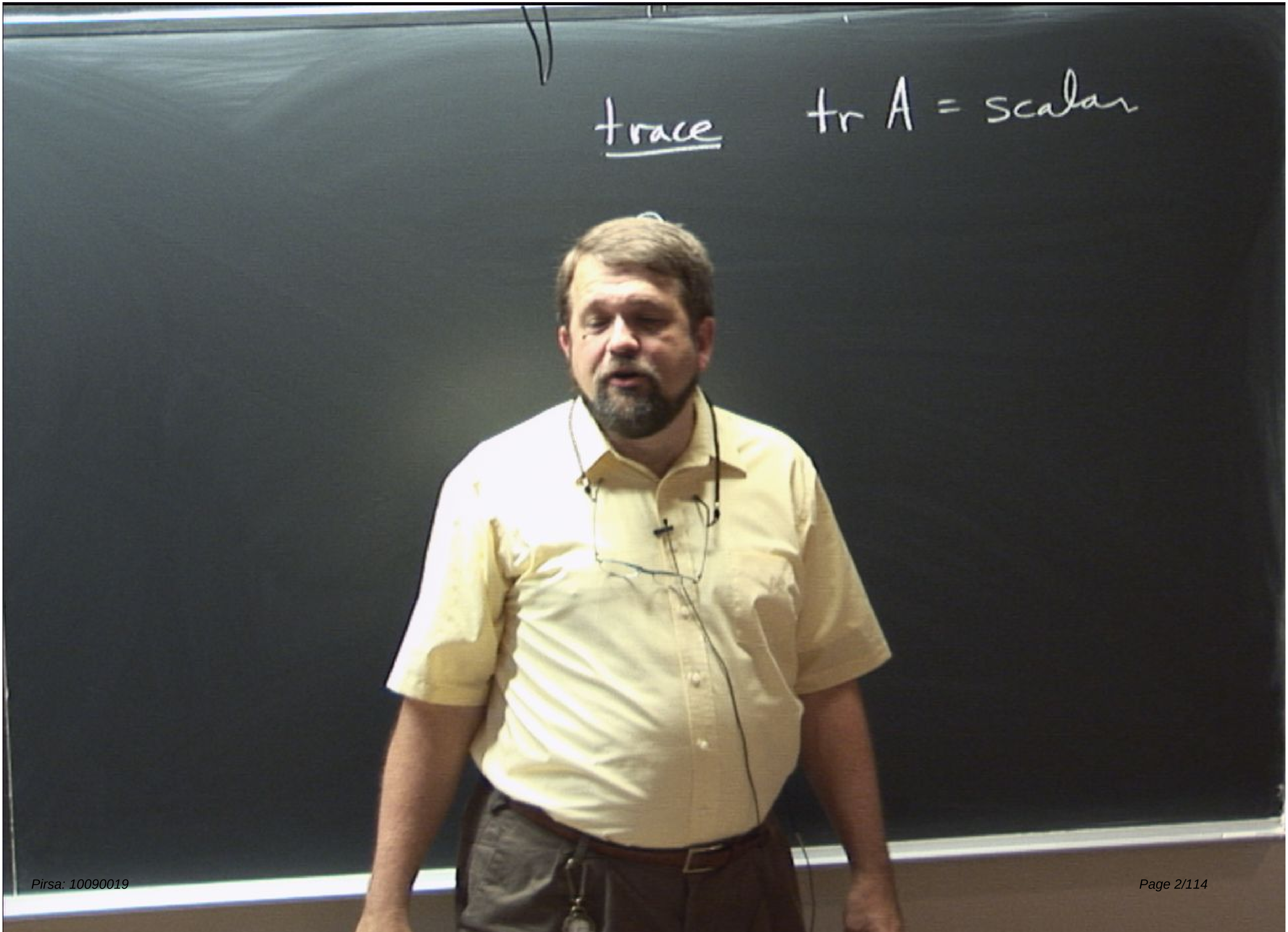
Title: Quantum Theory (PHYS 605) - Lecture 8

Date: Sep 22, 2010 09:00 AM

URL: http://pirsa.org/10090019

Abstract:


trace
tr A = scalar

<u>trace</u> $\operatorname{tr} A = \text{scalar}$

① trace : outer product
→ inner product

$$\operatorname{tr} |\alpha\rangle\langle\beta| = \langle\beta|\alpha\rangle$$

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① trace : outer product
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$$\text{tr}\, |\alpha\rangle\langle\beta| = \langle\beta|\alpha\rangle$$

② $\text{tr}(\cdot)$ is <u>linear</u>

trace $\quad \mathrm{tr}\, A = \text{scalar}$

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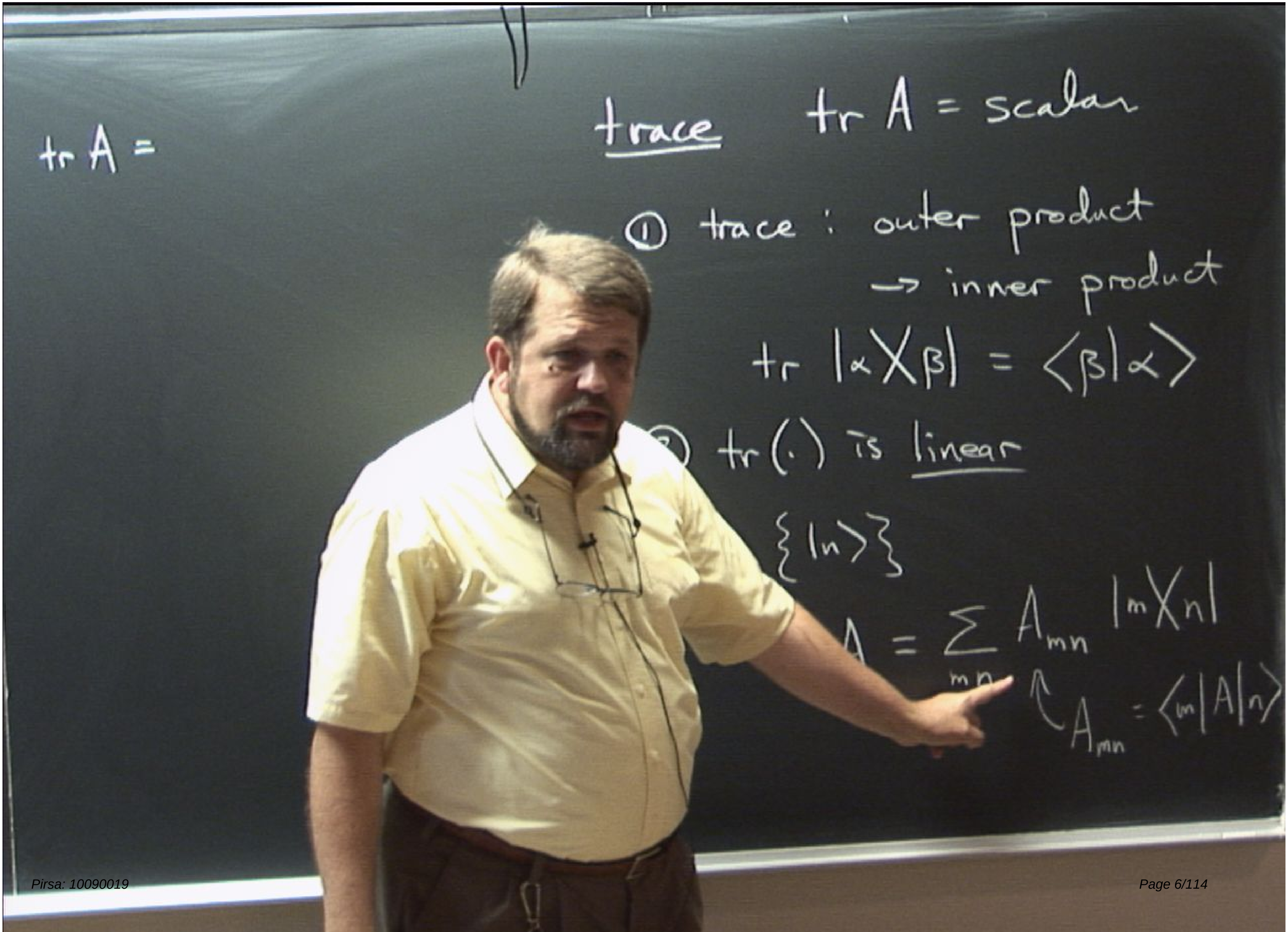
② $\mathrm{tr}(\cdot)$ is linear

Basis $\{|n\rangle\}$

$$\Rightarrow A = \sum_{mn} A_{mn} |m\rangle\langle n|$$

$$A_{mn} = \langle m|A|n\rangle$$


tr A =
trace tr A = scalar
① trace : outer product
→ inner product
tr |α⟩⟨β| = ⟨β|α⟩
tr(·) is linear
{|n⟩}
A = Σ A_mn |m⟩⟨n|
mn
A_mn = ⟨m|A|n⟩

$$\mathrm{tr}\, A = \sum_{mn} A_{mn} \langle n|m\rangle$$

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$$\mathrm{tr}\, A = \sum_{mn} A_{mn} \underbrace{\langle n | m \rangle}_{\delta_{nm}}$$

$$= \sum_{n} A_{nn}$$

<u>trace</u> $\mathrm{tr}\, A$ = scalar

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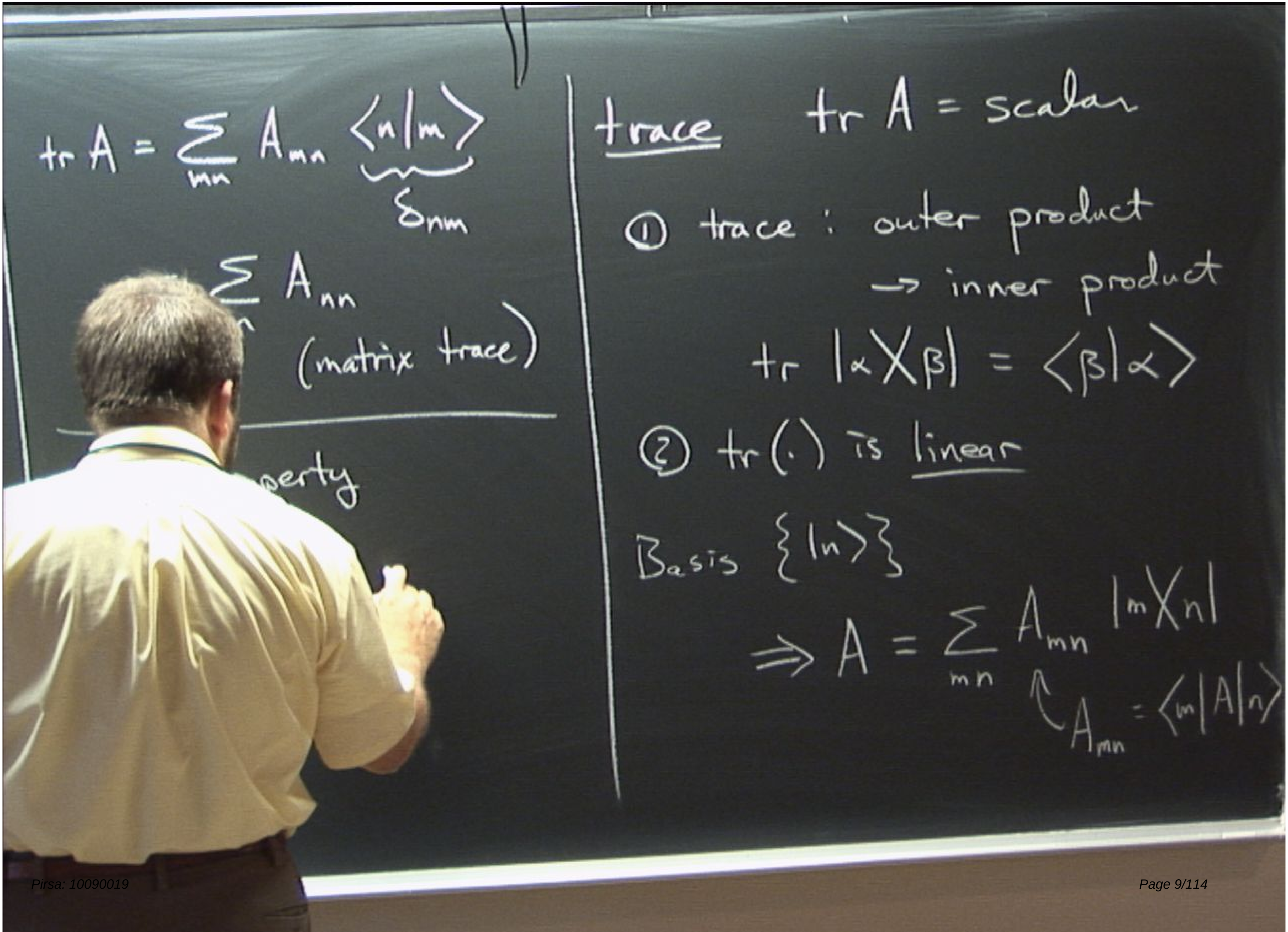
$\mathrm{tr}\, A = \sum_{mn} A_{mn} \underbrace{\langle n|m\rangle}_{\delta_{nm}}$
$\sum_n A_{nn}$
(matrix trace)
perty
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$$= \sum_n A_{nn}$$

(matrix trace)

Cyclic property

$$\mathrm{tr}\, AB = \mathrm{tr}\, BA$$

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(matrix trace)

Cyclic property

$$\operatorname{tr} AB = \operatorname{tr} BA$$

Beware!

$$\operatorname{tr} RST = \operatorname{tr} STR$$

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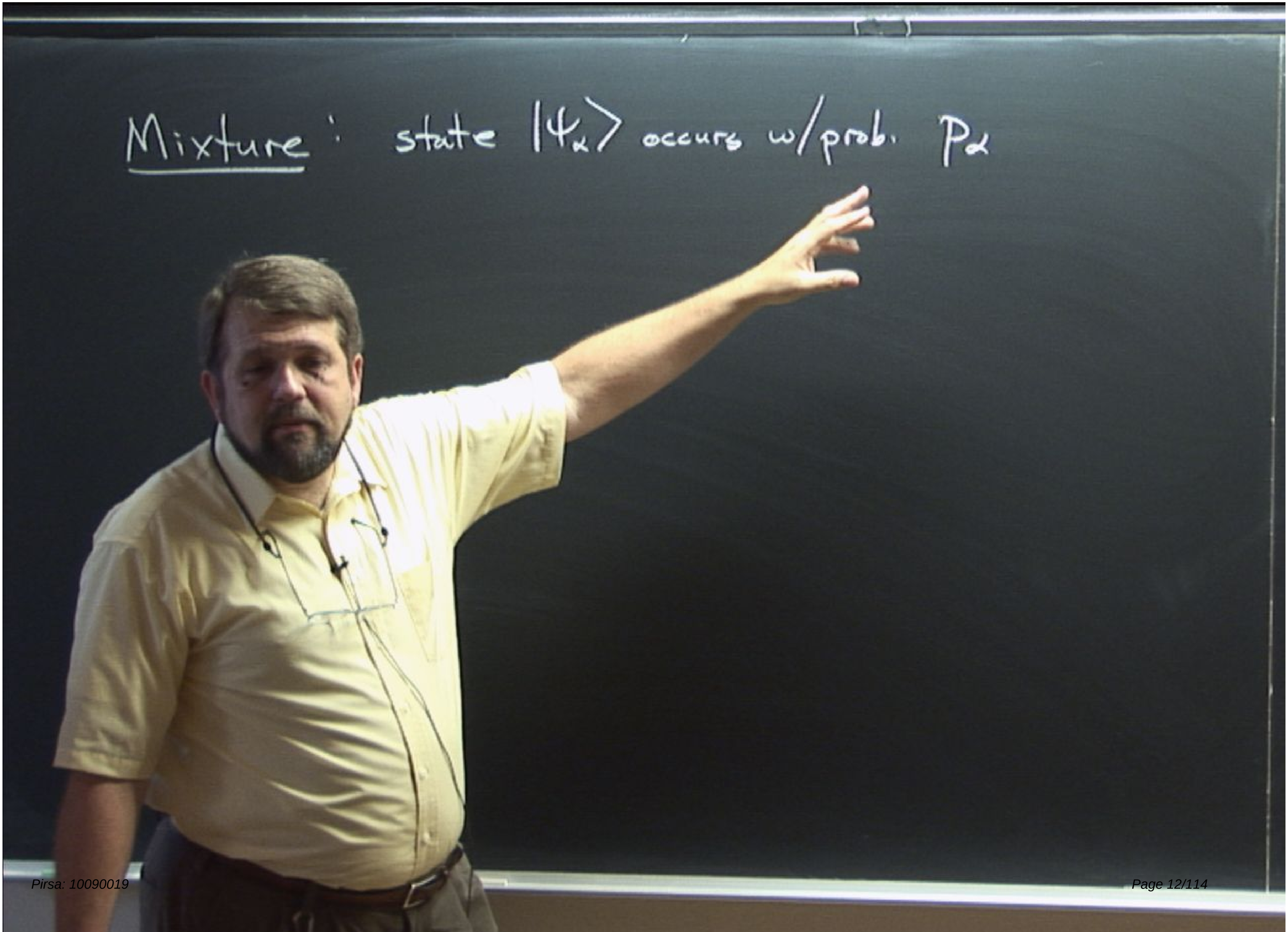
Mixture: state $|\psi_\alpha\rangle$ occurs w/prob. p_α

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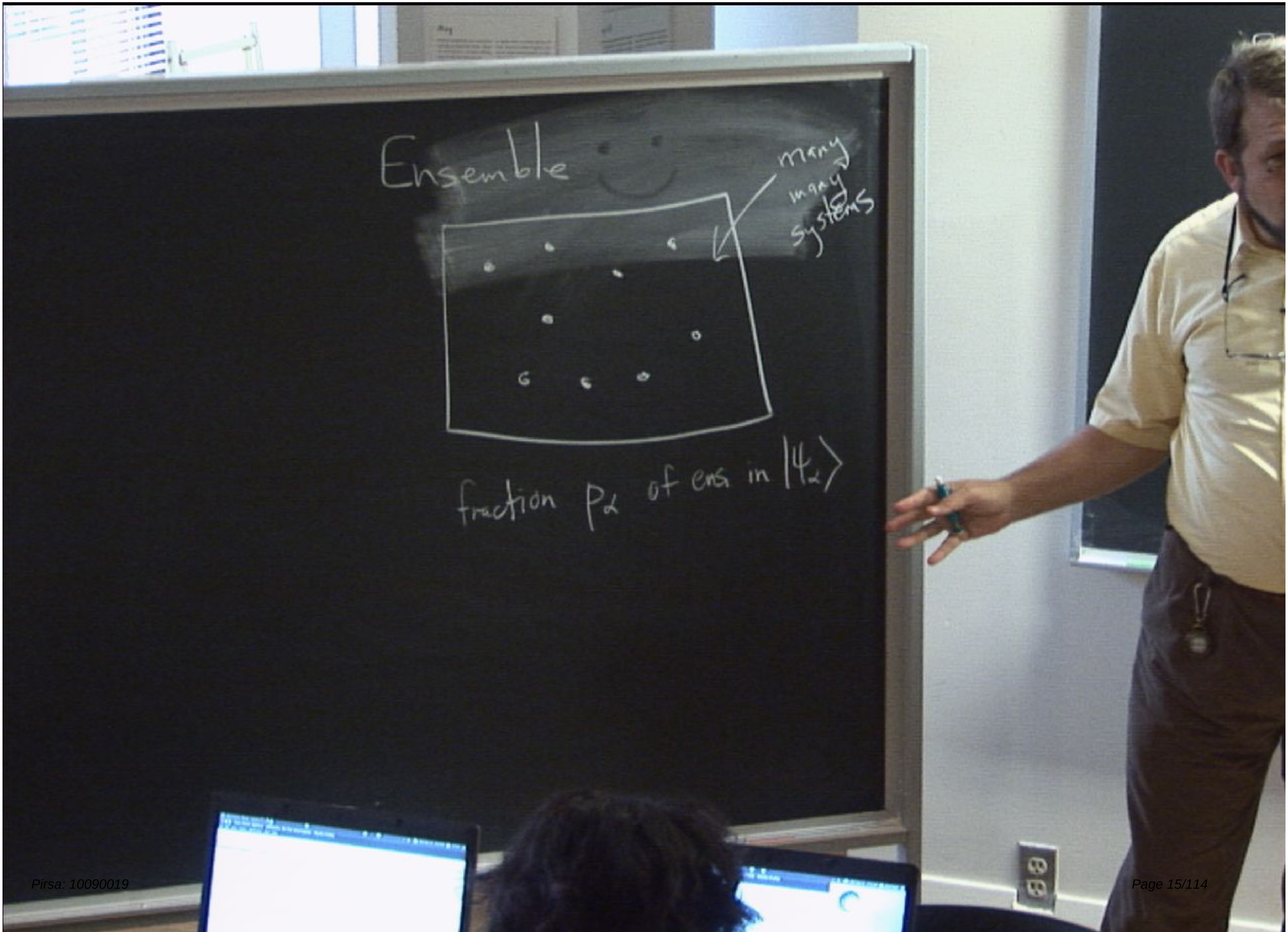
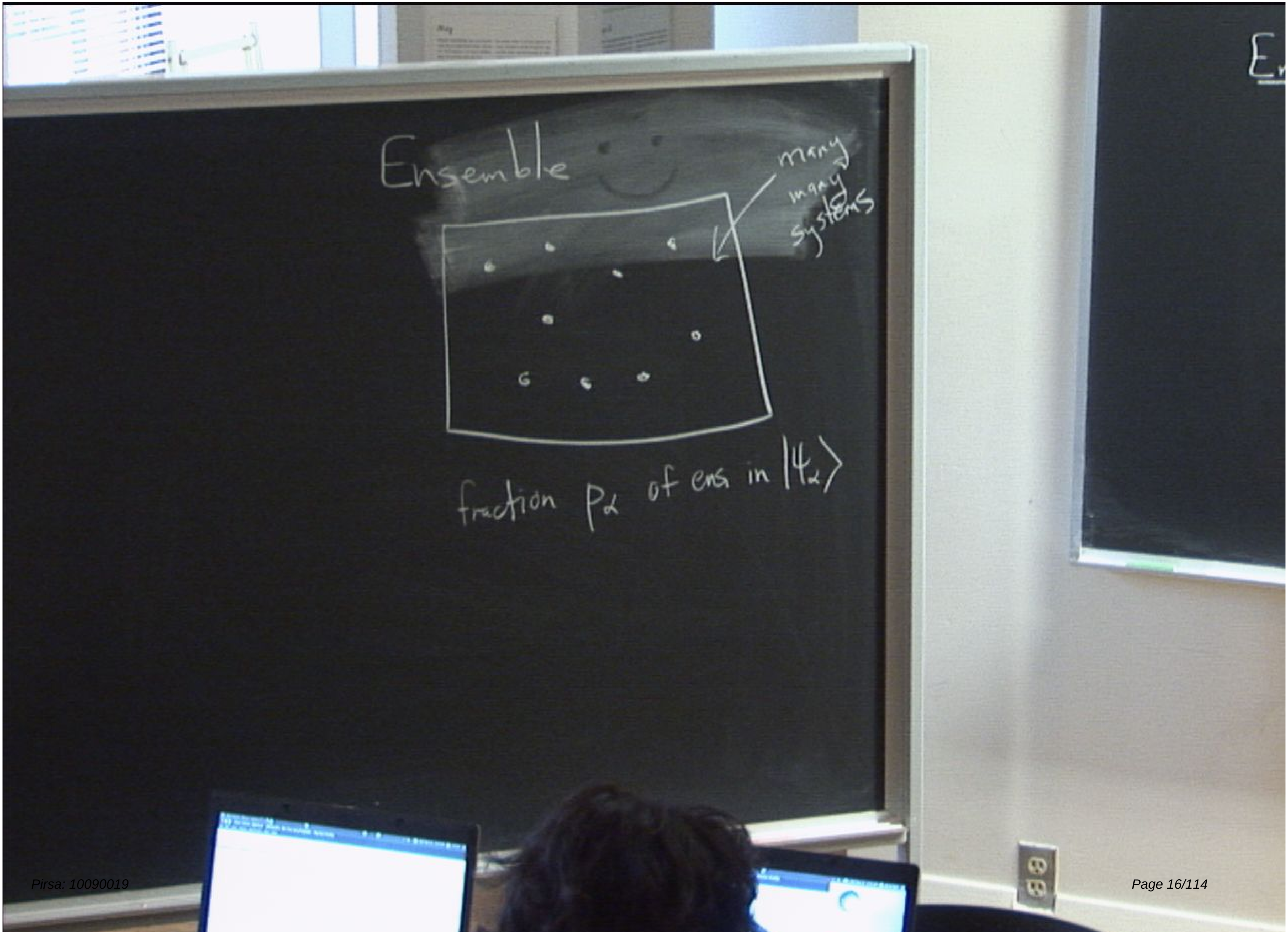
↑ need not be orthogonal

