordinate e.g. φ
$\frac{\partial S}{\partial \varphi}$


for a massive p*
Action is easier in this form: Lcan
set e=1 (to fix dλ=dτ) directly.
then vary w.r.t. x^μ to get geodesic equation
(Ex)
e=1
S' = m/2 ∫(g_μν ẋ^μ ẋ^ν dτ − 1)
ẍ^μ + Γ^μ_λκ ẋ^λ ẋ^κ = 0
if g_μν is independent of some coordinate e.g. φ
∂S/∂φ ⟹ d/dτ(g_φν ẋ^ν) = 0


Action is easier in this form: (for a massive pt)
set $e=1$ (to fix $d\lambda = d\tau$) directly.
then vary w.r.t. x^μ to get geodesic equation
(Ex)
$S = \frac{m}{2}\int (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau - 1)$
$\frac{\delta S}{\delta x} = 0 \Rightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
Note: if $g_{\mu\nu}$ is independent of some coordinate e.g. φ
$\frac{\delta S}{\delta \varphi} \Rightarrow \frac{d}{d\tau}(g_{\varphi\nu}\dot{x}^\nu) = 0$

Action is easier in this form: [for a massive p^μ can set $e=1$ ($\to$ fix $d\lambda = d\tau$) directly.

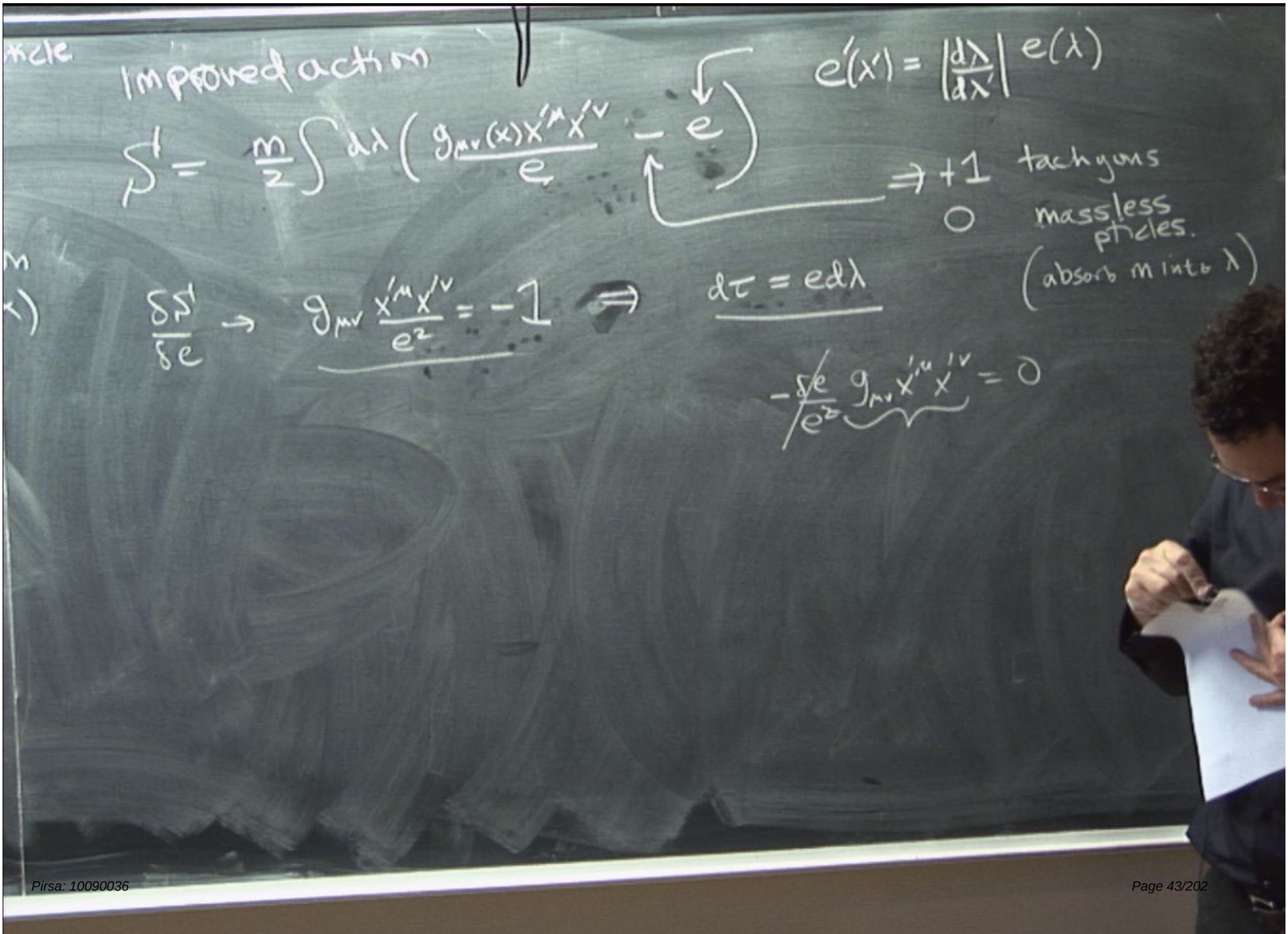
then vary w.r.t. x^μ to get geodesic equation (Ex)

$e=1$ $\quad S = \frac{m}{2}\int (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\tau - 1)$ $\quad \to g_{\varphi\nu}\delta\dot{\varphi}\dot{x}^\nu$

$$\frac{\delta S}{\delta x} = 0 \implies \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

Note: if $g_{\mu\nu}$ is independent of some coordinate e.g. φ

$$\frac{\delta S}{\delta \varphi} \implies \frac{d}{d\tau}\left(g_{\varphi\nu}\dot{x}^\nu\right) = 0$$


Improved action
$S' = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}(x) x'^\mu x'^\nu}{e} - e \right)$
$e'(\lambda') = \left|\frac{d\lambda}{d\lambda'}\right| e(\lambda)$
$\Rightarrow +1$ tachyons
0 massless ptcles.
(absorb m into λ)
$\frac{\delta S'}{\delta e} \rightarrow g_{\mu\nu}\frac{x'^\mu x'^\nu}{e^2} = -1 \Rightarrow$
$d\tau = e\,d\lambda$
$-\frac{\delta e}{e^2} g_{\mu\nu} x'^\mu x'^\nu = 0$

rticle

Improved action

$$S' = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}(x) x'^{\mu} x'^{\nu}}{e} - e \right)$$

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$\Rightarrow$ +1 tachyons

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$$-\frac{\delta e}{e^2} g_{\mu\nu} x'^{\mu} x'^{\nu} = 0$$

Action is easier in this form: L for a massive particle.

set $e=1$ (to fix $d\lambda = d\tau$) directly.

then vary w.r.t. x^μ to get geodesic equation (Ex)

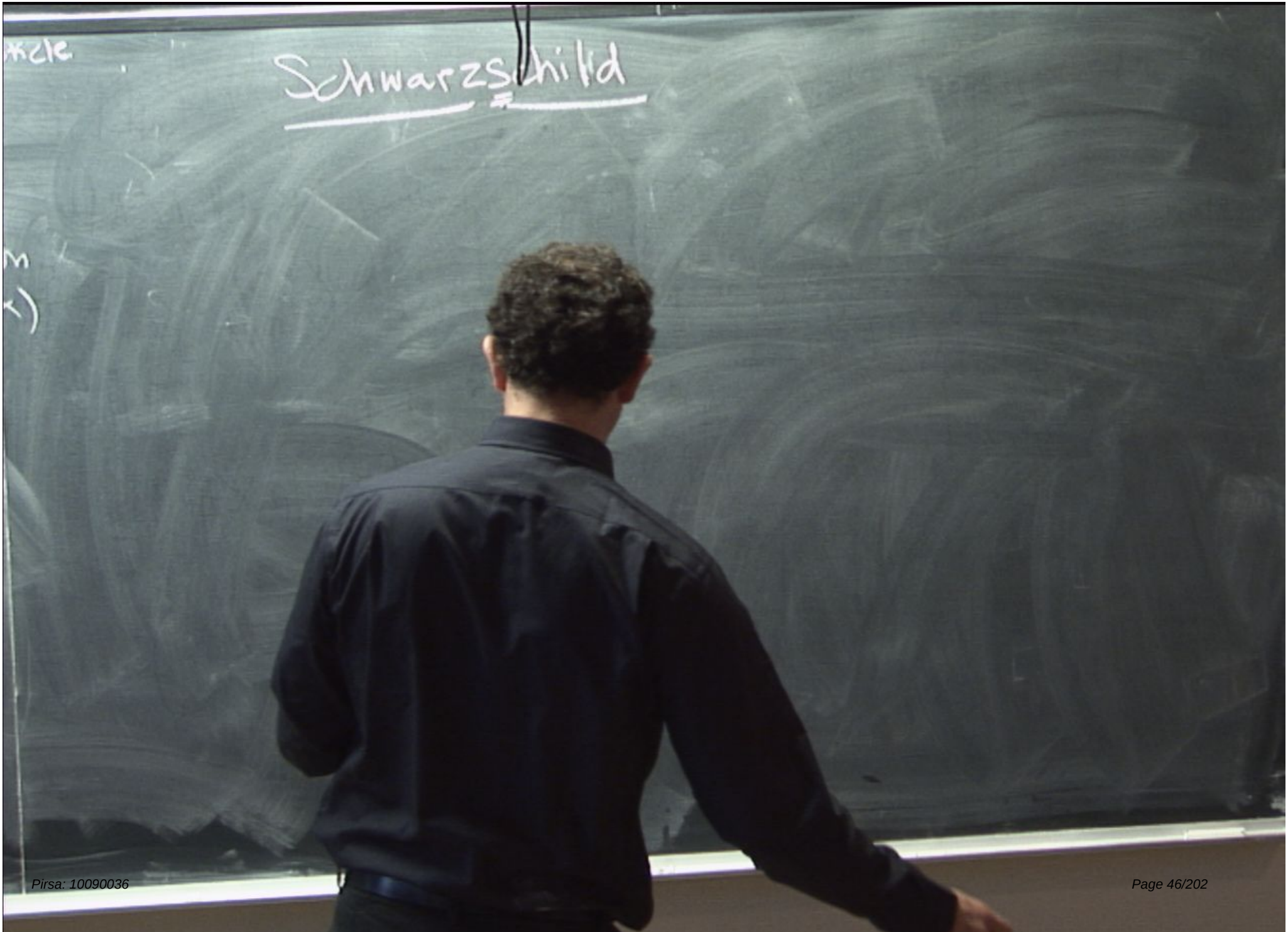
$e=1$ $\quad S = \frac{m}{2}\int \left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\tau - 1 \right)$ $\quad g_{\varphi\nu}\delta\dot{\varphi}\dot{x}^\nu$

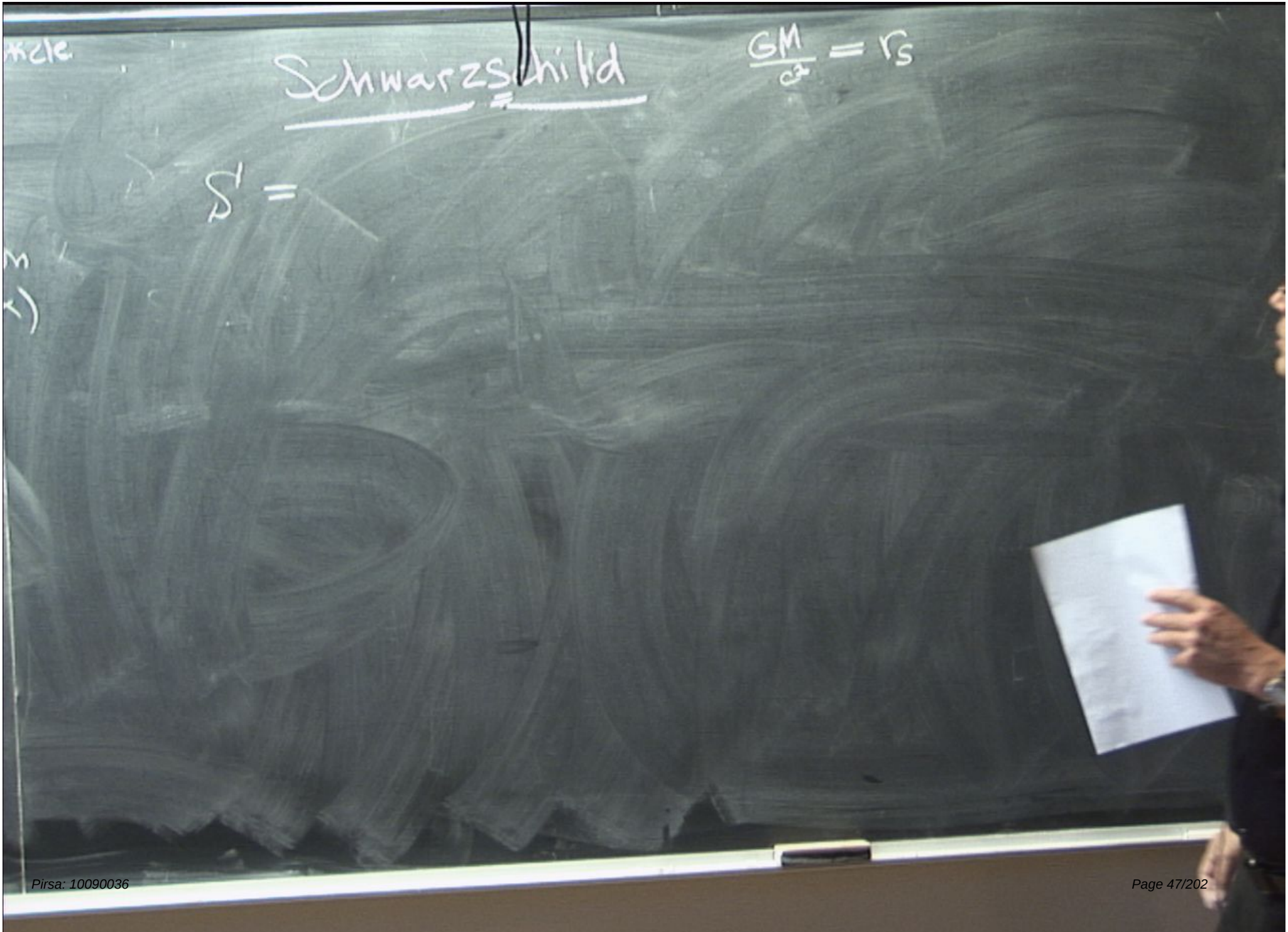
$$\frac{\delta S}{\delta x} = 0 \implies \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

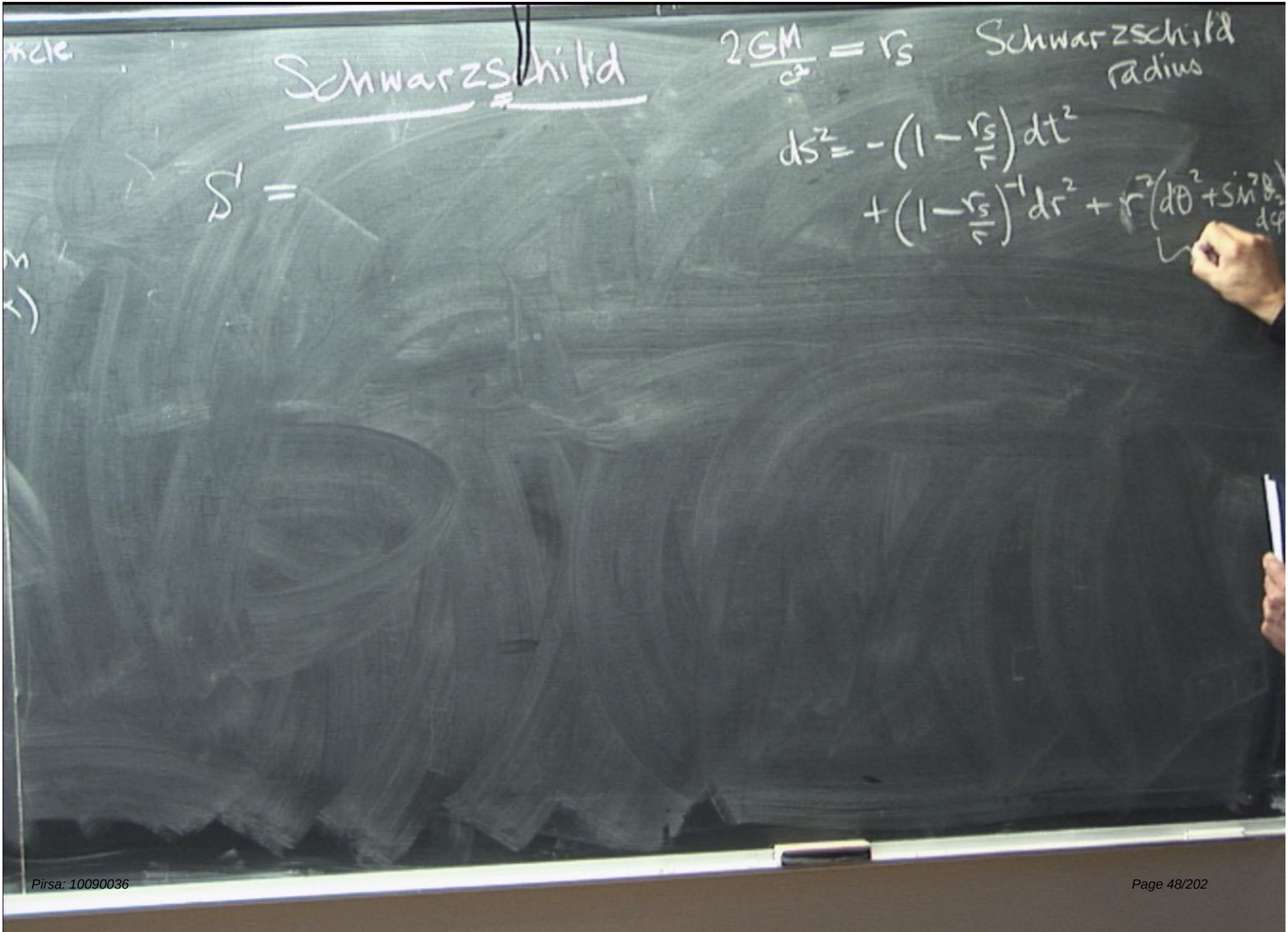
Note: if $g_{\mu\nu}$ is independent of same coordinate e.g. φ

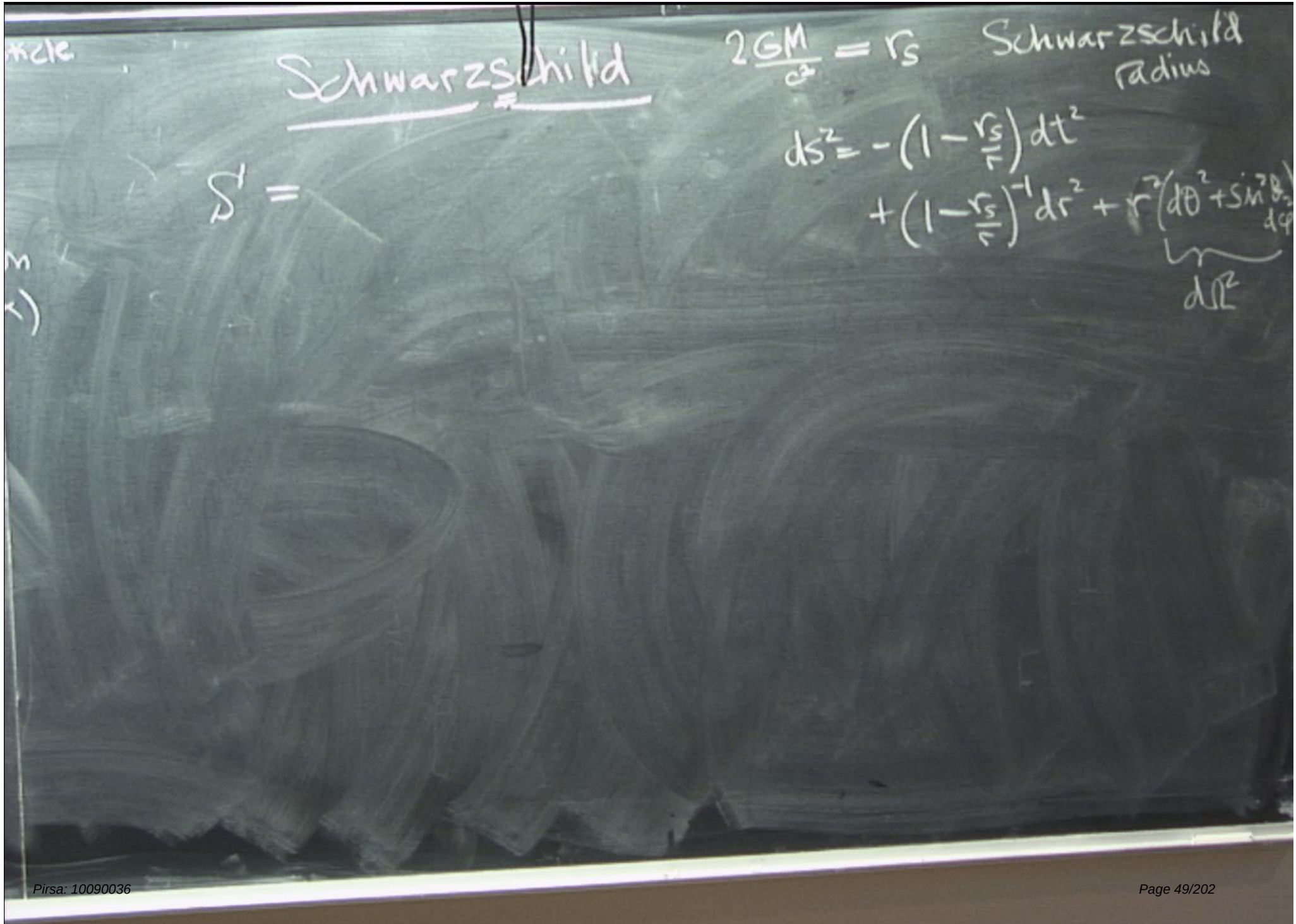
$$\frac{\delta S}{\delta \varphi} \implies \frac{d}{d\tau}\left(g_{\varphi\nu}\dot{x}^\nu \right) = 0 \qquad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$$

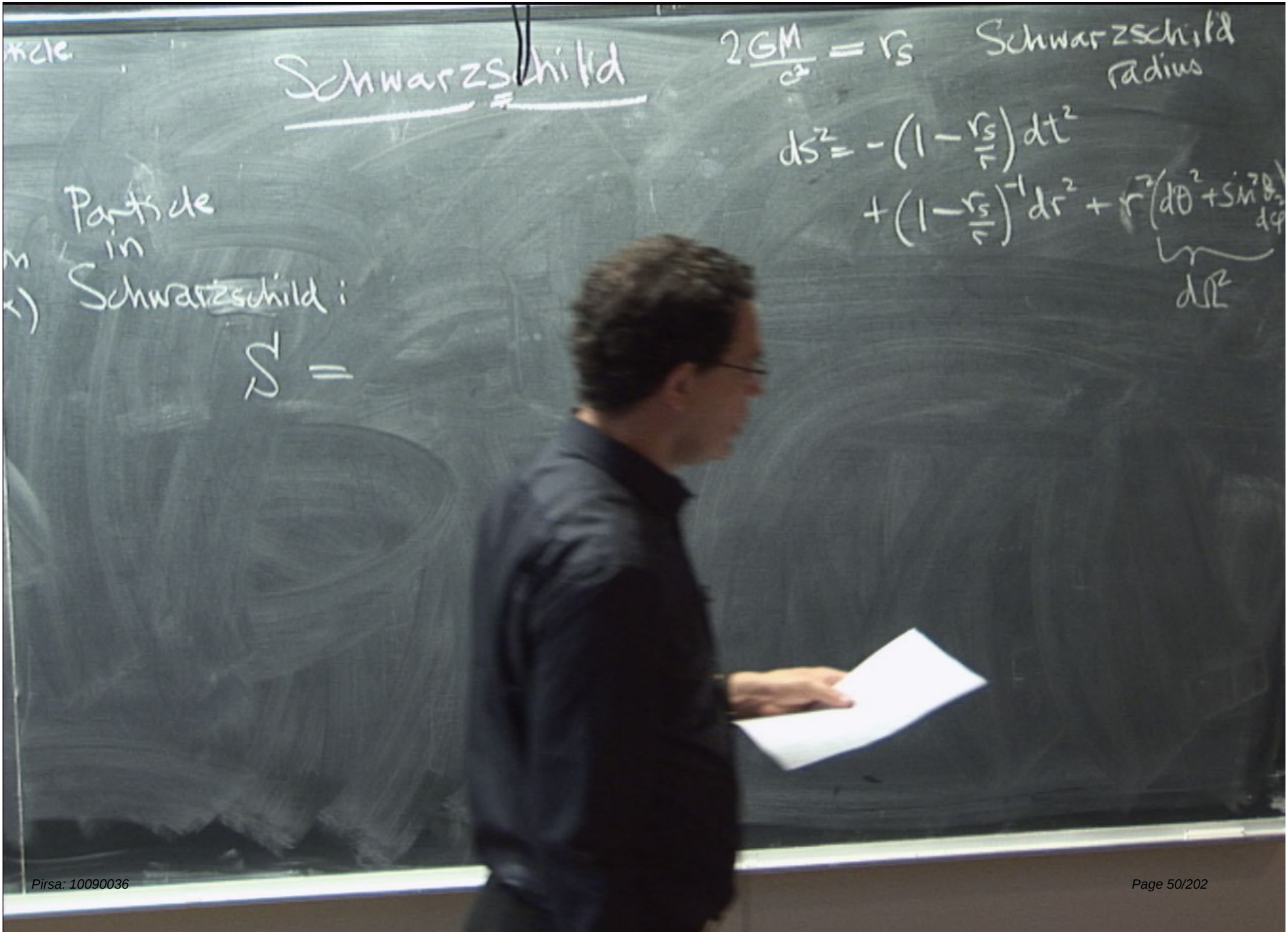
p_φ


Schwarzschild


Schwarzschild
$\frac{GM}{c^2} = r_s$
$S' =$


Schwarzschild
$2\frac{GM}{c^2} = r_s$ Schwarzschild radius
$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2$
$+\left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$


Schwarzschild
$\frac{2GM}{c^2} = r_s$ Schwarzschild radius
$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2$
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Particle in Schwarzschild:
$S =$

...cle

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...m Particle in

...) Schwarzschild:

$$S = \frac{m}{2}\int d\tau\left[-\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{r_s}{r}\right)} + r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)\right]$$

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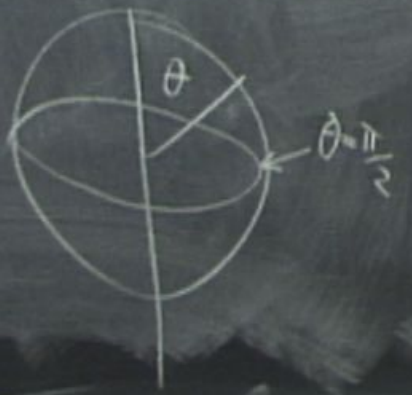
Particle in Schwarzschild:

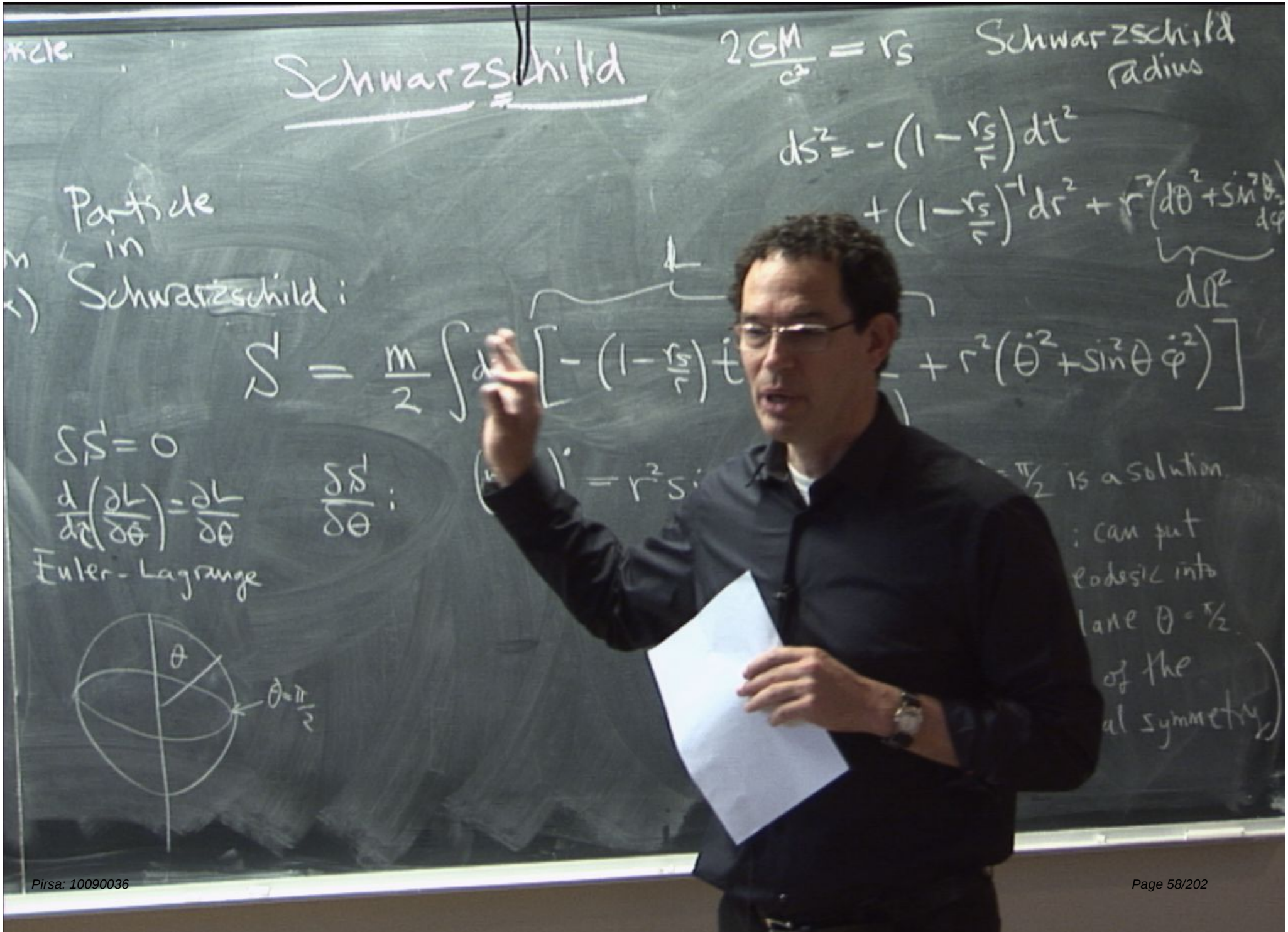
$$S = \frac{m}{2}\int d\tau \underbrace{\left[-\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{r_s}{r}\right)} + r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)\right]}_{L}$$

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Particle in Schwarzschild:
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$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$
$\frac{\delta S}{\delta \theta}$:
Euler-Lagrange
$\theta = \pi/2$ is a solution
: can put
geodesic into
plane $\theta = \pi/2$
of the
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Schwarzschild

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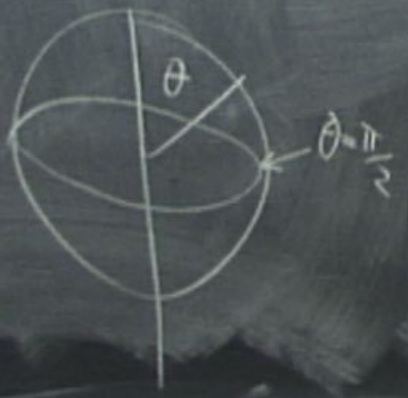
$\frac{\delta S}{\delta \theta}$: $\left(r^2\dot{\theta}\right)^{\cdot} = r^2\sin\theta\cos\theta\,\dot{\varphi}^2$; $\theta = \pi/2$ is a solution.