Entanglement: subsystem of a system in an entangled state $|\Psi\rangle$


Ensemble


Ensemble
many many systems
fraction pα of ens. in |ψα⟩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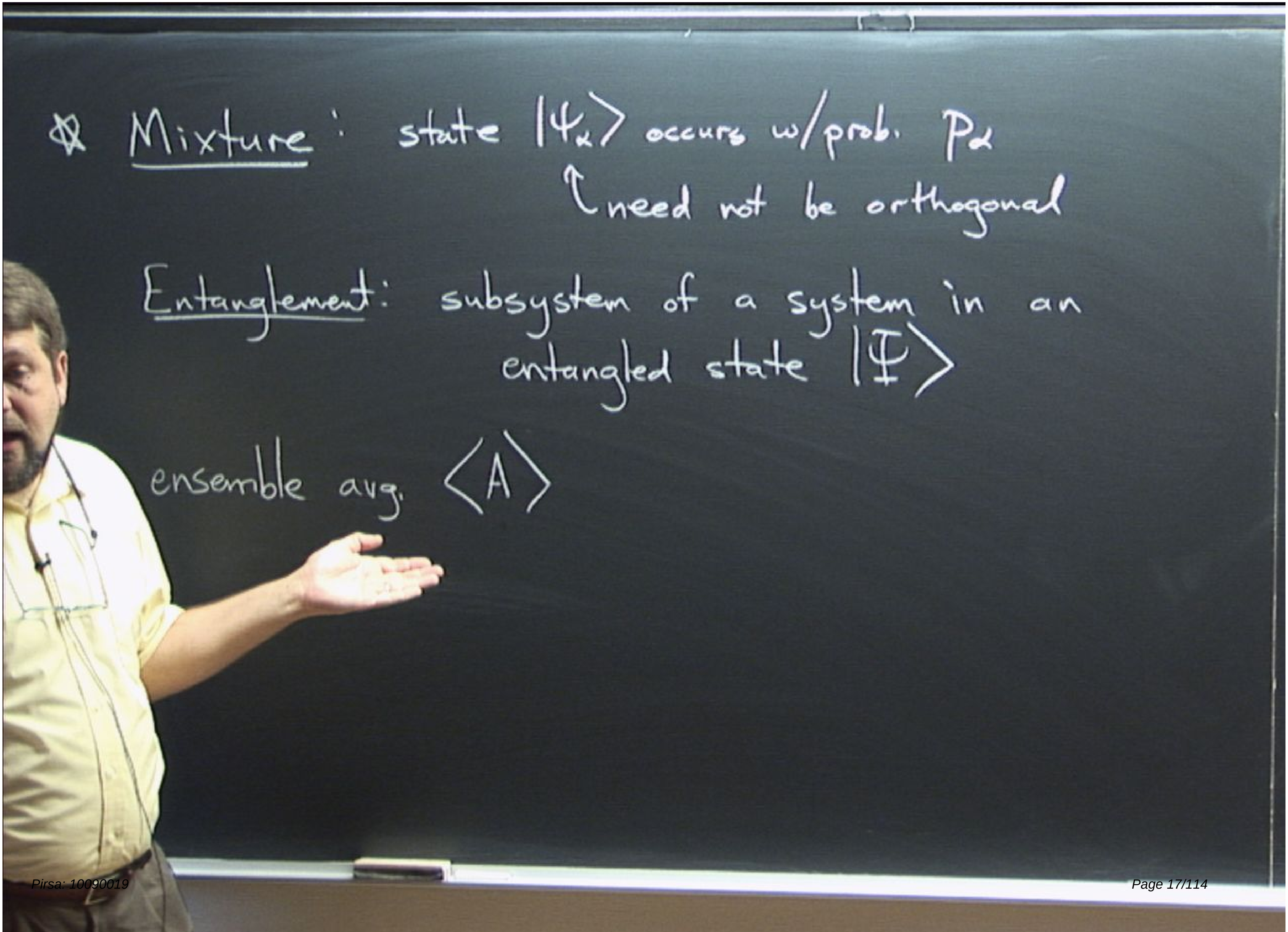

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ensemble avg. $\langle A\rangle$

✲ <u>Mixture</u>: state $|\psi_\alpha\rangle$ occurs w/prob. p_α

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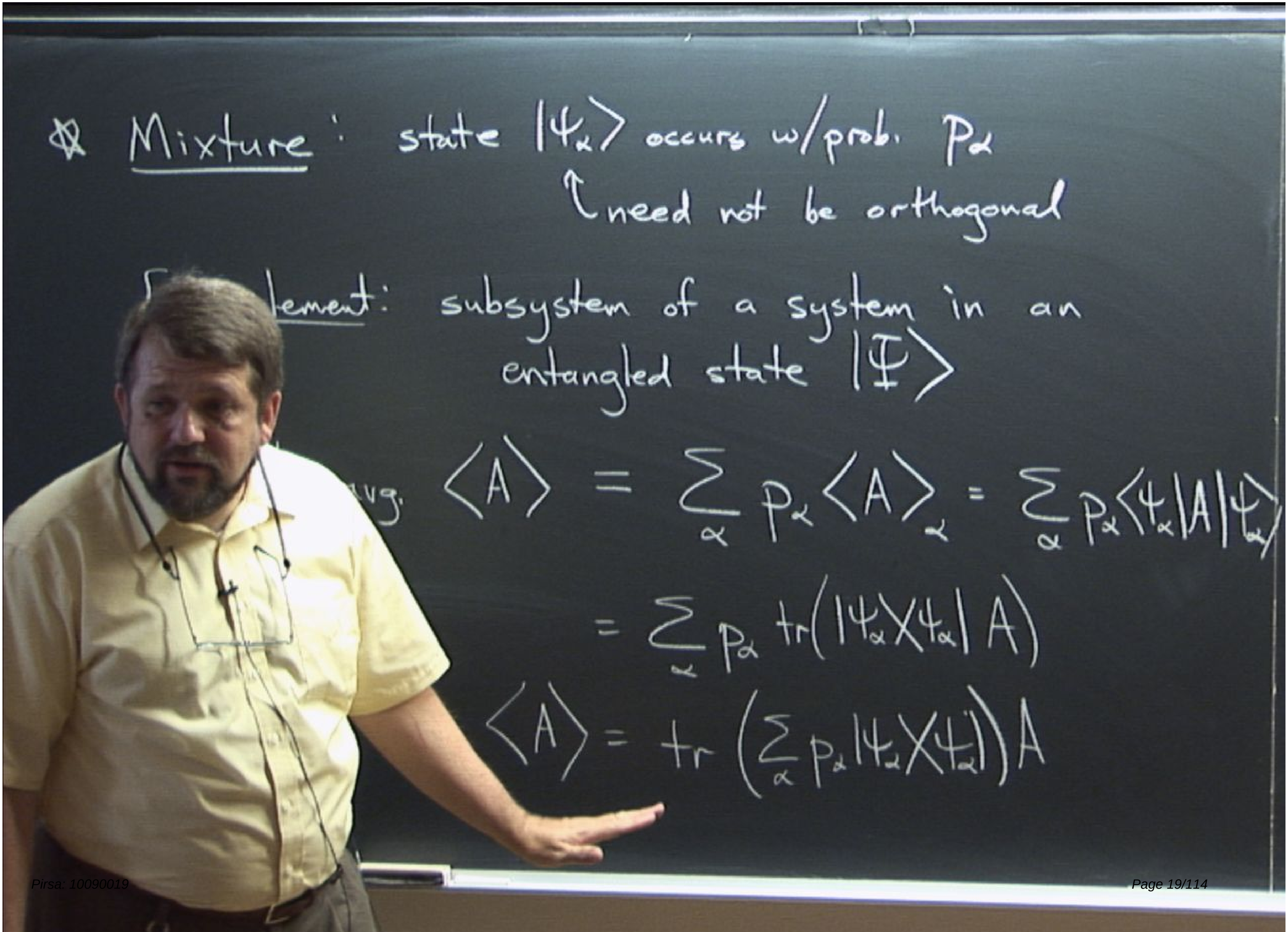
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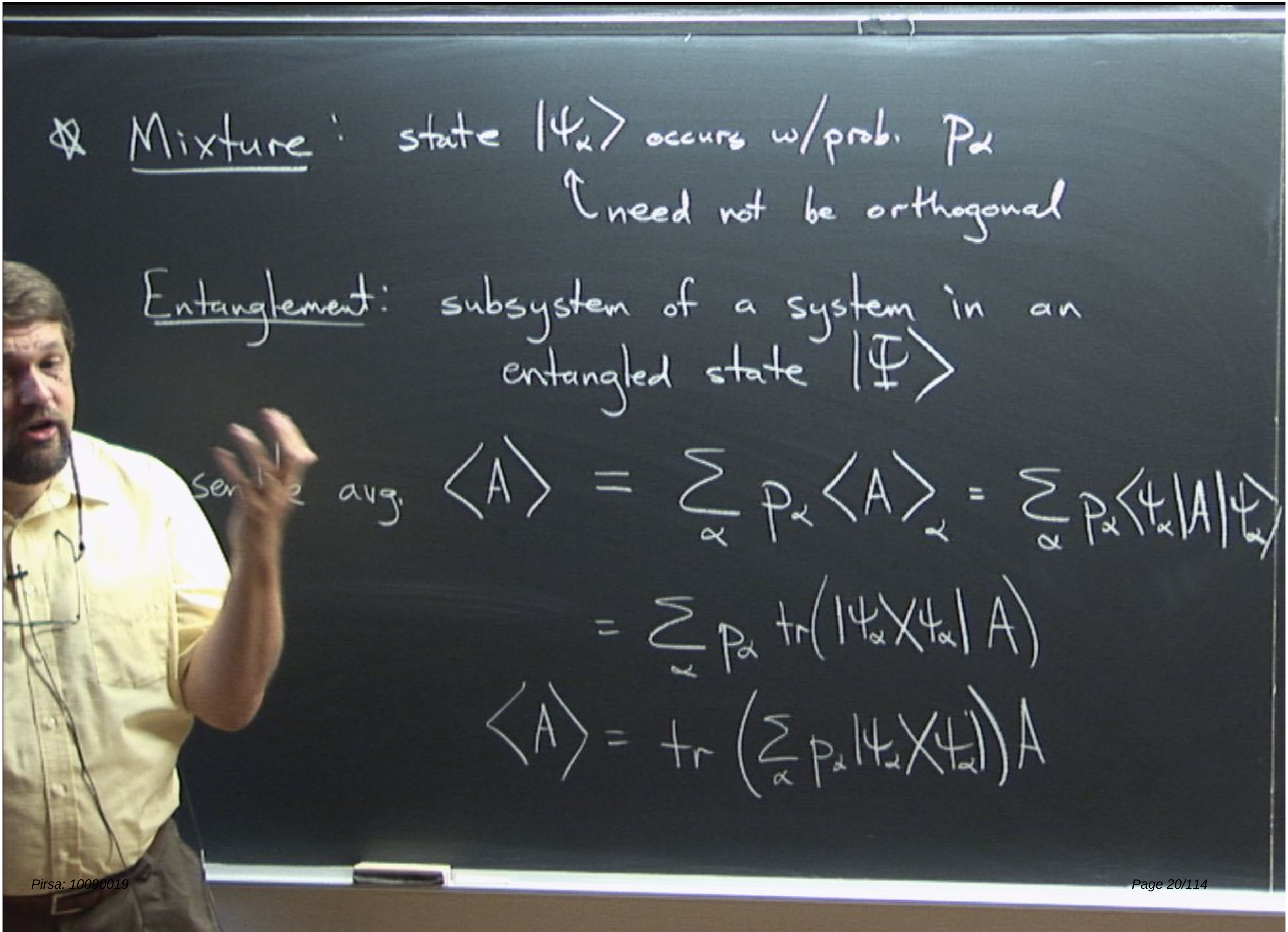
$= \sum_\alpha p_\alpha \, \mathrm{tr}(|\psi_\alpha\rangle\langle\psi_\alpha| A)$
$\langle A\rangle = \mathrm{tr}\left(\sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|\right) A$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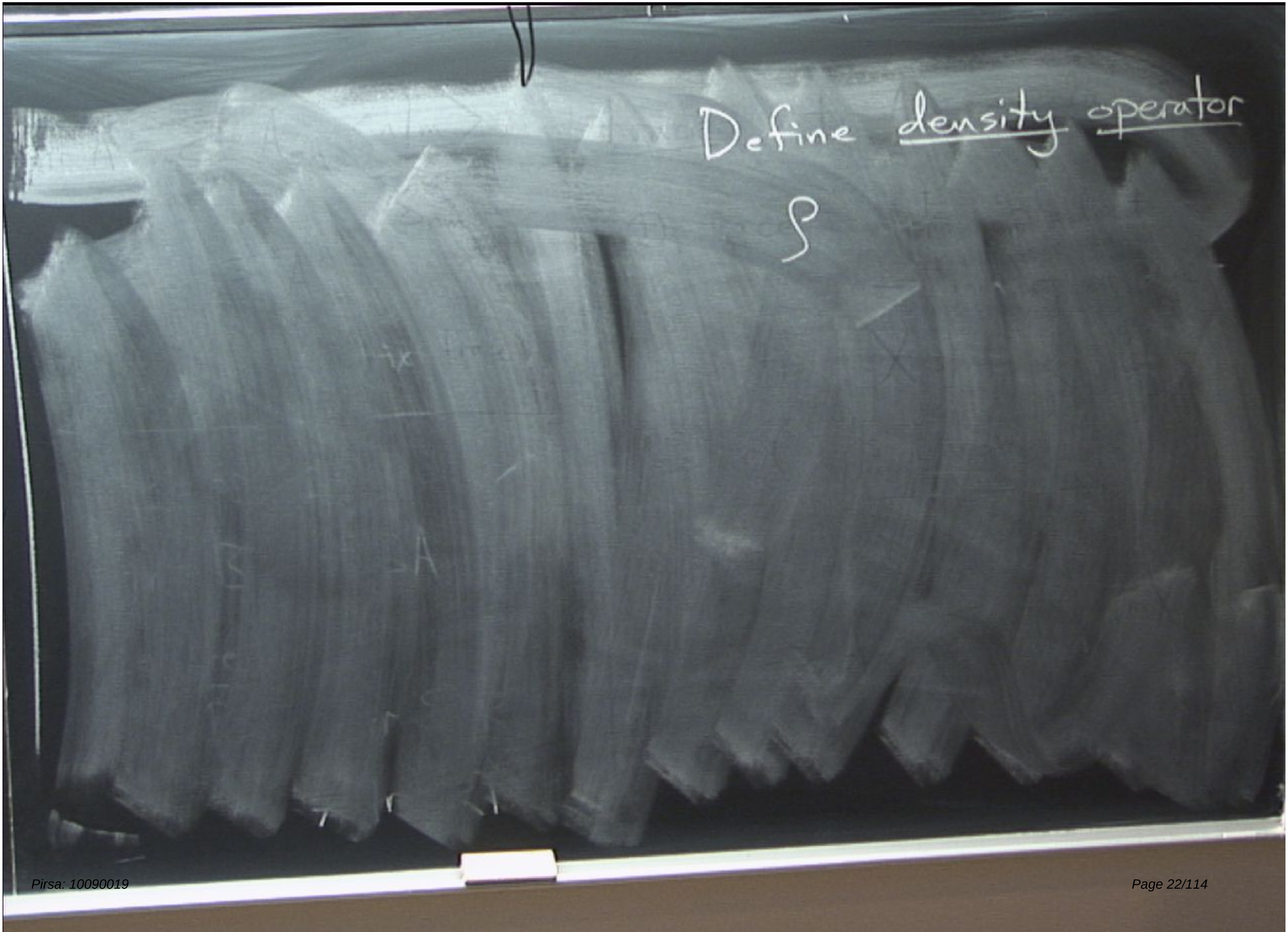
Define density operator
ρ

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$\langle A\rangle = \mathrm{tr}\left(\sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|\right) A = \mathrm{tr}\,\rho A$

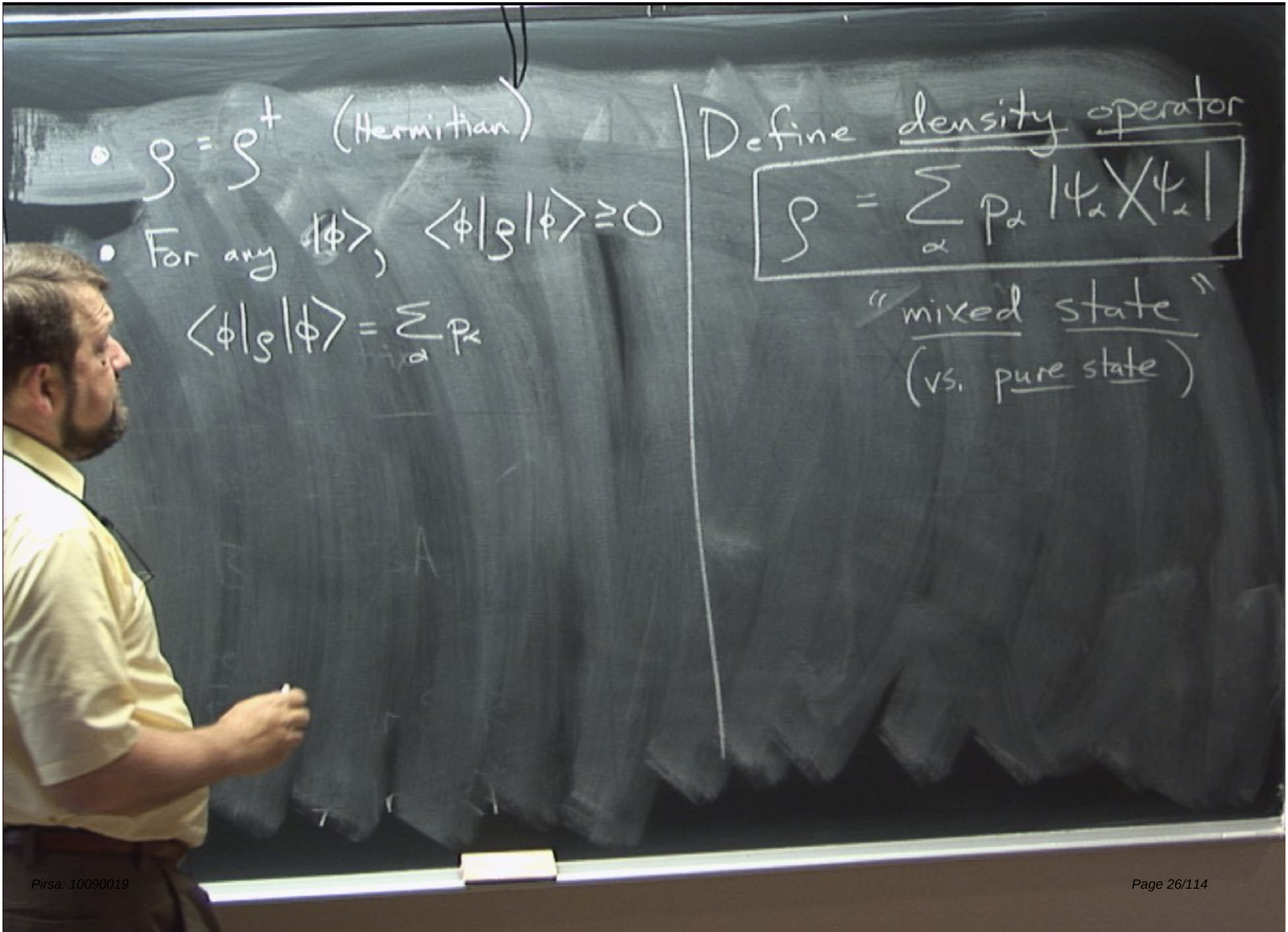
- $\rho = \rho^\dagger$ (Hermitian)
-

Define <u>density</u> <u>operator</u>

$$\boxed{\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|}$$

"<u>mixed</u> <u>state</u>"

(vs. <u>pure</u> state)


• $\rho = \rho^\dagger$ (Hermitian)
• For any $|\phi\rangle$, $\langle\phi|\rho|\phi\rangle \geq 0$
$\langle\phi|\rho|\phi\rangle = \sum_\alpha p_\alpha$
Define density operator
$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$
"mixed state"
(vs. pure state)

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ρ is a positive op.

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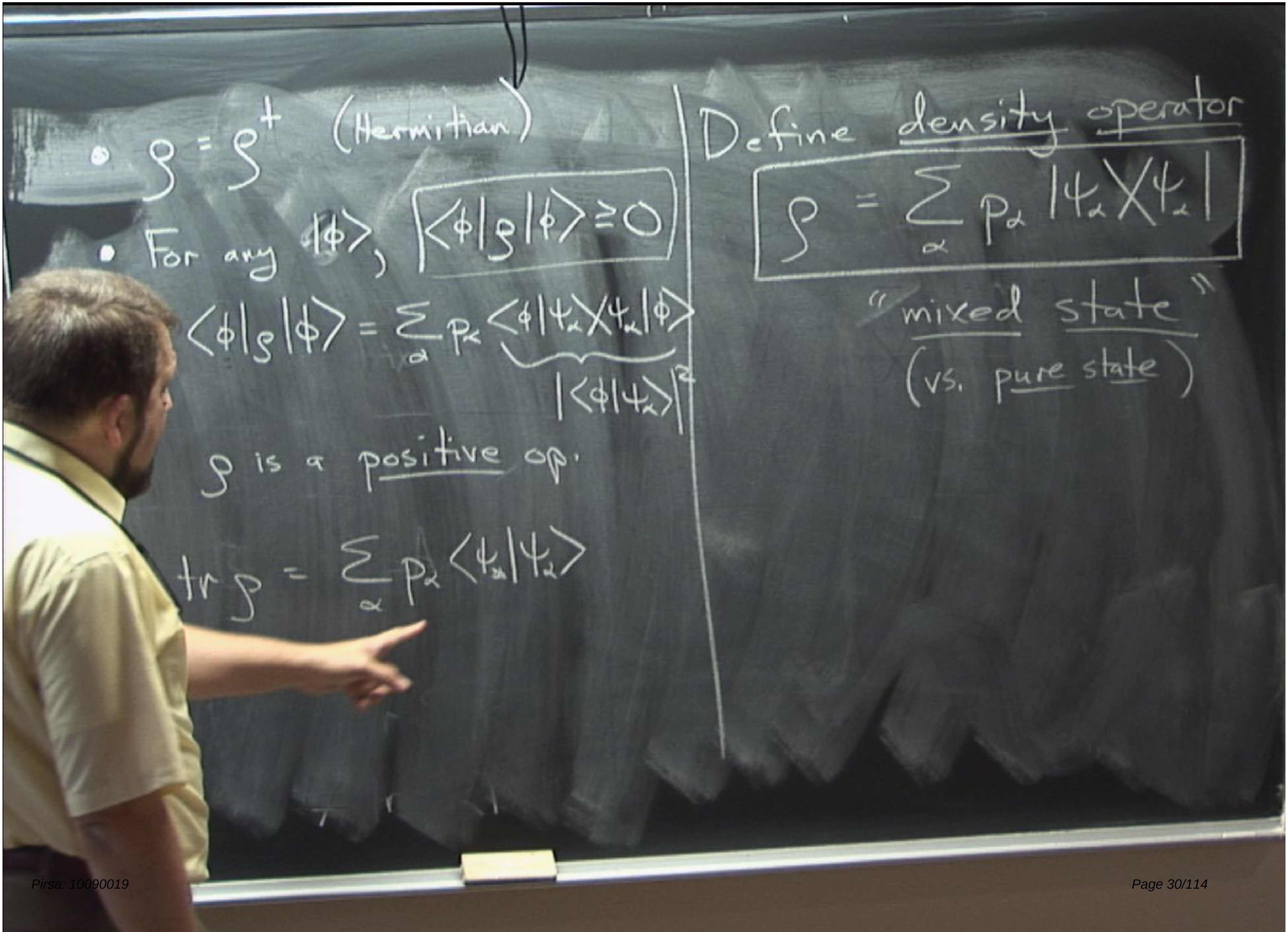
• ρ = ρ† (Hermitian)
• For any |φ⟩, ⟨φ|ρ|φ⟩ ≥ 0
⟨φ|ρ|φ⟩ = Σ_α p_α ⟨φ|ψ_α⟩⟨ψ_α|φ⟩
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ρ is a positive op.
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Define density operator
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ρ is a positive op.