WLOG: can put any geodesic into this plane $\theta = \pi/2$ (because of the spherical symmetry)

$\frac{\delta S}{\delta \varphi}$: $\frac{d}{d\tau}\left(r^2\sin^2\theta\,\dot{\varphi}\right) = 0$

$\Rightarrow r^2\sin^2\theta\,\dot{\varphi} = L_z = \text{const}$

$= r^2\dot{\varphi}$ since $\theta = \pi/2$.



Schwarzschild

$2\frac{GM}{c^2} = r_s$ Schwarzschild radius

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$\delta S = 0$

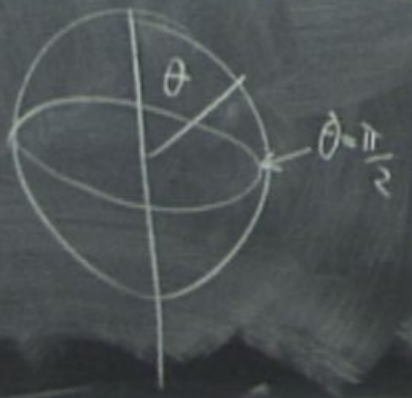
$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta}$ Euler-Lagrange

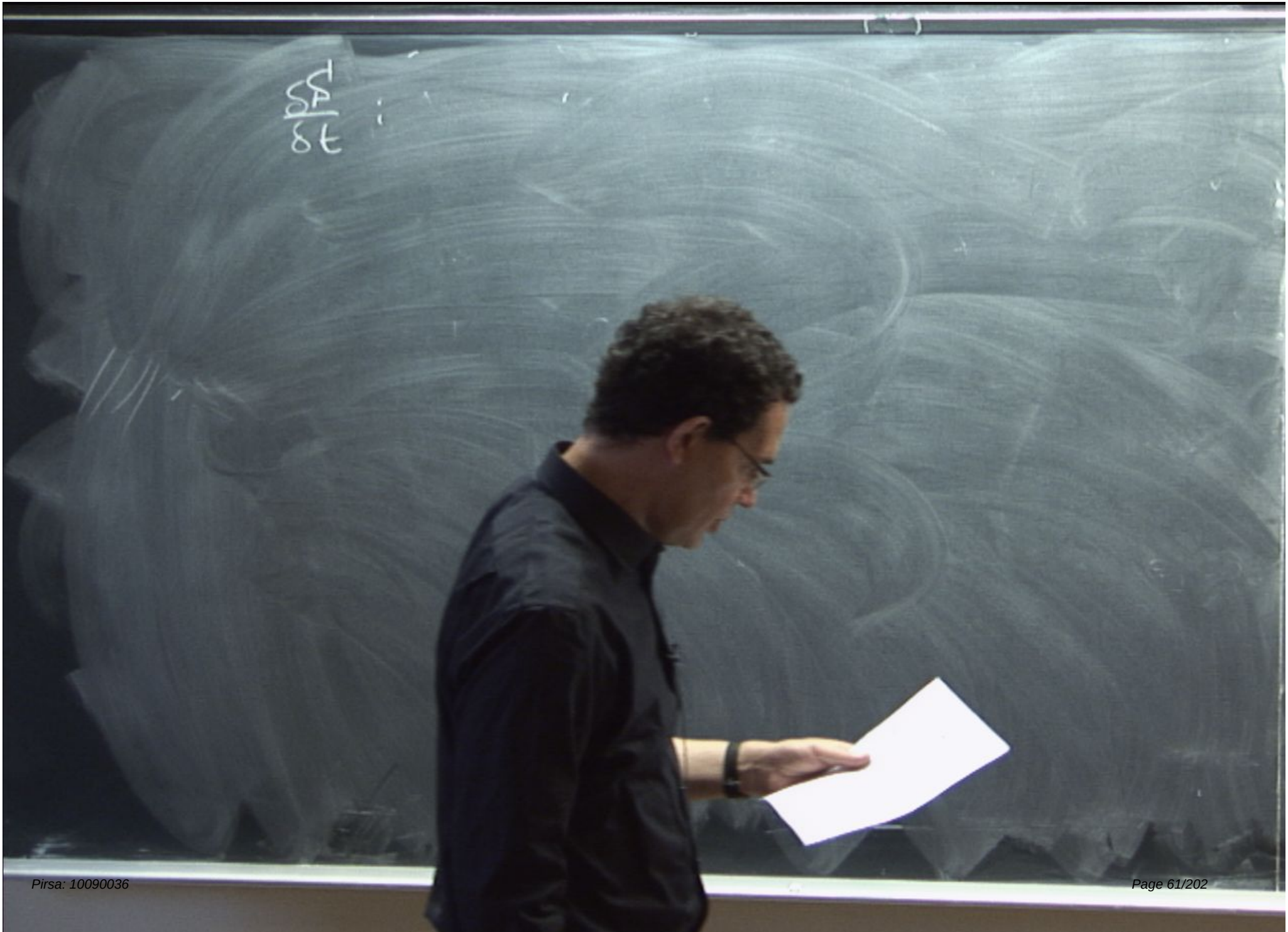
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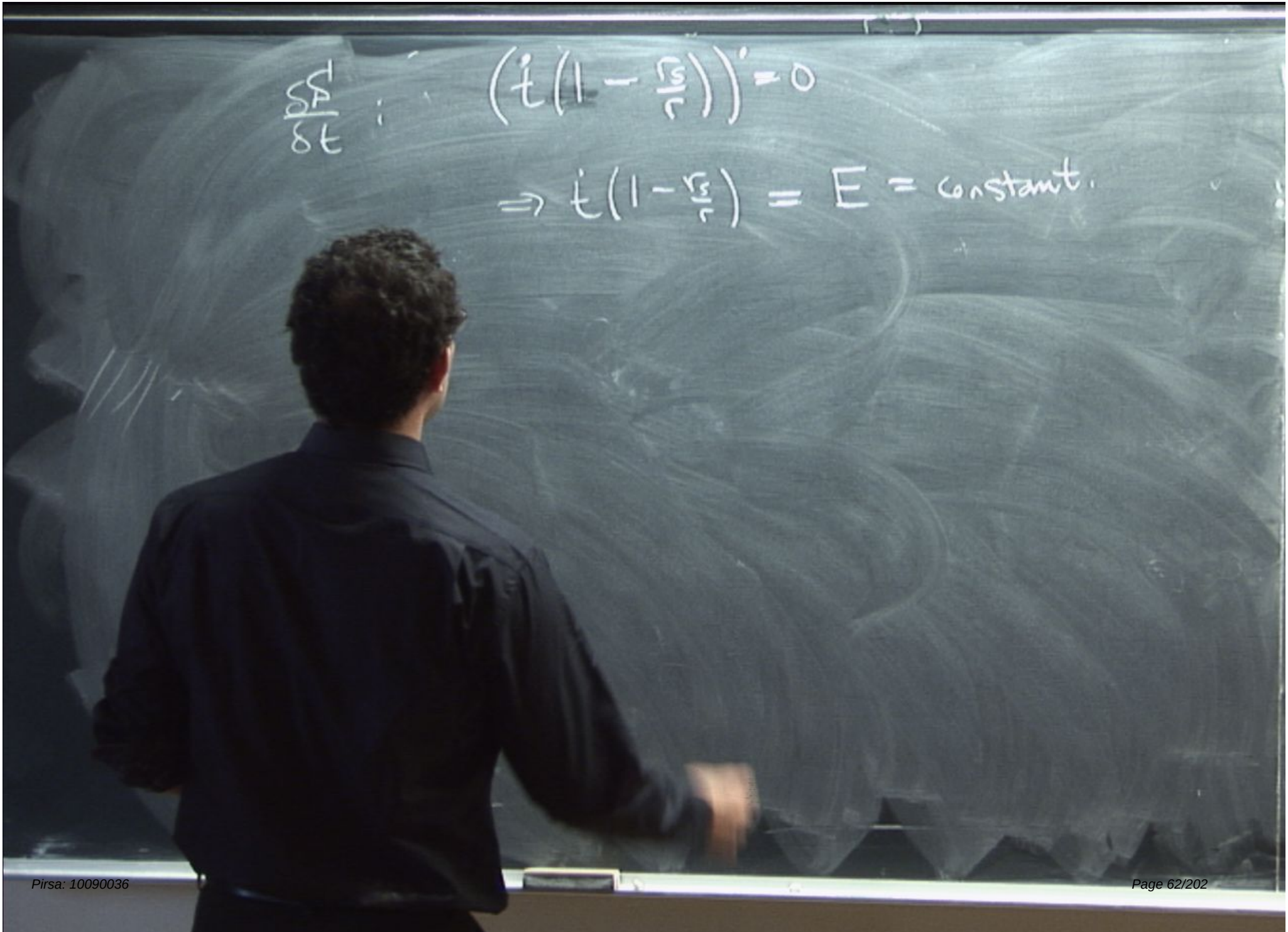
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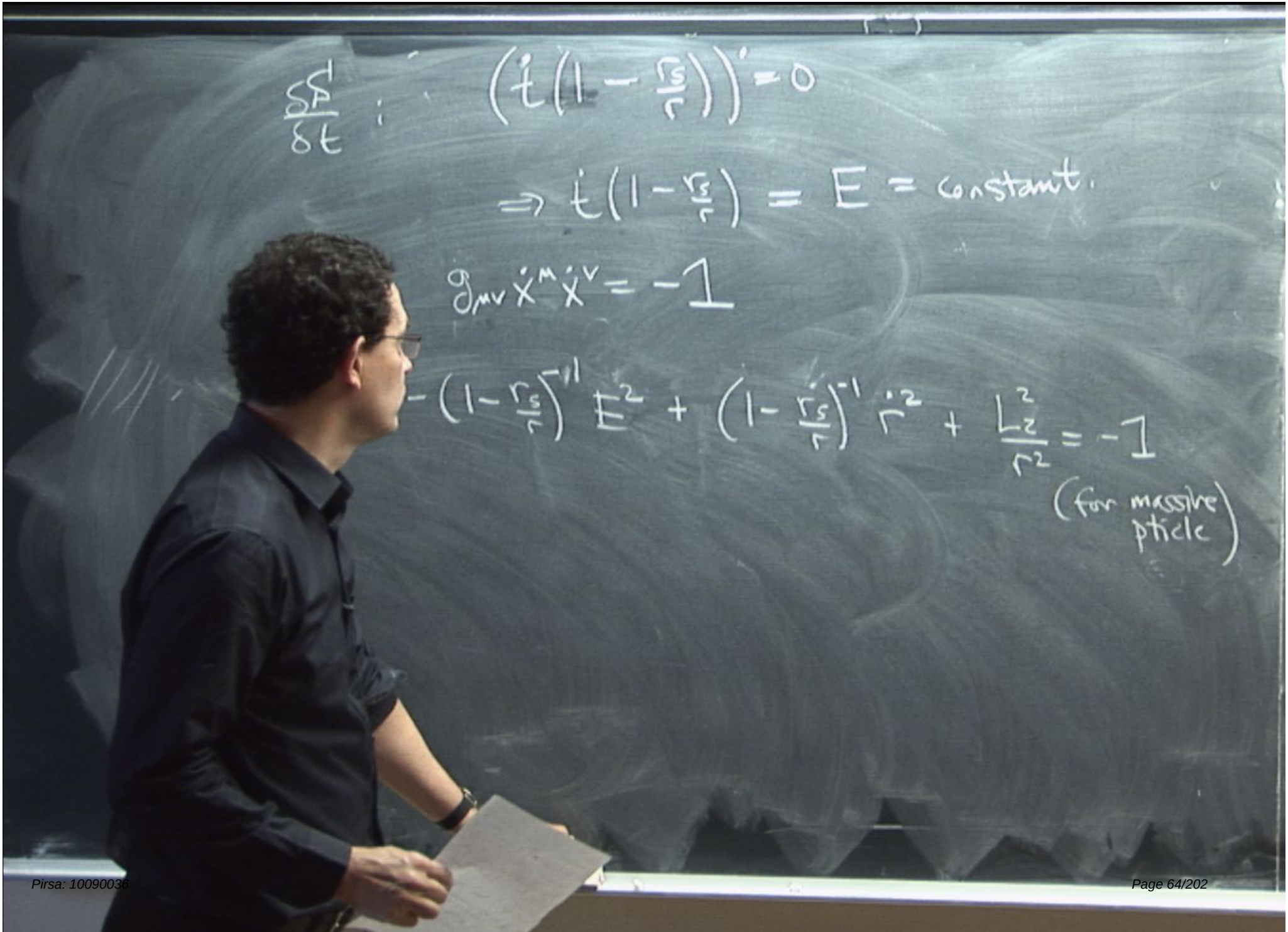

$\frac{\delta \mathcal{L}}{\delta t}$: $\left(\dot{t}\left(1-\frac{r_s}{r}\right)\right)' = 0$
$\Rightarrow \dot{t}\left(1-\frac{r_s}{r}\right) = E = \text{constant}.$

$$\frac{\delta \mathcal{L}}{\delta t} : \quad \left(\dot{t}\left(1-\frac{r_s}{r}\right)\right)' = 0$$

$$\Rightarrow \dot{t}\left(1-\frac{r_s}{r}\right) = E = \text{constant}.$$

now use $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$

$$-\left(1-\frac{r_s}{r}\right)^{-1} E^2 + \left(1-\frac{r_s}{r}\right)^{-1} \dot{r}^2 + \frac{L_z^2}{r^2} = -1$$


$\frac{\delta A}{\delta t} : \left(\dot{t}\left(1-\frac{r_s}{r}\right)\right)' = 0$
$\Rightarrow \dot{t}\left(1-\frac{r_s}{r}\right) = E = \text{constant}.$
$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$
$-\left(1-\frac{r_s}{r}\right)^{-1}E^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + \frac{L_z^2}{r^2} = -1$
(for massive ptiicle)

$\frac{\delta S}{\delta t}$: $\left(\dot{t}\left(1-\frac{r_s}{r}\right)\right)^{\cdot} = 0$

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$\Rightarrow$ straight forward to analyse (for massive ptcle)

(this pm)

$$\frac{\delta \mathcal{L}}{\delta t} : \quad \left(\dot{t}\left(1-\frac{r_s}{r}\right)\right)' = 0$$

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now use $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$

$$-\left(1-\frac{r_s}{r}\right)^{-1} E^2 + \left(1-\frac{r_s}{r}\right)^{-1} \dot{r}^2 + \frac{L_z^2}{r^2} = -1$$

$\Rightarrow$ straight forward to analyse (for massive ptiele)

(this pm)

Bending of light $S = \int \frac{d\lambda}{e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$

$$S = \frac{m}{2} \int d\tau \underbrace{\left[-\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right) \right]}_{L}$$

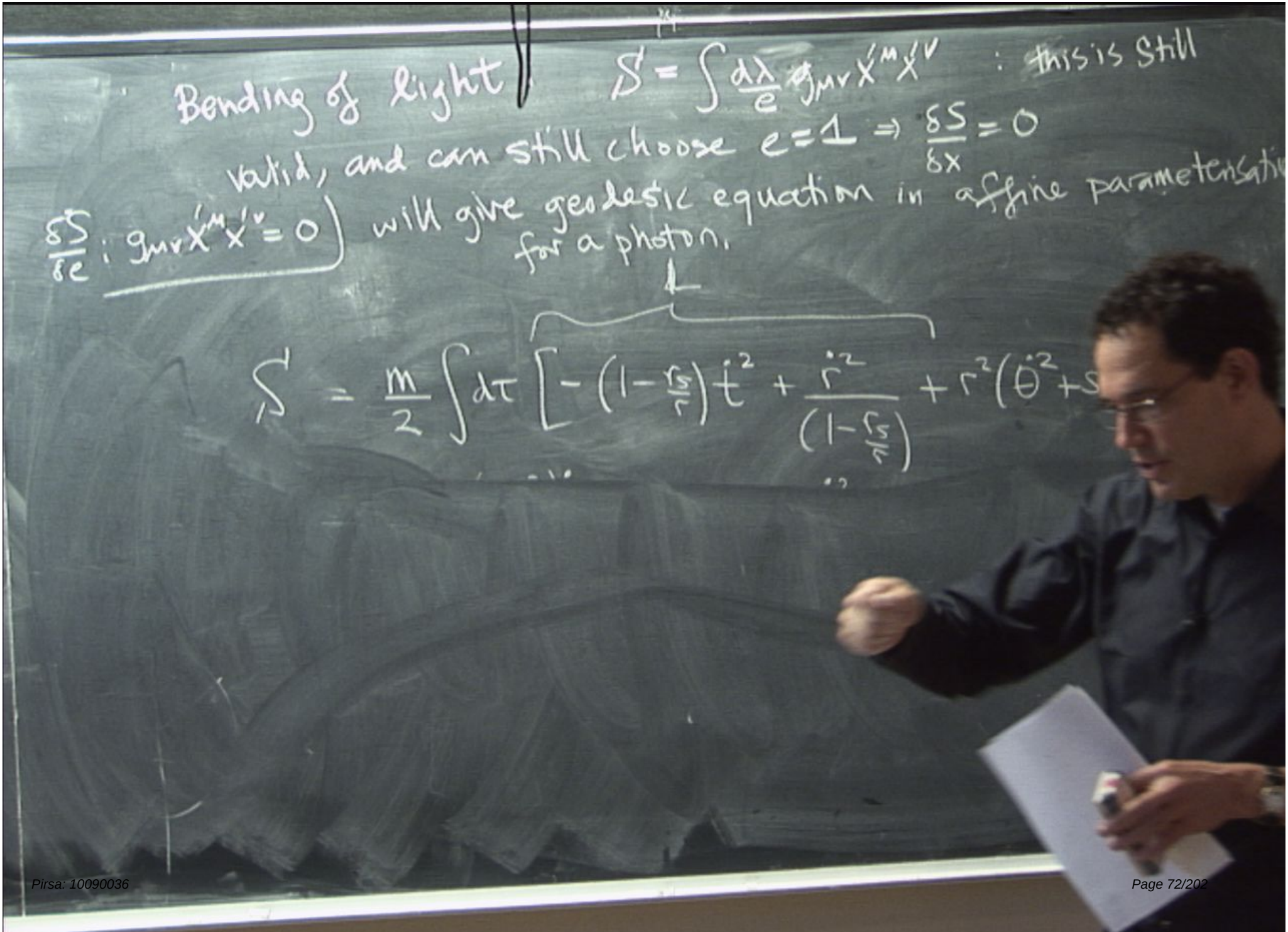
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$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$

$e \to \Omega^2(e)$

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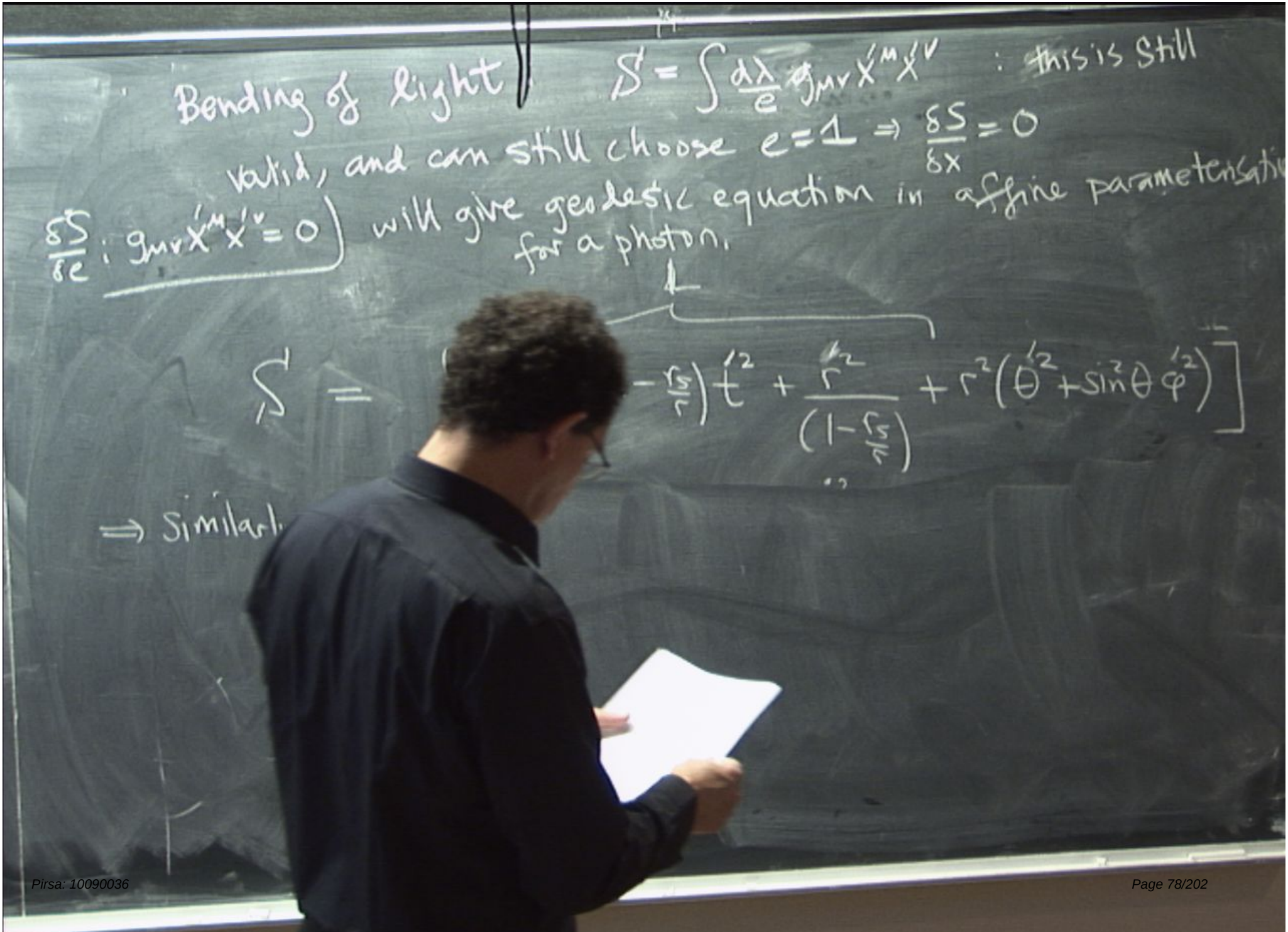
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$\Rightarrow$ Similarl

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$\Rightarrow$ Similarly can set $\theta = \pi/2$

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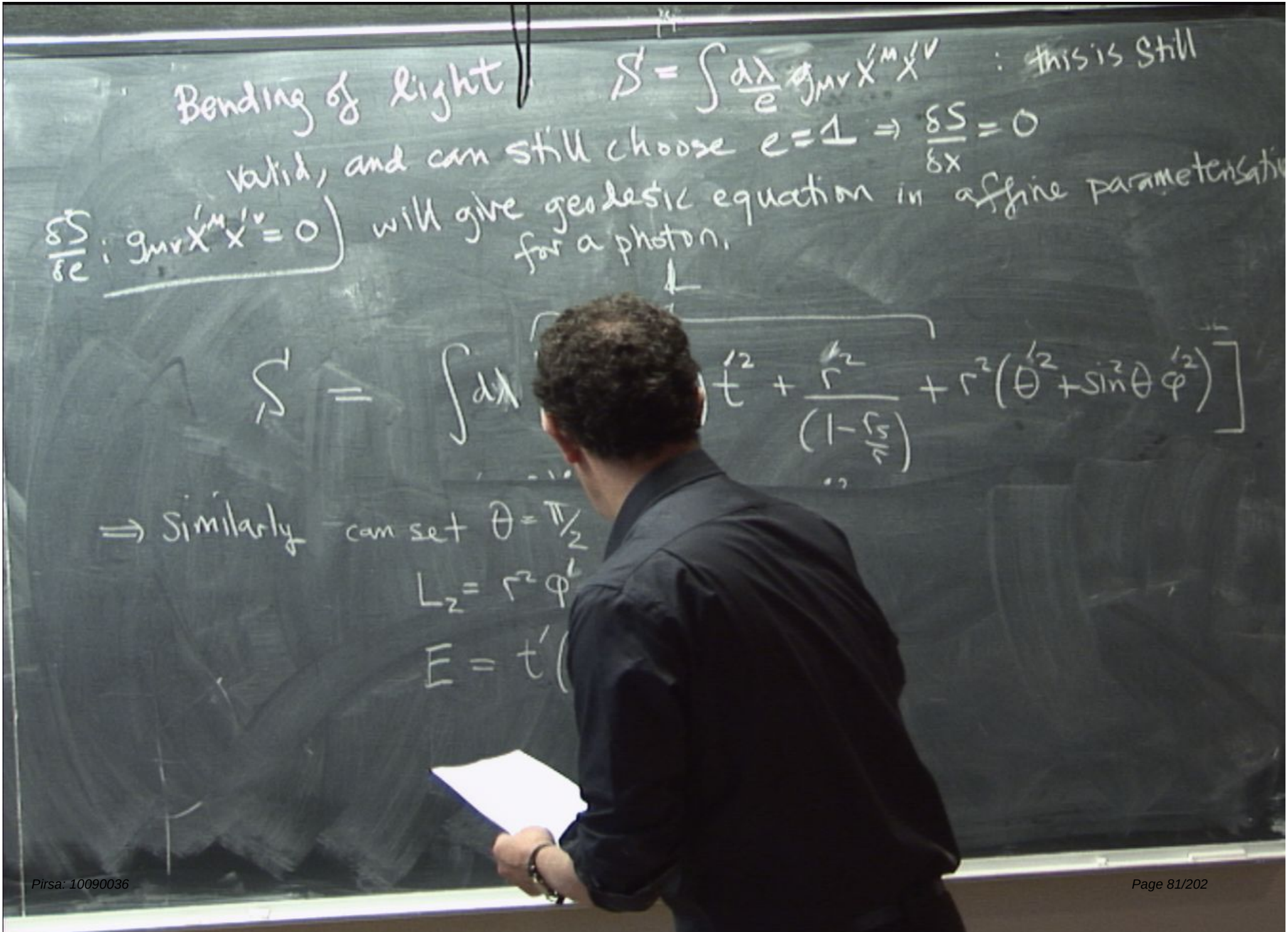
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$$L_z = r^2 \varphi' = 0$$


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$E = t'$

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$L_z = r^2 \dot{\varphi} = $ constant

$E = \dot{t}\left(1 - \frac{r_s}{r}\right) = $ constant.

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(actual value of E is not physical :)

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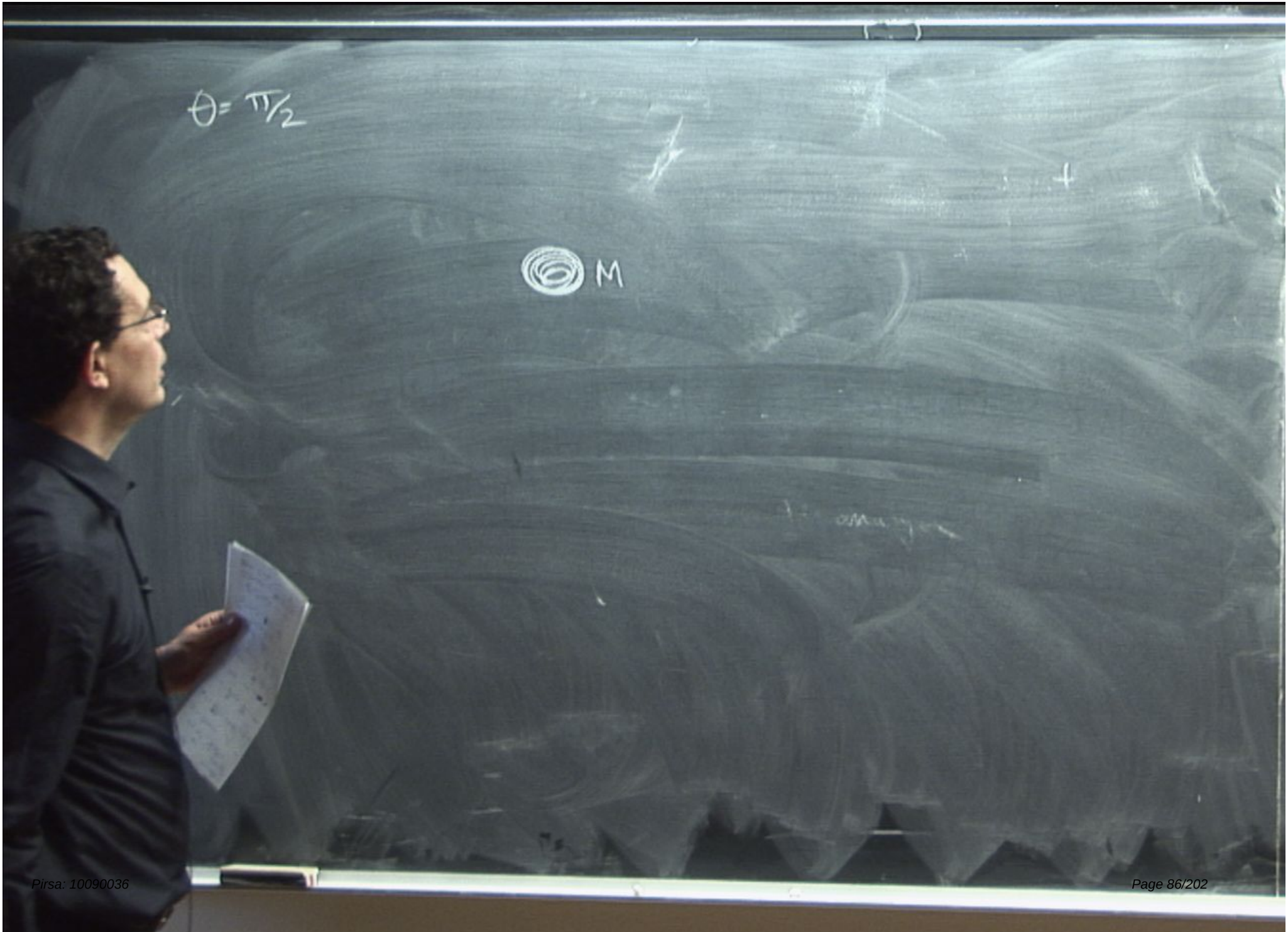
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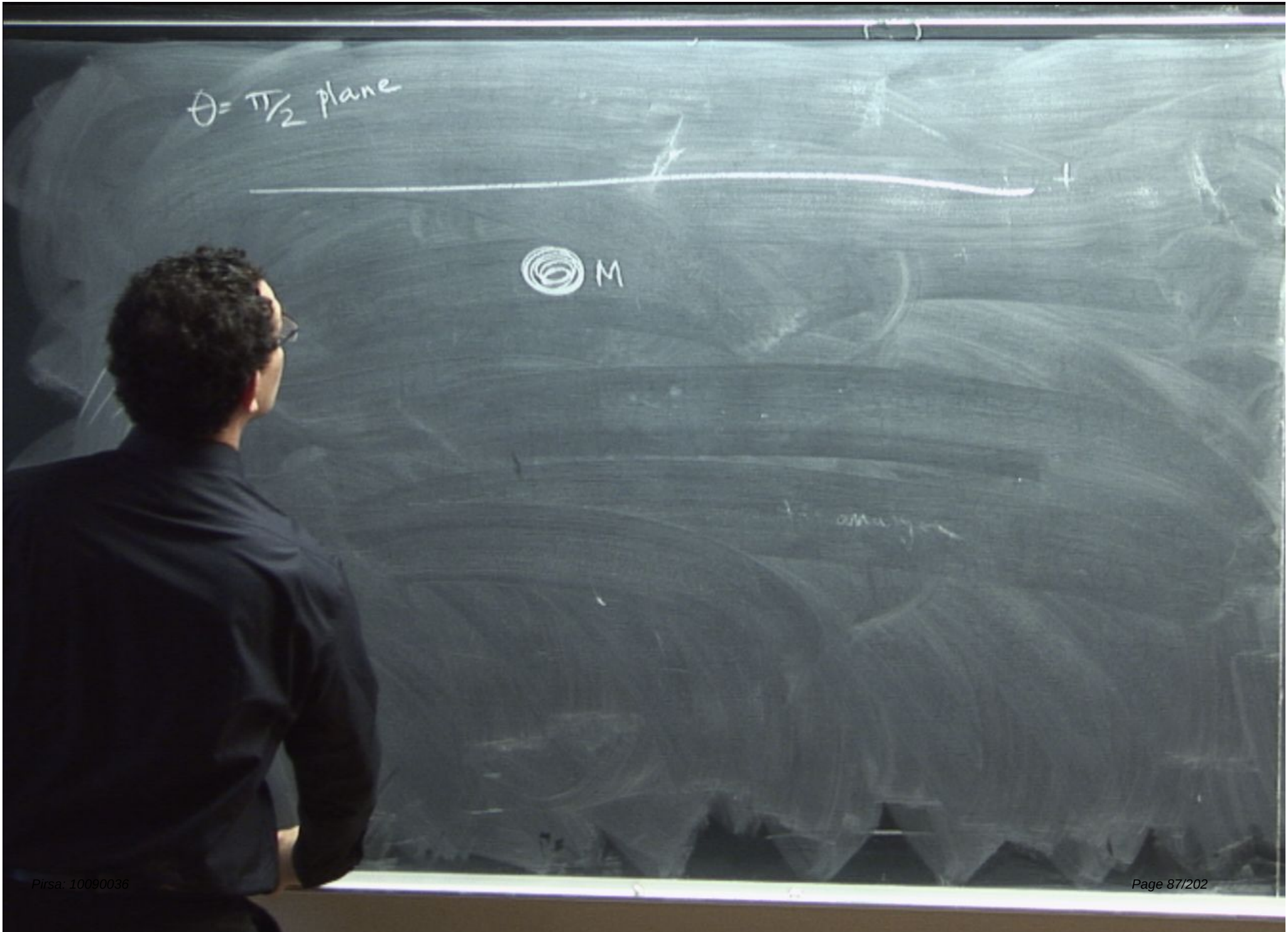
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θ = π/2
M


θ = π/2 plane
M

$\theta = \pi/2$ plane



$\theta = \pi/2$ plane



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$\theta = \pi/2$ plane



Bending of light

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$\theta = \pi/2$ plane



$$r' = \left(\frac{dr}{d\lambda}\right) = \frac{d\varphi}{d\lambda}\frac{dr}{d\varphi} = \frac{L_z}{r^2}\frac{dr}{d\varphi}.$$

$\theta = \pi/2$ plane



$$u = \frac{1}{r}$$

$$r' = \left(\frac{dr}{d\lambda}\right) = \frac{d\varphi}{d\lambda}\frac{dr}{d\varphi} = \frac{L_z}{r^2}\frac{dr}{d\varphi} = -L_z\frac{du}{d\varphi}$$

$\theta = \pi/2$ plane



$u = \frac{1}{r}$ $du = -\frac{1}{r^2} dr$

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$\theta = \pi/2$ plane



$u = \frac{1}{r}$ $du = -\frac{1}{r^2} dr$ $\frac{dS}{dr}$

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$g_{\mu\nu}x$

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$\theta = \pi/2$ plane



$u = \frac{1}{r}$ $du = -\frac{1}{r^2} dr$

$$r' = \left(\frac{dr}{d\lambda}\right) = \frac{d\varphi}{d\lambda}\frac{dr}{d\varphi} = \frac{L_z}{r^2}\frac{dr}{d\varphi} = -L_z\frac{du}{d\varphi}$$

$$(*) \Rightarrow \quad L_z^2\left(\frac{du}{d\varphi}\right)^2 = E^2$$

$\theta = \pi/2$ plane



$u = \frac{1}{r}$ $du = -\frac{1}{r^2} dr$

$r' = \left(\frac{dr}{d\lambda}\right) = \frac{d\varphi}{d\lambda}\frac{dr}{d\varphi} = \frac{L_z}{r^2}\frac{dr}{d\varphi} = -L_z$

$(*) \Rightarrow$ $L_z^2\left(\frac{du}{d\varphi}\right)^2 = E^2$

$\theta = \pi/2$ plane



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$\theta = \pi/2$ plane

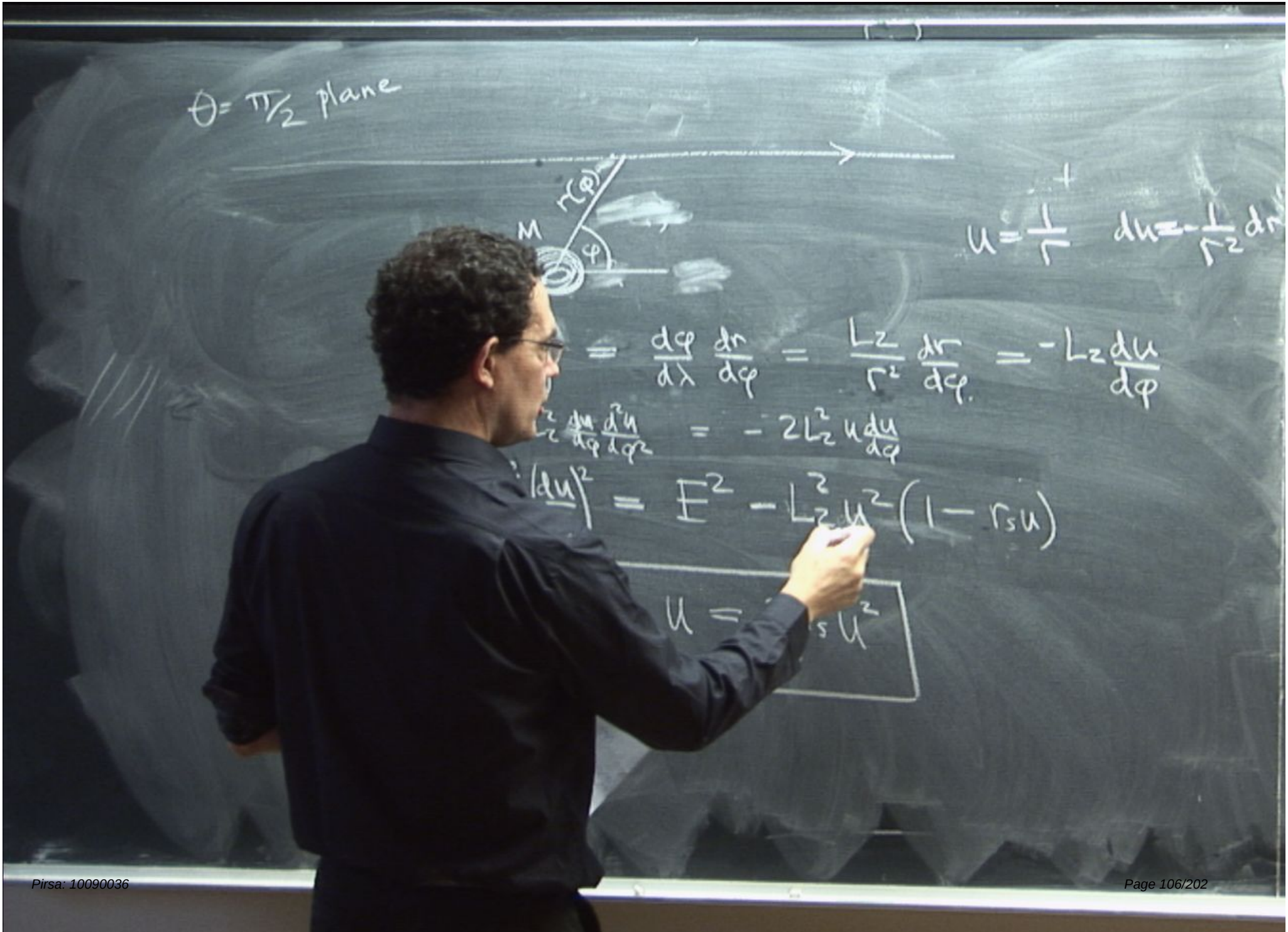


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$$\frac{d}{d\varphi} \Rightarrow \quad \frac{d^2u}{d\varphi^2}$$


θ = π/2 plane
M
r(φ)
φ
u = 1/r
du = -1/r² dr
= dφ/dλ dr/dφ = L_z/r² dr/dφ = -L_z du/dφ
L_z² du/dφ d²u/dφ² = -2L_z² u du/dφ
(du/dφ)² = E² - L_z² u² (1 - r_s u)
u =
s u²

$\theta = \pi/2$ plane



$u = \frac{1}{r}$ $du = -\frac{1}{r^2} dr$

$$r' = \left(\frac{dr}{d\lambda}\right) = \frac{d\varphi}{d\lambda}\frac{dr}{d\varphi} = \frac{L_z}{r^2}\frac{dr}{d\varphi} = -L_z\frac{du}{d\varphi}$$

$$2L_z^2 u\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} = -2L_z^2 u\frac{du}{d\varphi} + 3L_z^2 r_s u^2\frac{du}{d\varphi}$$

$$(*) \Rightarrow \quad L_z^2\left(\frac{du}{d\varphi}\right)^2 = E^2 - L_z^2 u^2(1 - r_s u)$$

$$\frac{d}{d\varphi} \Rightarrow \quad \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{3}{2}r_s u^2}$$

$\theta = \pi/2$ plane



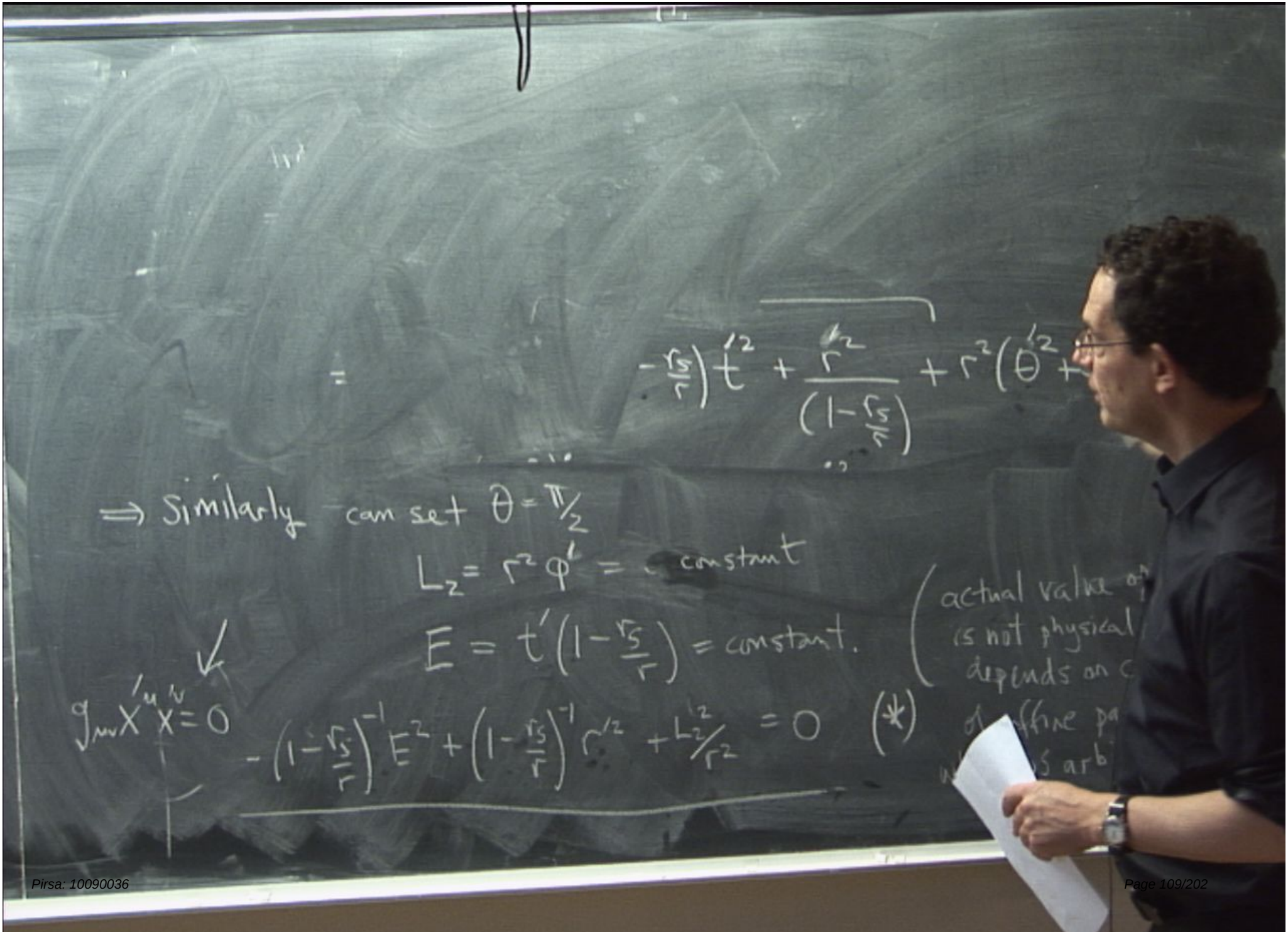
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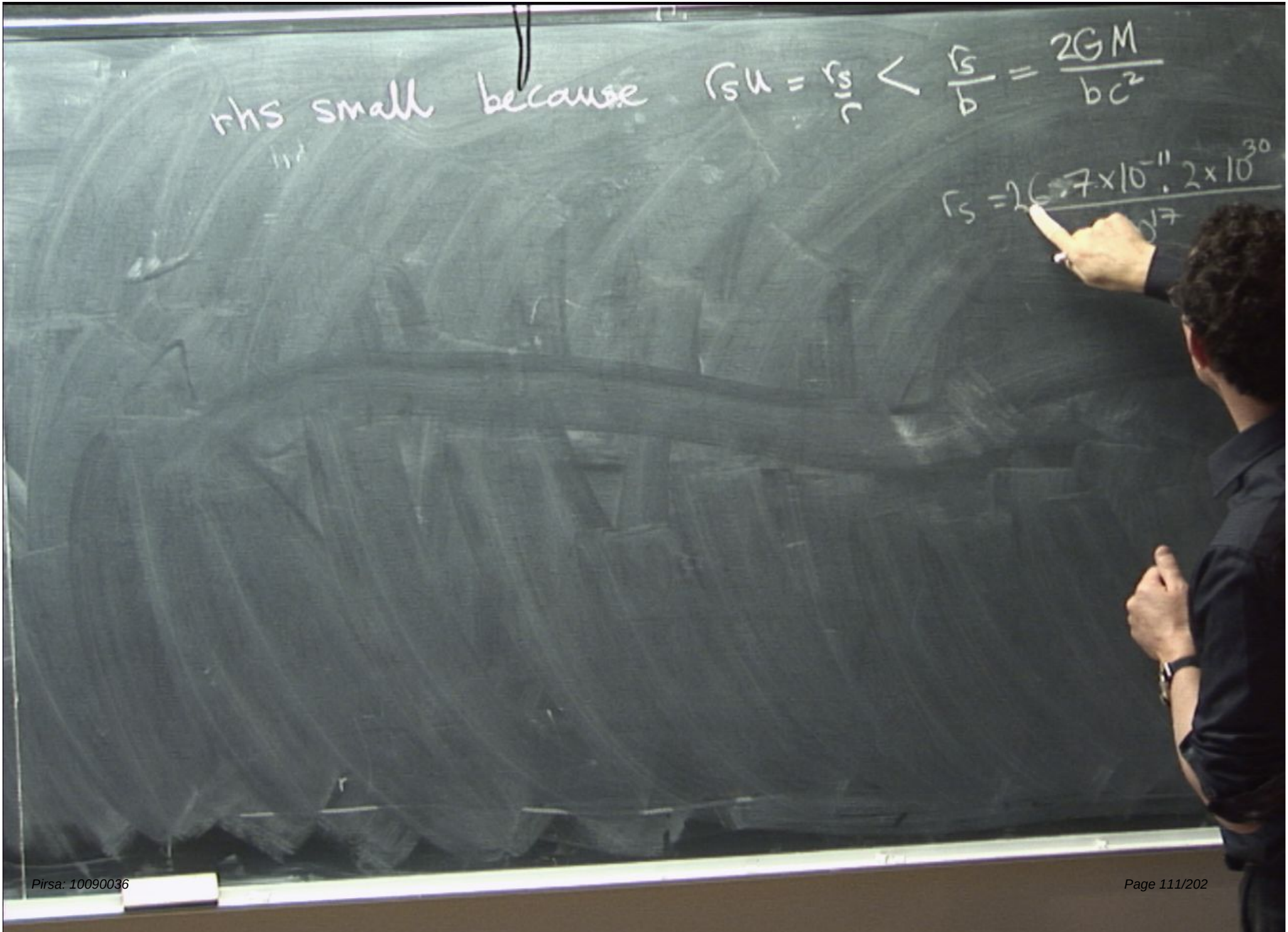
$$2L_z^2 u\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} = -2L_z^2 u\frac{du}{d\varphi} + 3L_z^2 r_s u^2\frac{du}{d\varphi}$$

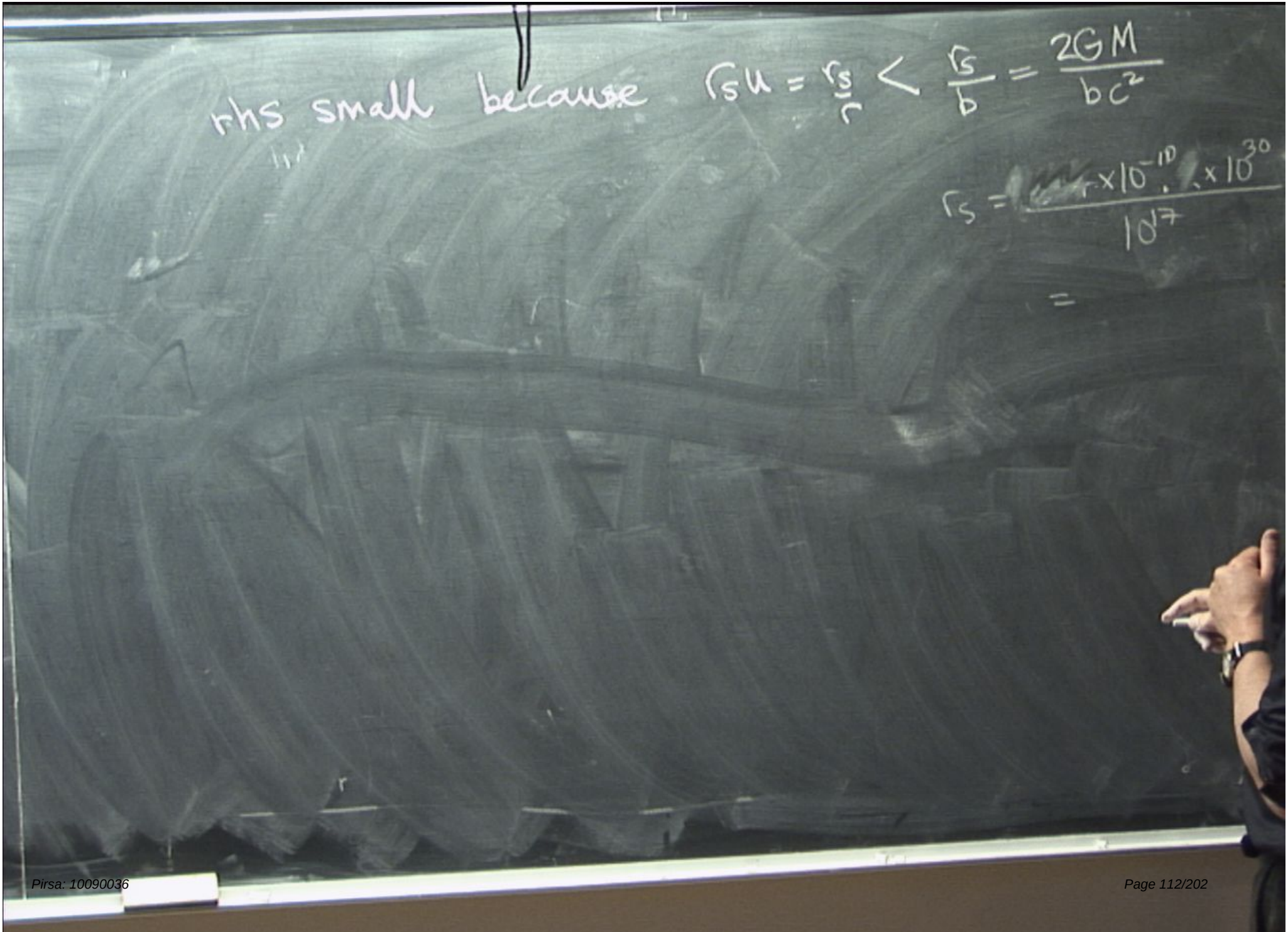
$$(*) \Rightarrow \quad L_z^2\left(\frac{du}{d\varphi}\right)^2 = E^2 - L_z^2 u^2(1 - r_s u)$$

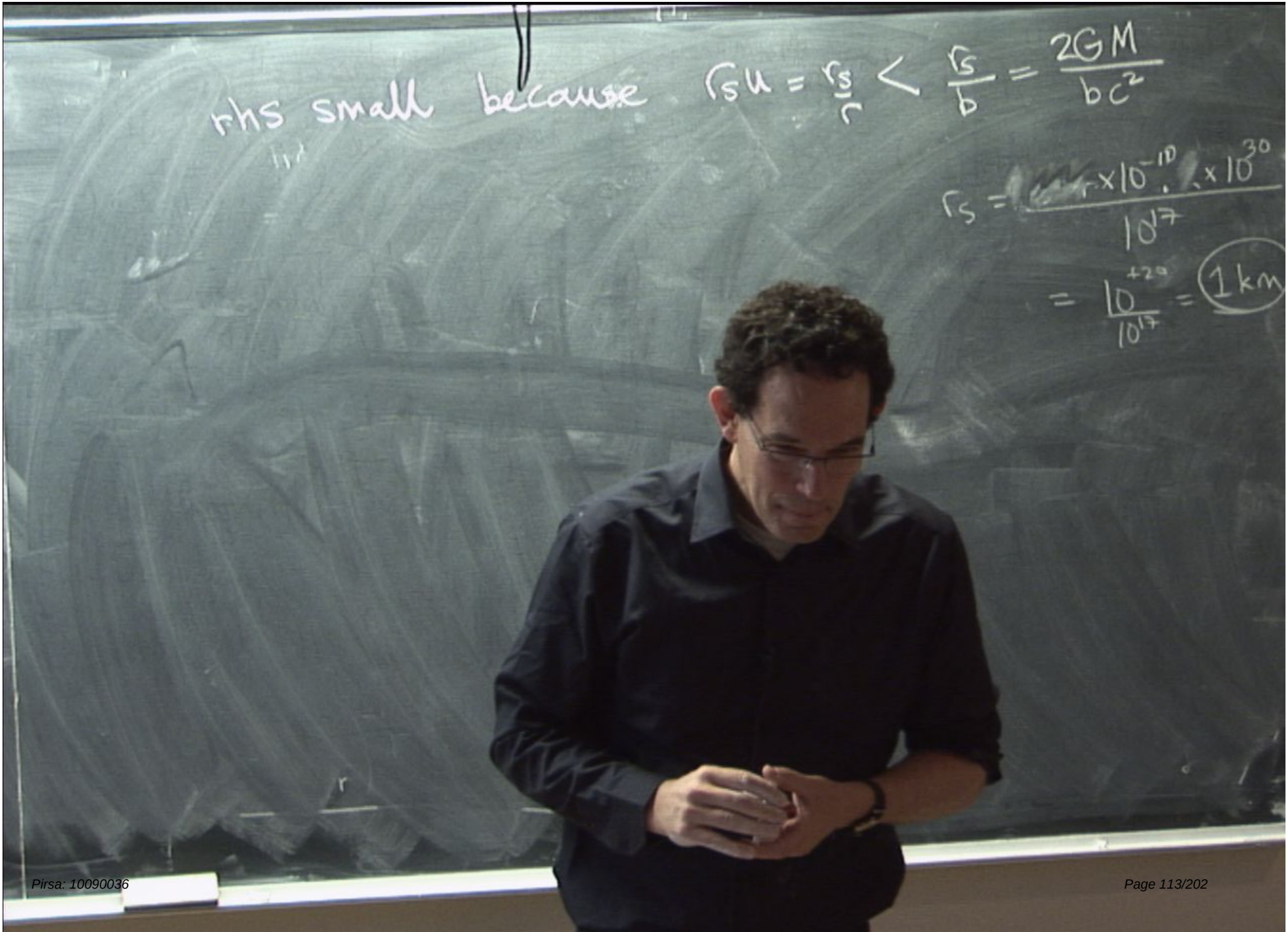
$$\frac{d}{d\varphi} \Rightarrow \quad \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{3}{2} r_s u^2}$$


$= -\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{r_s}{r}\right)} + r^2(\dot{\theta}^2 +$
$\Rightarrow$ Similarly can set $\theta = \pi/2$
$L_z = r^2\dot{\varphi} =$ constant
$E = t'\left(1-\frac{r_s}{r}\right) =$ constant.
(actual value of
is not physical
depends on c
affine pa
s arb
$g_{\mu\nu}X'^{\mu}X'^{\nu} = 0$
$-\left(1-\frac{r_s}{r}\right)^{-1}E^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + L_z^2/r^2 = 0$ (*)

rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b}$


rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2}$
$r_s = 2(\quad .7\times10^{-11}\,.\,2\times10^{30}$


rhs small because $r_{SU} = \frac{r_S}{r} < \frac{r_S}{b} = \frac{2GM}{bc^2}$
$r_S = \frac{\times 10^{-10} \times 10^{30}}{10^7}$
=


rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2}$
$r_s = \frac{\times 10^{-10} \times 10^{30}}{10^{17}}$
$= \frac{10^{+20}}{10^{17}} =$ 1 km

rhs small because $r_{SU} = \frac{r_S}{r} < \frac{r_S}{b} = \frac{2GM}{bc^2} \ll 1$

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→ can solve perturbatively

$r_s = \frac{\sim 1 \times 10^{-10} \times 10^{30}}{10^{17}}$

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$= \frac{10^{+29}}{10^{17}} =$ 1 km

first, neglect rhs.