- $\operatorname{tr}\rho = \sum_{\alpha} p_{\alpha} \langle\psi_{\alpha}|\psi_{\alpha}\rangle = 1$

- $\boxed{\langle A\rangle = \operatorname{tr}\rho A}$

Define density operator

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"mixed state"

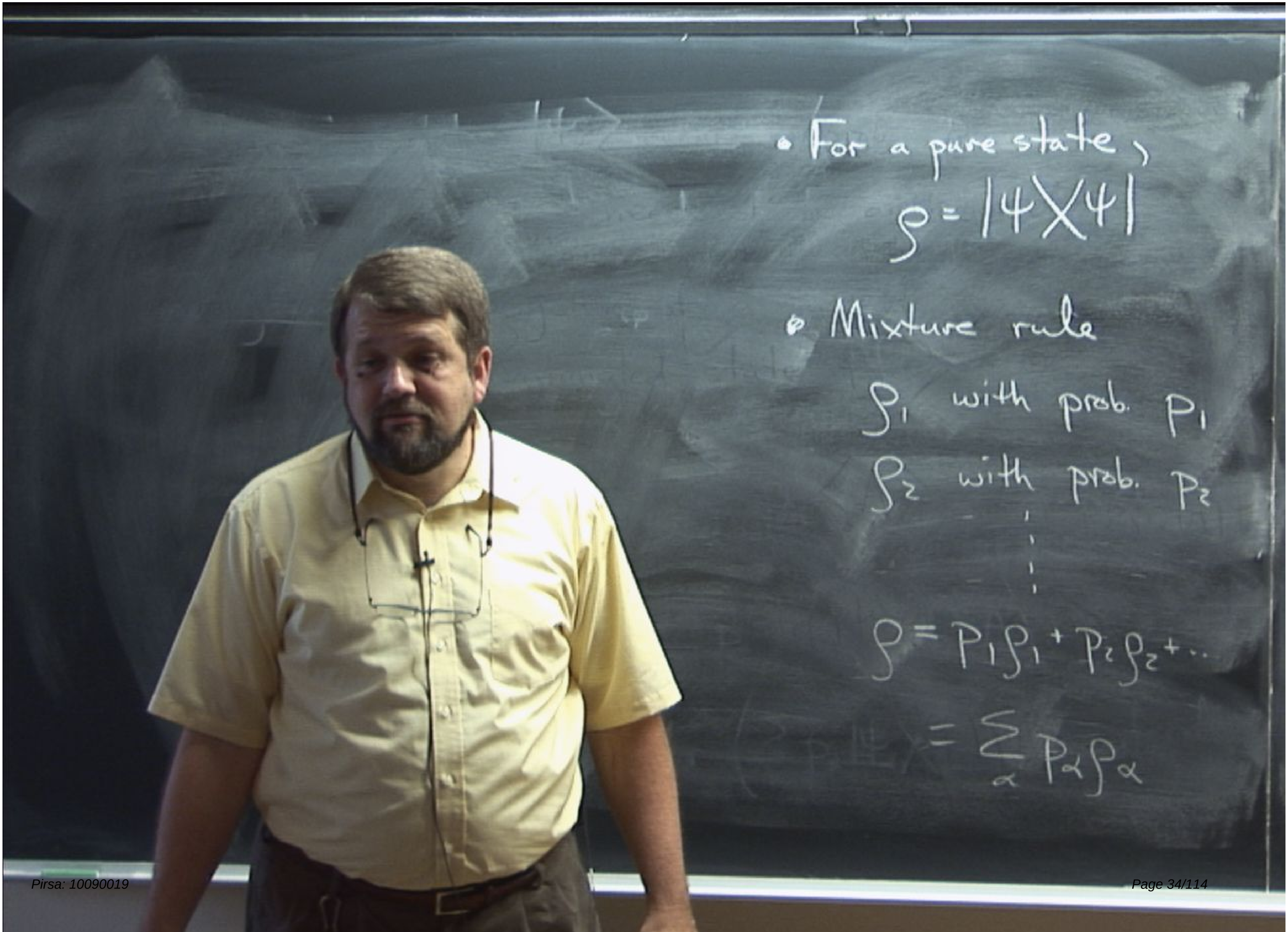
(vs. pure state)

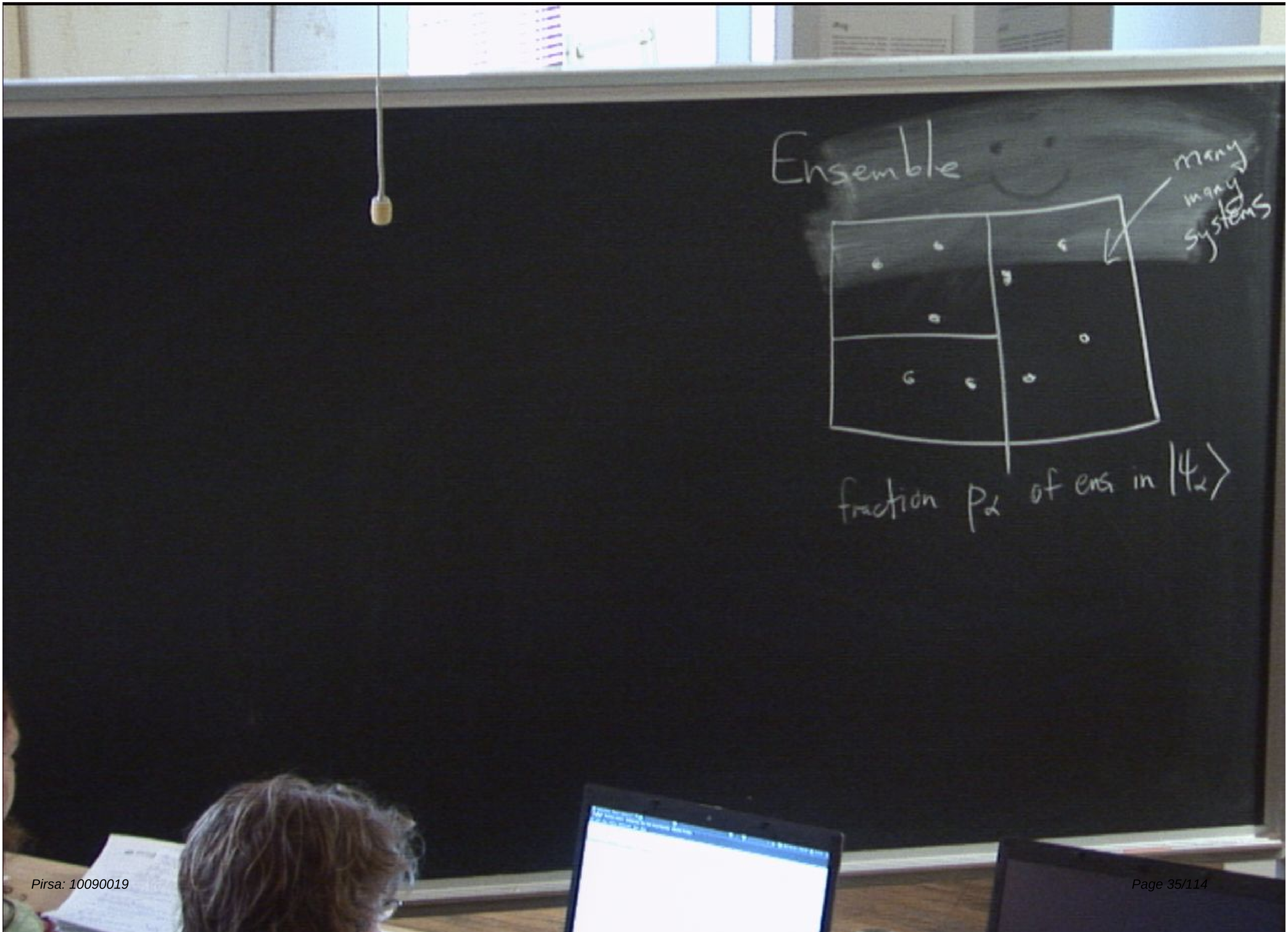
• For a pure state

- For a pure state,

$$\rho = |\psi\rangle\langle\psi|$$

$$|\psi\rangle \sim e^{i\phi}|\psi\rangle$$


• For a pure state,
ρ = |ψ⟩⟨ψ|
• Mixture rule
ρ1 with prob. p1
ρ2 with prob. p2
ρ = p1ρ1 + p2ρ2 + …
= Σα pα ρα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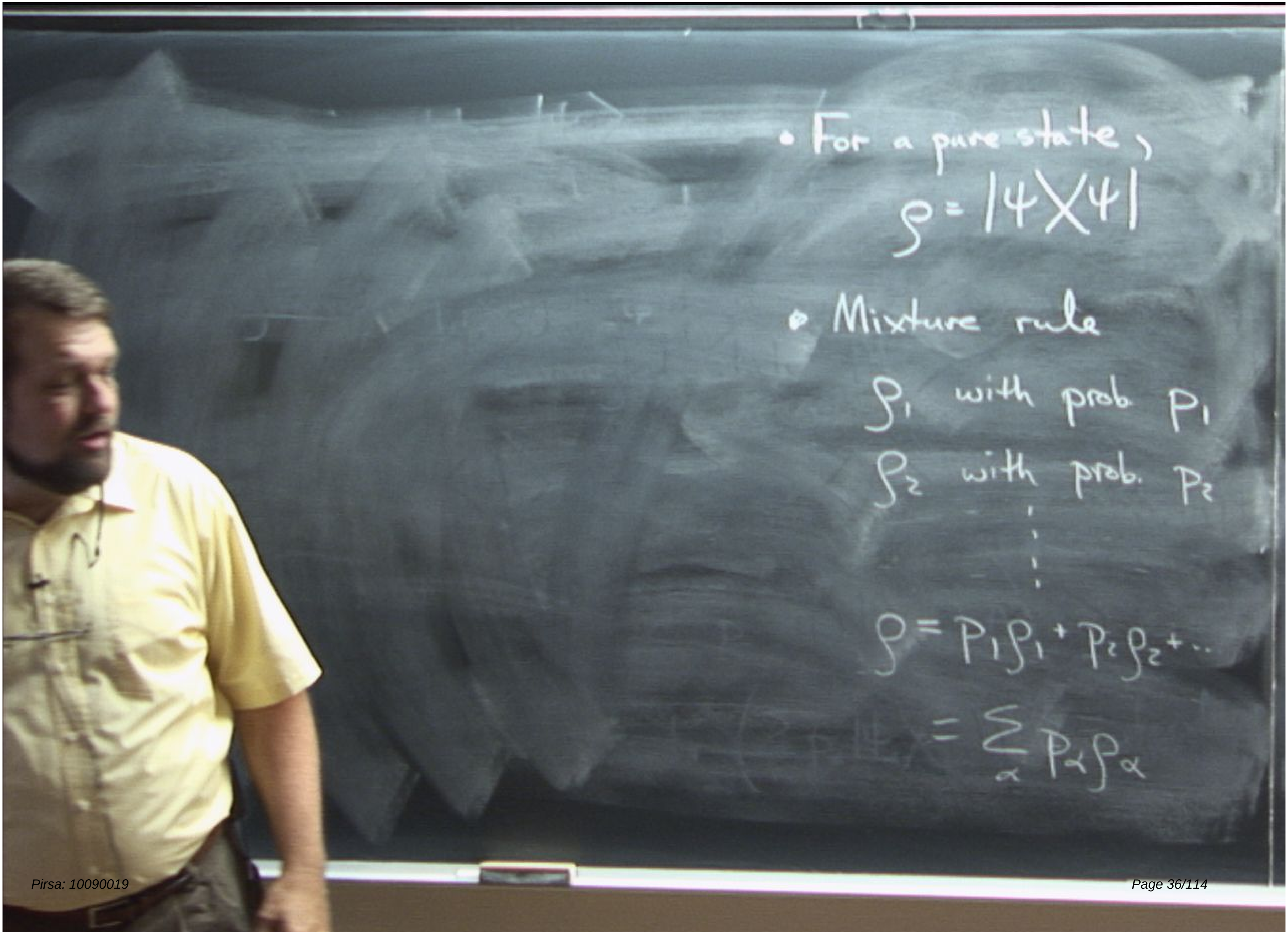
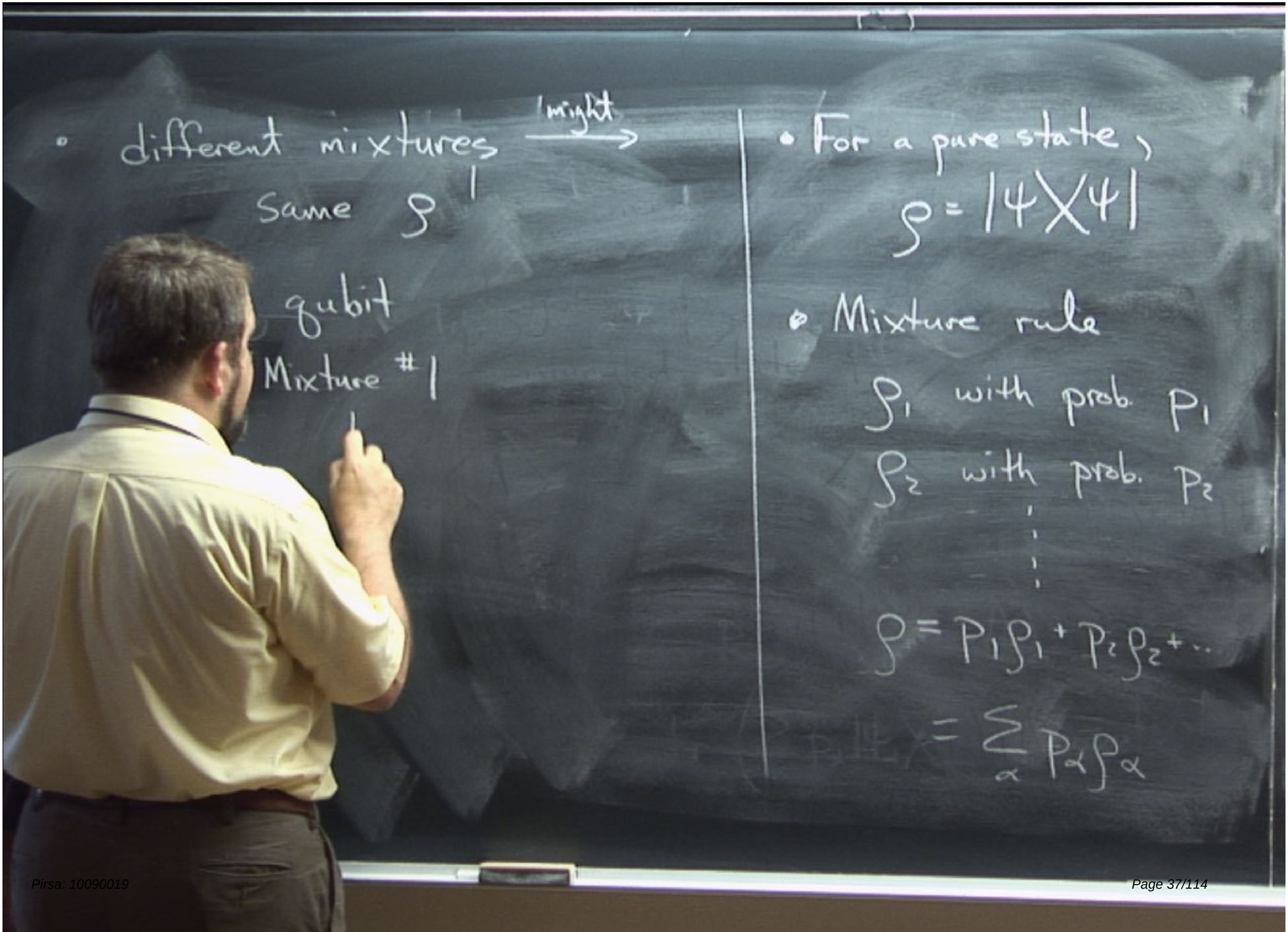
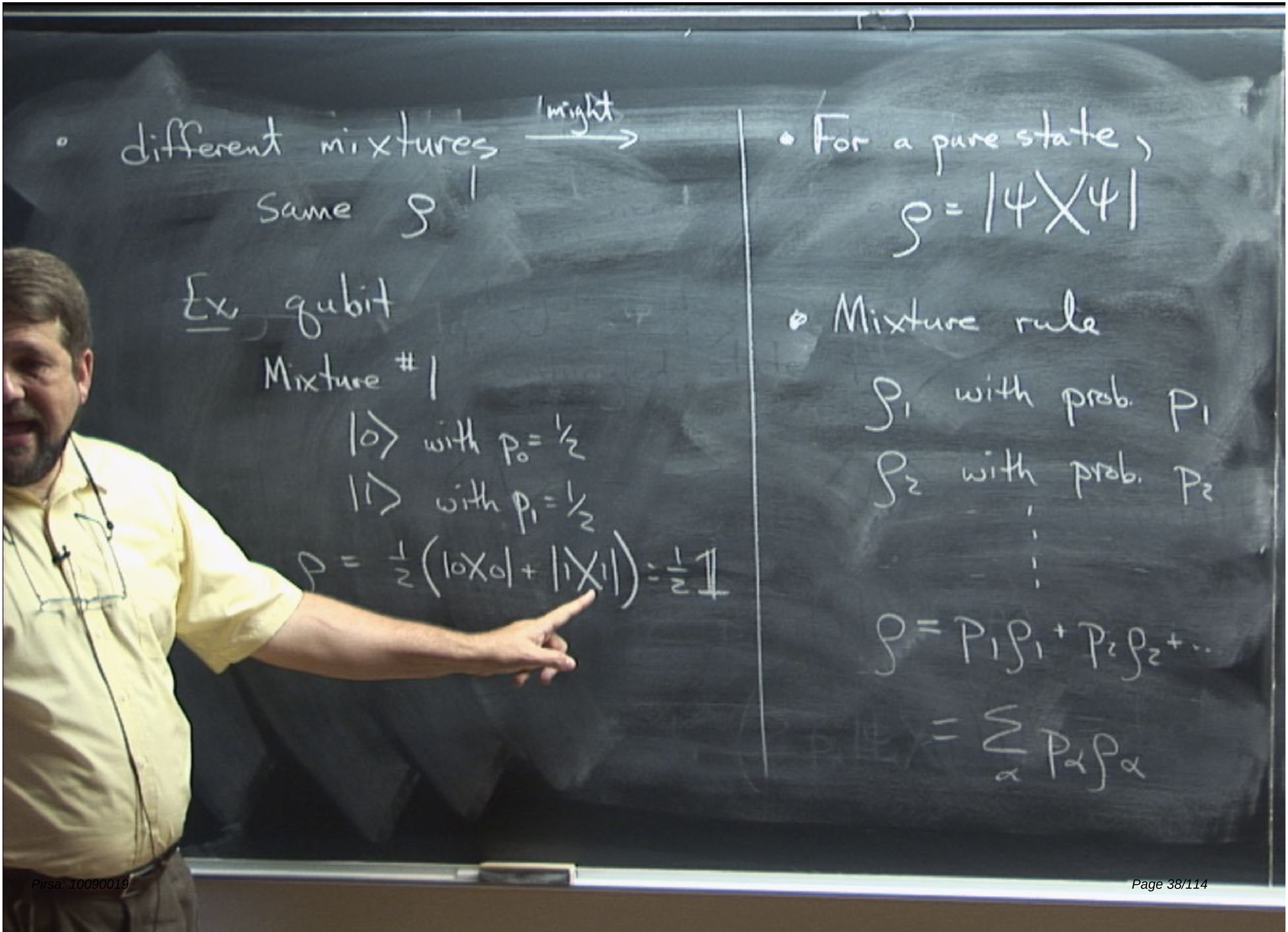
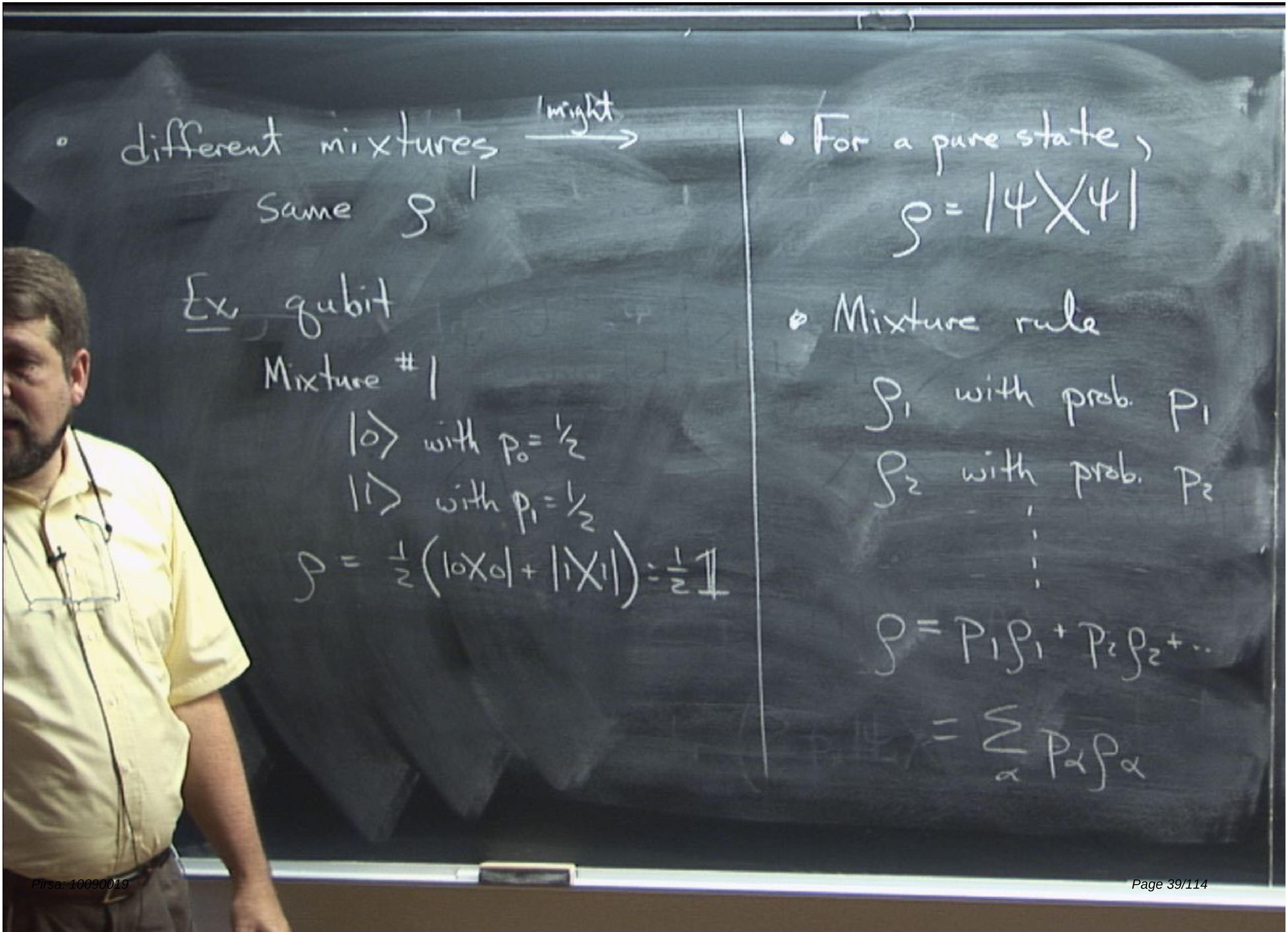
many many systems
fraction p_α of ens in $|\psi_\alpha\rangle$


• For a pure state,
$\rho = |\psi\rangle\langle\psi|$
• Mixture rule
ρ_1 with prob. p_1
ρ_2 with prob. p_2
$\vdots$
$\rho = p_1\rho_1 + p_2\rho_2 + \dots$
$= \sum_\alpha p_\alpha \rho_\alpha$


• different mixtures —might→ same ρ
qubit
Mixture #1
• For a pure state,
ρ = |ψ⟩⟨ψ|
• Mixture rule
ρ₁ with prob. p₁
ρ₂ with prob. p₂
ρ = p₁ρ₁ + p₂ρ₂ + ...
= Σ_α p_α ρ_α


• different mixtures —'might'→ same ρ
Ex. qubit
Mixture #1
|0⟩ with p₀ = ½
|1⟩ with p₁ = ½
ρ = ½(|0⟩⟨0| + |1⟩⟨1|) = ½ 𝟙
• For a pure state,
ρ = |ψ⟩⟨ψ|
• Mixture rule
ρ₁ with prob. p₁
ρ₂ with prob. p₂
ρ = p₁ρ₁ + p₂ρ₂ + ...
= Σ_α p_α ρ_α



• different mixtures 'might→ same ρ'
Ex. qubit
Mixture #1
$|0\rangle$ with $p_0 = \frac{1}{2}$
$|1\rangle$ with $p_1 = \frac{1}{2}$
$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}$
• For a pure state,
$\rho = |\psi\rangle\langle\psi|$
• Mixture rule
ρ_1 with prob. p_1
ρ_2 with prob. p_2
$\rho = p_1\rho_1 + p_2\rho_2 + \cdots$
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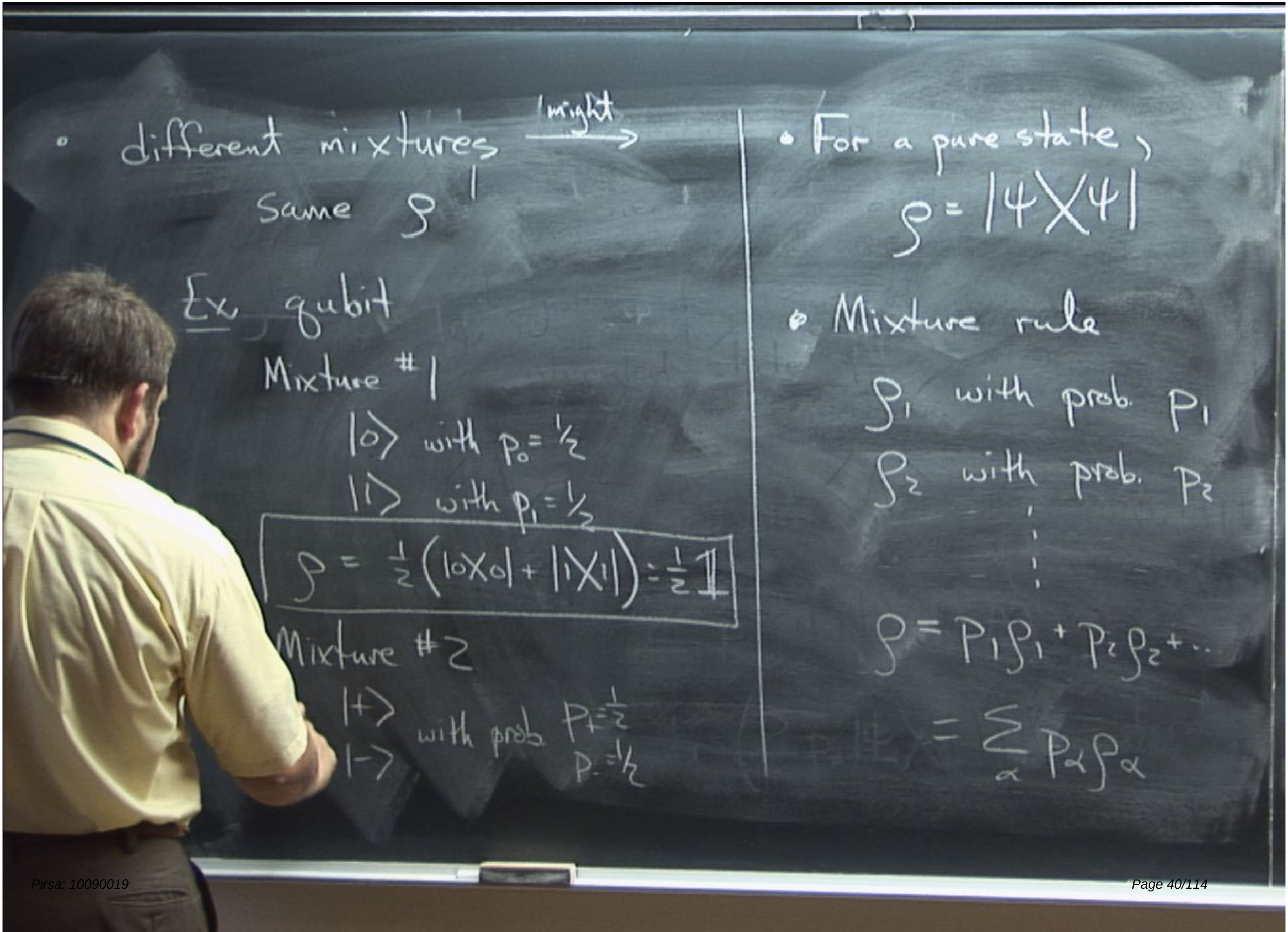
• different mixtures (might) → same ρ
Ex. qubit
Mixture #1
|0⟩ with p0 = 1/2
|1⟩ with p1 = 1/2
ρ = 1/2 (|0⟩⟨0| + |1⟩⟨1|) = 1/2 𝟙
Mixture #2
|+⟩ with prob. p+ = 1/2
|−⟩ p− = 1/2
• For a pure state,
ρ = |ψ⟩⟨ψ|
• Mixture rule
ρ1 with prob. p1
ρ2 with prob. p2
ρ = p1ρ1 + p2ρ2 + …
= Σα pα ρα

- different mixtures $\xrightarrow{\text{might}}$ same ρ

Ex qubit

Mixture #1

$|0\rangle$ with $p_0 = \frac{1}{2}$

$|1\rangle$ with $p_1 = \frac{1}{2}$

SAME

$$\boxed{\rho = \tfrac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \tfrac{1}{2}\mathbb{1}}$$

Mixture #2

$|+\rangle$ with prob $p_+ = \frac{1}{2}$

$|-\rangle$ $p_- = \frac{1}{2}$

- For a pure state,

$$\rho = |\psi\rangle\langle\psi|$$

- Mixture rule

ρ_1 with prob. p_1

ρ_2 with prob. p_2

$\vdots$

$$\rho = p_1\rho_1 + p_2\rho_2 + \dots$$

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- different mixtures $\xrightarrow{\text{might}}$ same ρ

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$|0\rangle$ with $p_0 = \frac{1}{2}$

$|1\rangle$ with $p_1 = \frac{1}{2}$

SAME

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Mixture #2

$|+\rangle$ with prob. $p_+ = \frac{1}{2}$

$|-\rangle$ $p_- = \frac{1}{2}$

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- Mixture rule

ρ_1 with prob. p_1

ρ_2 with prob. p_2

⋮

$$\rho = p_1\rho_1 + p_2\rho_2 + \dots$$

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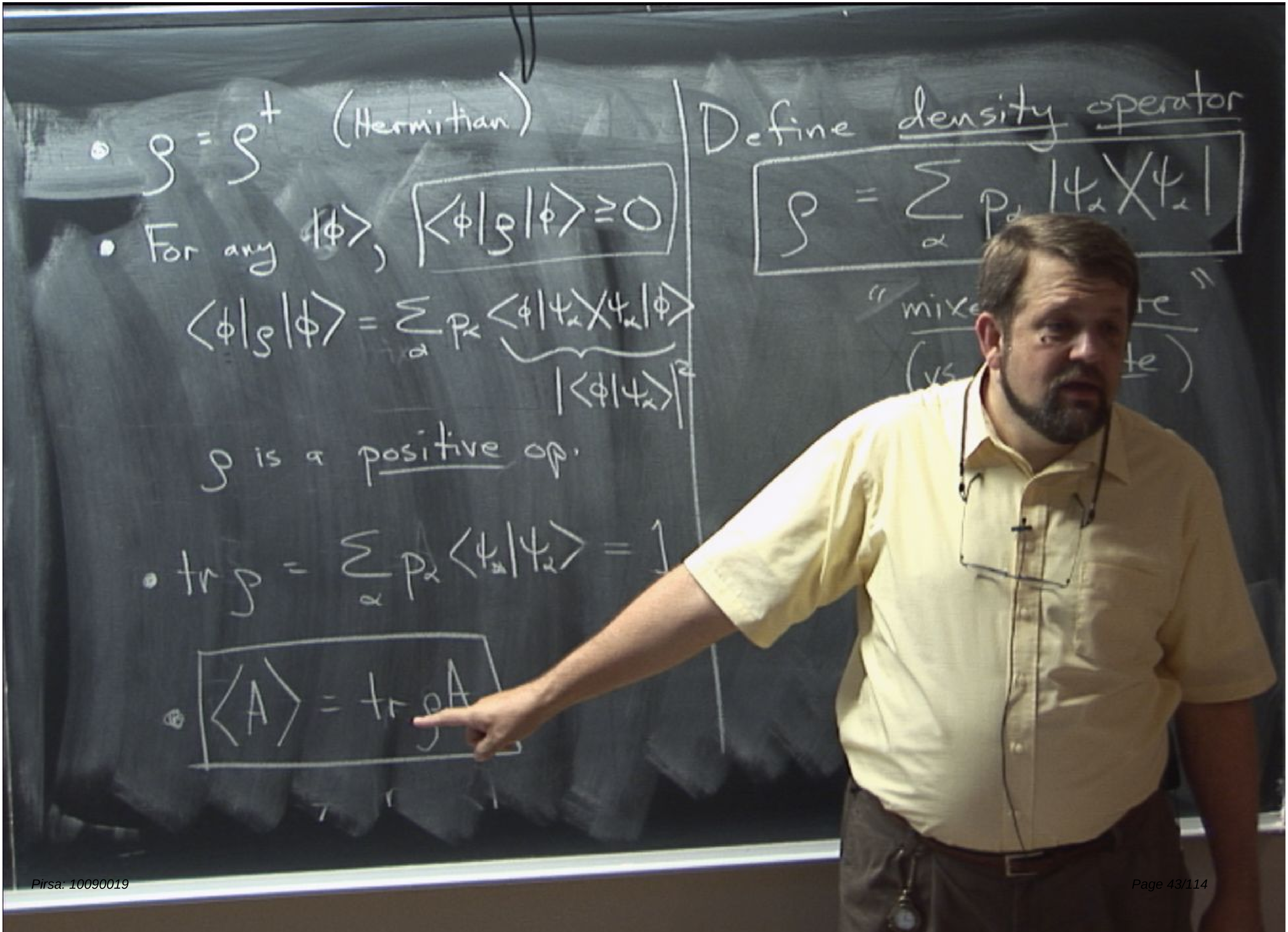

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ρ is a positive op.
• tr ρ = Σ_α p_α ⟨ψ_α|ψ_α⟩ = 1
• ⟨A⟩ = tr ρA
Define density operator
ρ = Σ_α p_α |ψ_α⟩⟨ψ_α|

- $\rho = \rho^\dagger$ (Hermitian)
- For any $|\phi\rangle$, $\boxed{\langle\phi|\rho|\phi\rangle \geq 0}$

$$\langle\phi|\rho|\phi\rangle = \sum_\alpha p_\alpha \underbrace{\langle\phi|\psi_\alpha\rangle\langle\psi_\alpha|\phi\rangle}_{|\langle\phi|\psi_\alpha\rangle|^2}$$

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- $\mathrm{tr}\,\rho = \sum_\alpha p_\alpha \langle\psi_\alpha|\psi_\alpha\rangle = 1$
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Define density operator

$$\boxed{\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|}$$

"mixed state"

(vs. pure state)

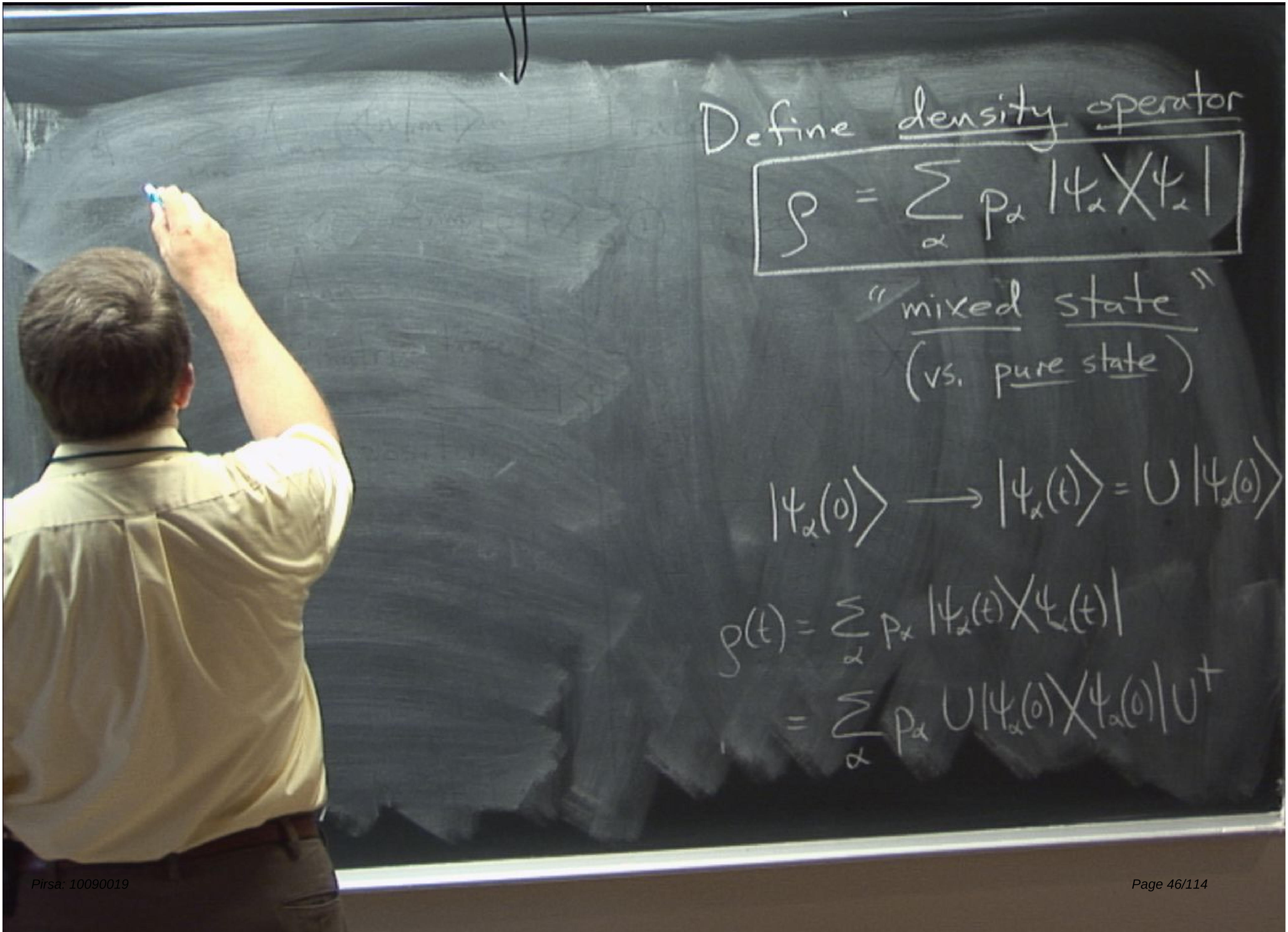
Define <u>density</u> <u>operator</u>

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

"<u>mixed</u> <u>state</u>"

(vs. <u>pure</u> <u>state</u>)

$$|\psi_\alpha(0)\rangle \longrightarrow |\psi_\alpha(t)\rangle = U|\psi_\alpha(0)\rangle$$


Define density operator
$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$
"mixed state"
(vs. pure state)
$|\psi_\alpha(0)\rangle \longrightarrow |\psi_\alpha(t)\rangle = U|\psi_\alpha(0)\rangle$
$\rho(t) = \sum_\alpha p_\alpha |\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|$
$= \sum_\alpha p_\alpha U|\psi_\alpha(0)\rangle\langle\psi_\alpha(0)|U^+$

$$\rho(t) = U\rho(0)U^\dagger$$

Define density operator

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"mixed state"

(vs. pure state)

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$$= \sum_\alpha p_\alpha U|\psi_\alpha(0)\rangle\langle\psi_\alpha(0)|U^\dagger$$

$$\boxed{\rho(t) = U\rho(0)U^\dagger}$$

$$\frac{d}{dt}|\psi_\alpha\rangle = \frac{1}{i\hbar} H|\psi_\alpha\rangle$$

$$\frac{d\rho}{dt} =$$

Define <u>density</u> <u>operator</u>

$$\boxed{\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|}$$

"<u>mixed</u> <u>state</u>"

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$$= \sum_\alpha p_\alpha U|\psi_\alpha(0)\rangle\langle\psi_\alpha(0)|U^\dagger$$

$$\rho(t) = U \rho(0) U^\dagger$$

$$\frac{d}{dt}|\psi_\alpha\rangle = \frac{1}{i\hbar} H |\psi_\alpha\rangle$$

$$\frac{d\rho}{dt} = \sum_\alpha p_\alpha \left[\left(\frac{d}{dt}|\psi_\alpha\rangle\right)\langle\psi_\alpha| + |\psi_\alpha\rangle\left(\frac{d}{dt}\langle\psi_\alpha|\right)\right]$$

Define <u>density</u> <u>operator</u>

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

"<u>mixed</u> <u>state</u>"

(vs. <u>pure</u> <u>state</u>)

$$|\psi_\alpha(0)\rangle \longrightarrow |\psi_\alpha(t)\rangle = U|\psi_\alpha(0)\rangle$$

$$\rho(t) = \sum_\alpha p_\alpha |\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|$$

$$= \sum_\alpha p_\alpha U|\psi_\alpha(0)\rangle\langle\psi_\alpha(0)|U^\dagger$$

$$\boxed{\rho(t) = U \rho(0) U^{+}}$$

$$\frac{d}{dt}|\psi_\alpha\rangle = \frac{1}{i\hbar} H |\psi_\alpha\rangle$$

$$\frac{d\rho}{dt} = \sum_\alpha p_\alpha \left[\left(\frac{d}{dt}|\psi_\alpha\rangle\right)\langle\psi_\alpha| + |\psi_\alpha\rangle\left(\frac{d}{dt}\langle\psi_\alpha|\right) \right]$$

$$\frac{1}{i\hbar} H|\psi_\alpha\rangle \qquad -\frac{1}{i\hbar}\langle\psi_\alpha| H$$

Define density operator

$$\boxed{\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|}$$

"mixed state"

(vs. pure state)

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$$\frac{1}{i\hbar} H|\psi_\alpha\rangle \qquad -\frac{1}{i\hbar}\langle\psi_\alpha|H$$

$$= \sum_\alpha p_\alpha \left(H|\psi_\alpha\rangle\langle\psi_\alpha| - |\psi_\alpha\rangle\langle\psi_\alpha|H \right)$$

Define density operator

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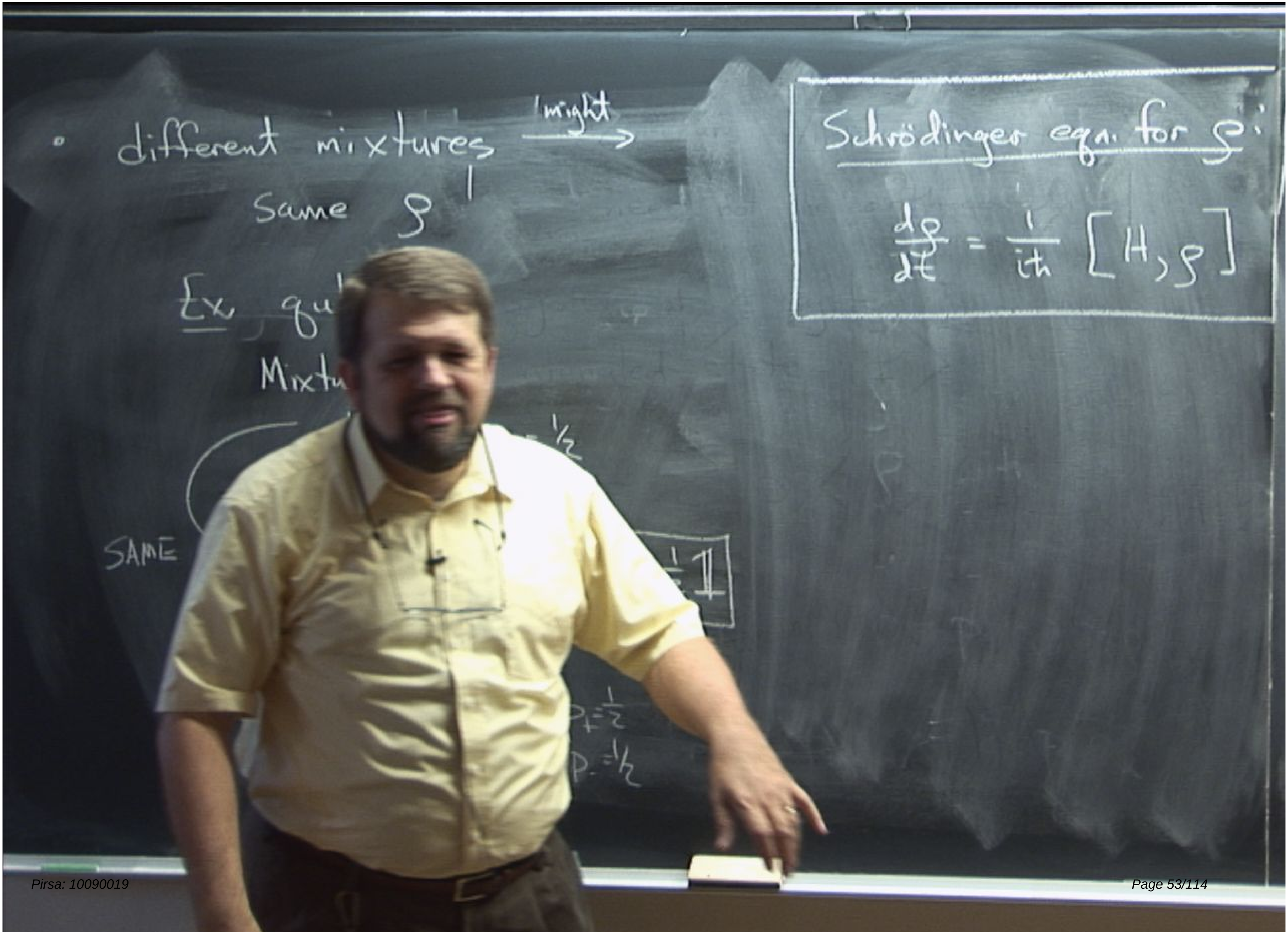

different mixtures
might
same ρ
Ex. qu
Mixtu
SAME
Schrödinger eqn. for ρ:
dρ/dt = 1/iħ [H, ρ]

Observable A = const.

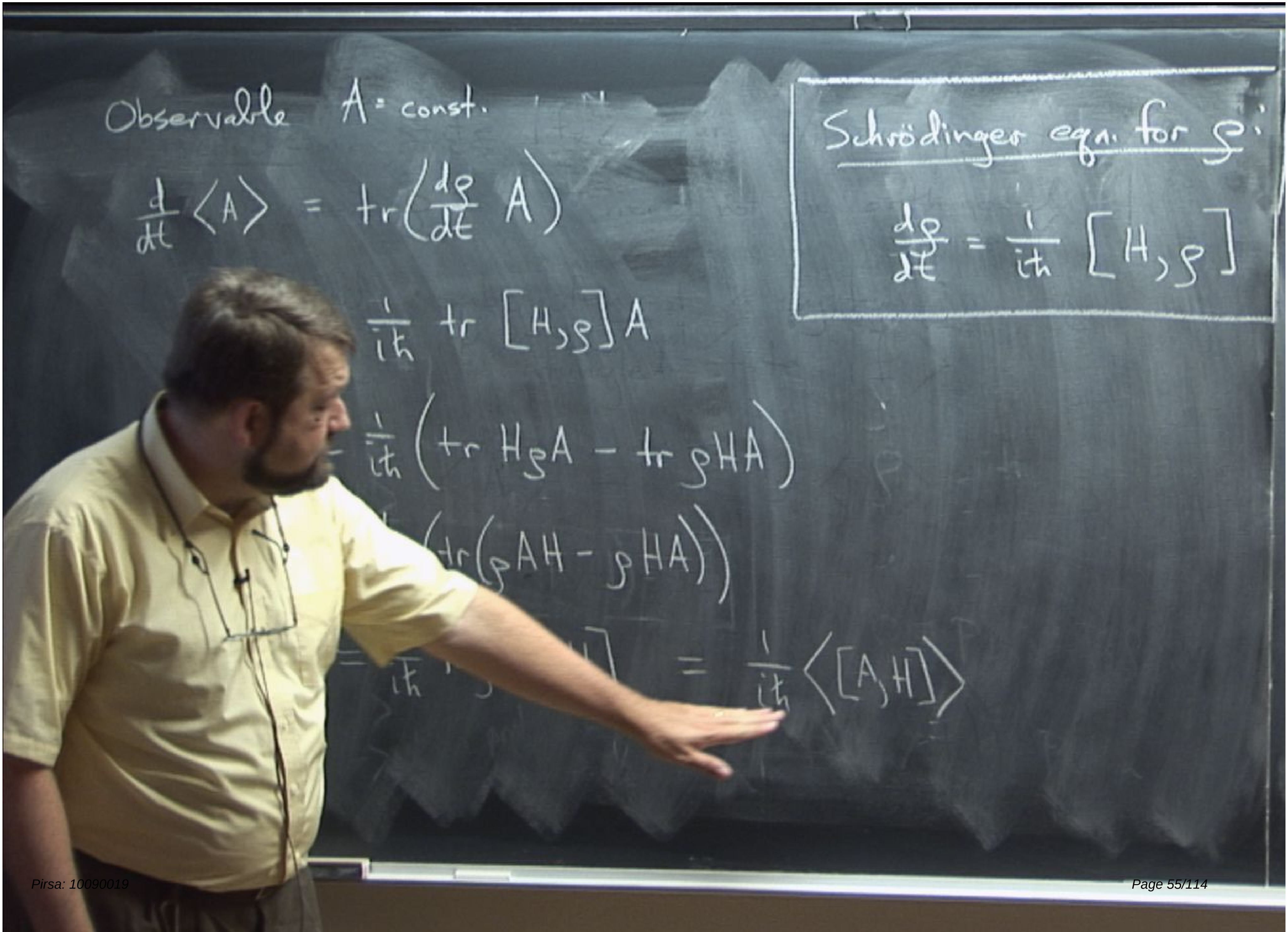
$$\frac{d}{dt}\langle A\rangle = \mathrm{tr}\left(\frac{d\rho}{dt} A\right)$$