$\Rightarrow \quad u = \frac{1}{b}\sin\varphi$

rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2} \ll 1$

→ can solve perturbatively

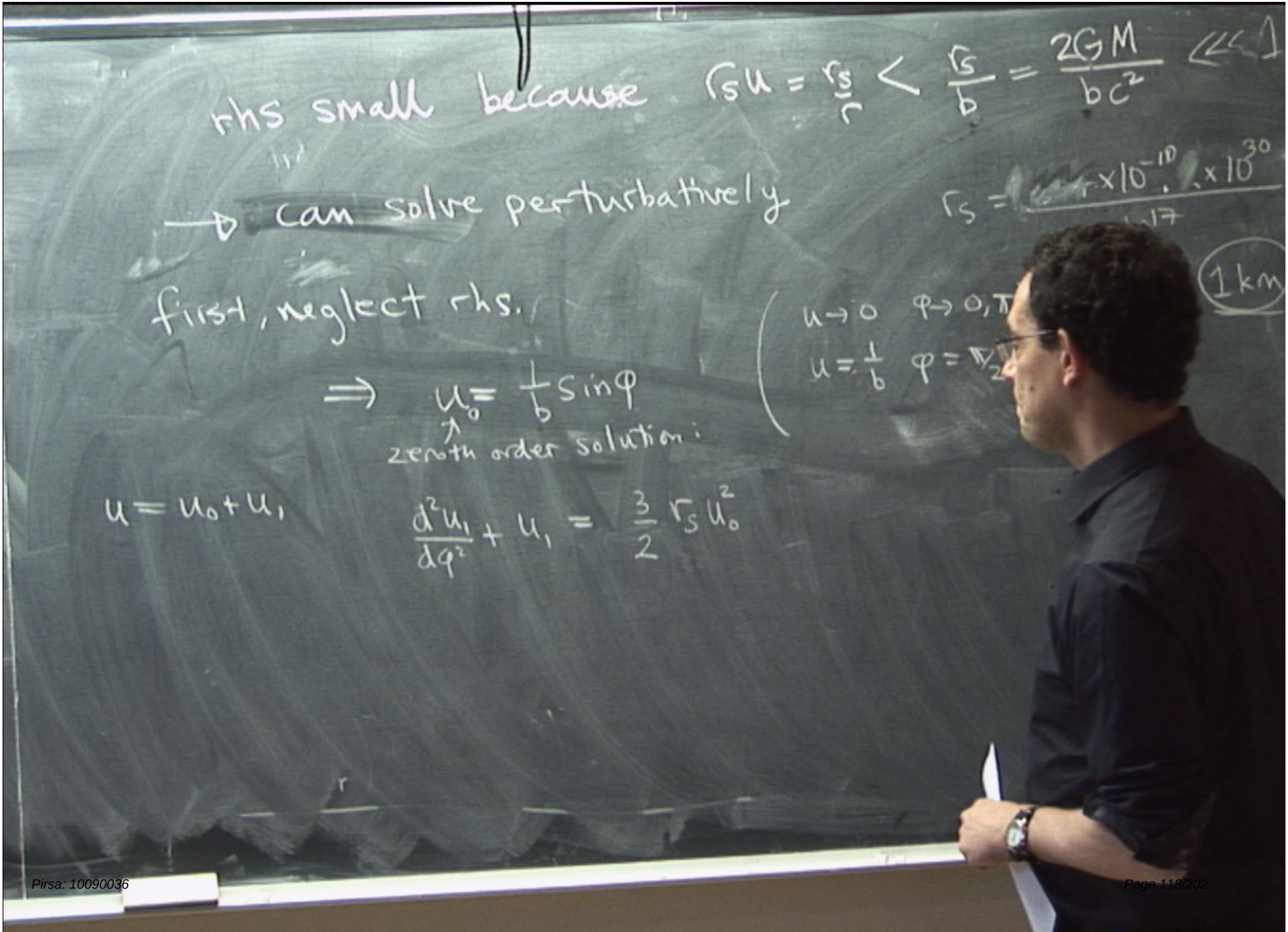
$r_s = \frac{\sim 1 \times 10^{-10} \times 10^{30}}{10^{17}}$

first, neglect rhs.

$\left(\begin{array}{l} u \to 0 \quad \varphi \to 0, \pi \\ u = \frac{1}{b} \quad \varphi = \pi/2 \end{array} \right)$

$\Rightarrow u = \frac{1}{b} \sin\varphi$

$= \frac{10^{+29}}{10^{17}} =$ 1 km


rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2} \ll 1$
can solve perturbatively
first, neglect rhs.
$\Rightarrow u_0 = \frac{1}{b}\sin\varphi$
zeroth order solution:
$u = u_0 + u_1$
$\frac{d^2u_1}{d\varphi^2} + u_1 = \frac{3}{2} r_s u_0^2$
1 km

rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2} \ll 1$

$\Rightarrow$ can solve perturbatively

$r_s = \frac{\sim 1 \times 10^{-10} \times 10^{30}}{10^{17}}$

first, neglect rhs.

$\left(\begin{array}{l} u \to 0 \quad \varphi \to 0, \pi \\ u = \frac{1}{b} \quad \varphi = \pi/2 \end{array} \right) = \frac{10^{+29}}{10^{17}} =$ 1 km

$\Rightarrow u_0 = \frac{1}{b} \sin\varphi$

u_0: zeroth order solution:

$u = u_0 + u_1$ $\qquad \frac{d^2 u_1}{d\varphi^2} + u_1 = \frac{3}{2} r_s u_0^2$ to leading order in $r_s u$.

$\theta = \pi/2$ plane



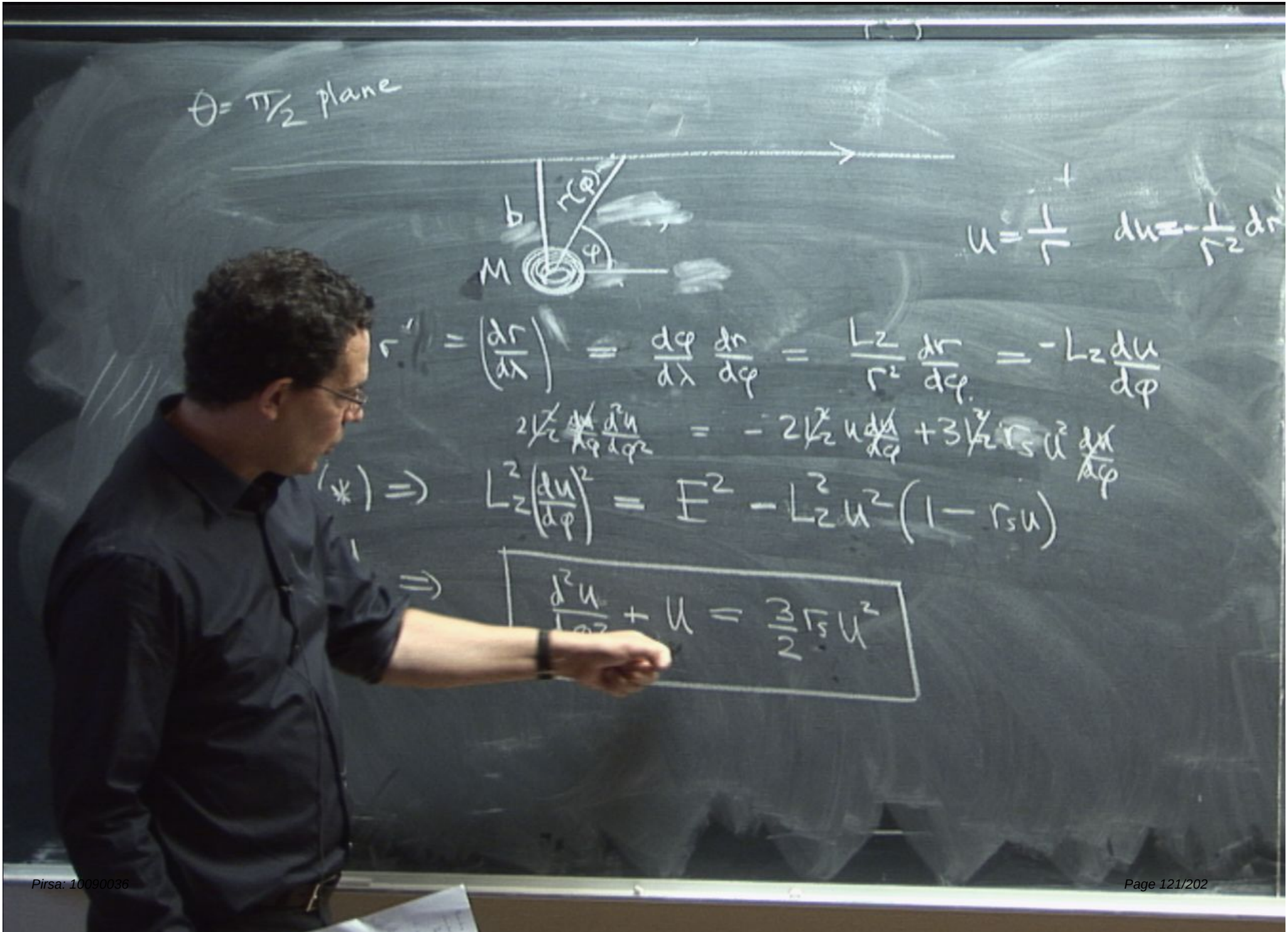
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$\frac{d}{d\varphi} \Rightarrow$ $\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{3}{2} r_s u^2}$


θ = π/2 plane
b
r(φ)
φ
M
u = 1/r
du = -1/r² dr
r' = (dr/dλ) = dφ/dλ dr/dφ = L_z/r² dr/dφ = -L_z du/dφ
2L_z² du/dφ d²u/dφ² = -2L_z² u du/dφ + 3L_z² r_s u² du/dφ
(*) ⇒ L_z² (du/dφ)² = E² - L_z² u² (1 - r_s u)
⇒ d²u/dφ² + u = 3/2 r_s u²

rhs small because $r_s u = \frac{r_s}{r} < \frac{r_s}{b} = \frac{2GM}{bc^2} \ll 1$

$\rightarrow$ can solve perturbatively

$r_s = \frac{\sim 3 \times 10^{-10} \times 10^{30}}{10^{17}}$

first, neglect rhs.

$\left(\begin{array}{l} u \to 0 \quad \varphi \to 0, \pi \\ u = \frac{1}{b} \quad \varphi = \pi/2 \end{array} \right) = \frac{10^{+29}}{10^{17}} =$ 1 km

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$= \frac{3}{2} \frac{1}{b^2} (1 - \cos 2\varphi)$

$\rightarrow u_1 = \frac{3 r_s}{2 b^2} \left(1 + \frac{1}{3} \cos 2\varphi \right)$

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$r_s = \frac{\sim 3 \times 10^{-10} \times 10^{30}}{10^{17}}$

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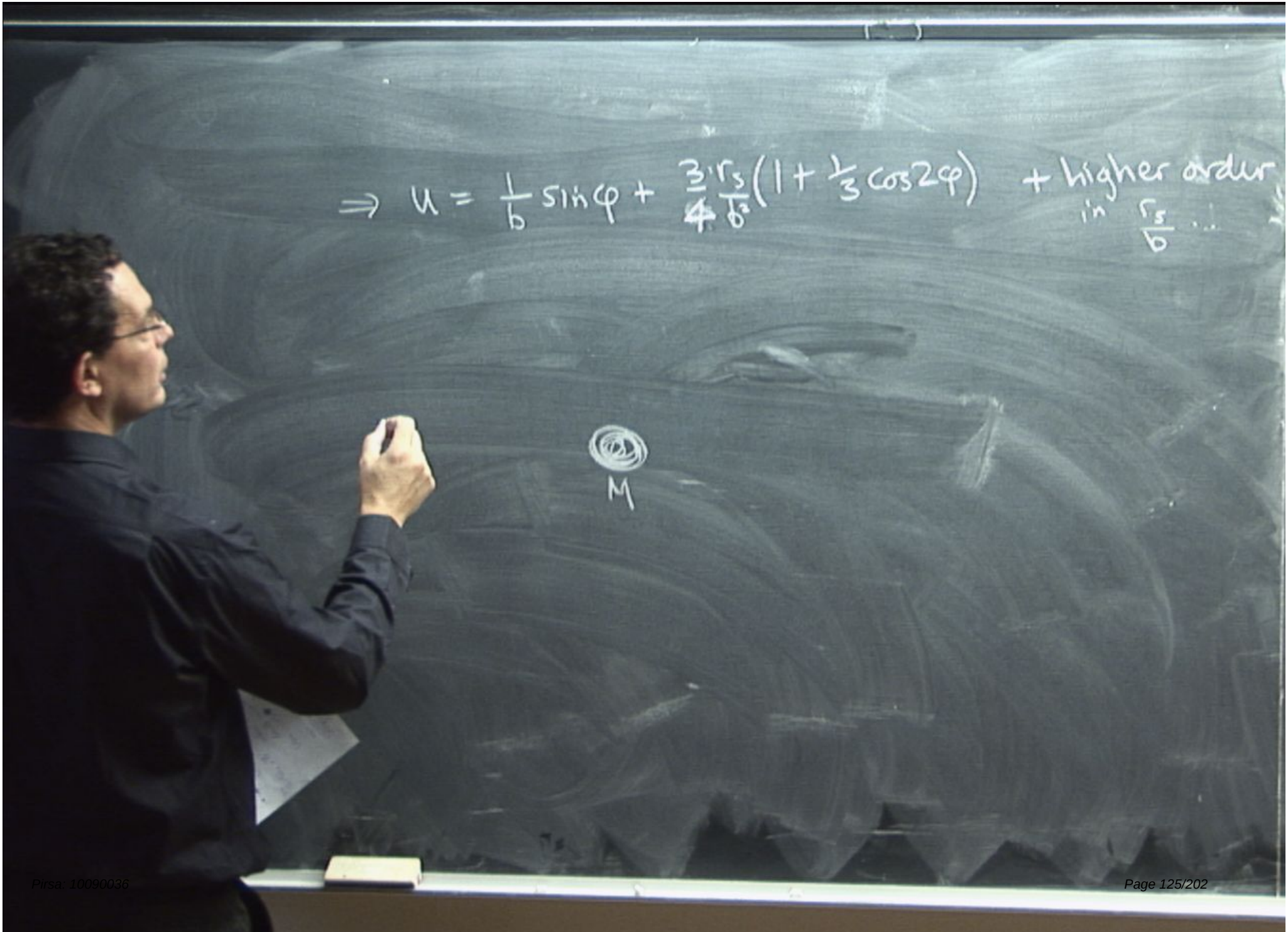
$u = u_0 + u_1$

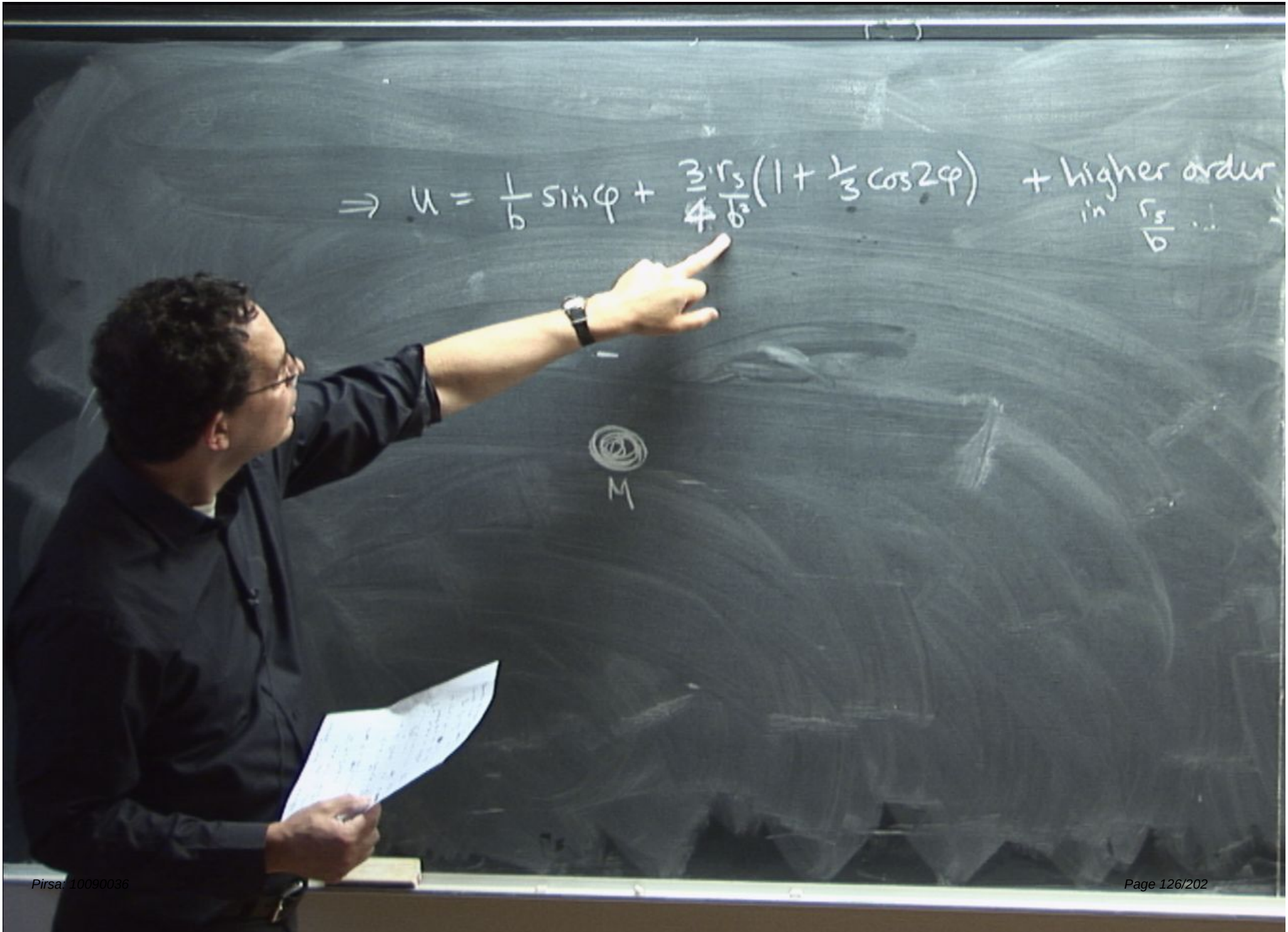
$\frac{d^2 u_1}{d\varphi^2} + u_1 = \frac{3}{2} r_s u_0^2$ to leading order in $r_s u$.

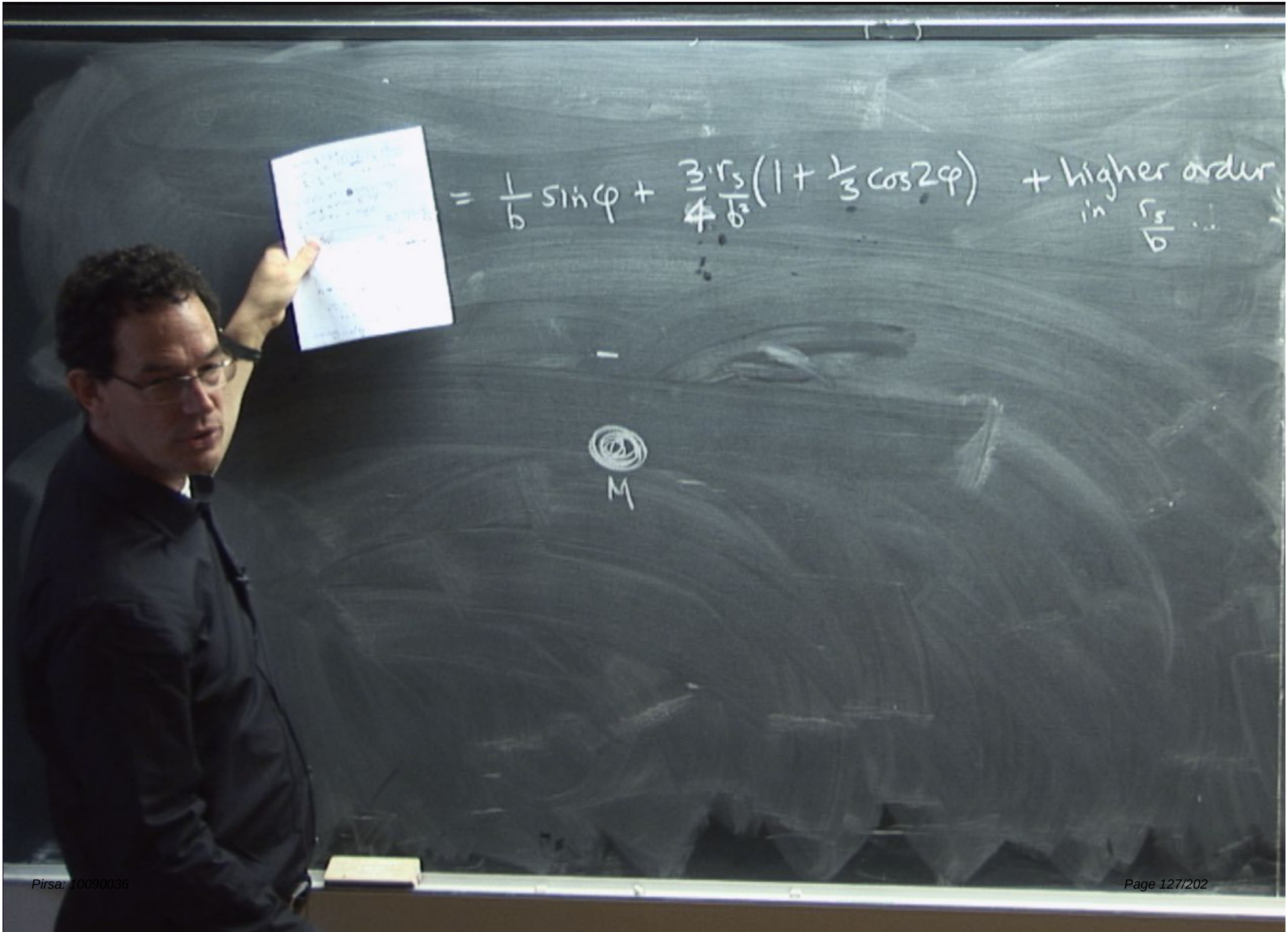
$= \frac{3}{2} \frac{1}{b^2} (1 - \cos 2\varphi)$

$\rightarrow u_1 = \frac{3 r_s}{2 b^2} \left(1 + \frac{1}{3} \cos 2\varphi \right)$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{2}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$


$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ...
M

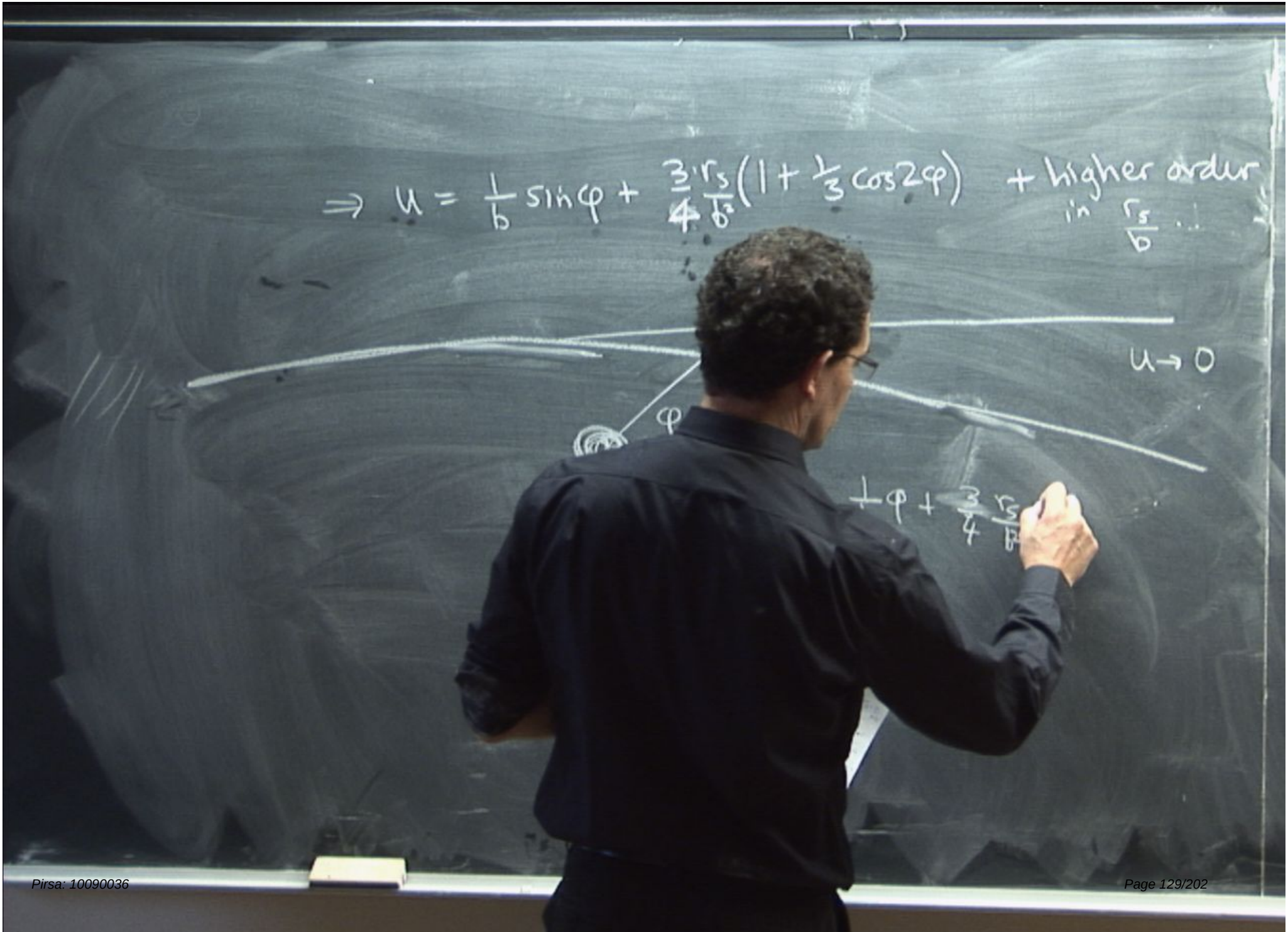

$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ...
M

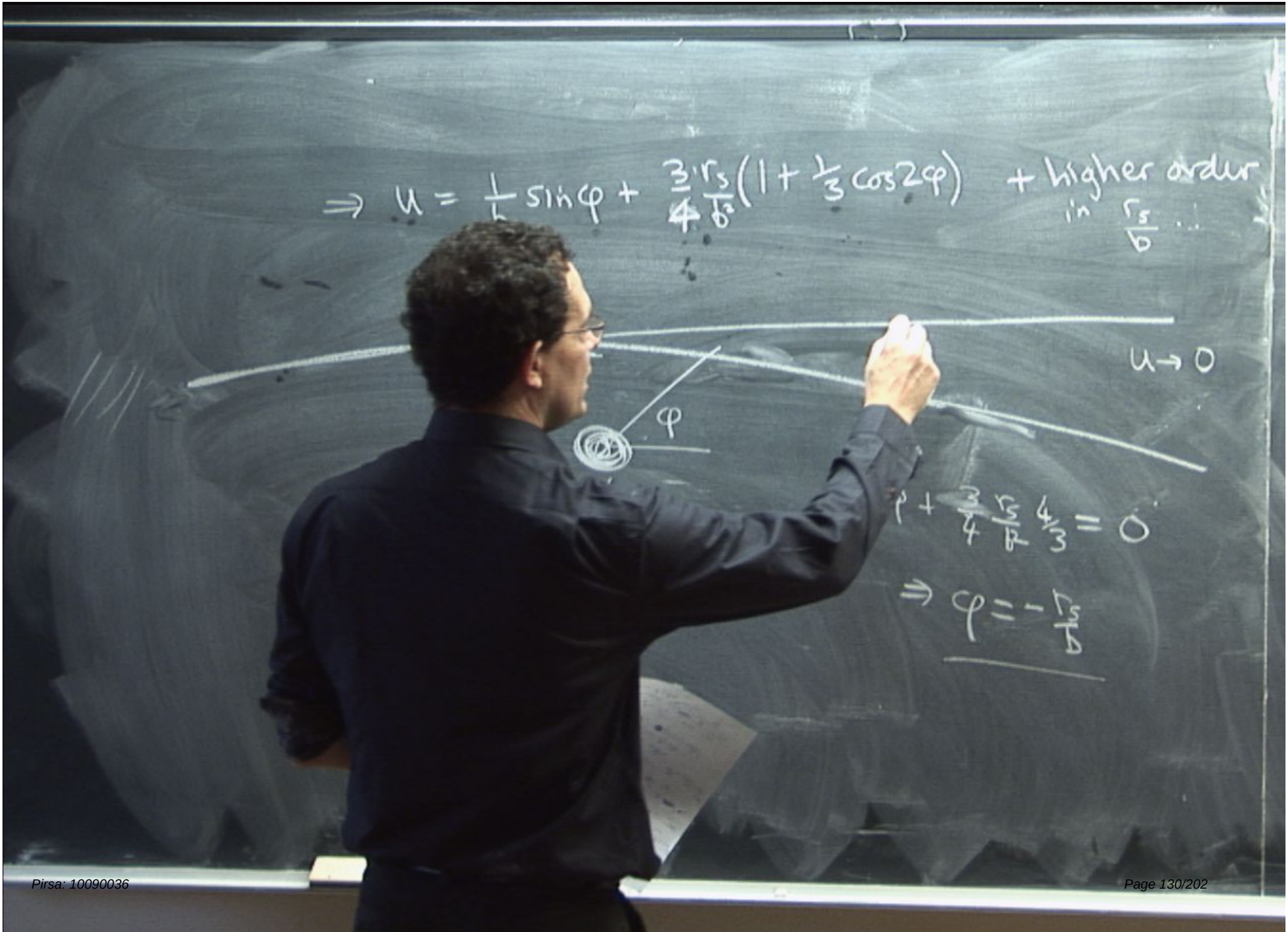

$= \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ...
M

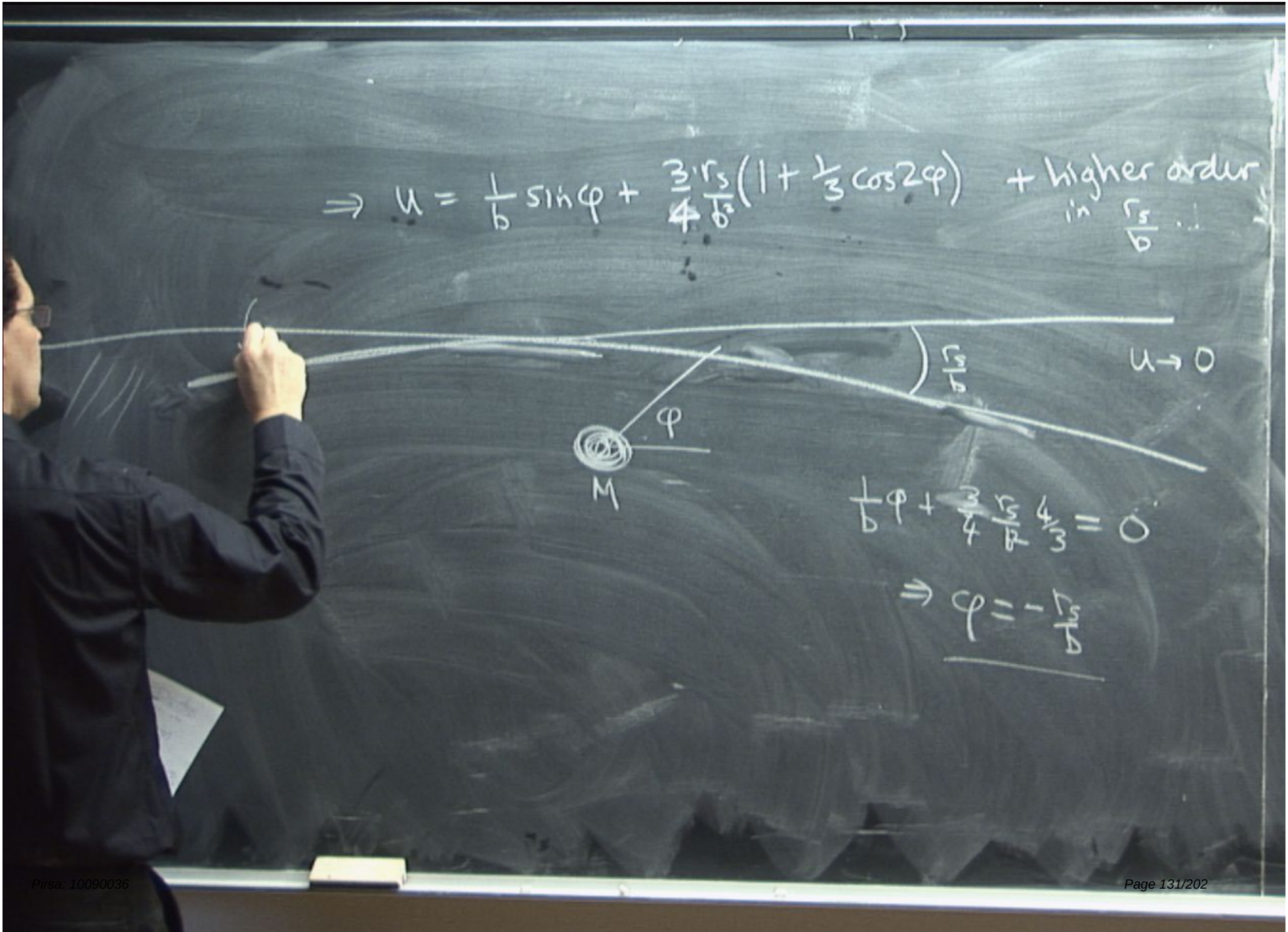
$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$

$u \to 0$

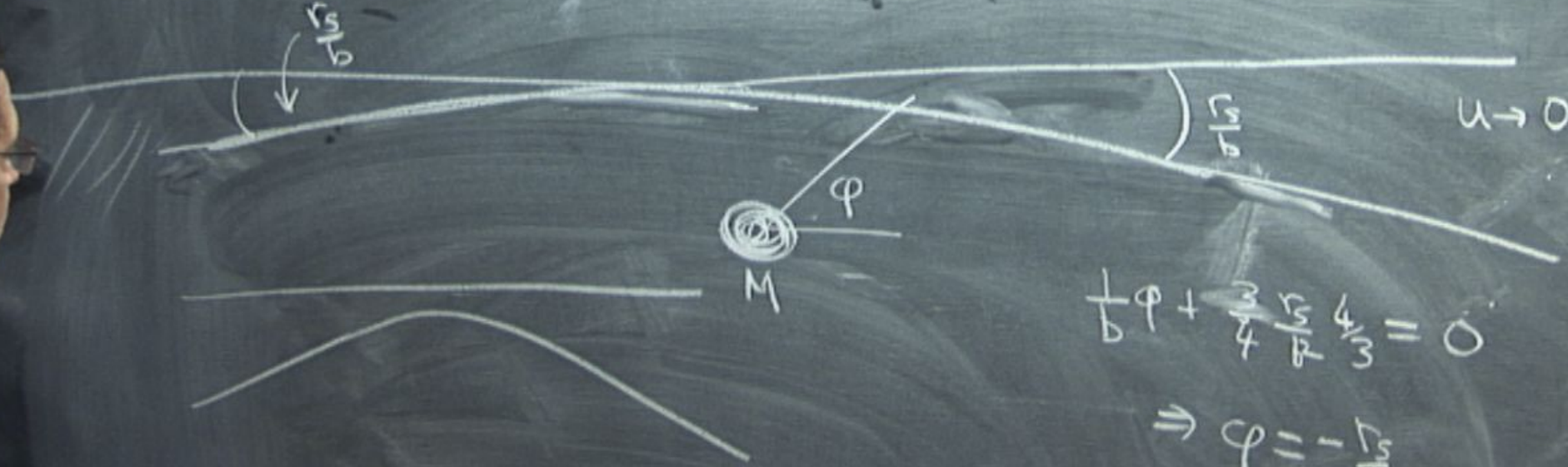
M


$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ..
$u \to 0$
φ
$\perp \varphi + \frac{3}{4}\frac{r_s}{b}$


$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ..
$u \to 0$
φ
$\Rightarrow \varphi = -\frac{r_s}{b}$

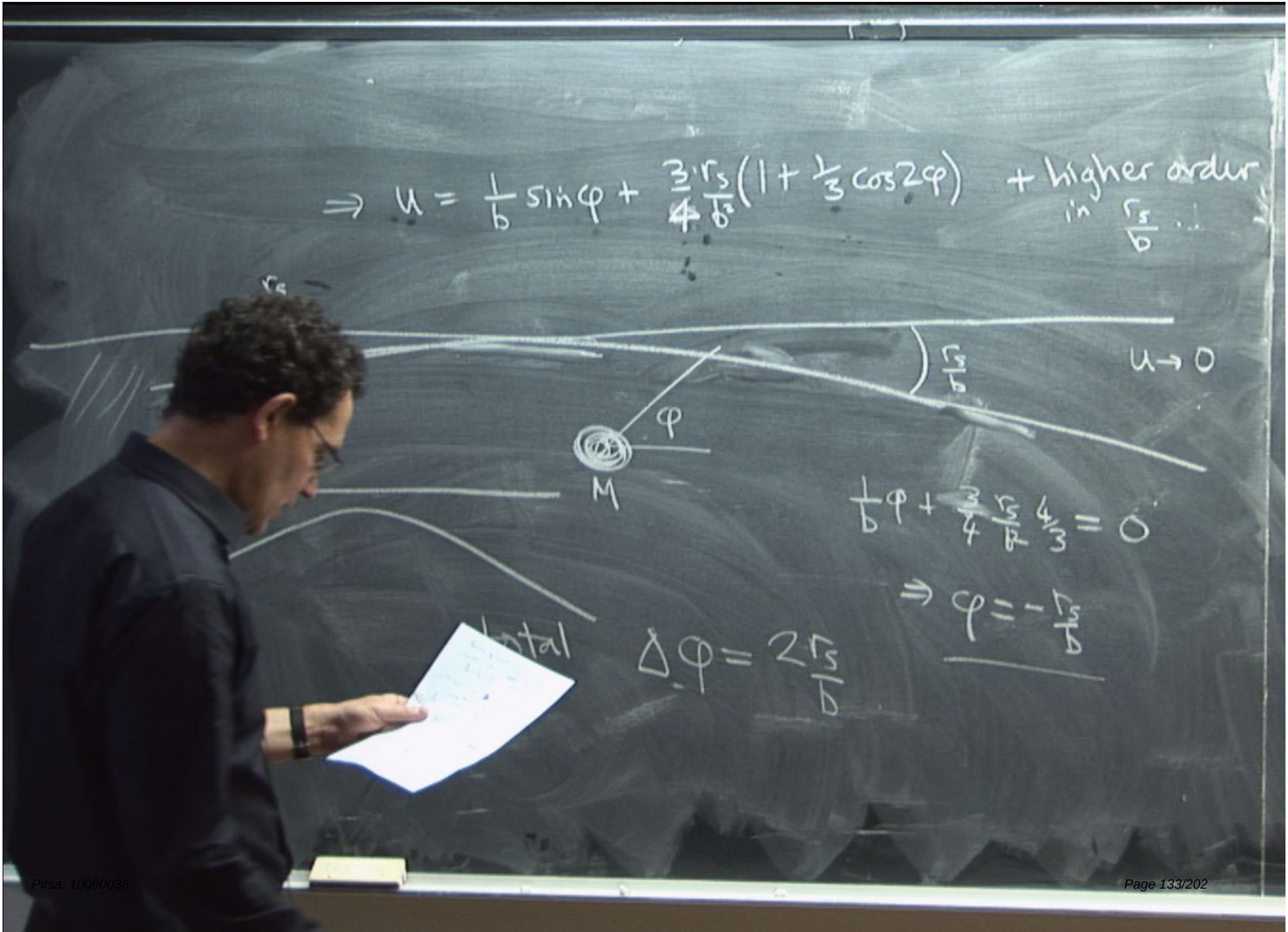

$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ...
$\frac{r_s}{b}$
$u \to 0$
φ
M
$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b}\frac{4}{3} = 0$
$\Rightarrow \varphi = -\frac{r_s}{b}$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher ordur in } \frac{r_s}{b} \ldots$$



$$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b}\frac{4}{3} = 0$$

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$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$ + higher ordur in $\frac{r_s}{b}$ ..
r_s
$\frac{r_s}{b}$
$u \to 0$
φ
M
$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b}\frac{4}{3} = 0$
$\Rightarrow \varphi = -\frac{r_s}{b}$
total $\Delta\varphi = 2\frac{r_s}{b}$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$



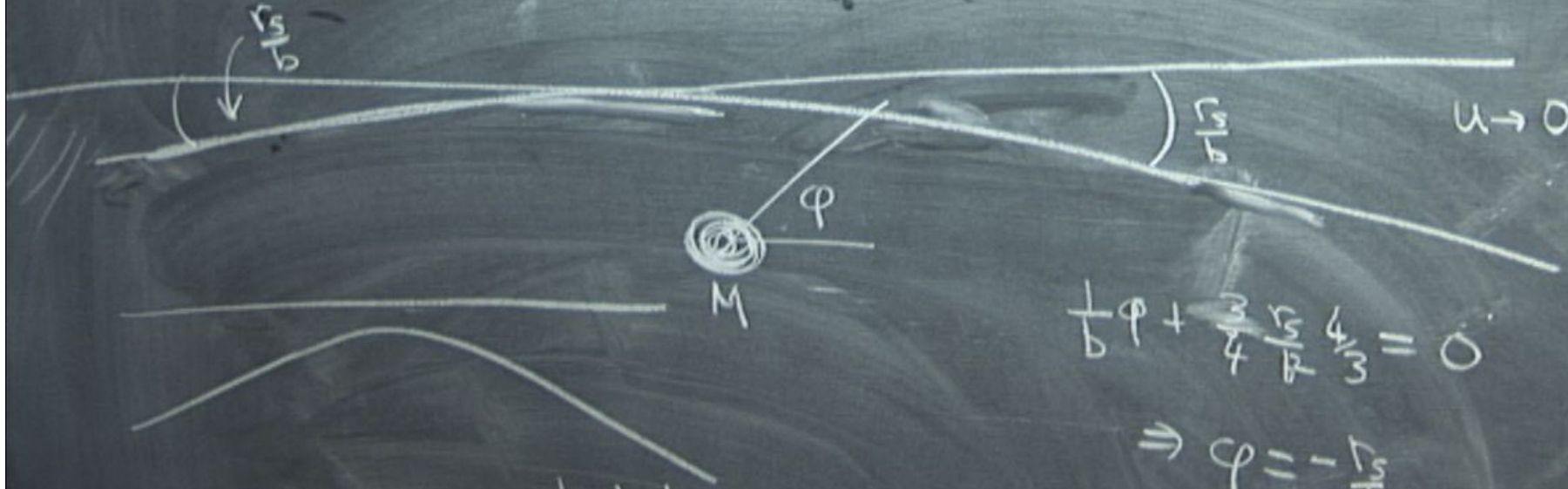
$$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b}\frac{4}{3} =$$

$$\Rightarrow \varphi = -$$

total $\Delta\varphi = 2\frac{r_s}{b}$

$$= \frac{4GM}{bc^2} = 1.75$$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$



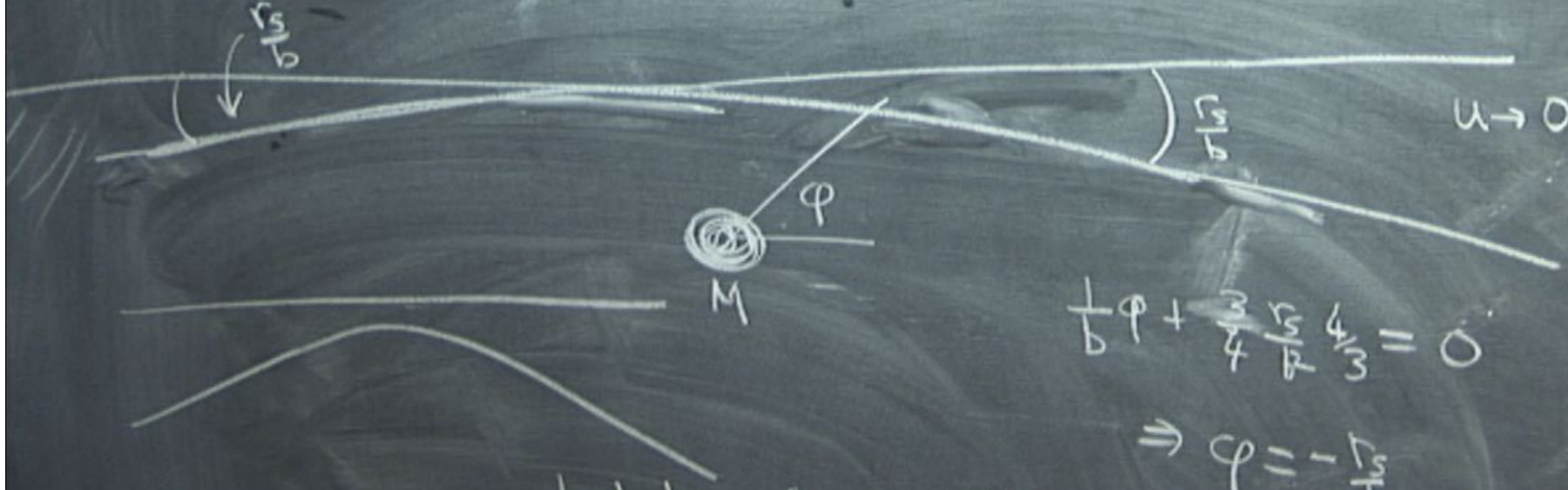
$$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b^2}\frac{4}{3} = 0$$

$$\Rightarrow \varphi = -\frac{r_s}{b}$$

total $\Delta\varphi = 2\frac{r_s}{b}$

$$= \frac{4GM}{bc^2} = 1.75'' \left(\frac{\quad}{b}\right)$$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$



$$\frac{1}{b}\varphi + \frac{3}{4}\frac{r_s}{b^2}\frac{4}{3} = 0$$

$$\Rightarrow \varphi = -\frac{r_s}{b}$$

total $\Delta\varphi = 2\frac{r_s}{b}$

$$= \frac{4GM}{bc^2} = 1.75'' \left(\frac{R_\odot}{b}\right)$$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$

zeroth order

$u = \frac{1}{b}\sin\varphi$



$$\Rightarrow \varphi = -\frac{r_s}{b}$$

$$\Delta\varphi = 2\frac{r_s}{b}$$

$$= \frac{4GM}{bc^2} = 1.75'' \left(\frac{R_\odot}{b}\right)$$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$

zeroth order

$u = \frac{1}{b}\sin\varphi$



$$\Rightarrow \varphi = -\frac{r_s}{b}$$

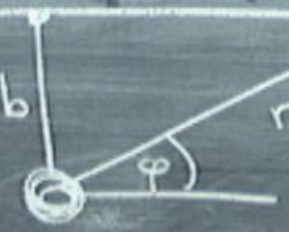
total $\Delta\varphi = 2\frac{r_s}{b}$

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zeroth order

$u = \frac{1}{b}\sin\varphi$



$$\sin\varphi = \frac{b}{r}$$

$$\Rightarrow \varphi = -\frac{r_s}{b}$$

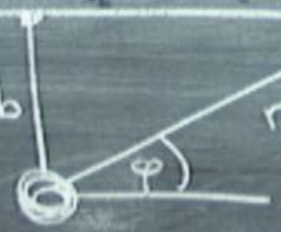
total $\Delta\varphi = 2\frac{r_s}{b}$

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zeroth order

$u = \frac{1}{b}\sin\varphi$



$\sin\varphi = \frac{b}{r} \Rightarrow u = \frac{1}{b}\sin\varphi$

$$\Rightarrow \varphi = -\frac{r_s}{b}$$

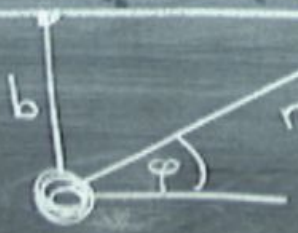
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zeroth order

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$\sin\varphi = \frac{b}{r} \Rightarrow u = \frac{1}{b}\sin\varphi$

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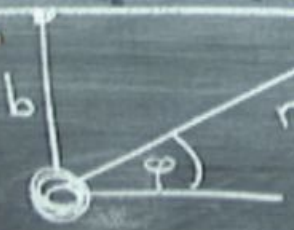
total $\Delta\varphi = 2\frac{r_s}{b}$

$$= \frac{4GM}{bc^2} = 1.75'' \left(\frac{R_\odot}{b}\right)$$

$$\Rightarrow u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \text{higher order in } \frac{r_s}{b} \ldots$$

zeroth order

$u = \frac{1}{b}\sin\varphi$



$$\sin\varphi = \frac{b}{r} \Rightarrow u = \frac{1}{b}\sin\varphi$$

$$\Rightarrow \varphi = -\frac{r_s}{b}$$

total $\Delta\varphi = 2\frac{r_s}{b}$

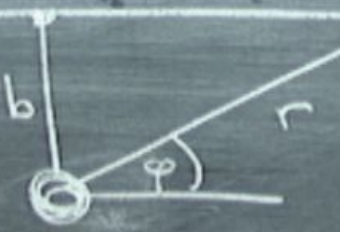
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$\frac{1}{b}\sin\varphi_1$ $\qquad$ 4/3

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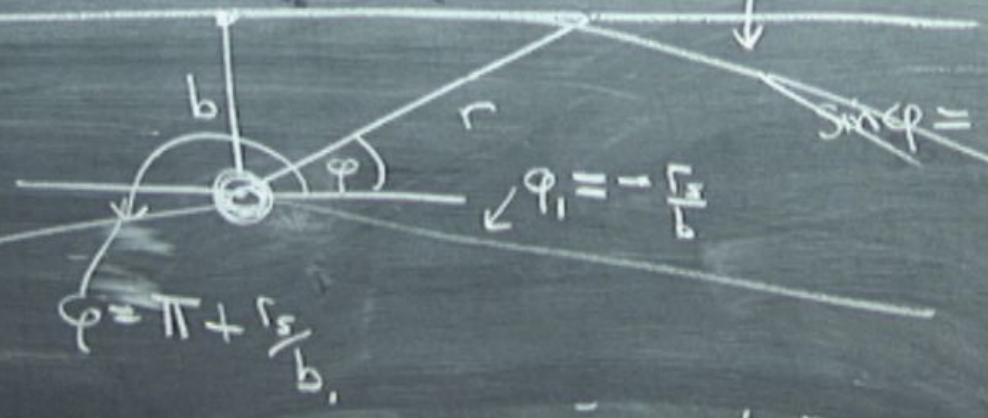
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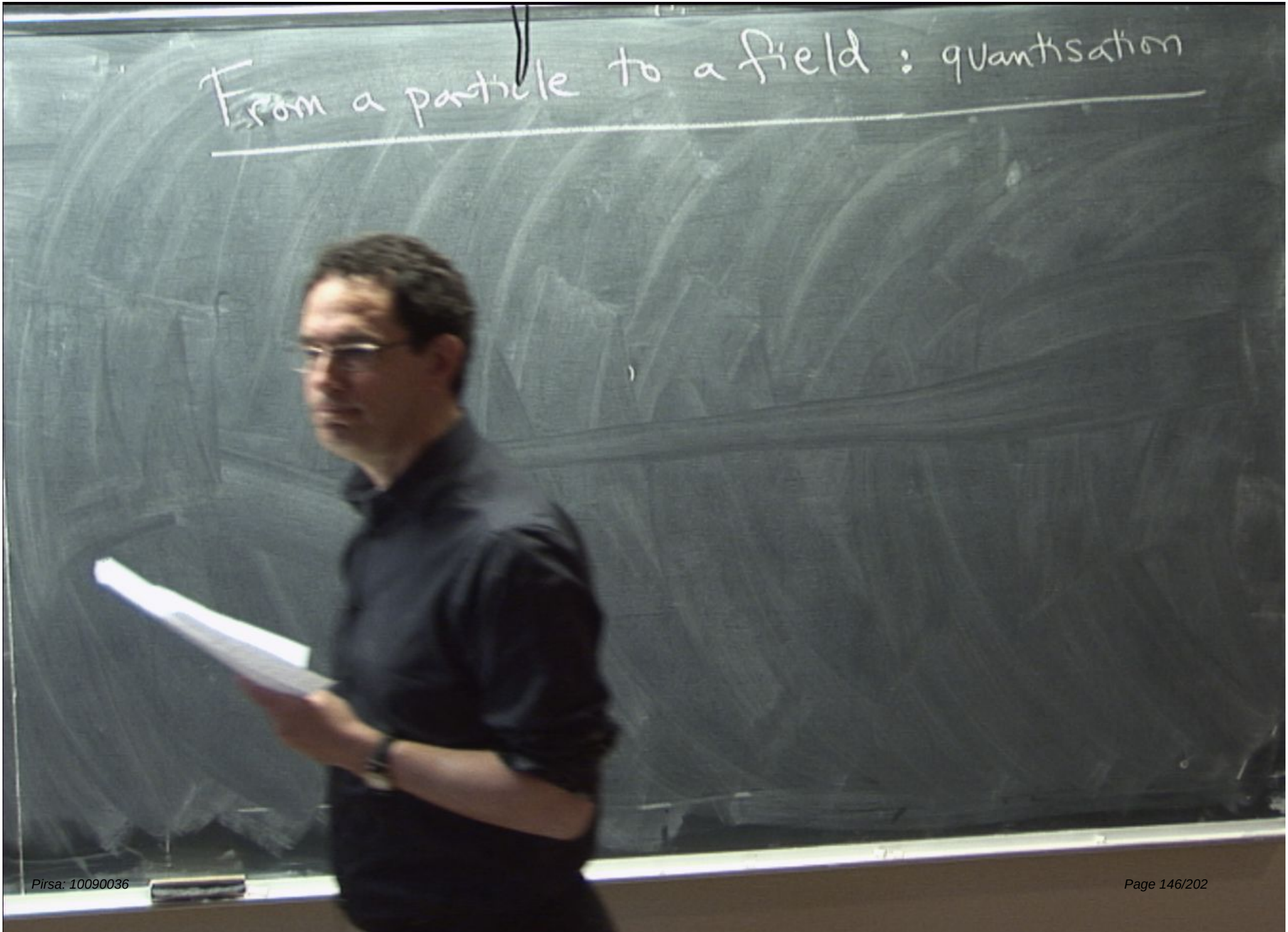


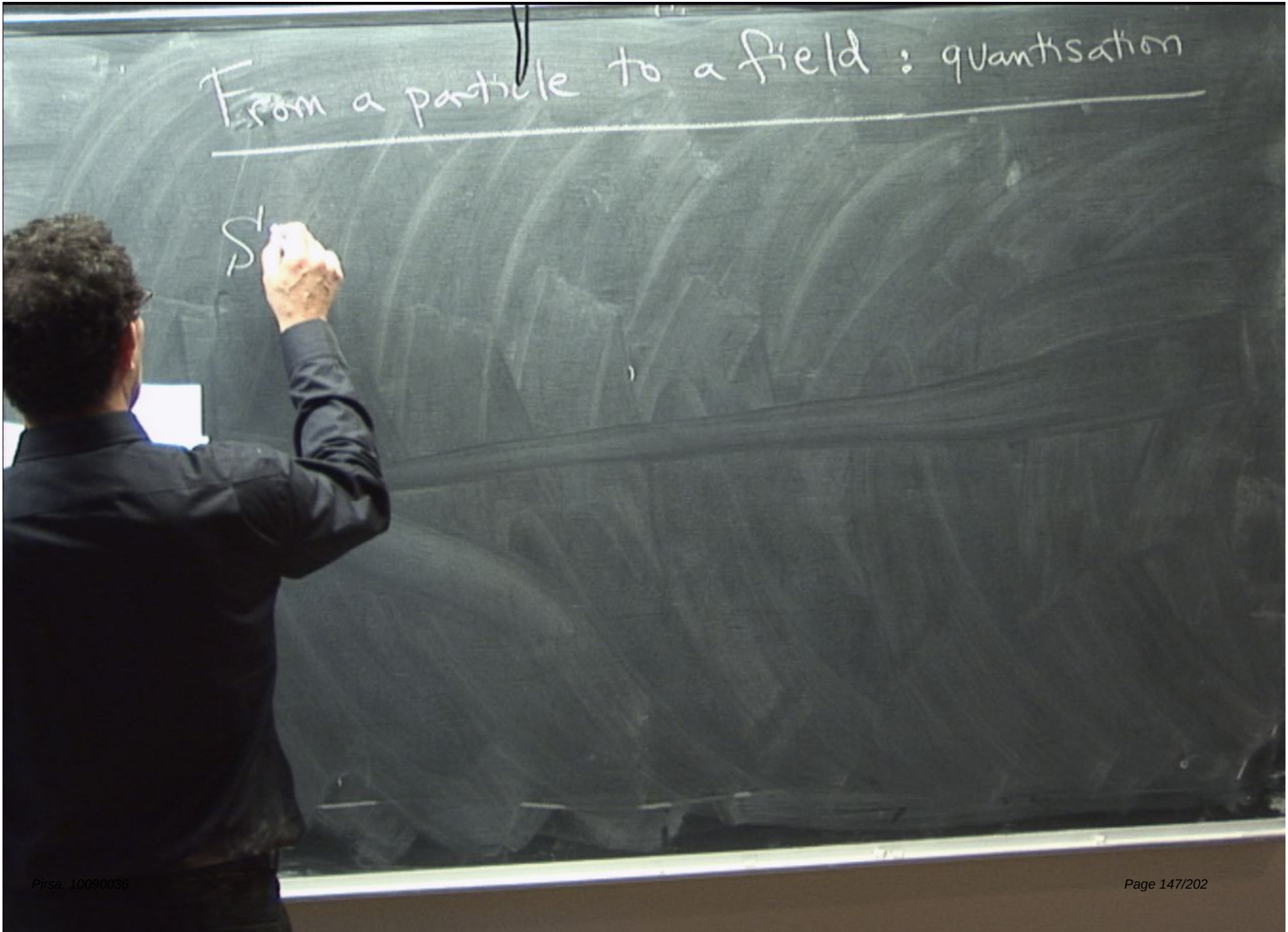
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From a particle to a field : quantisation