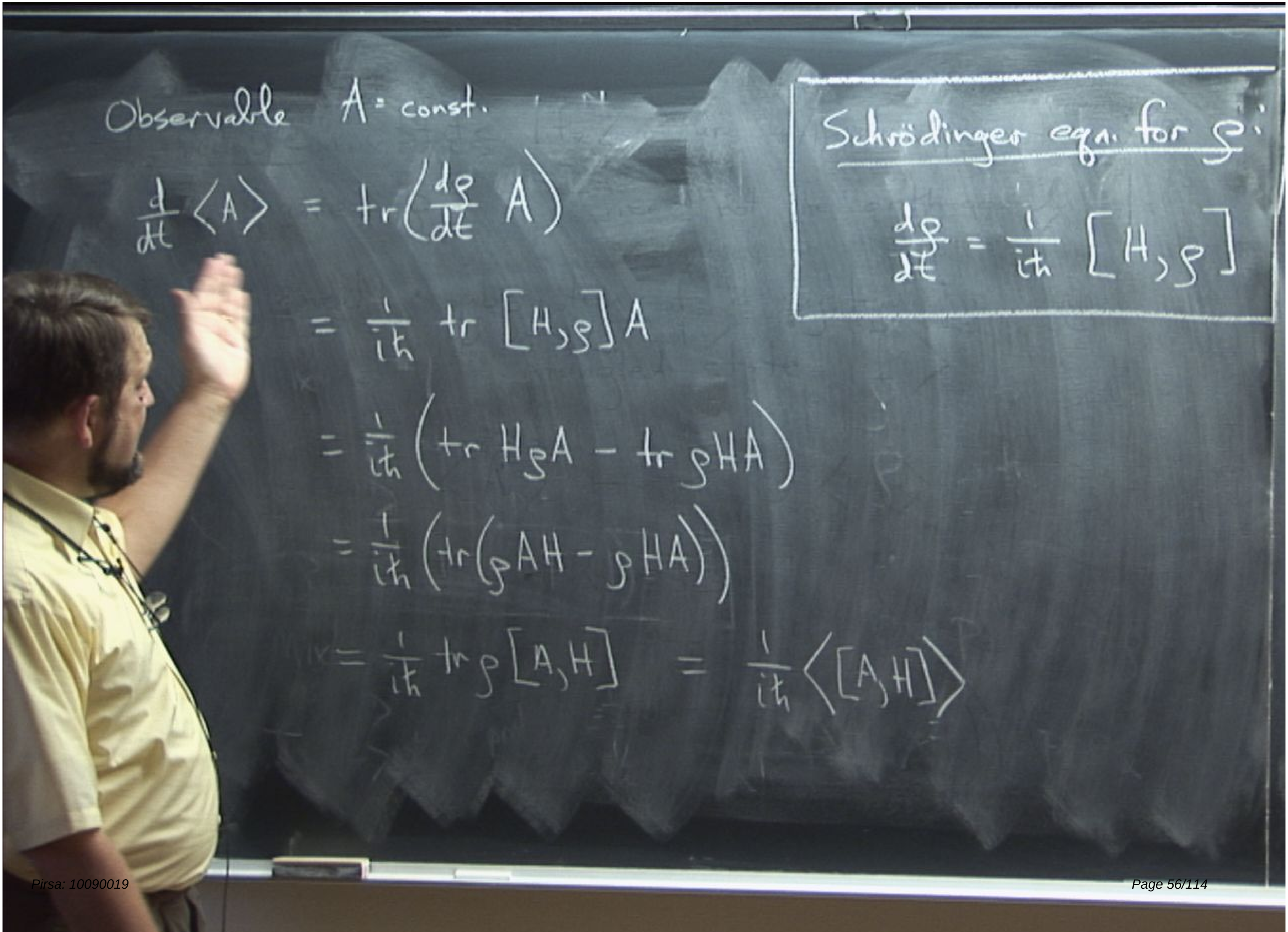
$$= \frac{1}{i\hbar}\,\mathrm{tr}\,[H,\rho]A$$

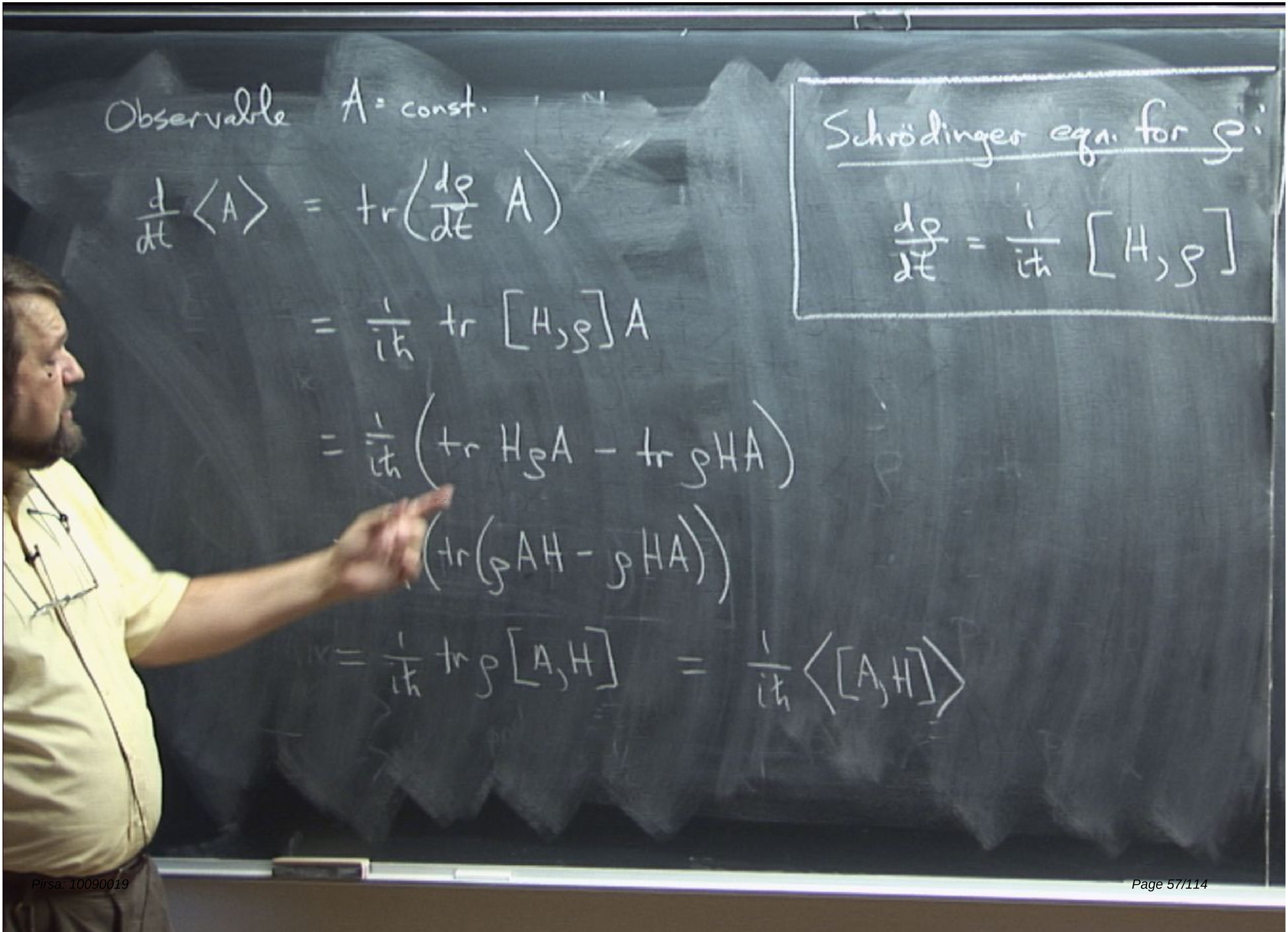
$$= \frac{1}{i\hbar}\left(\mathrm{tr}\, H\rho A - \mathrm{tr}\,\rho H A\right)$$

Schrödinger eqn. for ρ:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$$


Observable A = const.
d/dt ⟨A⟩ = tr(dρ/dt A)
= 1/iħ tr [H, ρ] A
= 1/iħ (tr HρA − tr ρHA)
tr(ρAH − ρHA)
= 1/iħ ⟨[A, H]⟩
Schrödinger eqn. for ρ:
dρ/dt = 1/iħ [H, ρ]


Observable A = const.
$\frac{d}{dt}\langle A\rangle = \mathrm{tr}\left(\frac{d\rho}{dt} A\right)$
$= \frac{1}{i\hbar}\,\mathrm{tr}\,[H,\rho]A$
$= \frac{1}{i\hbar}\left(\mathrm{tr}\,H\rho A - \mathrm{tr}\,\rho H A\right)$
$= \frac{1}{i\hbar}\left(\mathrm{tr}\left(\rho A H - \rho H A\right)\right)$
$= \frac{1}{i\hbar}\,\mathrm{tr}\,\rho\,[A,H] = \frac{1}{i\hbar}\langle [A,H]\rangle$
Schrödinger eqn. for ρ:
$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$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$= \frac{1}{i\hbar}\left(tr\, H\rho A - tr\, \rho H A\right)$
$\left(tr\left(\rho A H - \rho H A\right)\right)$
$= \frac{1}{i\hbar} tr\,\rho\,[A,H] = \frac{1}{i\hbar}\langle [A,H]\rangle$
Schrödinger eqn. for ρ:
$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$

Observable A = const.

$$\frac{d}{dt}\langle A\rangle = \mathrm{tr}\left(\frac{d\rho}{dt}A\right)$$

$$= \frac{1}{i\hbar}\,\mathrm{tr}\,[H,\rho]A$$

$$= \frac{1}{i\hbar}\left(\mathrm{tr}\,H\rho A - \mathrm{tr}\,\rho HA\right)$$

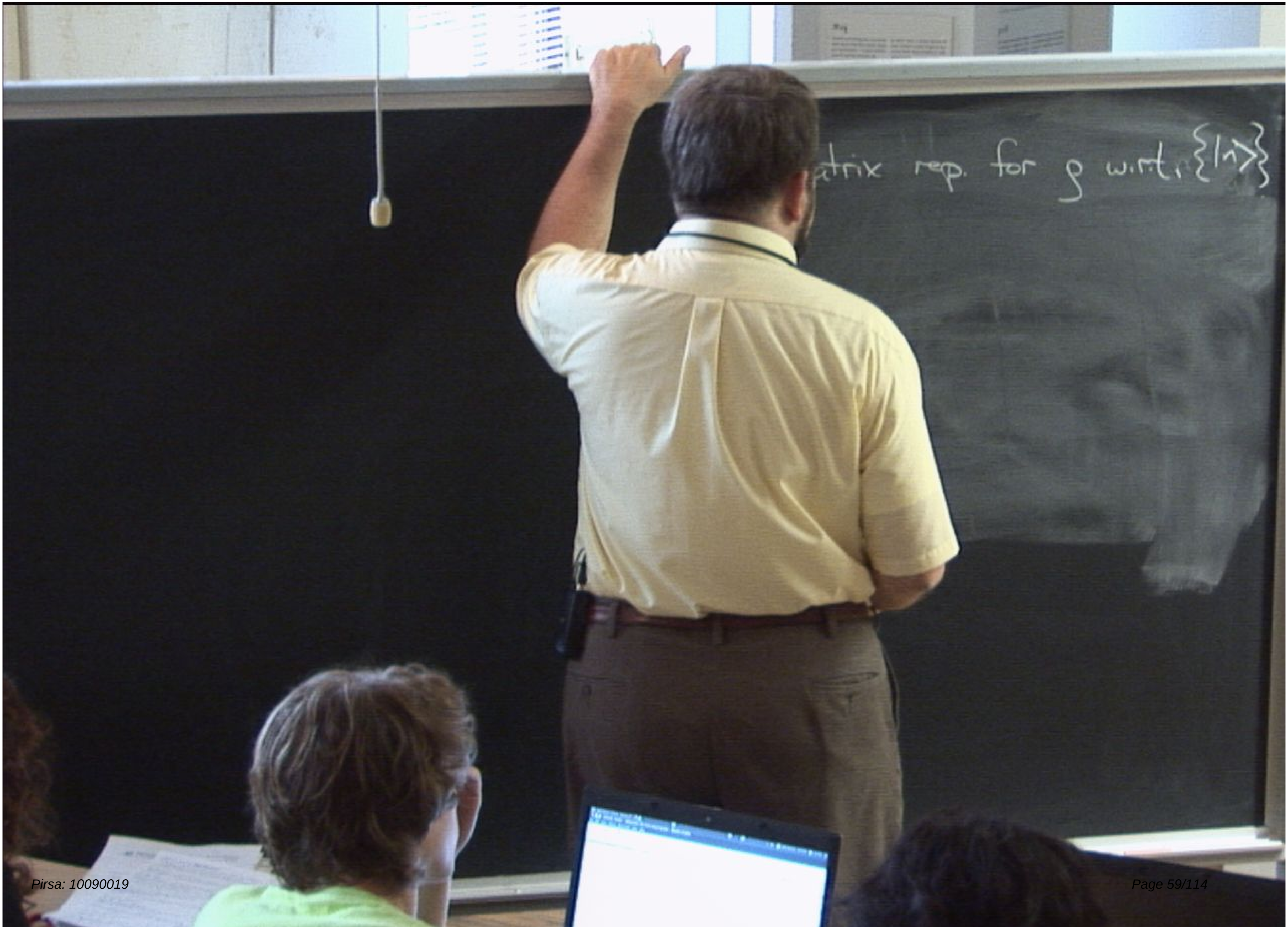
cyclic prop!

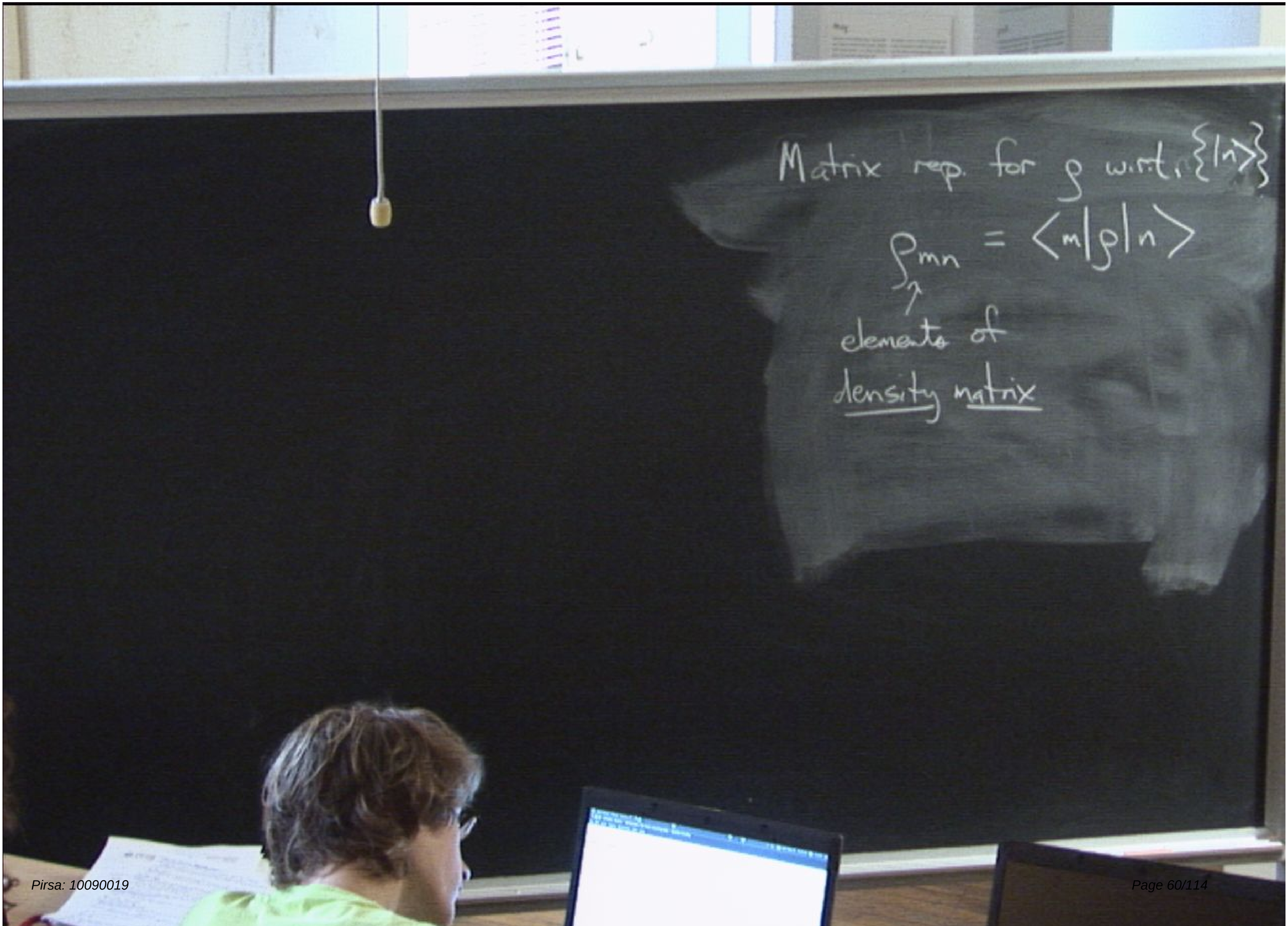
$$= \frac{1}{i\hbar}\left(\mathrm{tr}\left(\rho AH - \rho HA\right)\right)$$

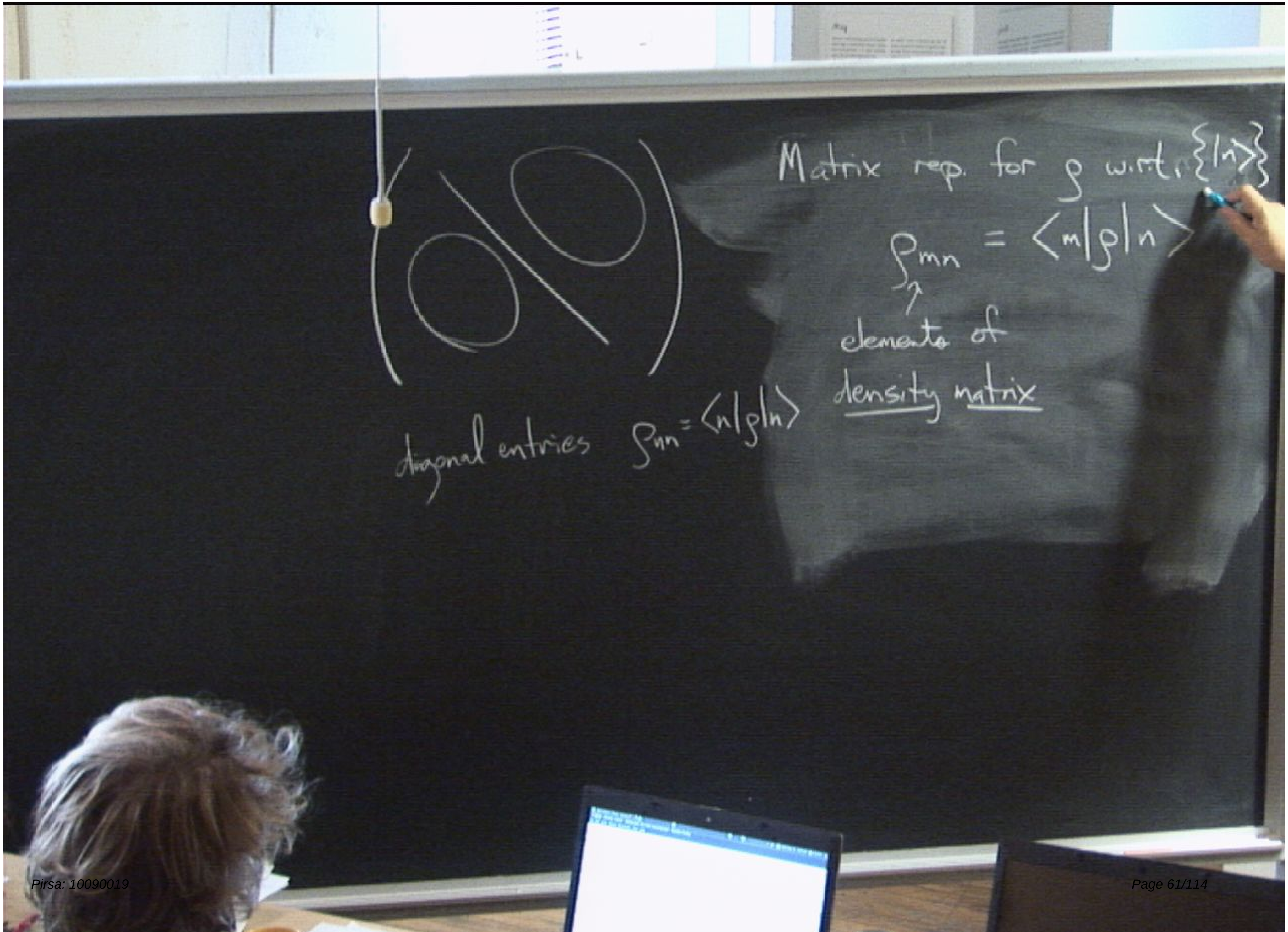
$$= \frac{1}{i\hbar}\,\mathrm{tr}\,\rho\,[A,H] = \frac{1}{i\hbar}\langle [A,H]\rangle$$

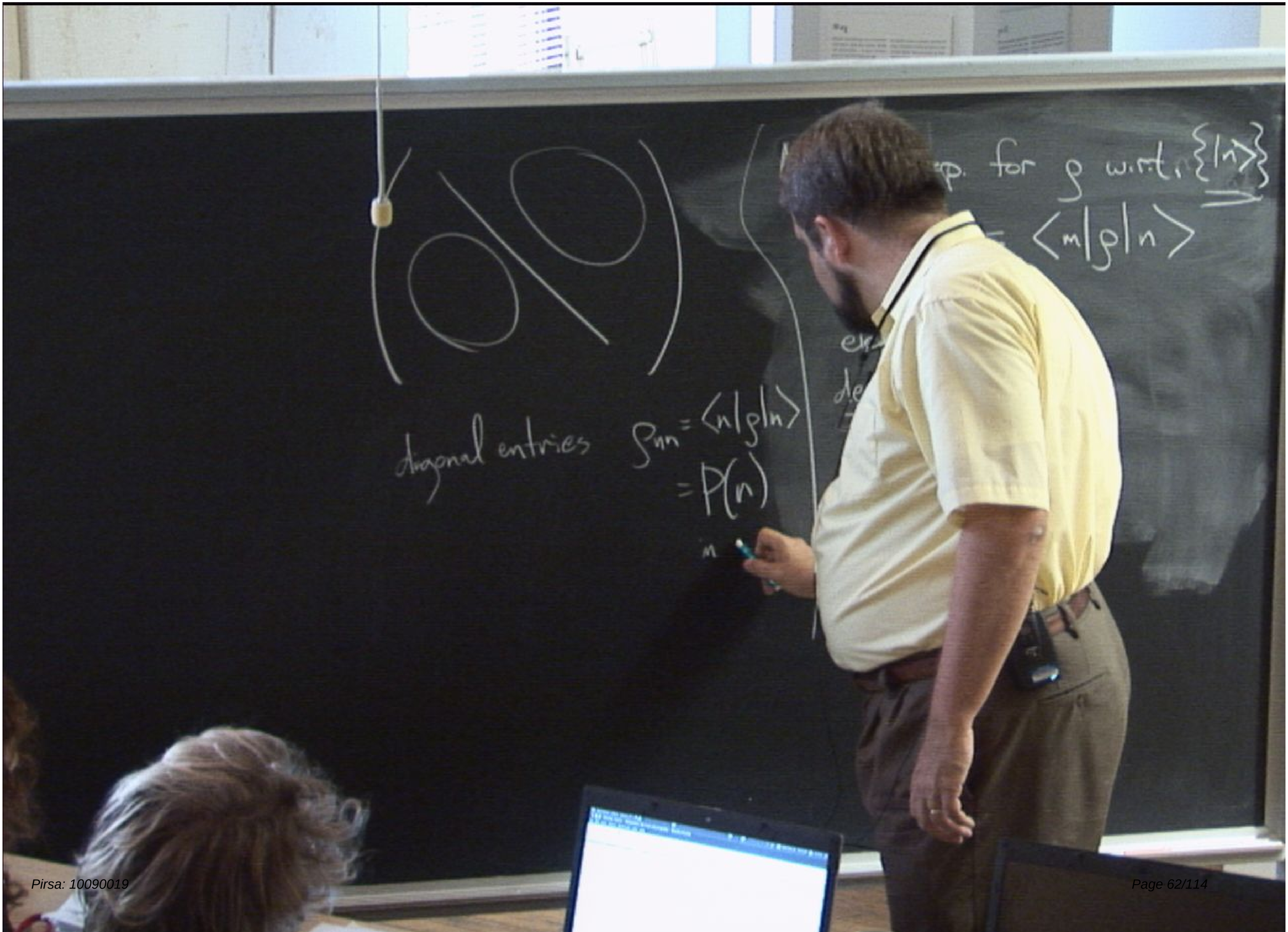
Schrödinger eqn. for ρ:

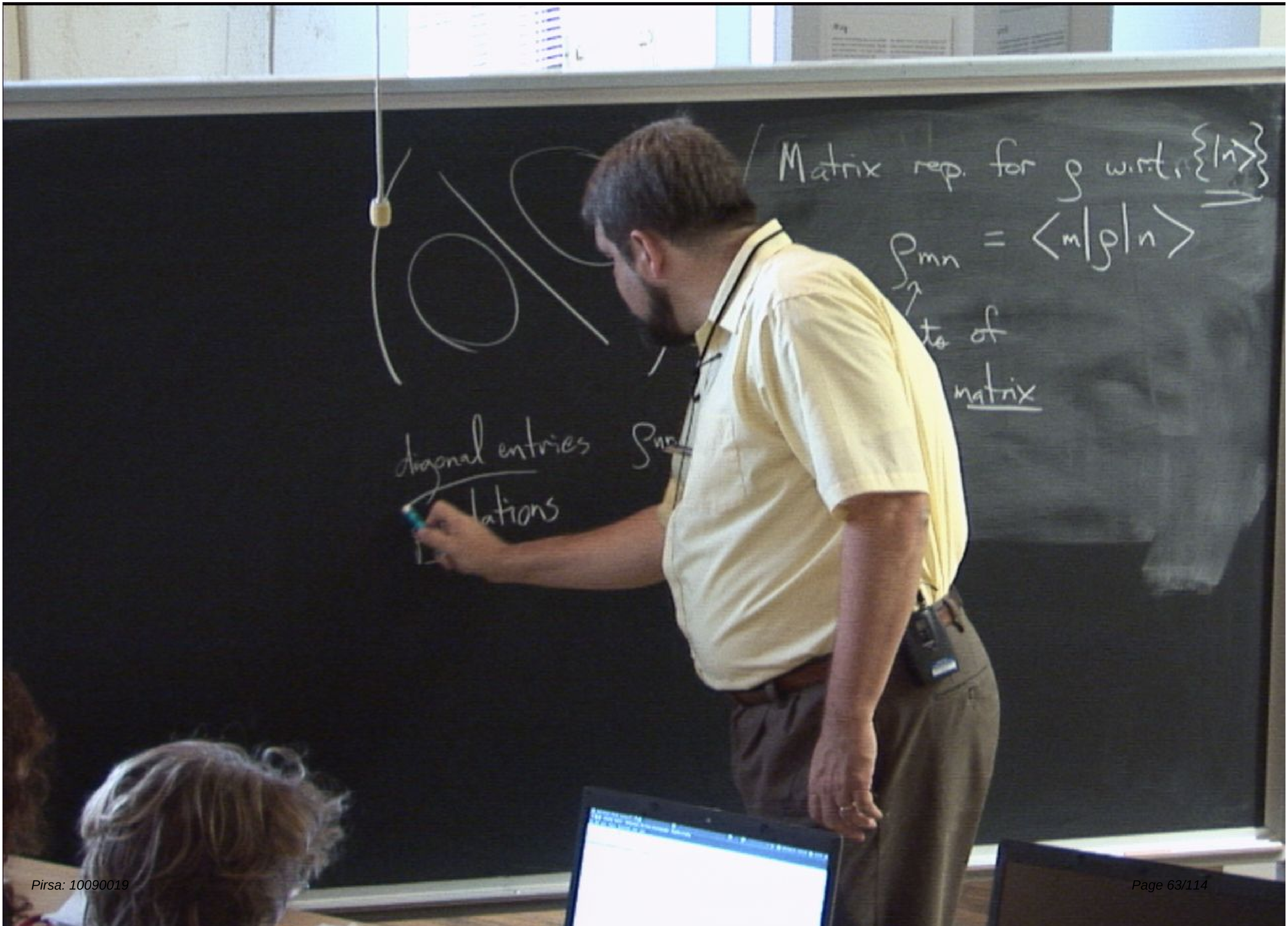
$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$$

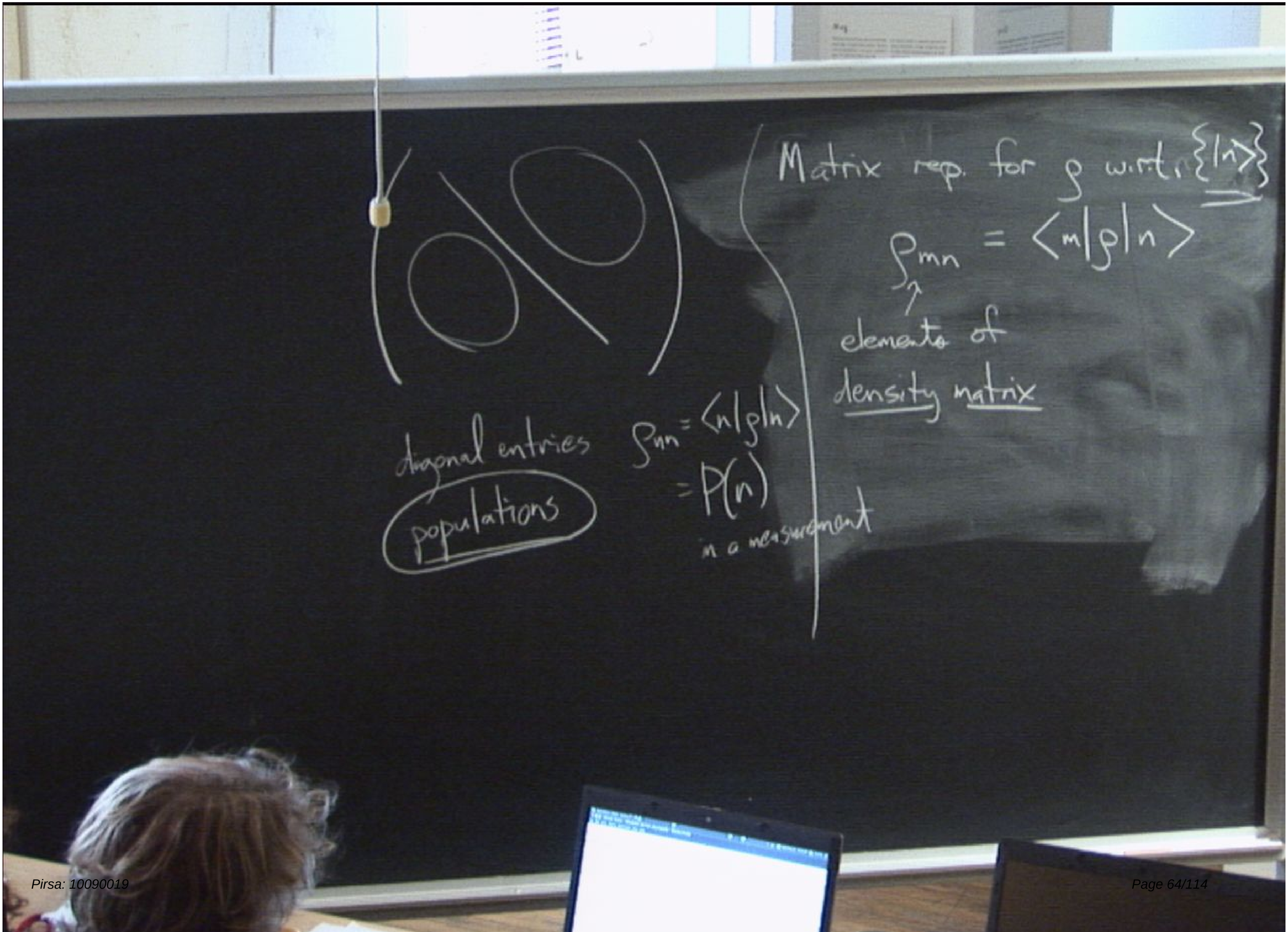

atrix rep. for ρ w.r.t. {|n⟩}

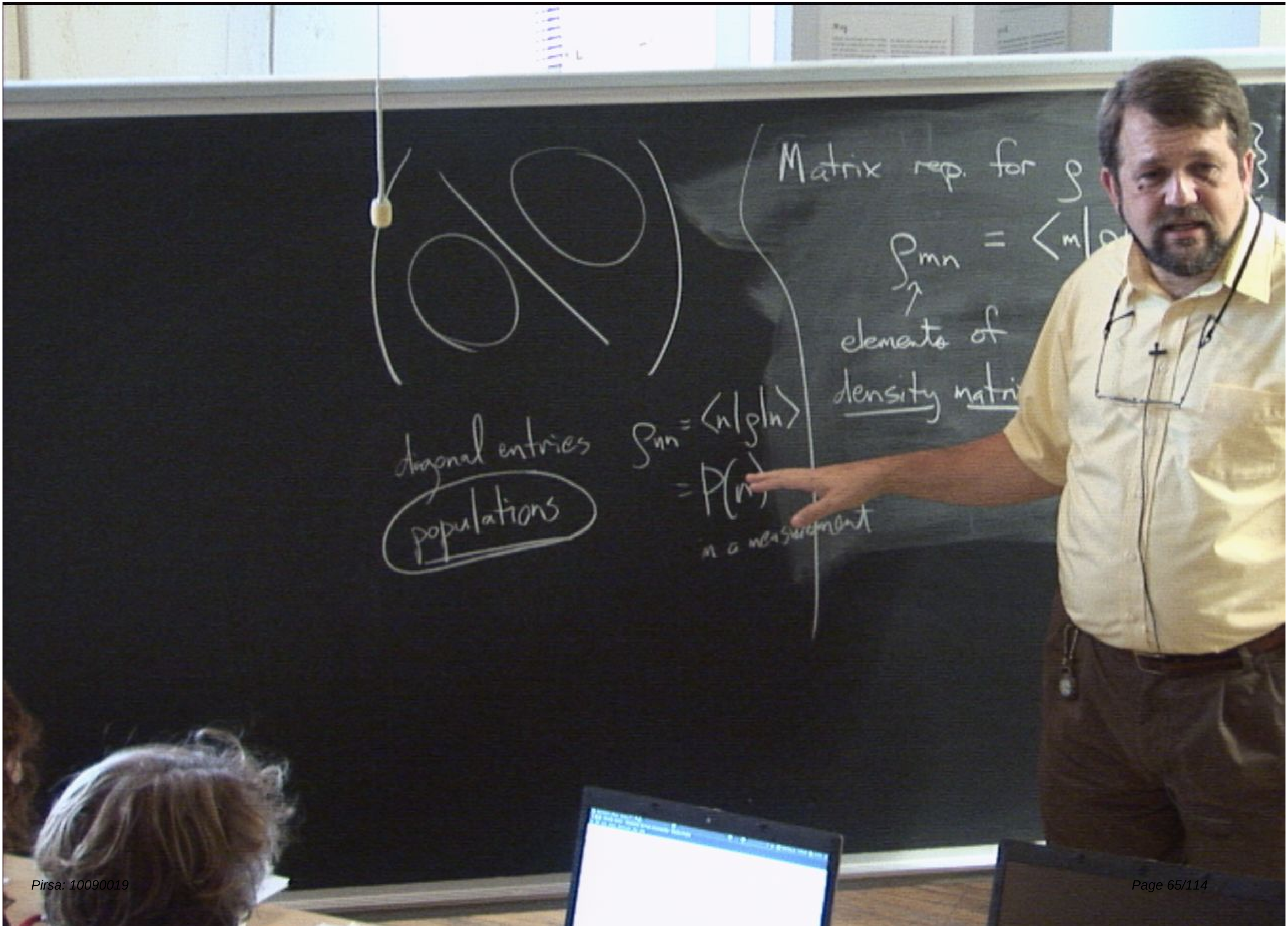

Matrix rep. for ρ w.r.t. {|n⟩}
ρmn = ⟨m|ρ|n⟩
elements of
density matrix

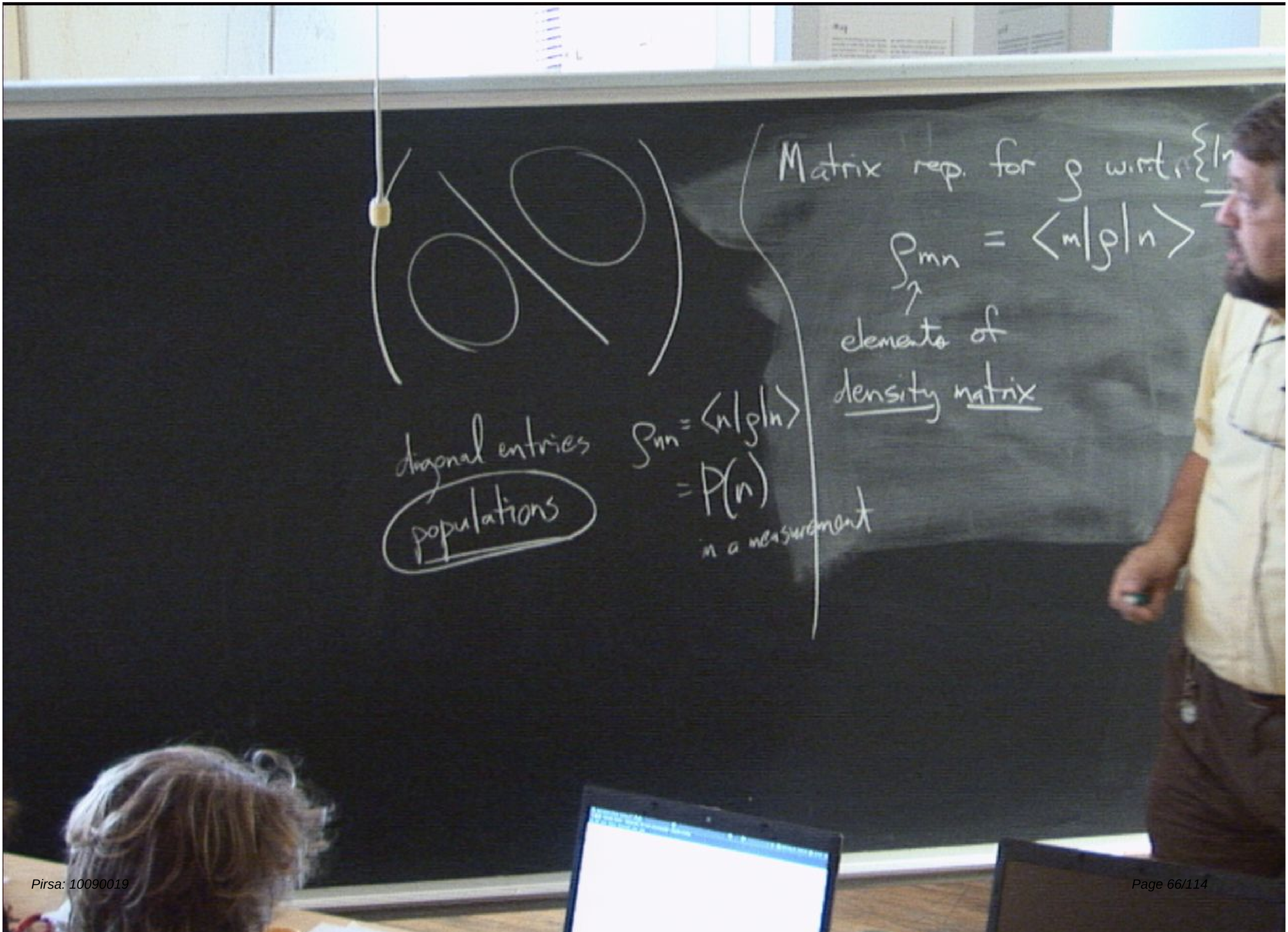

Matrix rep. for ρ w.r.t. {|n⟩}
ρmn = ⟨m|ρ|n⟩
elements of
density matrix
diagonal entries ρnn = ⟨n|ρ|n⟩


diagonal entries $\rho_{nn} = \langle n|\rho|n\rangle$
$= P(n)$
rep. for ρ w.r.t. $\{|n\rangle\}$
$\langle m|\rho|n\rangle$


Matrix rep. for ρ w.r.t. {|n⟩}
ρmn = ⟨m|ρ|n⟩
matrix
diagonal entries


Matrix rep. for ρ w.r.t. {|n⟩}
ρmn = ⟨m|ρ|n⟩
elements of
density matrix
diagonal entries ρnn = ⟨n|ρ|n⟩
= P(n)
in a measurement
populations


Matrix rep. for ρ
ρmn =
elements of
density
diagonal entries
ρnn = ⟨n|ρ|n⟩
= P(n)
in a measurement
populations


Matrix rep. for ρ w.r.t.
ρmn = ⟨m|ρ|n⟩
elements of
density matrix
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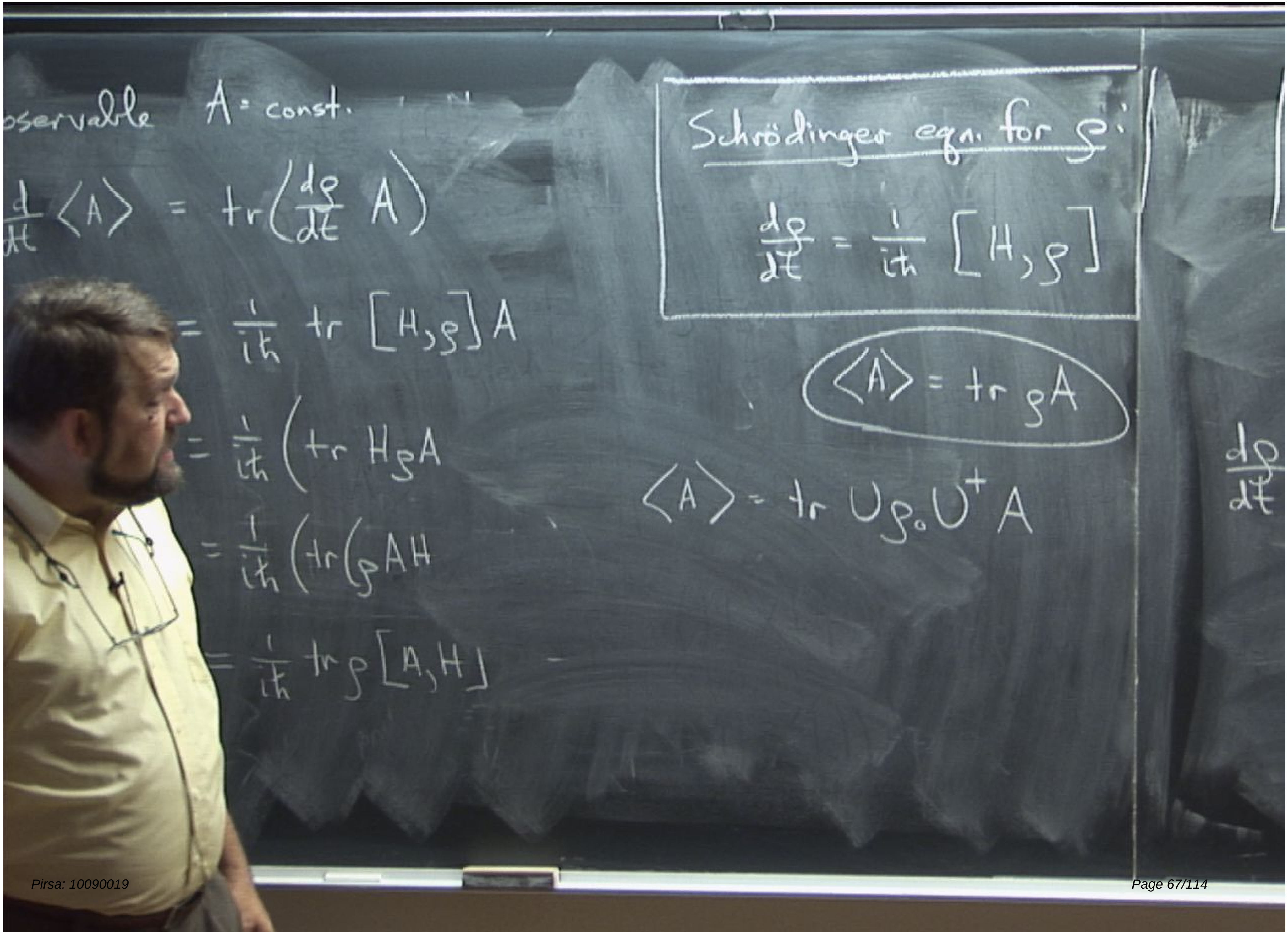
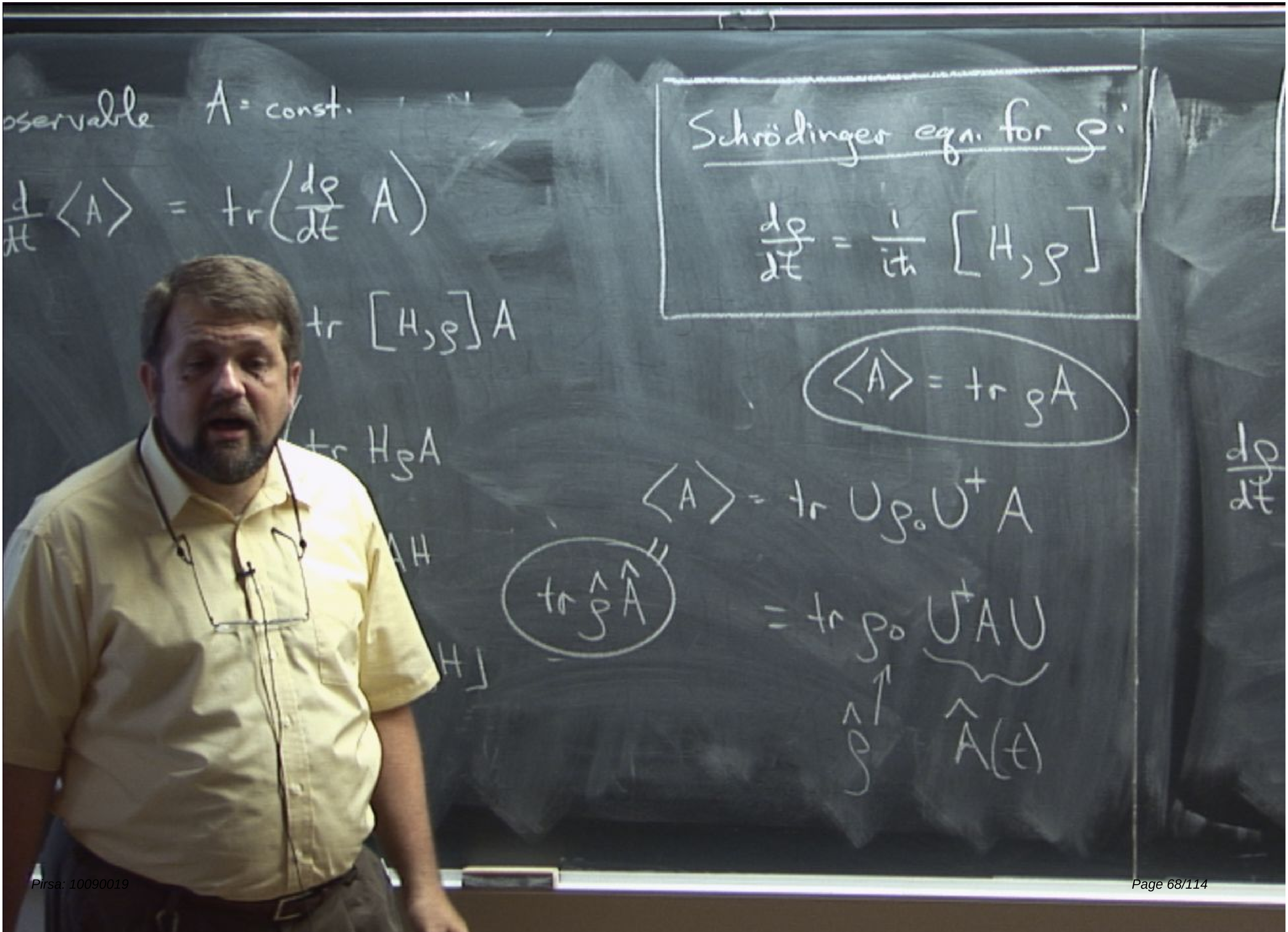
populations
ρnn = ⟨n|ρ|n⟩
= P(n)
in a measurement



bservable A = const.
$\frac{d}{dt}\langle A\rangle = tr\left(\frac{d\rho}{dt} A\right)$
$= \frac{1}{i\hbar}\, tr\, [H,\rho] A$
$= \frac{1}{i\hbar}\left(tr\, H\rho A\right.$
$= \frac{1}{i\hbar}\left(tr(\rho A H\right.$
$= \frac{1}{i\hbar}\, tr\, \rho [A,H]$
Schrödinger eqn. for ρ:
$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$
$\langle A\rangle = tr\, \rho A$
$\langle A\rangle = tr\, U\rho_0 U^+ A$
$\frac{d\rho}{dt}$