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$$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{\dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

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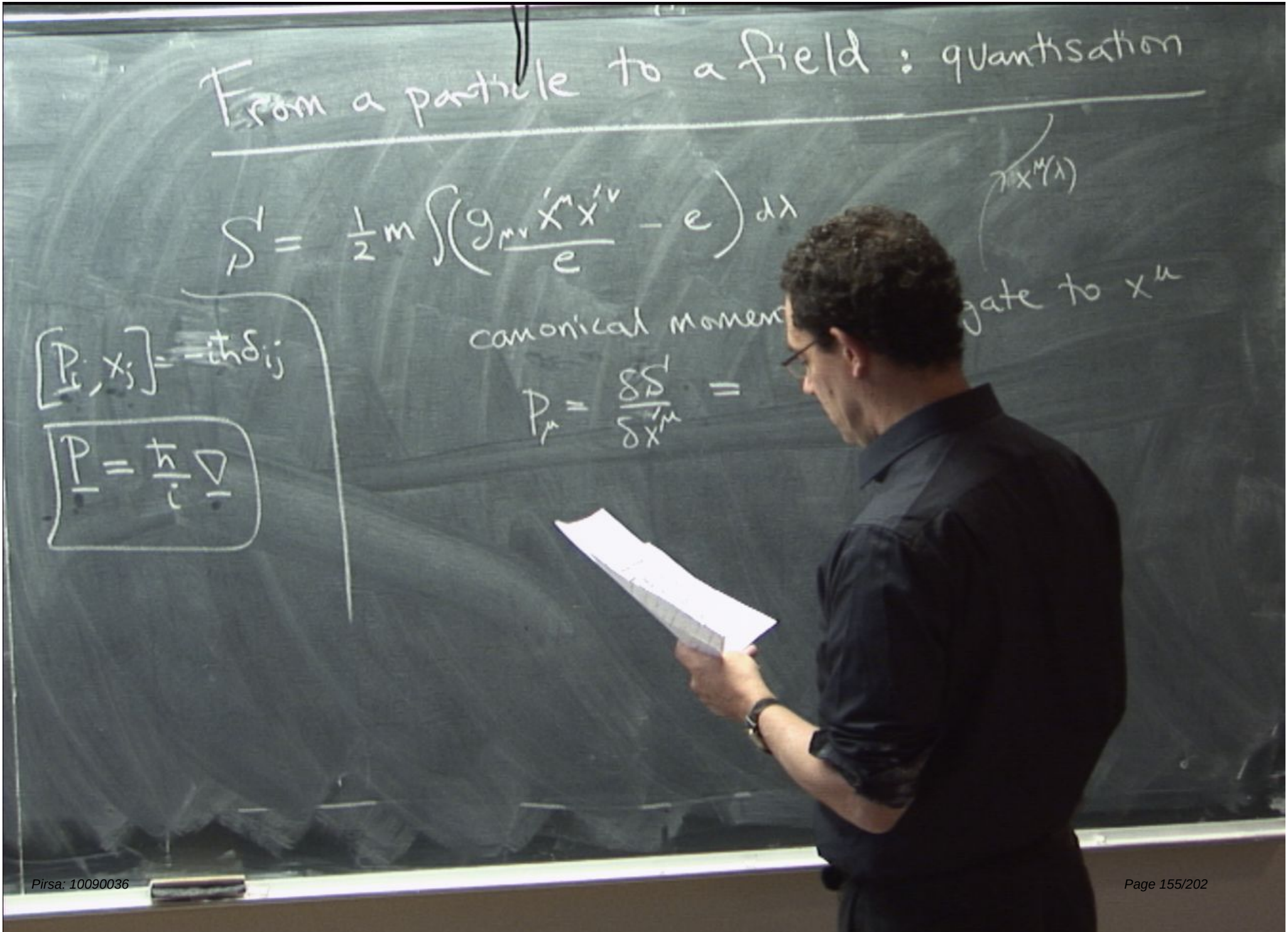
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$x^\mu(\lambda)$
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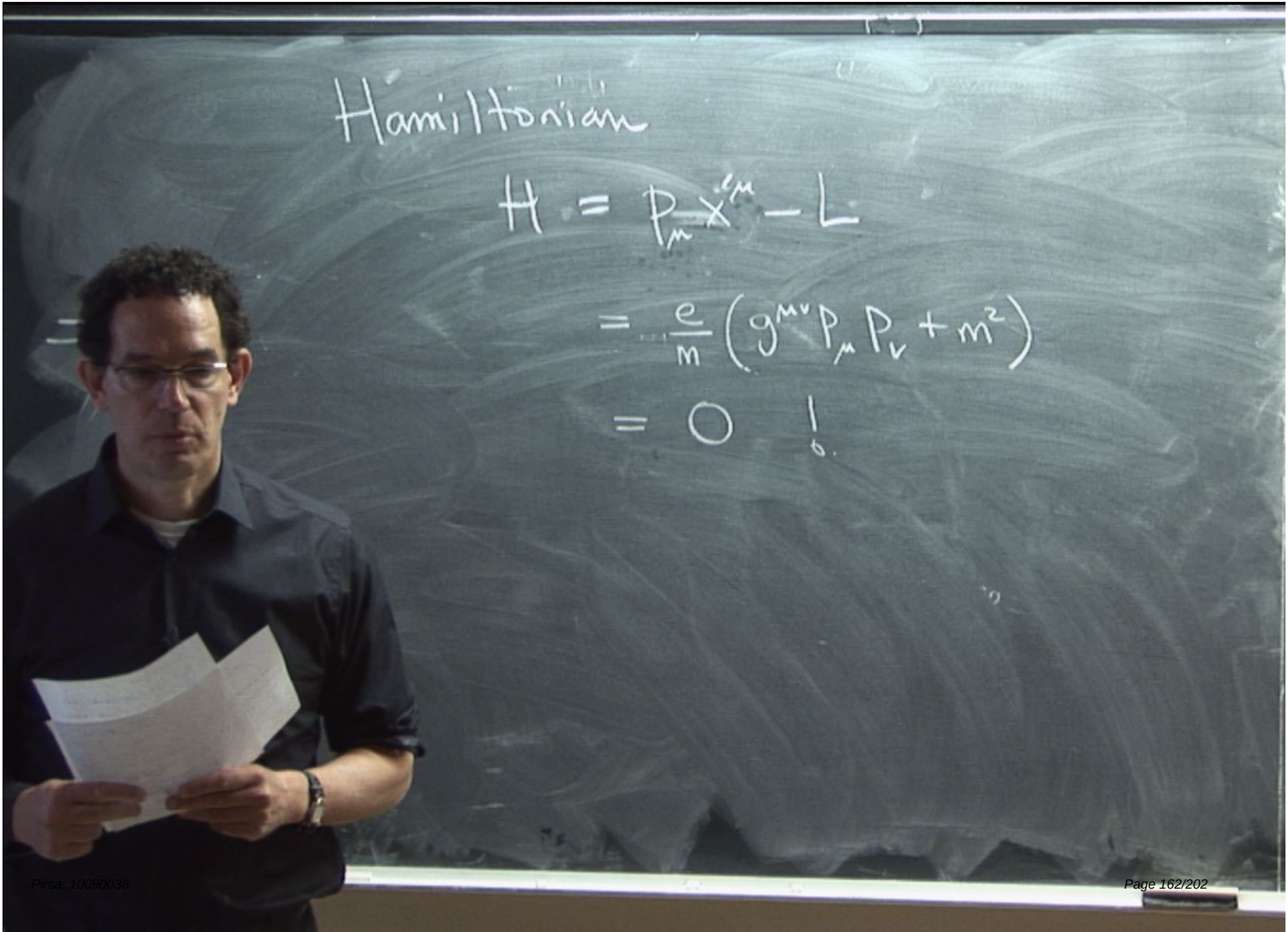
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Hamiltonian

$$H = P_\mu \dot{X}^\mu - L$$

$$= \frac{e}{m}\left(g^{\mu\nu} P_\mu P_\nu + m^2\right)$$

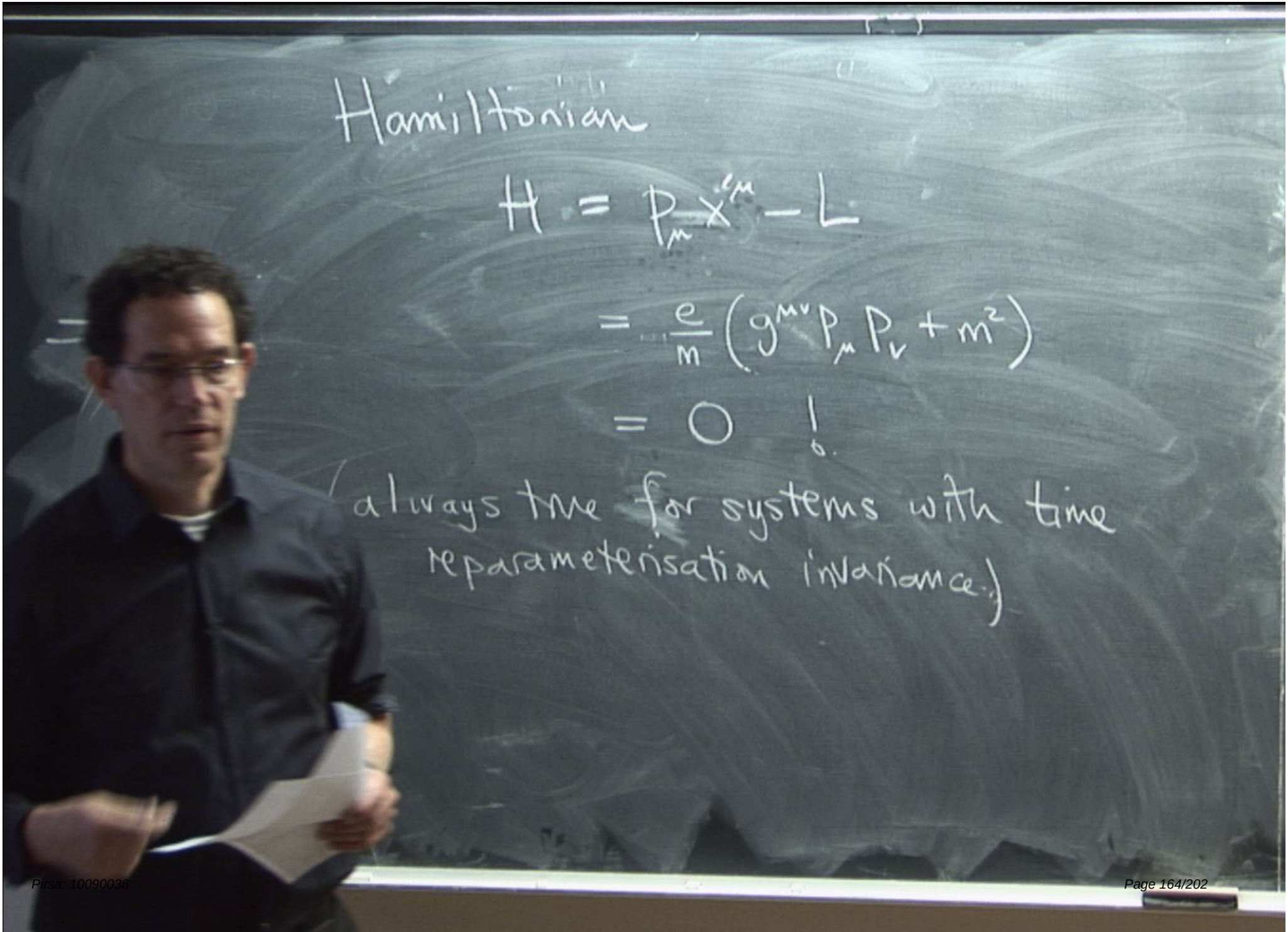

Hamiltonian
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(always true for systems with time reparameterisation invariance.)

From a particle to a field : quantisation

$$S' = \frac{1}{2} m \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right) d\lambda$$

time

$x^\mu(\lambda)$

canonical momentum conjugate to x^μ

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mass-shell condition

$$[p_i, x_j] = -i\hbar \delta_{ij}$$

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$$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{x'^\mu x'^\nu}{e} - e \right) d\lambda$$

λ ← time

$x^\mu(\lambda)$

$\int d\lambda \sqrt{g_{\mu\nu} x'^\mu x'^\nu}$

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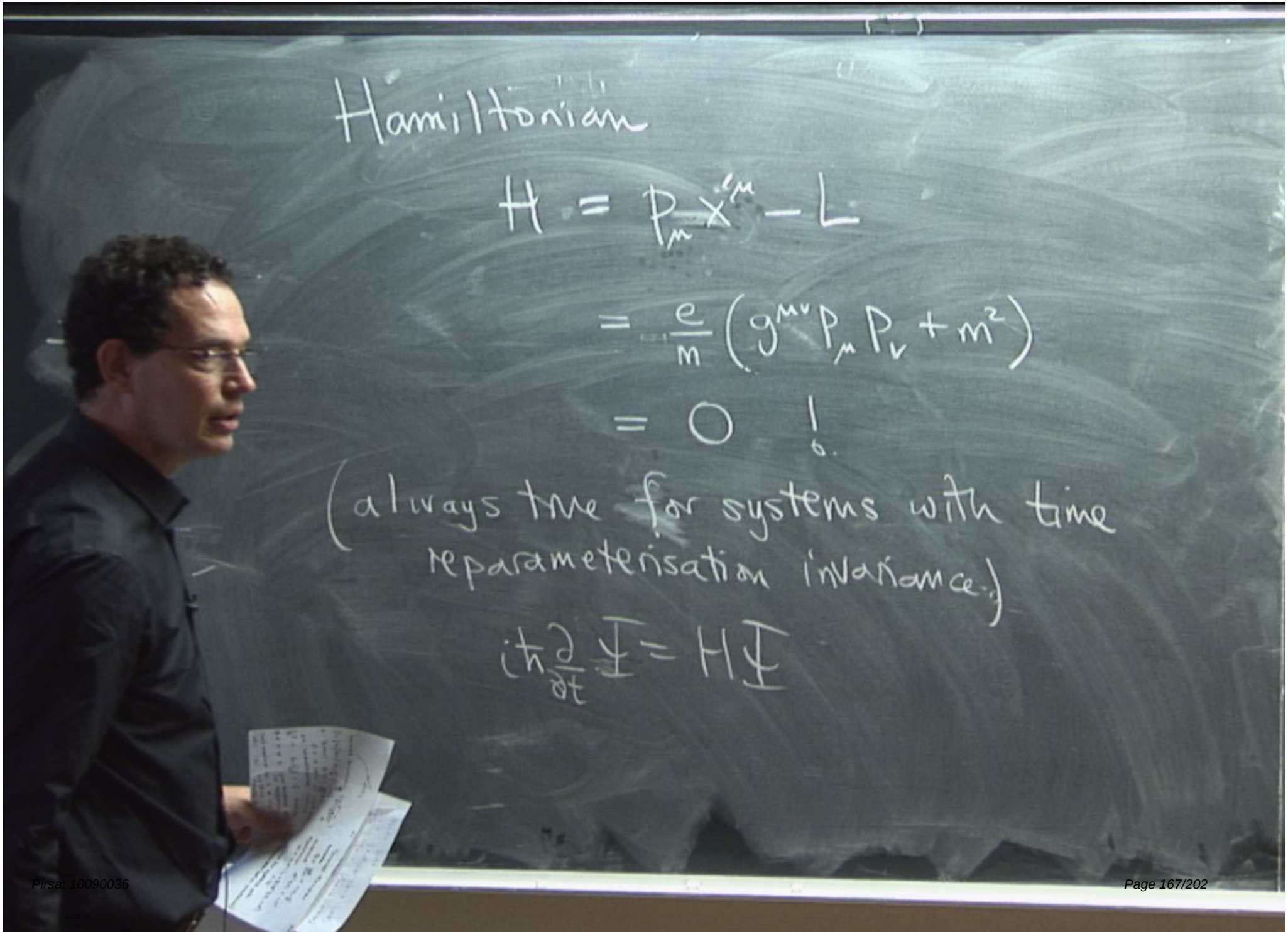
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$$\int dt \left(\frac{1}{2} m \underline{\dot{x}}^2 - V(\underline{x}) \right)$$


Hamiltonian
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$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi$

Hamiltonian

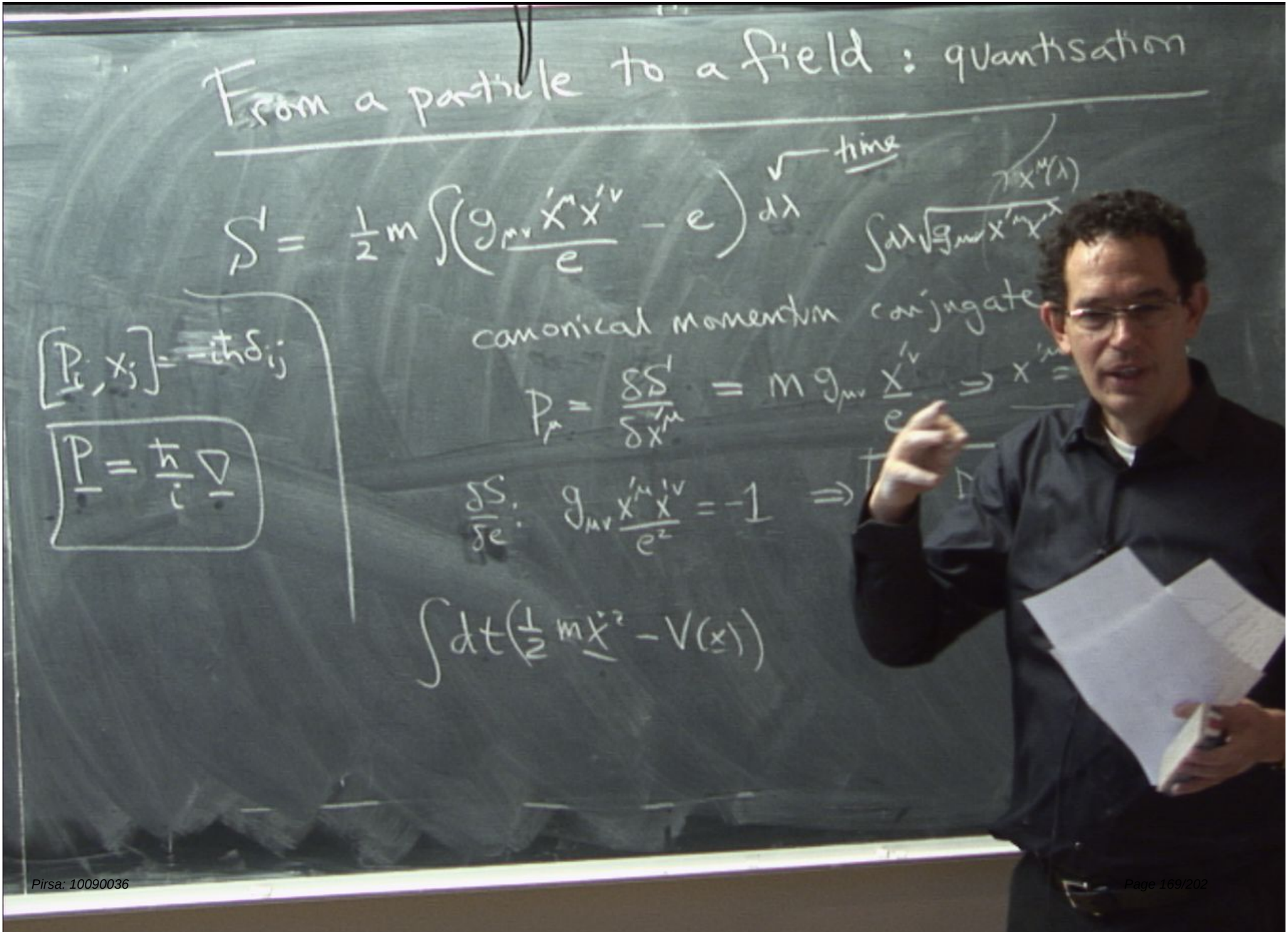
$$H = P_\mu \dot{x}^\mu - L$$

$$= \frac{e}{m}\left(g^{\mu\nu} P_\mu P_\nu + m^2\right)$$

$$= 0 \quad !$$

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$$i\hbar \frac{\partial}{\partial \lambda} \Psi = H\Psi = 0.$$


From a particle to a field : quantisation
$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{x'^\mu x'^\nu}{e} - e \right) d\lambda$
time
$\int d\lambda \sqrt{g_{\mu\nu} x'^\mu x'^\nu}$
$x^\mu(\lambda)$
canonical momentum conjugate
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$[P_i, x_j] = -i\hbar \delta_{ij}$
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From a particle to a field : quantis

$$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{x'^\mu x'^\nu}{e} - e \right) d\lambda$$

time

$x^\mu(\lambda)$

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canonical momentum conjugate to

$$P_\mu = \frac{\delta S'}{\delta x'^\mu} = m g_{\mu\nu} \frac{x'^\nu}{e} \Rightarrow x'^\mu = e$$

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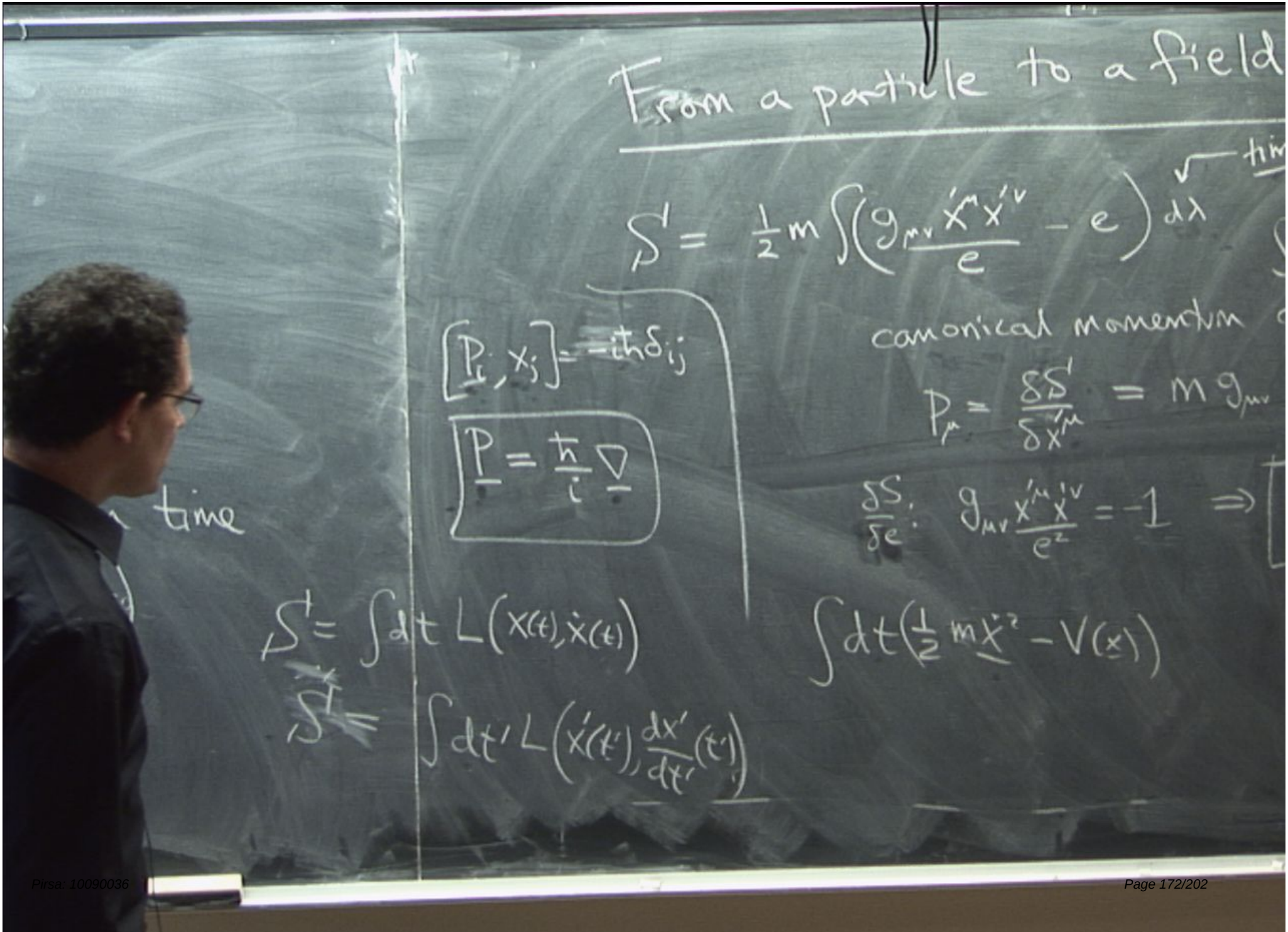
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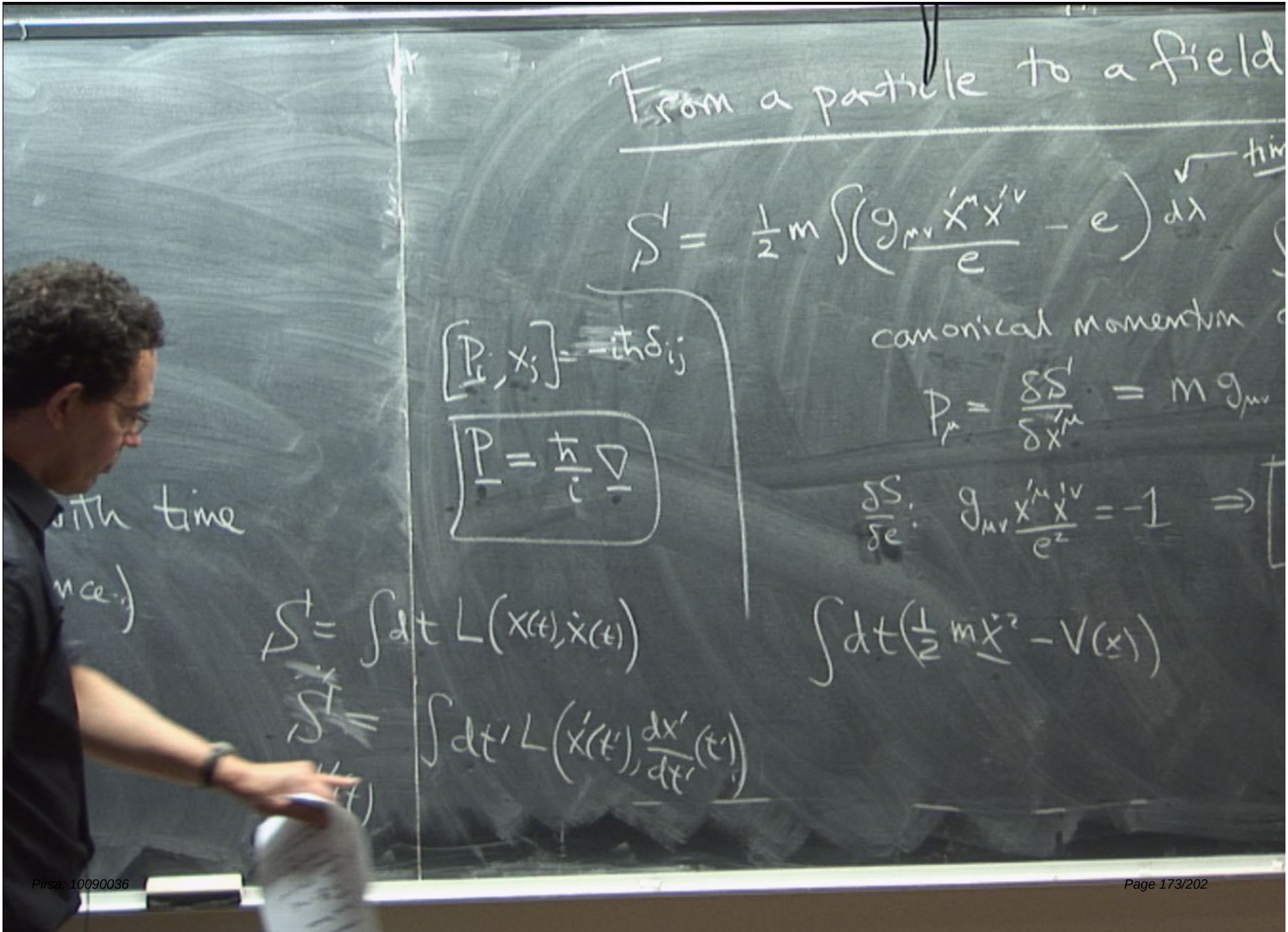
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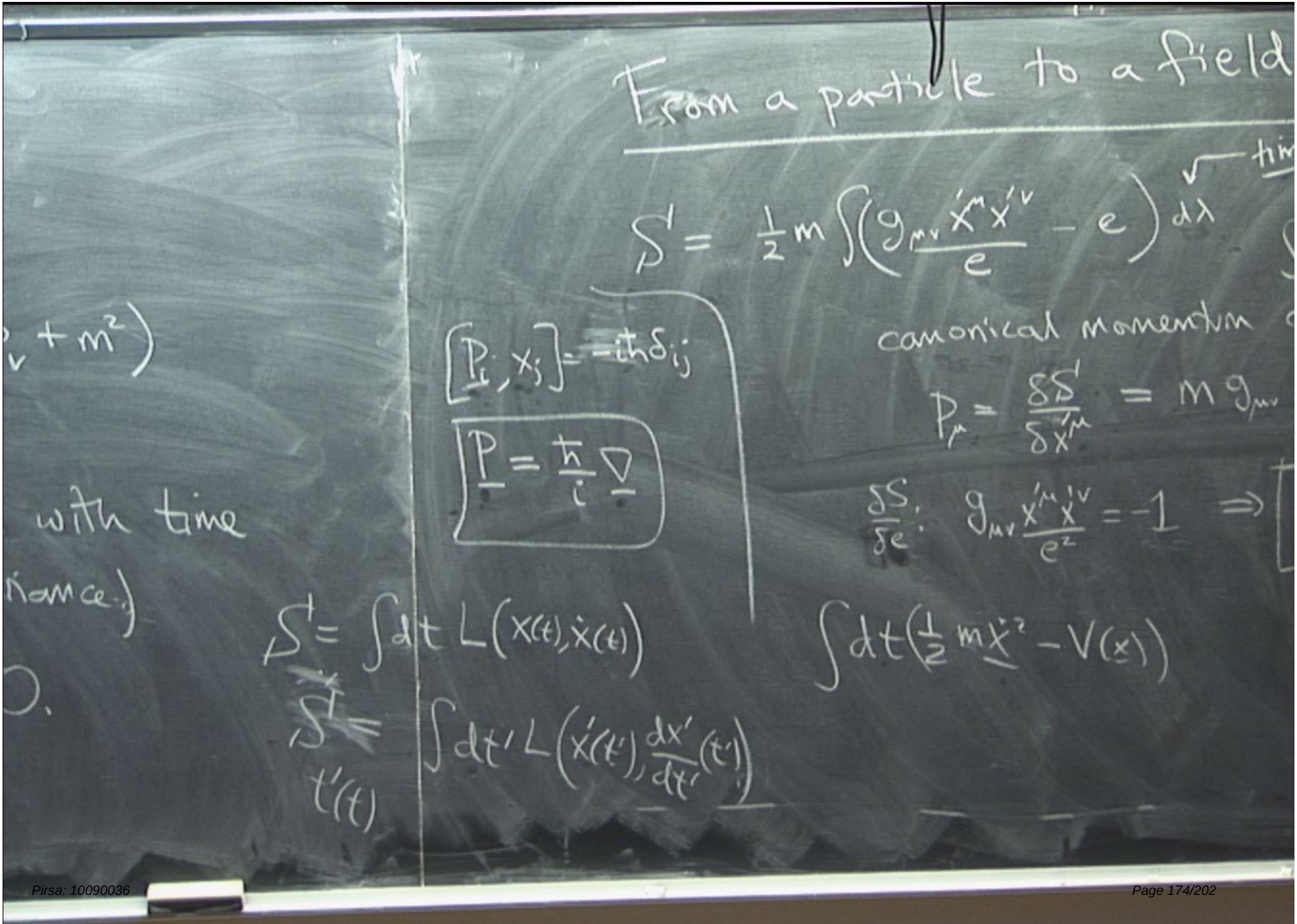
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$x(t) \to x$