bservable A = const.
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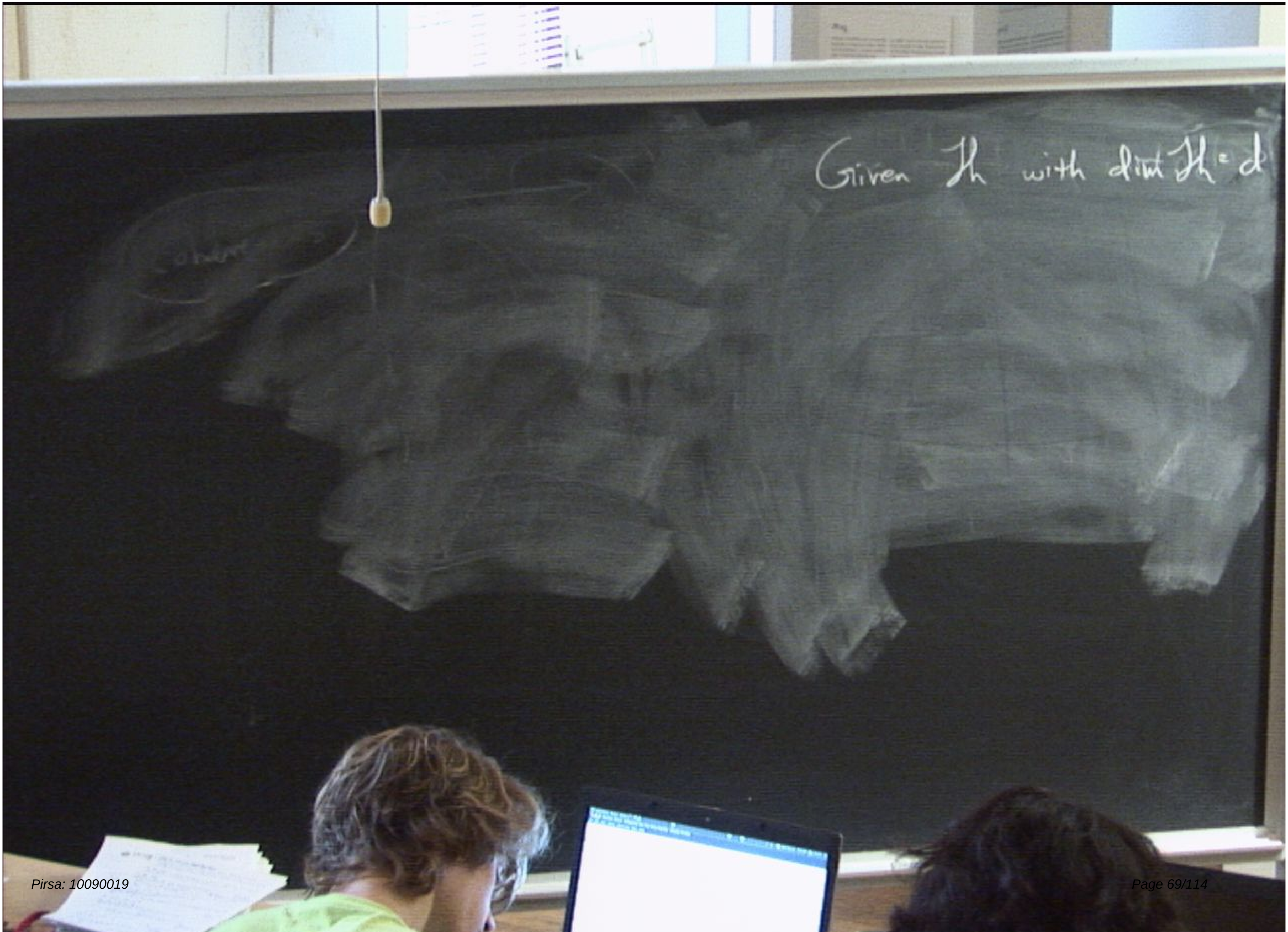
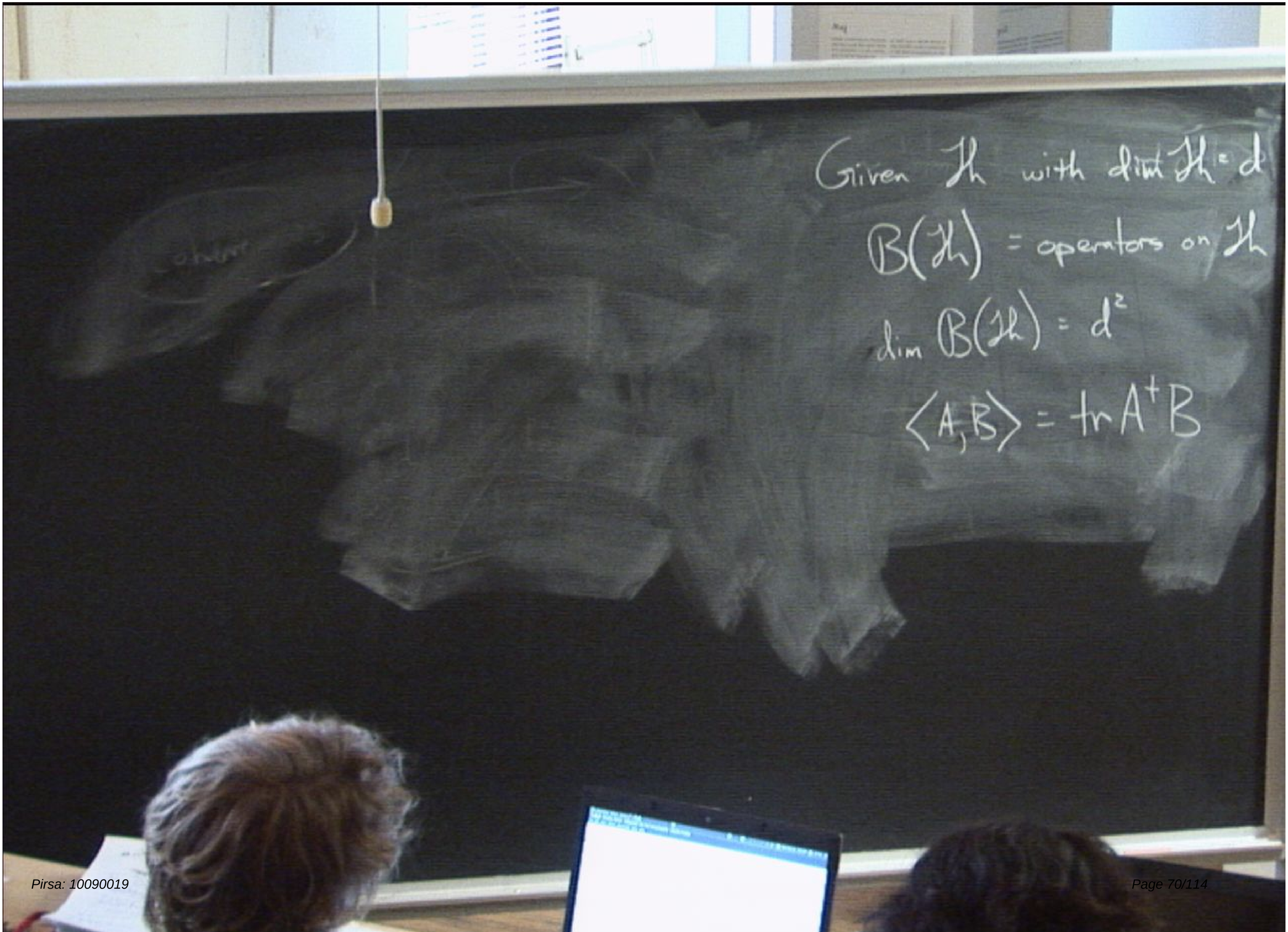
tr $[H,\rho]$ A
tr $H\rho A$
AH
H]
Schrödinger eqn. for ρ:
$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$
$\langle A\rangle = \mathrm{tr}\, \rho A$
$\langle A\rangle = \mathrm{tr}\, U\rho_0 U^{+} A$
$= \mathrm{tr}\, \rho_0 U^{+} A U$
$\mathrm{tr}\, \hat{\rho}\hat{A}$
$\hat{\rho}$
$\hat{A}(t)$


Given $\mathcal{H}$ with dim $\mathcal{H}$ = d

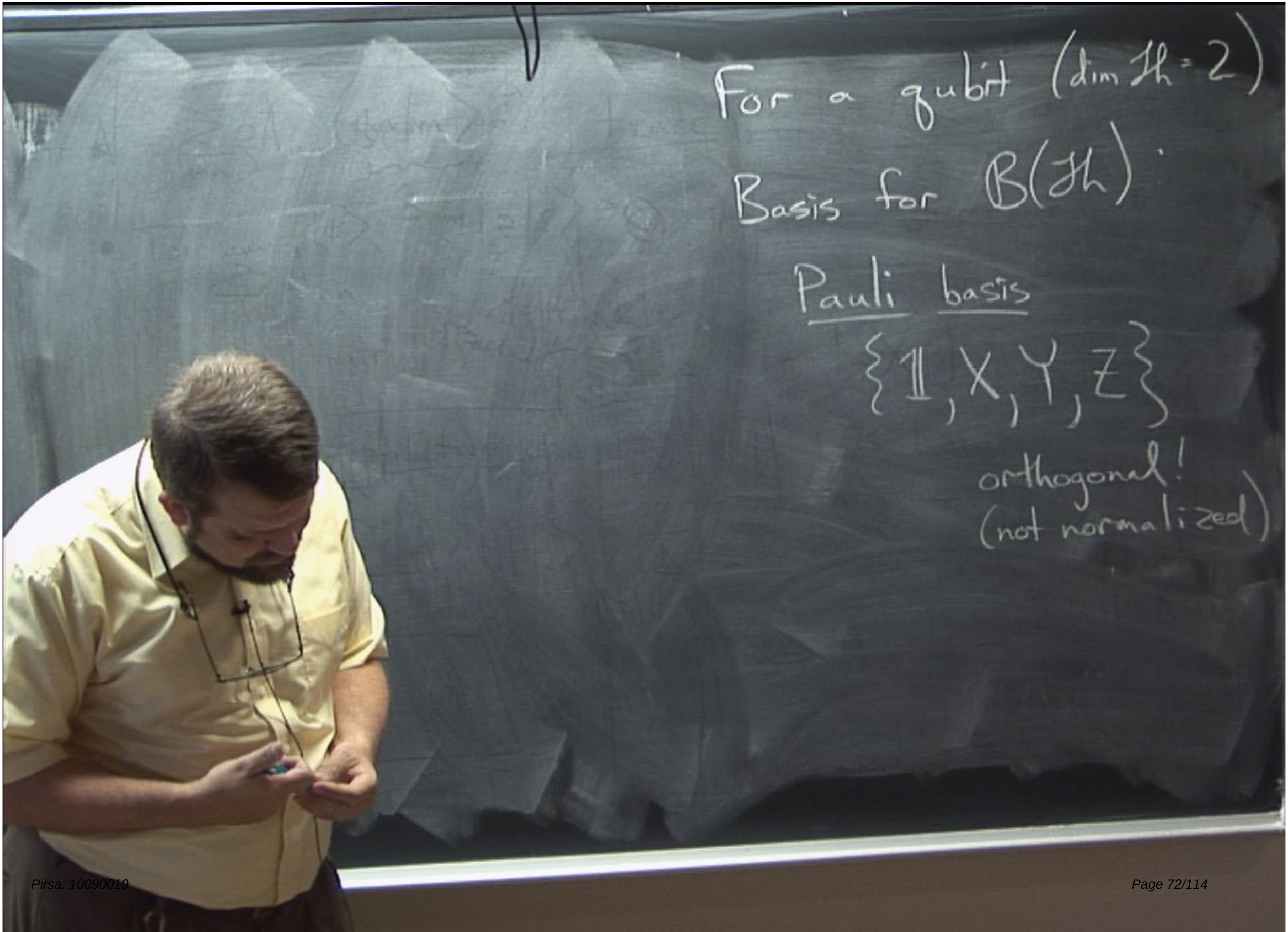

Given $\mathcal{H}$ with $\dim \mathcal{H} = d$
$\mathcal{B}(\mathcal{H})$ = operators on $\mathcal{H}$
$\dim \mathcal{B}(\mathcal{H}) = d^2$
$\langle A, B \rangle = \operatorname{tr} A^{\dagger} B$

For a qubit ($\dim \mathcal{H} = 2$)

Basis for $\mathcal{B}(\mathcal{H})$:

<u>Pauli basis</u>

$$\{\mathbb{1}, X, Y, Z\}$$


For a qubit (dim ℋ = 2)
Basis for 𝔹(ℋ):
Pauli basis
{𝟙, X, Y, Z}
orthogonal!
(not normalized)

$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$

For a qubit ($\dim \mathcal{H} = 2$)

Basis for $\mathcal{B}(\mathcal{H})$:

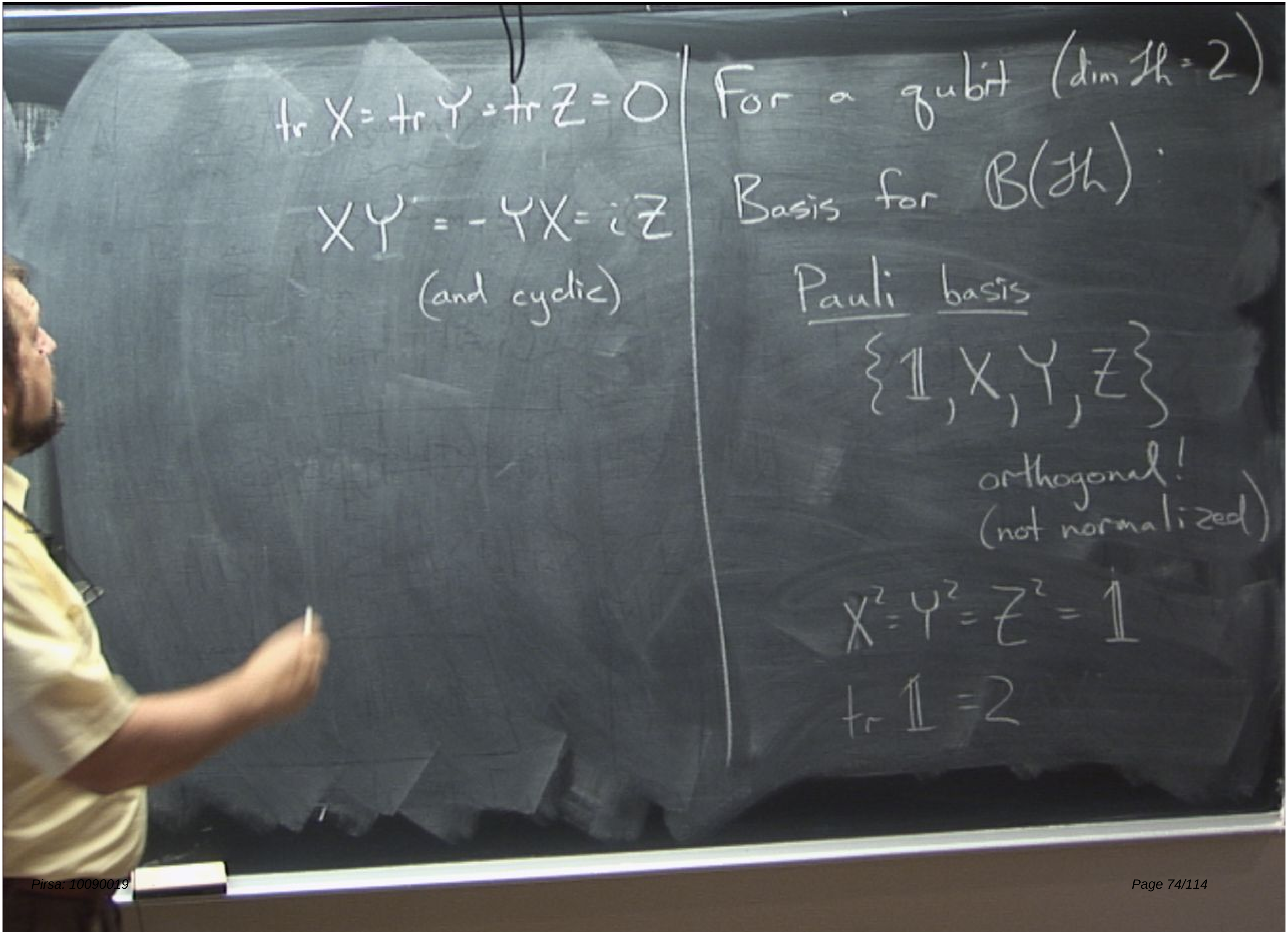
Pauli basis

$$\{\mathbb{1}, X, Y, Z\}$$

orthogonal!
(not normalized)

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

$$\mathrm{tr}\, \mathbb{1} = 2$$


$\mathrm{tr}\,X = \mathrm{tr}\,Y = \mathrm{tr}\,Z = 0$
$XY = -YX = iZ$
(and cyclic)
For a qubit ($\dim \mathcal{H} = 2$)
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Pauli basis
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(and cyclic)

$\rho =$

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(not normalized)

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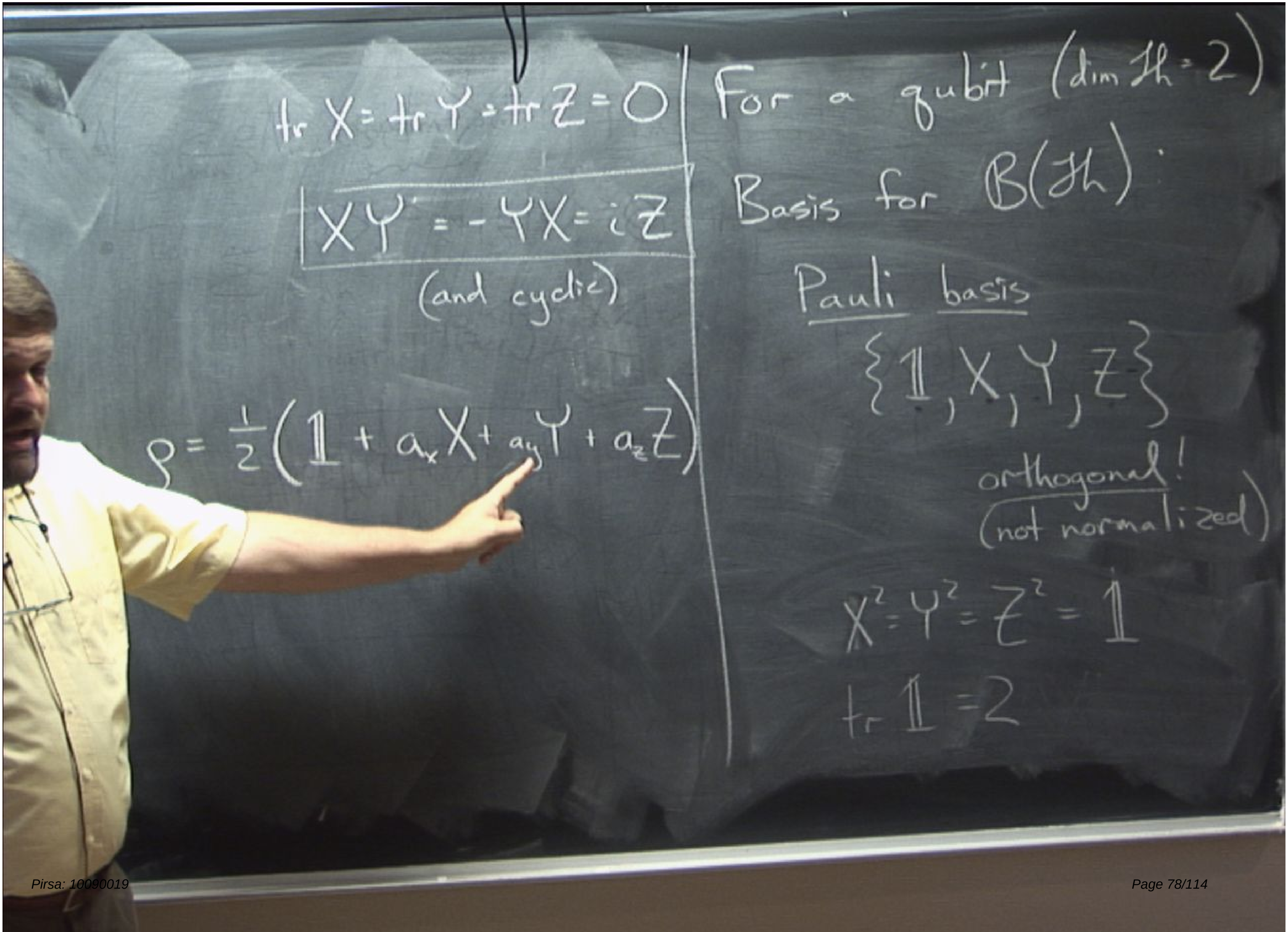

$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$
$XY = -YX = iZ$
(and cyclic)
$\rho = \frac{1}{2}(\mathbb{1} + a_x X + a_y Y + a_z Z)$
For a qubit ($\dim \mathcal{H} = 2$)
Basis for $\mathcal{B}(\mathcal{H})$:
Pauli basis
$\{\mathbb{1}, X, Y, Z\}$
orthogonal!
(not normalized)
$X^2 = Y^2 = Z^2 = \mathbb{1}$
$\mathrm{tr}\, \mathbb{1} = 2$

$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$

$$\boxed{XY = -YX = iZ}$$

(and cyclic)

$$\rho = \frac{1}{2}\left(\mathbb{1} + a_x X + a_y Y + a_z Z\right)$$

$\uparrow$ remember, $\mathrm{tr}\, \rho = 1$

For a qubit $(\dim \mathcal{H} = 2)$

Basis for $\mathcal{B}(\mathcal{H})$:

Pauli basis

$$\{\mathbb{1}, X, Y, Z\}$$

orthogonal!

(not normalized)

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

$$\mathrm{tr}\, \mathbb{1} = 2$$

$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$

$$XY = -YX = iZ$$

(and cyclic)

$$\rho = \frac{1}{2}\left(\mathbb{1} + a_x X + a_y Y + a_z Z\right)$$

remember, $\mathrm{tr}\, \rho = 1$

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$$

$\vec{\sigma} = (X, Y, Z)$

For a qubit ($\dim \mathcal{H} = 2$)

Basis for $\mathcal{B}(\mathcal{H})$:

Pauli basis

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orthogonal!
(not normalized)

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

$$\mathrm{tr}\, \mathbb{1} = 2$$

or ρ

$\operatorname{tr} X = \operatorname{tr} Y = \operatorname{tr} Z = 0$

$$\boxed{XY = -YX = iZ}$$

(and cyclic)

$$\rho = \frac{1}{2}\left(\mathbb{1} + a_x X + a_y Y + a_z Z\right)$$

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For a qubit ($\dim \mathcal{H} = 2$)

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$X^2 = Y^2 = Z^2 = \mathbb{1}$

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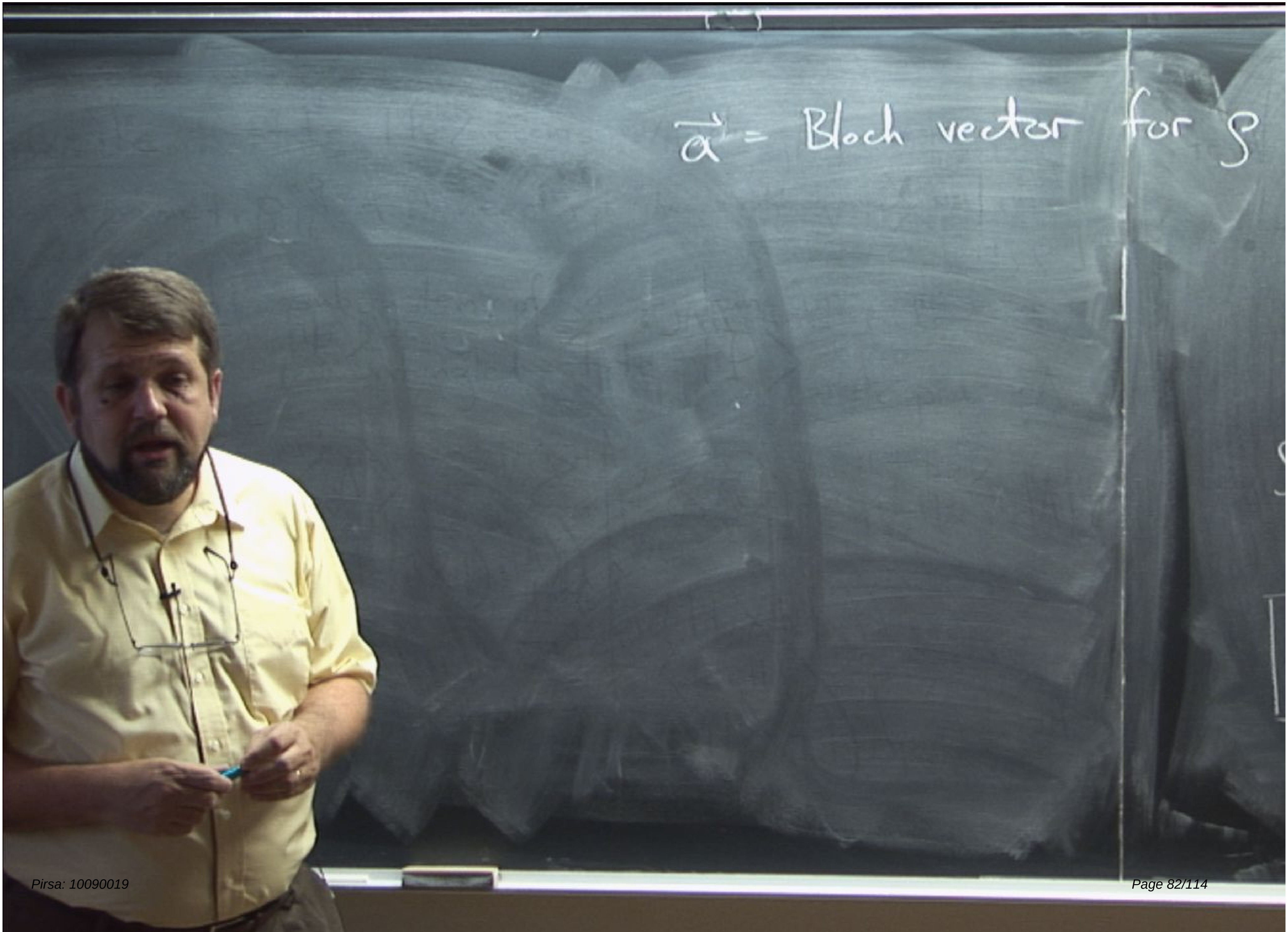
$\vec{a}$ = Bloch vector for ρ