From a particle to a field
$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{\dot{x}^\mu \dot{x}^\nu}{e} - e \right) d\lambda$
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$P_\mu = \frac{\delta S'}{\delta \dot{x}^\mu} = m g_{\mu\nu}$
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$+ m^2)$
with time
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0.
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From a particle to a field : quantisation

$$S' = \frac{1}{2} m \int \left(g_{\mu\nu} \frac{x'^\mu x'^\nu}{e} - e \right) d\lambda$$

time

$x^\mu(\lambda)$

$\int d\lambda \sqrt{g_{\mu\nu} x'^\mu x'^\nu}$

canonical momentum conjugate to x^μ

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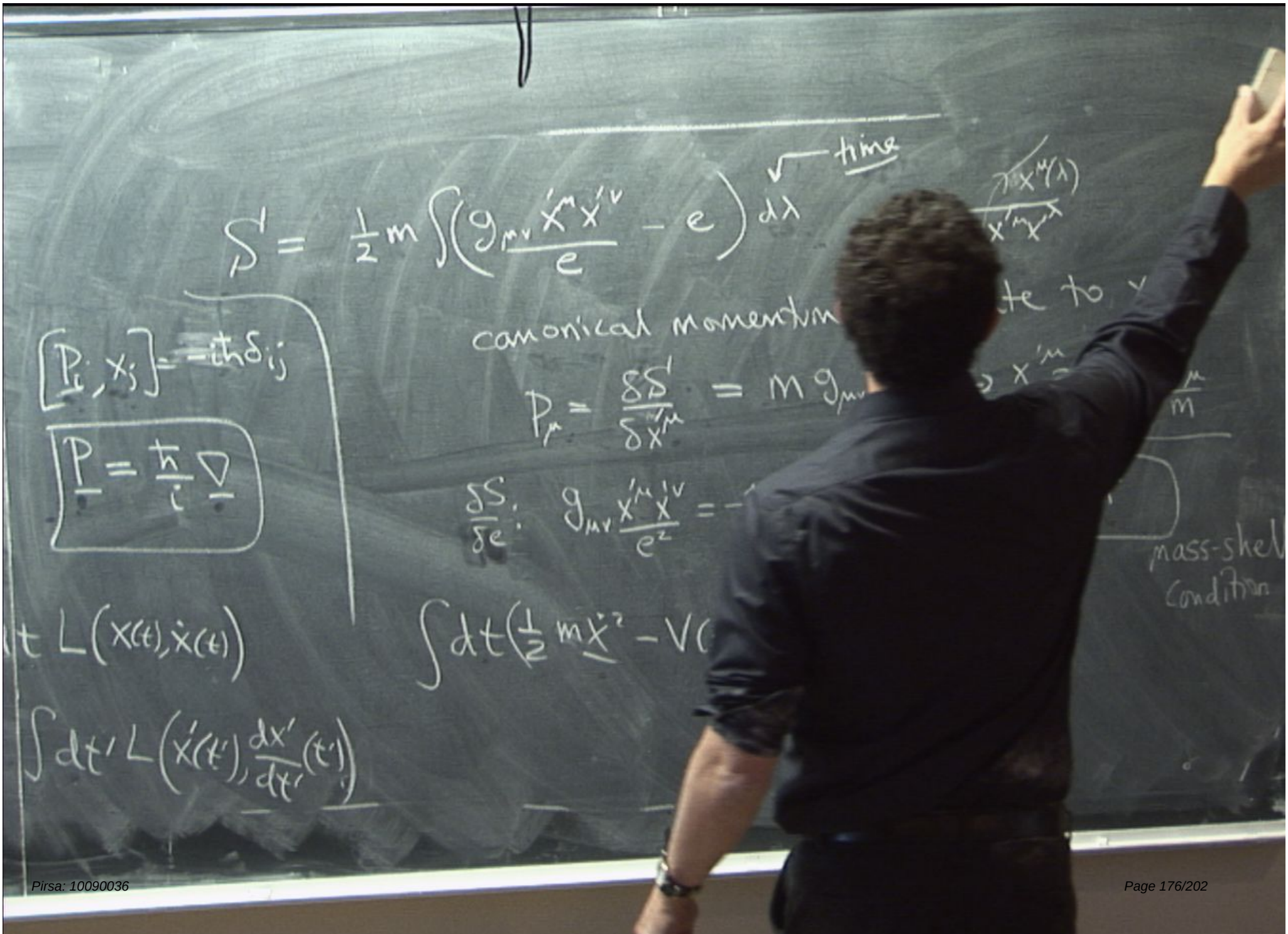
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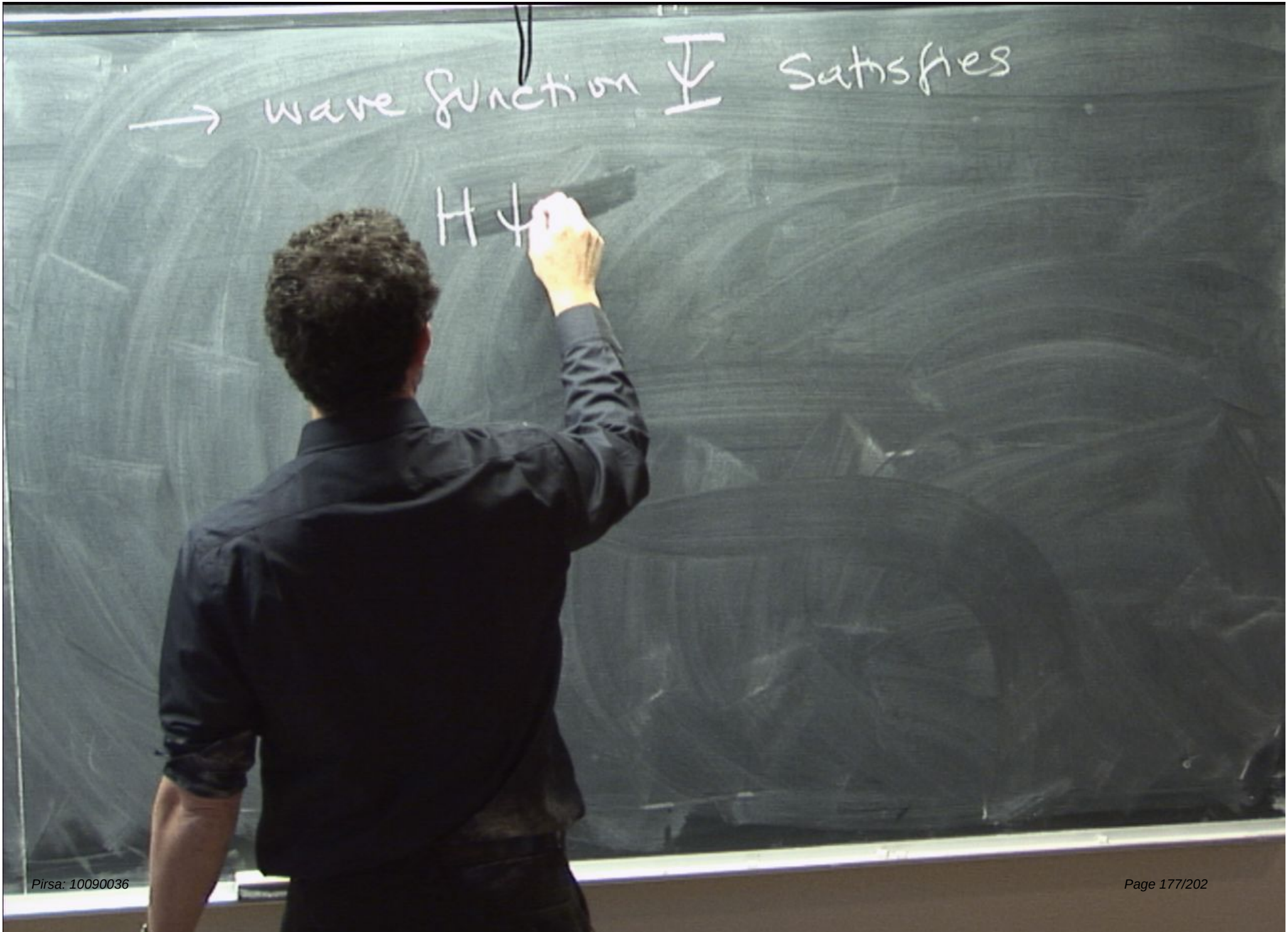
$\int dt\, L(x(t), \dot{x}(t))$

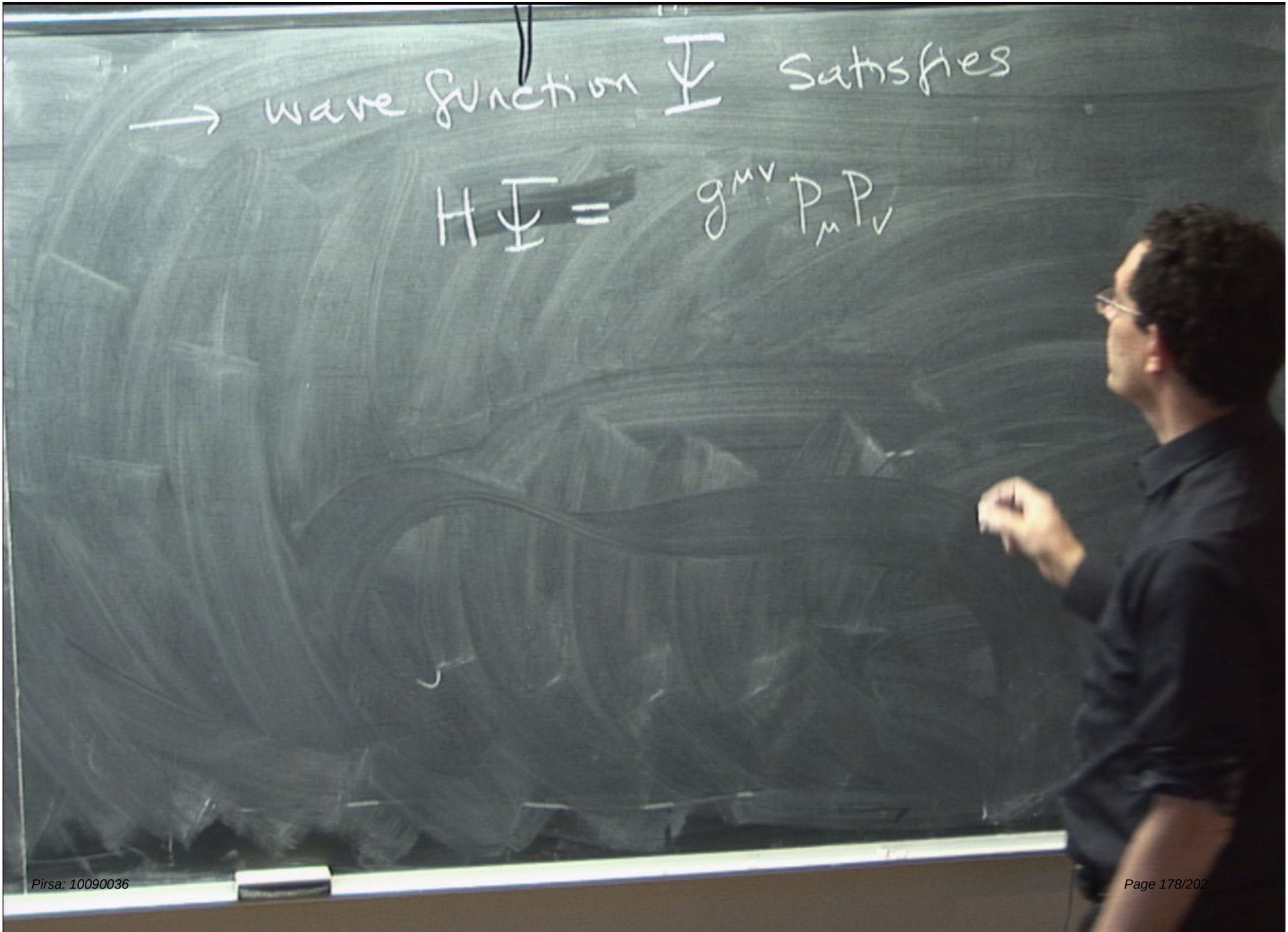
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$\int dt \left(\frac{1}{2} m \underline{\dot{x}}^2 - V(\underline{x}) \right)$

$x(t)$


time
canonical momentum
mass-shell
Condition


→ wave function Ψ satisfies
H ψ


→ wave function Ψ satisfies
HΨ = g^μν P_μ P_ν

→ wave function Ψ satisfies

$$H\Psi = \left(g^{\mu\nu} P_\mu P_\nu + m^2\right)\Psi = 0$$

Canonical quantisation:

$$\left[P_\mu, X^\nu\right] = i\hbar\delta_\mu^\nu$$

→ wave function Ψ satisfies

$$H\Psi = \left(g^{\mu\nu} P_\mu P_\nu + m^2\right)\Psi = 0$$

Canonical quantisation:

$$\left[P_\mu, X^\nu\right] = i\hbar\delta_\mu^\nu$$

$$\Rightarrow P_\mu = i\hbar \frac{\partial}{\partial x^\mu}$$

→ wave function Ψ satisfies

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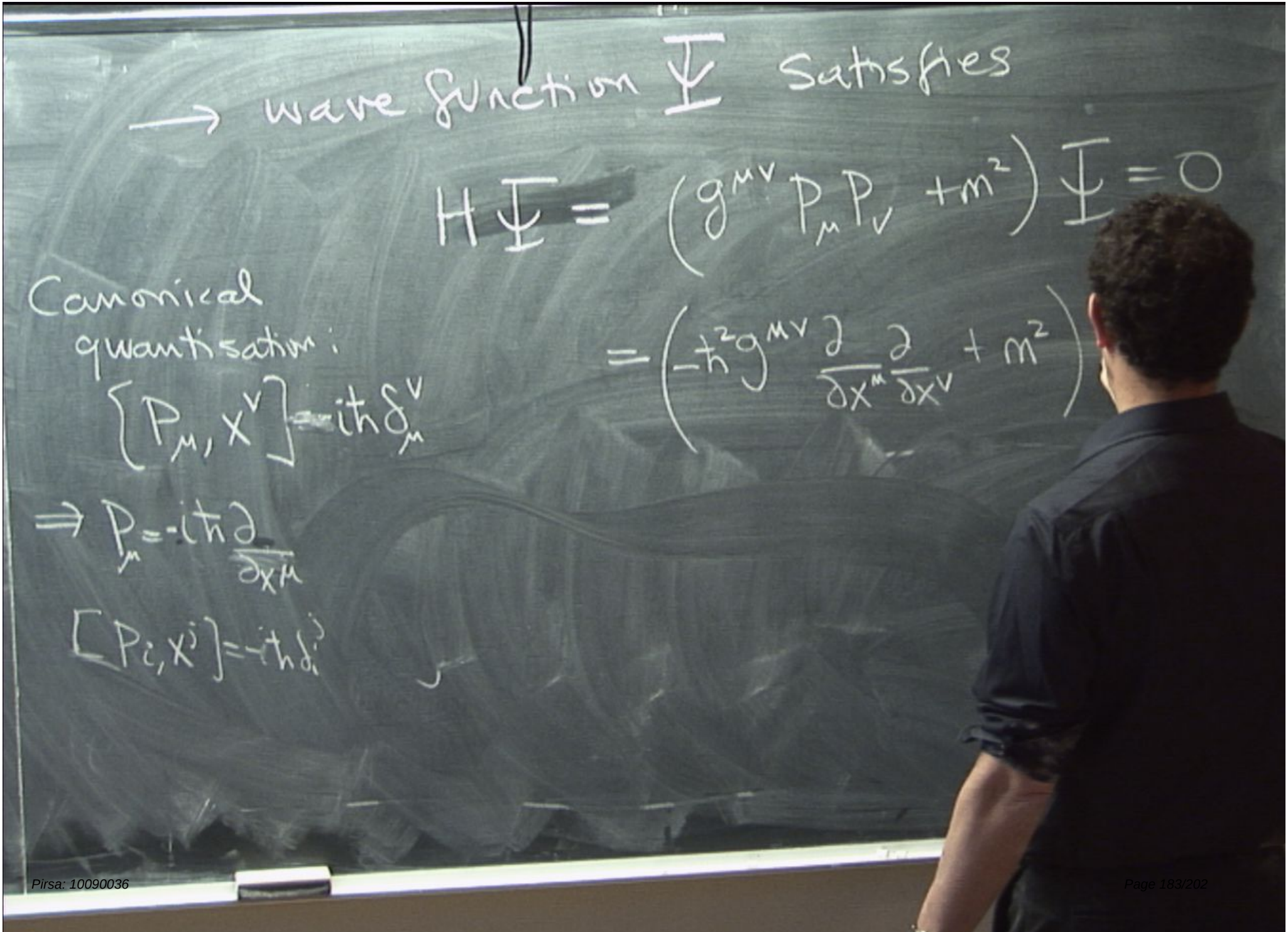
$$H\Psi = \left(g^{\mu\nu} P_\mu P_\nu + m^2\right)\Psi = 0$$

Canonical quantisation:

$$[P_\mu, X^\nu] = -i\hbar\delta_\mu^\nu$$

$$\Rightarrow P_\mu = -i\hbar \frac{\partial}{\partial x^\mu}$$

$$[P_i, X^j] = -i\hbar\delta_i^j$$


→ wave function Ψ satisfies
$H\Psi = (g^{\mu\nu} P_\mu P_\nu + m^2)\Psi = 0$
$= \left(-\hbar^2 g^{\mu\nu} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + m^2\right)$
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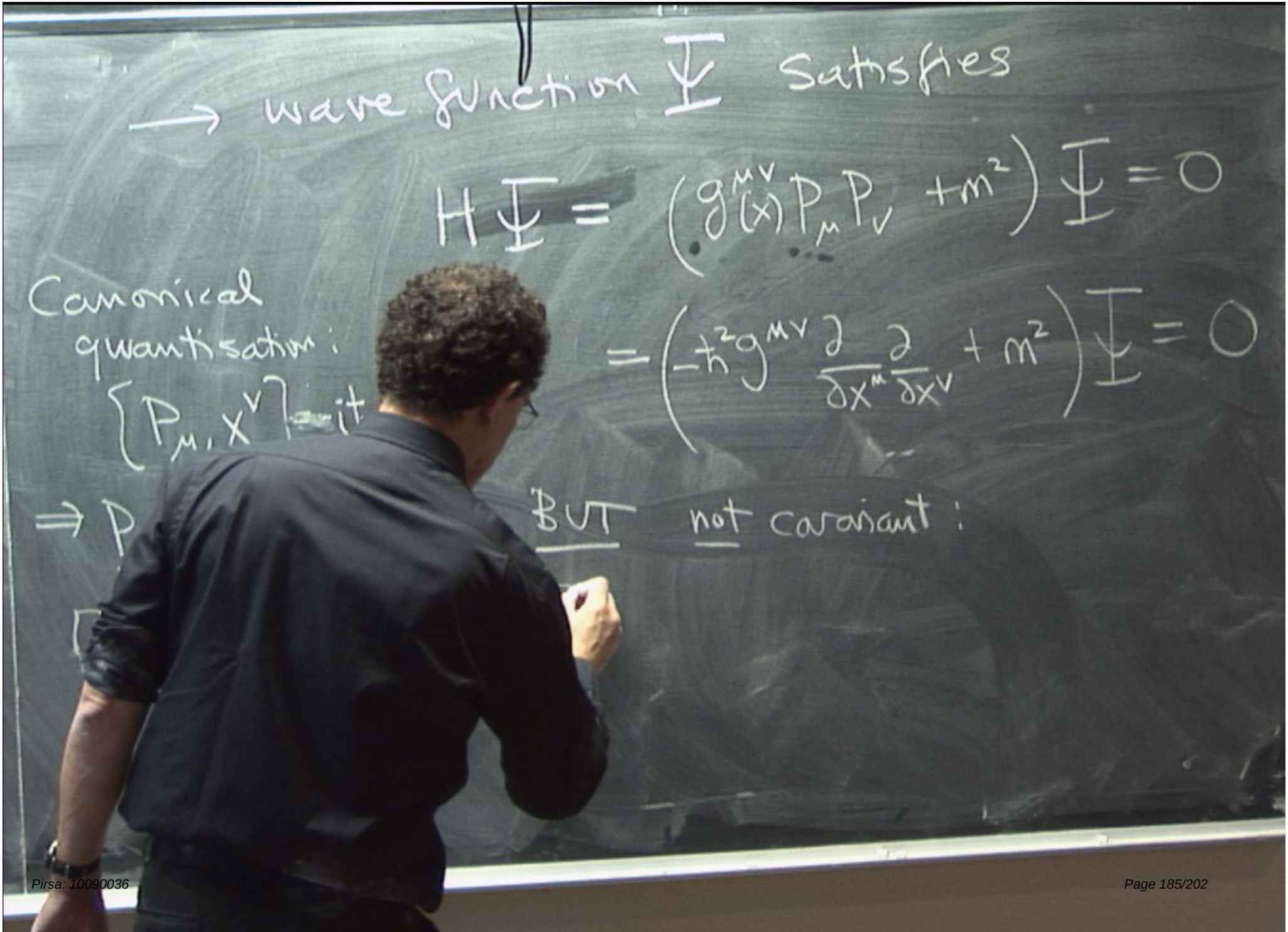
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→ wave function Ψ satisfies
$H\Psi = \left(g^{\mu\nu}(x) P_\mu P_\nu + m^2\right)\Psi = 0$
Canonical quantisation:
$[P_\mu, X^\nu] = -it$
$= \left(-\hbar^2 g^{\mu\nu} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + m^2\right)\Psi = 0$
BUT not covariant:

→ wave function Ψ satisfies

$$H\Psi = \left(g^{\mu\nu}(x) P_\mu P_\nu + m^2\right)\Psi = 0$$

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Canonical quantisation:

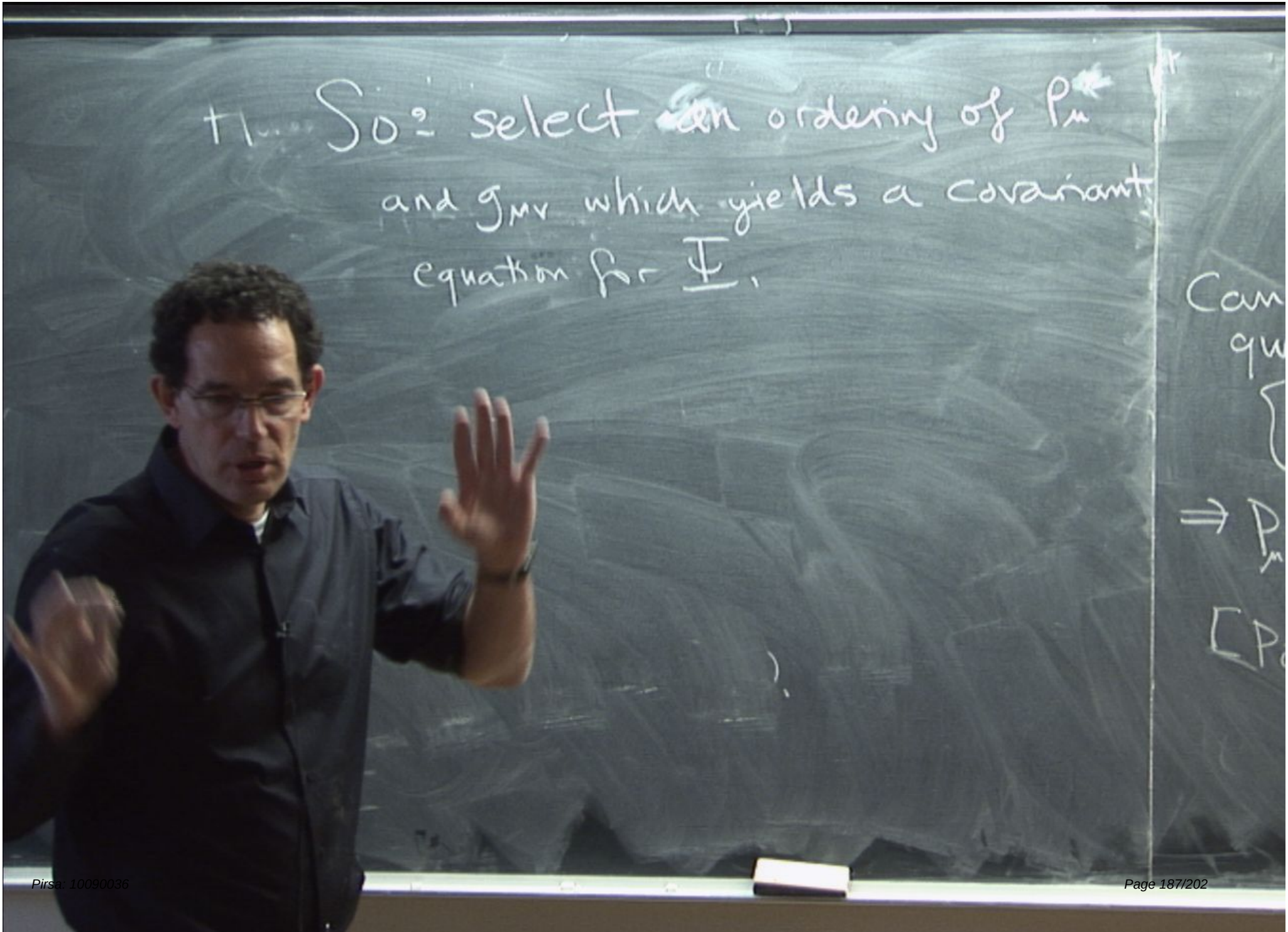
$$[P_\mu, x^\nu] = -i\hbar\delta_\mu^\nu$$

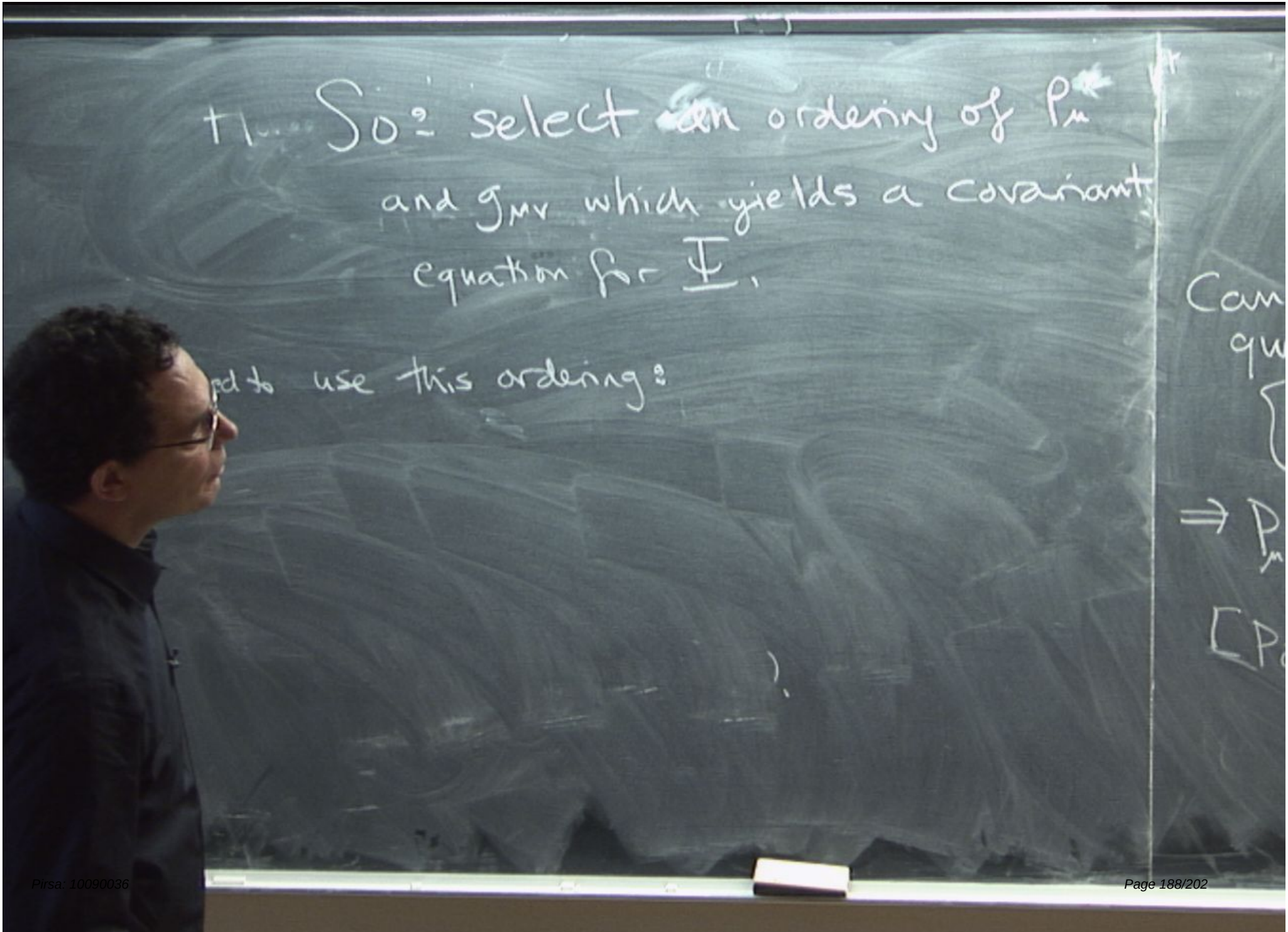
$$\Rightarrow P_\mu = -i\hbar\frac{\partial}{\partial x^\mu}$$

$$[P_i, x^j] = -i\hbar\delta_i^j$$

BUT not covariant:

ordering of P_μ and $g^{\mu\nu}$ is ambiguous

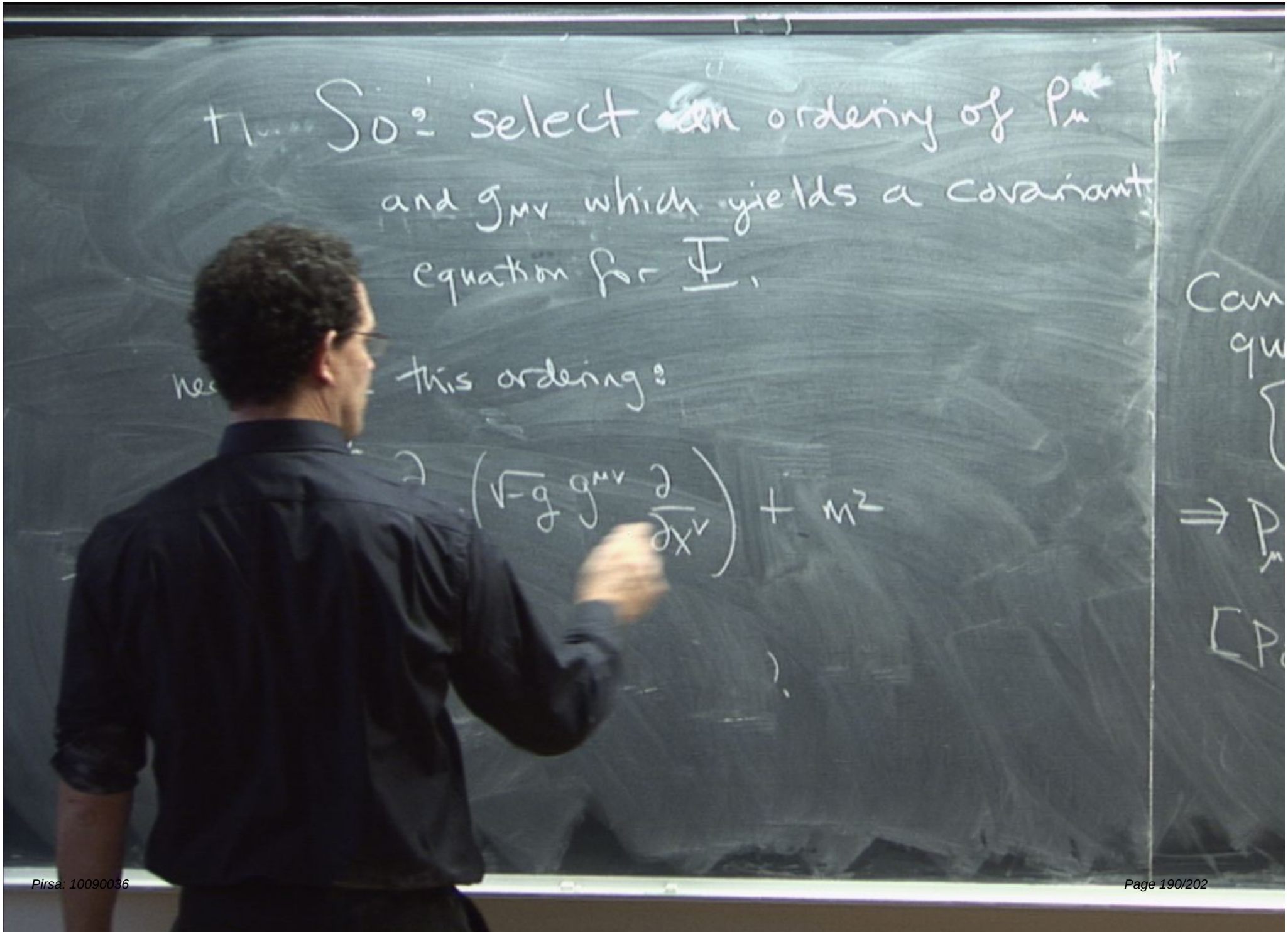

So: select an ordering of $\hat{p}_\mu$
and $g_{\mu\nu}$ which yields a covariant
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ed to use this ordering:

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need to use this ordering:

Pirsa: 10090036


So: select an ordering of p_μ
and $g_{\mu\nu}$ which yields a covariant
equation for Ψ.
this ordering:
$\partial\left(\sqrt{-g}\, g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) + m^2$

So: select an ordering of p_μ and $g_{\mu\nu}$ which yields a covariant equation for Ψ.

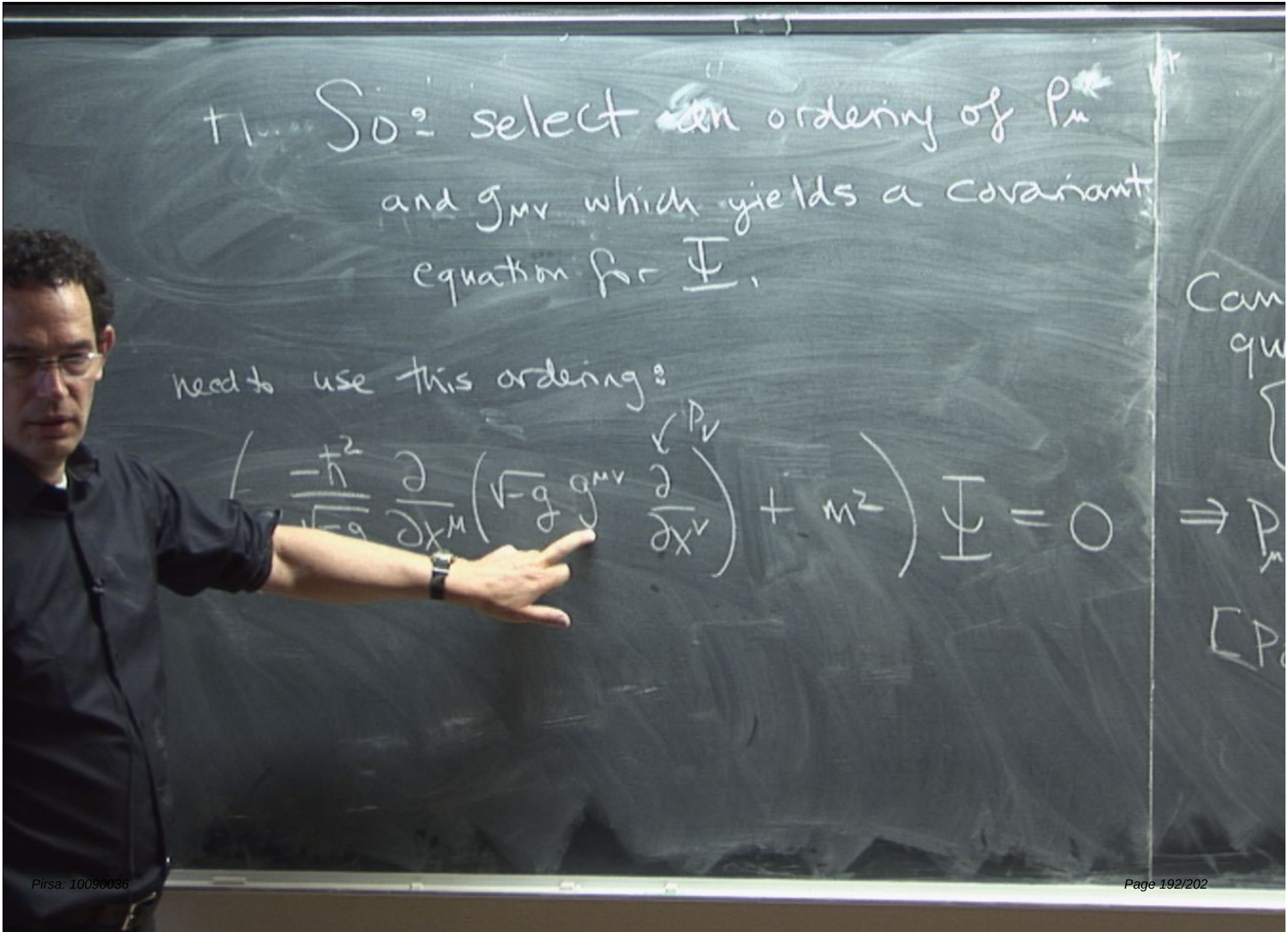
need to use this ordering:

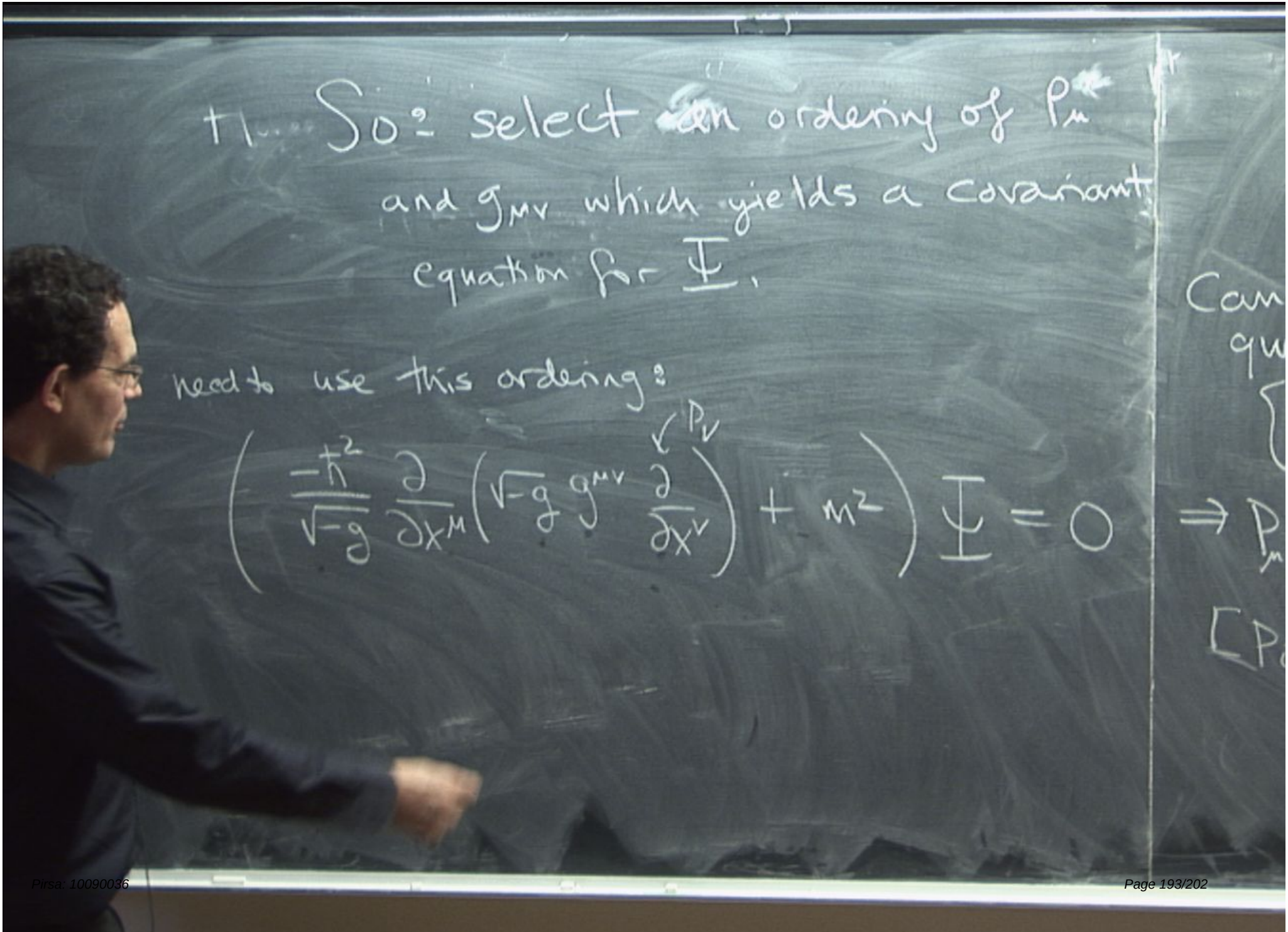
$$\left(\frac{-\hbar^2}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) + m^2 \right) \Psi = 0$$

Can
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$\Rightarrow p_\mu$

[P


So: select an ordering of p_μ and $g_{\mu\nu}$ which yields a covariant equation for Ψ.
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($\frac{\partial}{\partial x^\nu}$ ↙ p_ν)

$$-\hbar^2 g^{\mu\nu} \nabla_\mu \nabla_\nu$$

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$\frac{\partial}{\partial x^\nu} \leftarrow P_\nu$

$-\hbar^2 g^{\mu\nu} \nabla_\mu \nabla_\nu$

Klein-Gordon field equation.

- correct equation for a scalar field Ψ in curved spacetime.

→ w

Canonical quantisation

$[P_\mu, X^\nu] =$

$\Rightarrow P_\mu = -i\hbar \frac{\partial}{\partial x^\mu}$

$[P_i, X^j] = -i\hbar \delta_i^j$

So: select an ordering of P_μ → wave

and $g_{\mu\nu}$ which yields a covariant equation for Ψ.

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use this ordering:

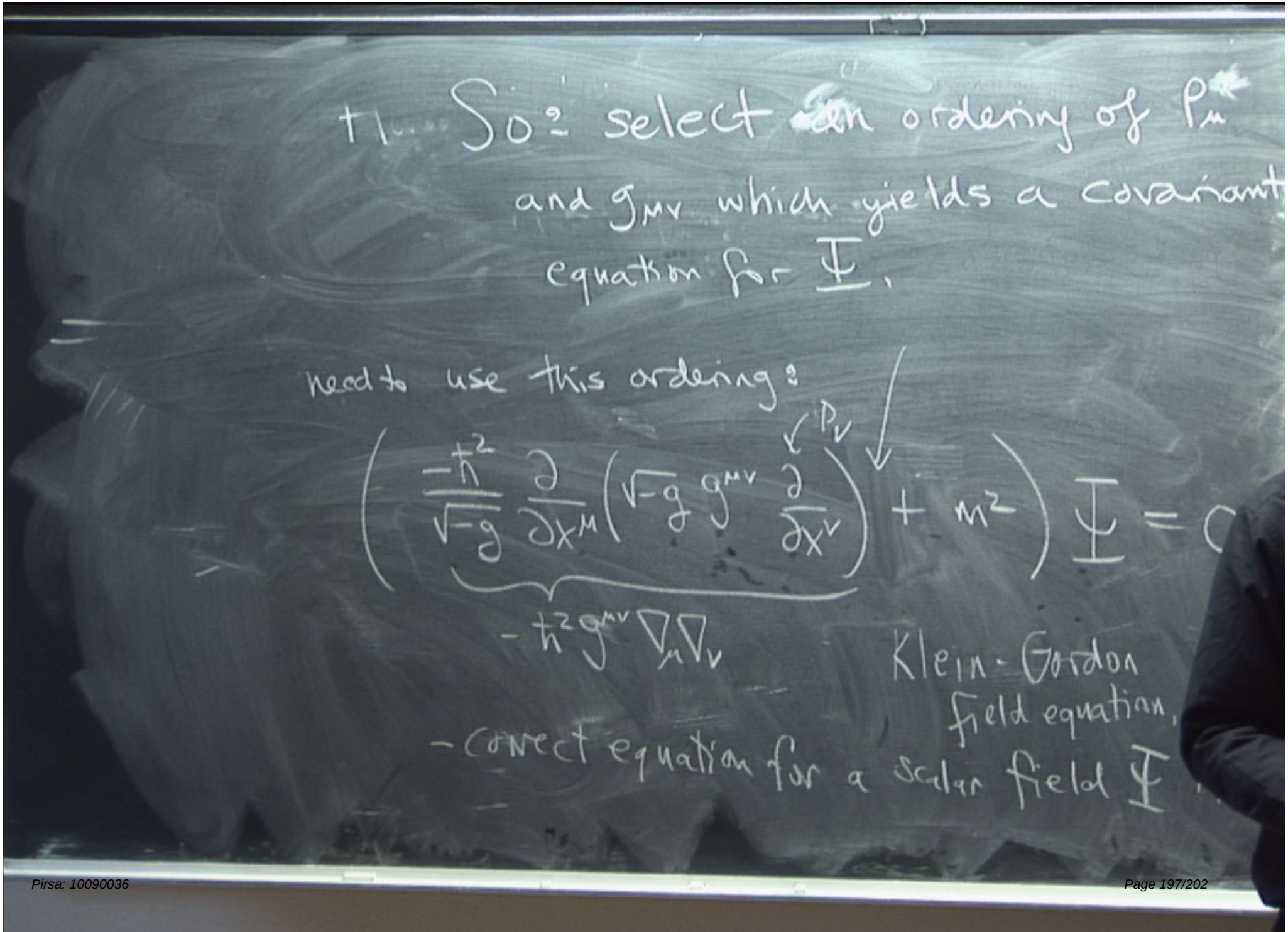
$$\left(\frac{\hbar^2}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) + m^2\right)\Psi = 0$$

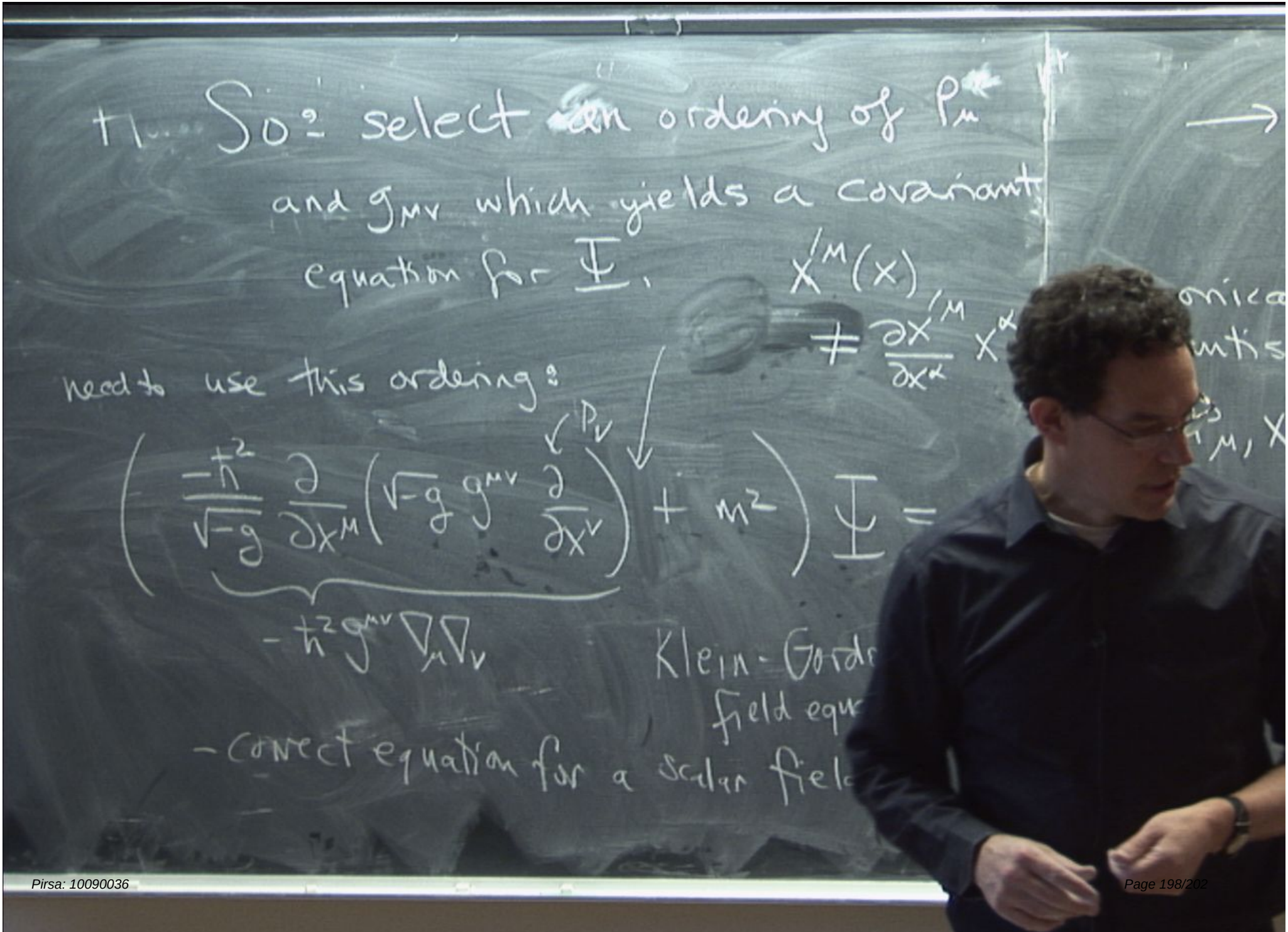
($\frac{\partial}{\partial x^\nu}$: P_ν)

$$-\hbar^2 g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Klein-Gordon field equation,

- correct equation for a scalar field Ψ in curved spacetime.


So: select an ordering of p_μ
and $g_{\mu\nu}$ which yields a covariant
equation for Ψ.
need to use this ordering:
$\left(\frac{-\hbar^2}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) + m^2\right)\Psi = 0$
p_ν
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Klein-Gordon
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So: select an ordering of p_μ
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$x'^\mu(x)$
$= \frac{\partial x'^\mu}{\partial x^\alpha} x^\alpha$
need to use this ordering:
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Klein-Gord
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$$x'^\mu(x) = \frac{\partial x'^\mu}{\partial x^\alpha} x^\alpha$$

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$\frac{\partial}{\partial x^\nu} \to P_\nu$

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Klein-Gordon field equation,

- correct equation for a scalar field Ψ in curved spacetime

Canonical quantis

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So: select an ordering of p_μ and $g_{\mu\nu}$ which yields a covariant equation for Ψ.

need to use this ordering:

$$\left(\frac{-\hbar^2}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) + \xi R + m^2 \right) \Psi = 0$$

(p_ν over the $\partial/\partial x^\nu$; $\#$ over ξR)

$$-\hbar^2 g^{\mu\nu} \nabla_\mu \nabla_\nu$$

Klein-Gordon field equation,

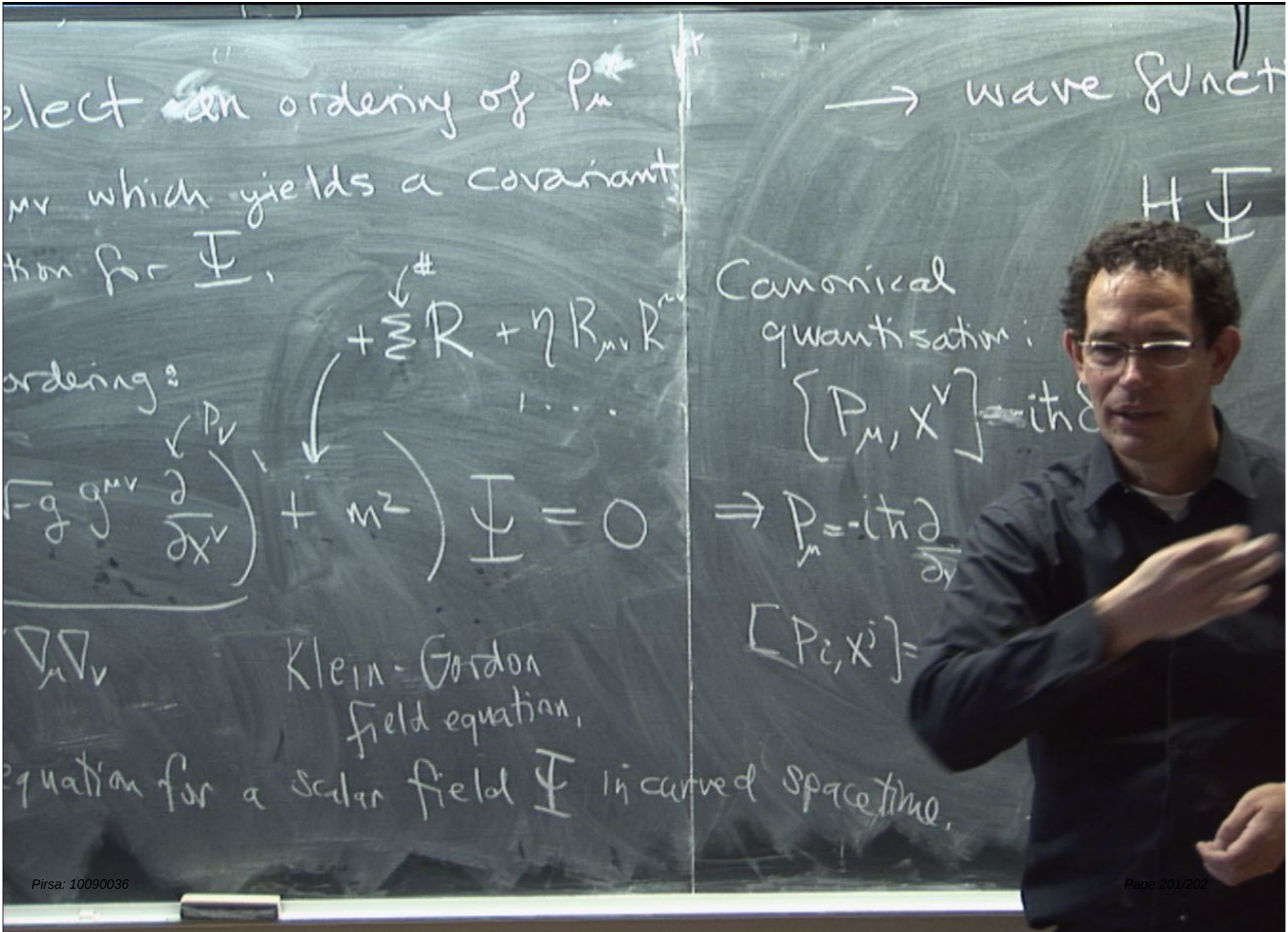
- correct equation for a scalar field Ψ in curved spacetime

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ordering:
$+\xi R + \eta R_{\mu\nu} R^{\mu\nu}$
$\dots$
$\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu}\right) + m^2 \Psi = 0$
$\nabla_\mu \nabla_\nu$
Klein-Gordon
field equation,
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$\longrightarrow$ wave funct
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$\nabla_\mu \nabla_\nu$

Klein-Gordon field equation,

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