...or ρ

$$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$$

$$\boxed{XY = -YX = iZ}$$

(and cyclic)

$$\rho = \frac{1}{2}\left(\mathbb{1} + a_x X + a_y Y + a_z Z\right)$$

↖ remember, $\mathrm{tr}\, \rho = 1$

$$\boxed{\rho = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)}$$

↖ $\vec{\sigma} = (X, Y, Z)$

For a qubit ($\dim \mathcal{H} = 2$)

Basis for $\mathcal{B}(\mathcal{H})$:

Pauli basis

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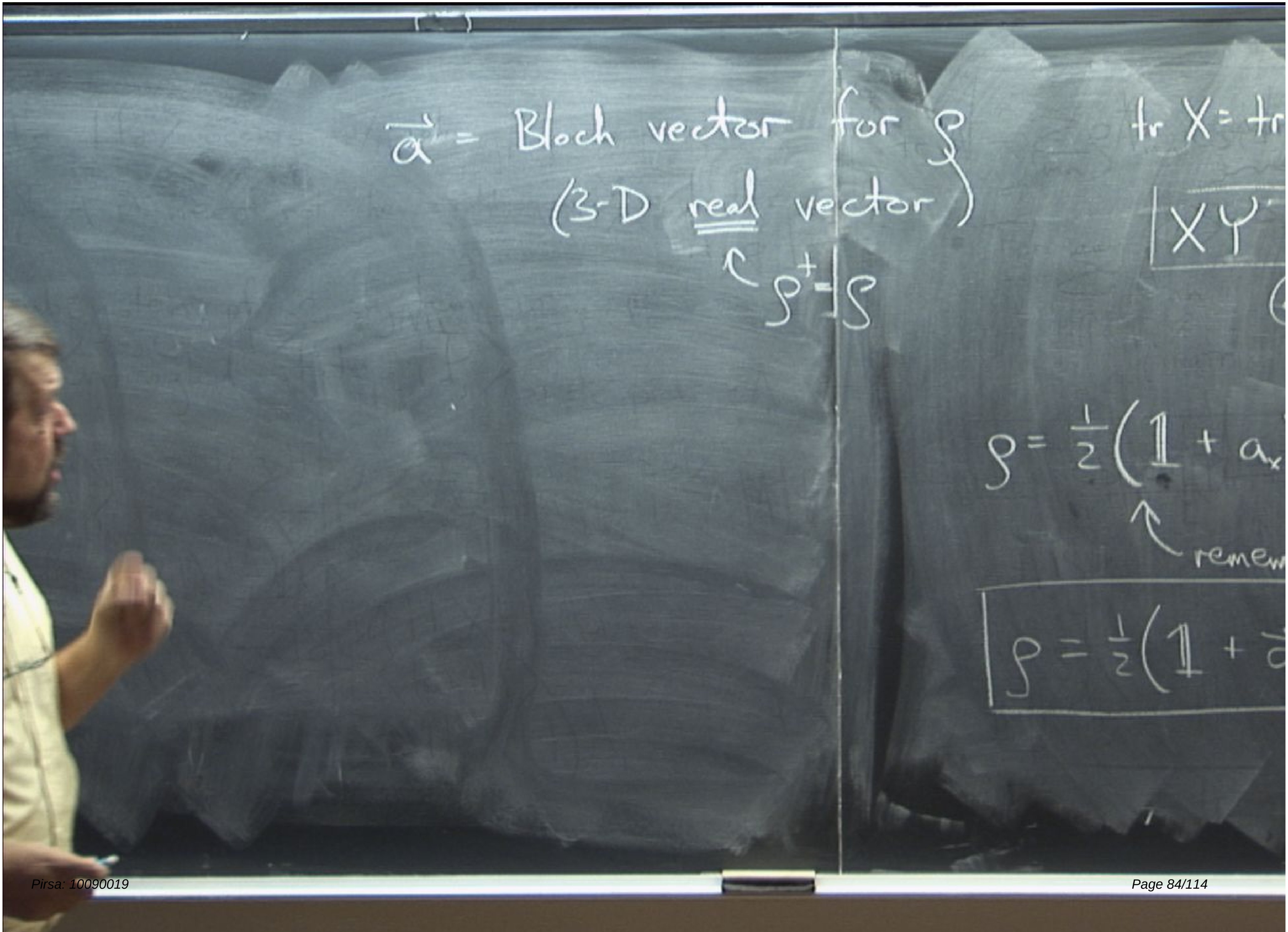
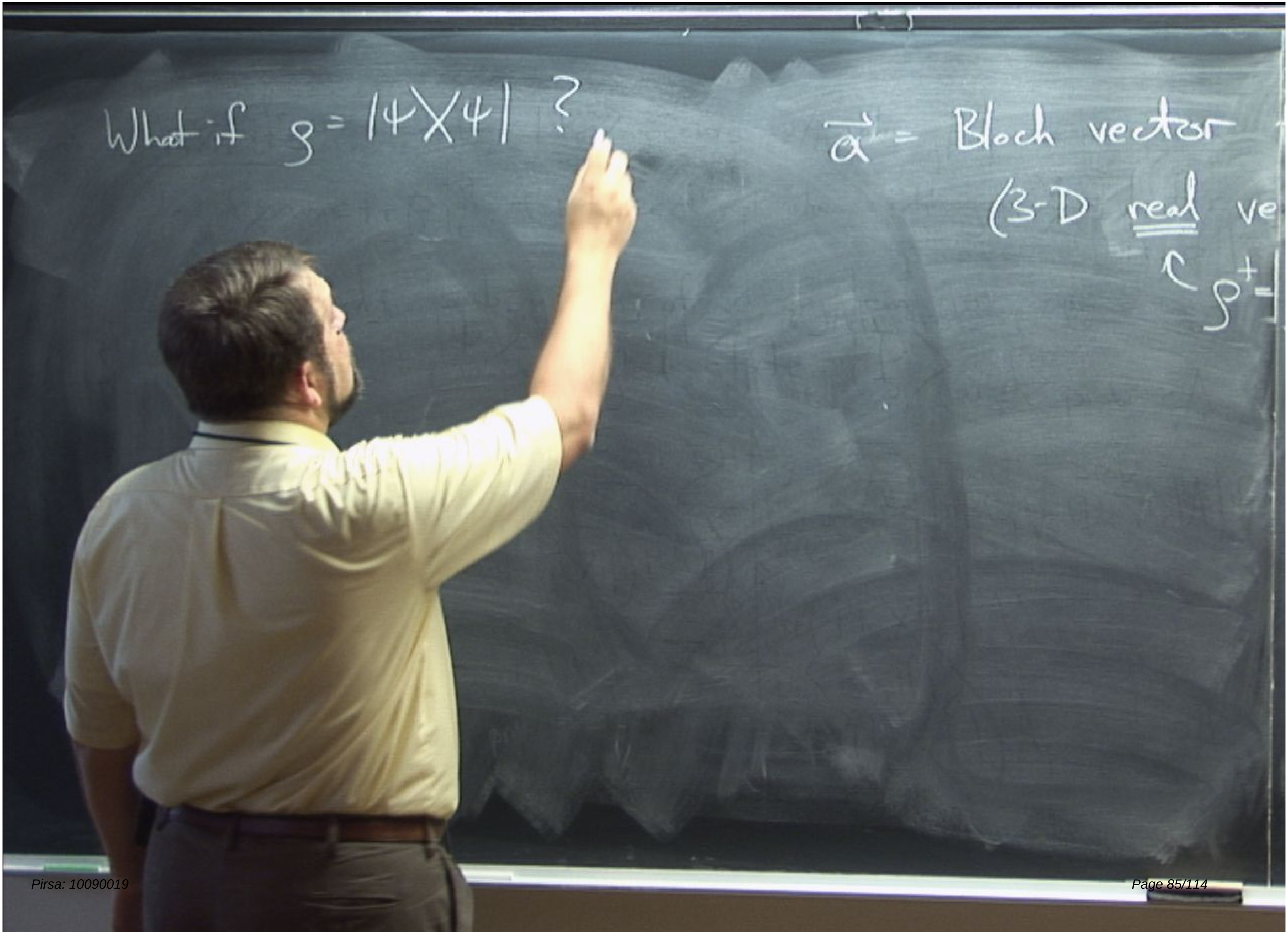
orthogonal!
(not normalized)

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

$$\mathrm{tr}\, \mathbb{1} = 2$$



$\vec{a}$ = Bloch vector for ρ
(3-D real vector)
$\rho^\dagger = \rho$
tr X = tr
XY
$\rho = \frac{1}{2}(\mathbb{1} + a_x$
remem
$\rho = \frac{1}{2}(\mathbb{1} + \vec{a}$


What if ρ = |ψ⟩⟨ψ| ?
α⃗ = Bloch vector
(3-D real ve
ρ† =

What if $\rho = |\psi\rangle\langle\psi|$? (pure)

$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \boxed{\mathrm{tr}\,\rho^2 = 1}$

$\rho^2 = \frac{1}{4}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$

$\vec{a}$ = Bloch vector

(3-D real ve

$\uparrow$ $\rho^{+}=$

What if $\rho = |\psi\rangle\langle\psi|$? (pure)

$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \text{tr}\,\rho^2 = 1$

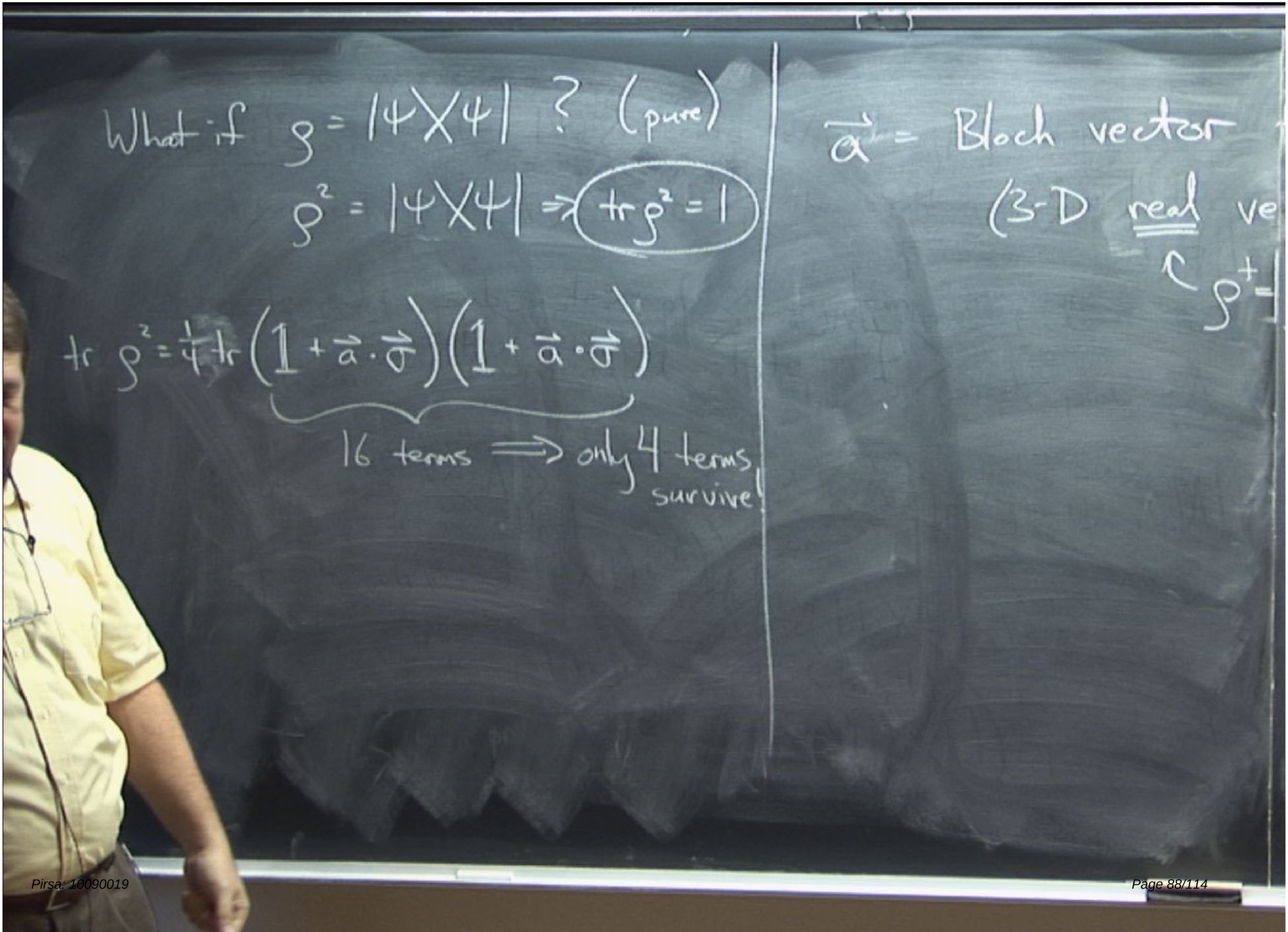
$\text{tr}\,\rho^2 = \frac{1}{4}\text{tr}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$

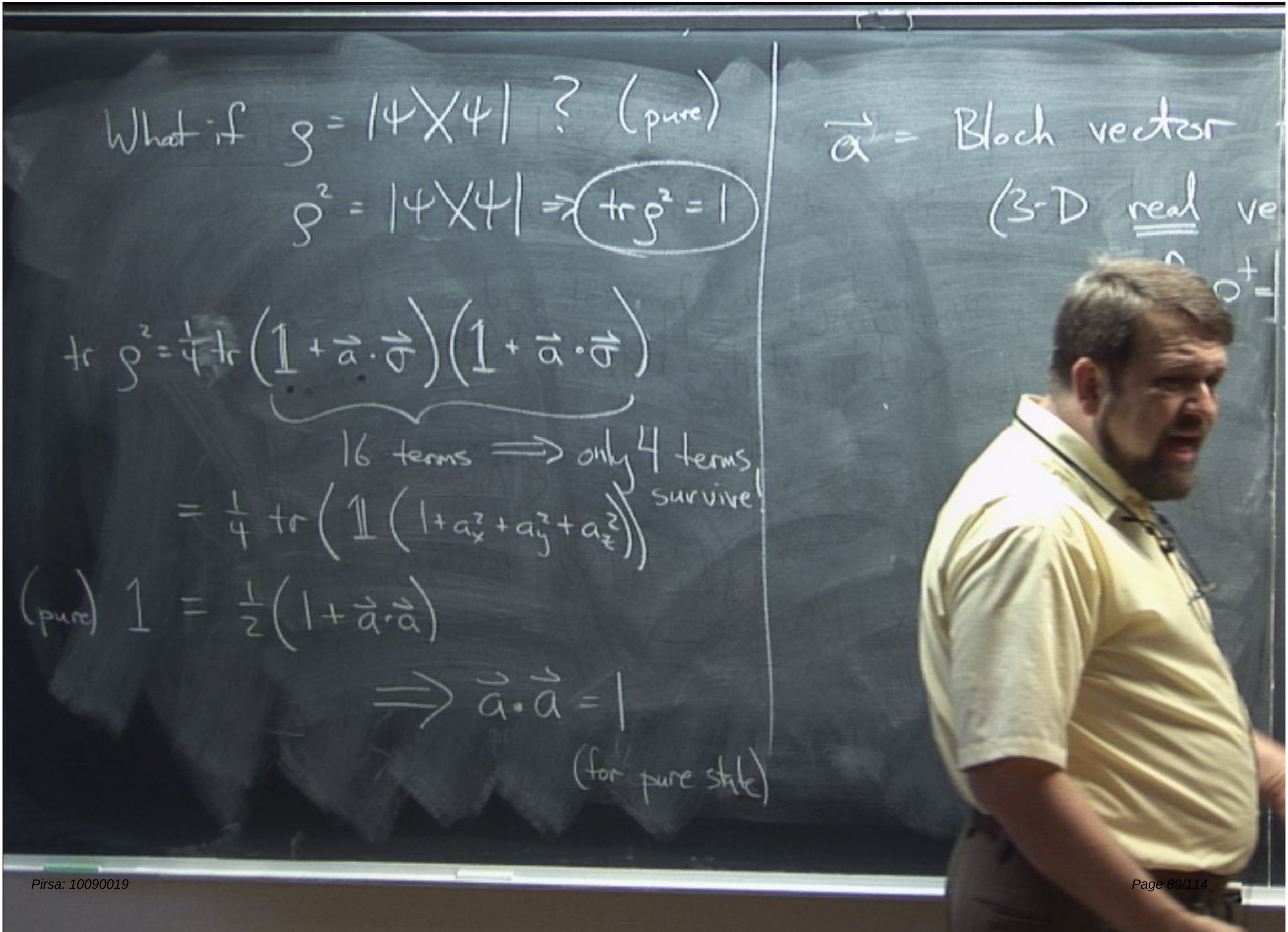
16 terms $\Longrightarrow$

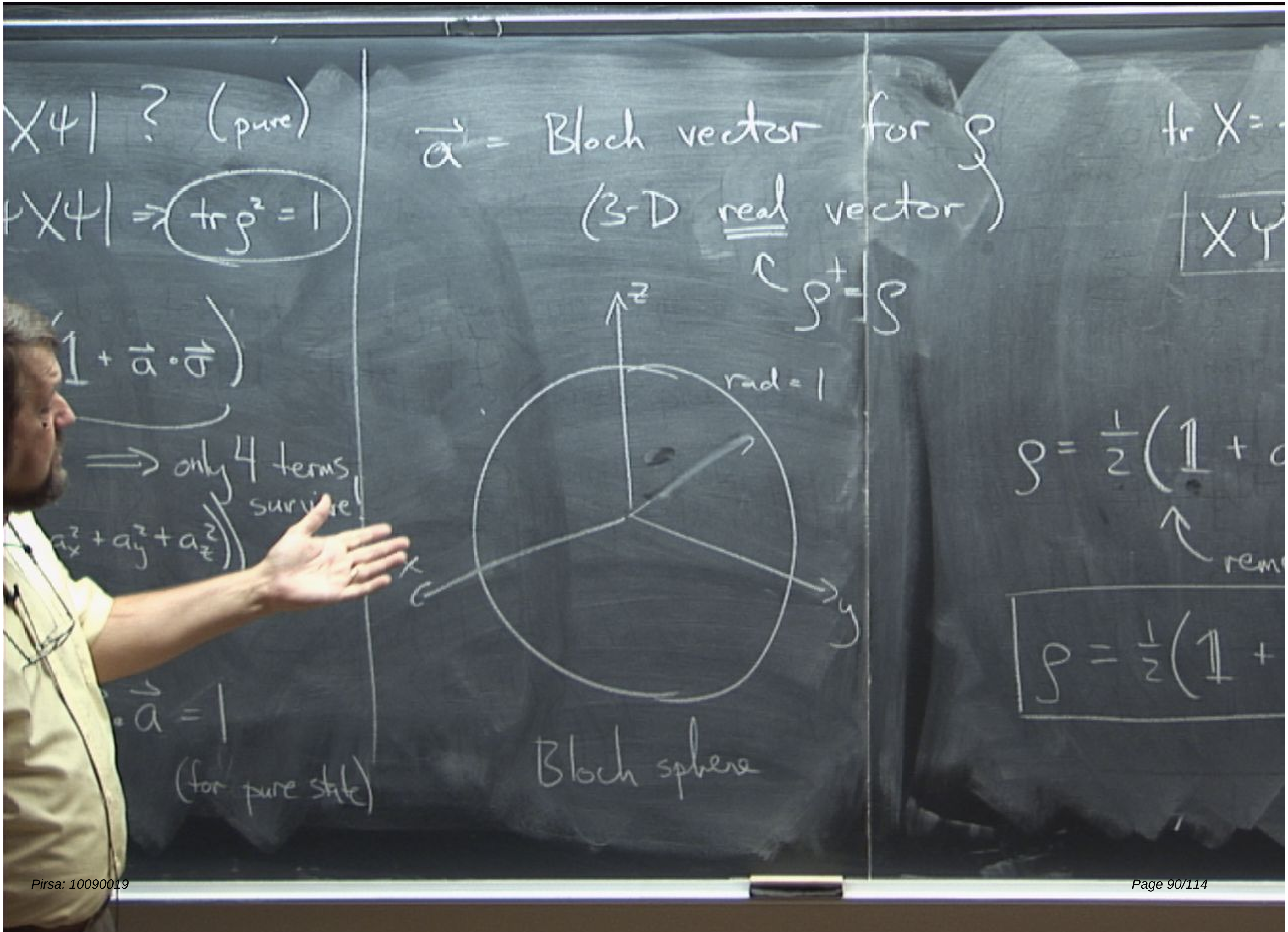
$\vec{a}$ = Bloch vector

(3-D real ve

$\uparrow$ $\rho^{+}=$


What if $\rho = |\psi\rangle\langle\psi|$? (pure)
$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \mathrm{tr}\,\rho^2 = 1$
$\mathrm{tr}\,\rho^2 = \frac{1}{4}\mathrm{tr}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$
16 terms ⟹ only 4 terms survive!
$\vec{a}$ = Bloch vector
(3-D real ve
$\rho^\dagger =$


What if $\rho = |\psi\rangle\langle\psi|$? (pure)
$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \mathrm{tr}\,\rho^2 = 1$
$\mathrm{tr}\,\rho^2 = \frac{1}{4}\mathrm{tr}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$
16 terms $\Longrightarrow$ only 4 terms survive!
$= \frac{1}{4}\mathrm{tr}\left(\mathbb{1}\left(1 + a_x^2 + a_y^2 + a_z^2\right)\right)$
(pure) $1 = \frac{1}{2}\left(1 + \vec{a}\cdot\vec{a}\right)$
$\Longrightarrow \vec{a}\cdot\vec{a} = 1$
(for pure state)
$\vec{a}$ = Bloch vector
(3-D real ve


a = Bloch vector for ρ
(3-D real vector)
ρ† = ρ
rad = 1
Bloch sphere
z
x
y
only 4 terms survive!
(for pure state)
tr ρ² = 1
(pure)
ρ = ½(𝟙 +
tr X =

$\rangle\langle\psi|$? (pure)

$\psi\rangle\langle\psi| \Rightarrow$ $\text{tr}\,\rho^2 = 1$

$)(\mathbb{1} + \vec{a}\cdot\vec{\sigma})$

erms $\Longrightarrow$ only 4 terms survive!

$1 + a_x^2 + a_y^2 + a_z^2))$

$\vec{a})$

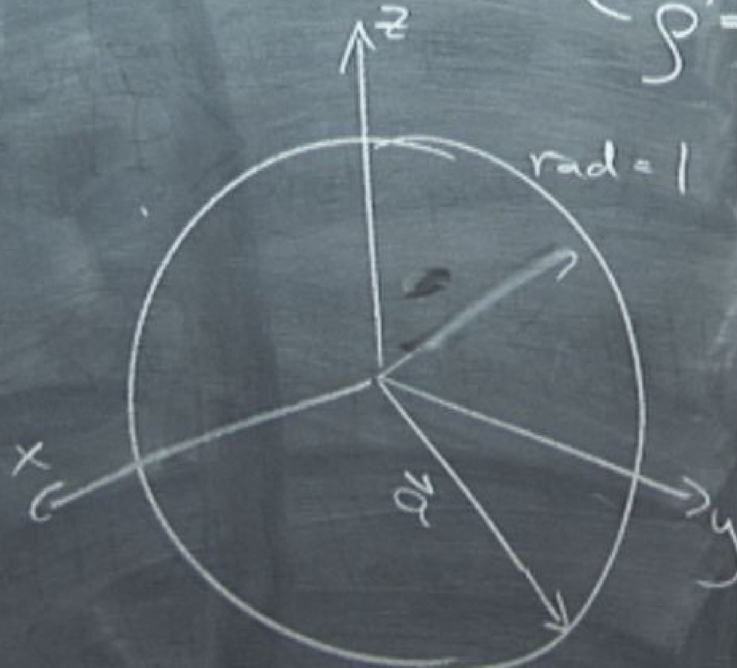
$\Rightarrow \vec{a}\cdot\vec{a} = 1$

(for pure state)

$\vec{a}$ = Bloch vector for ρ

(3-D real vector)

$\uparrow \rho^\dagger = \rho$



Bloch sphere

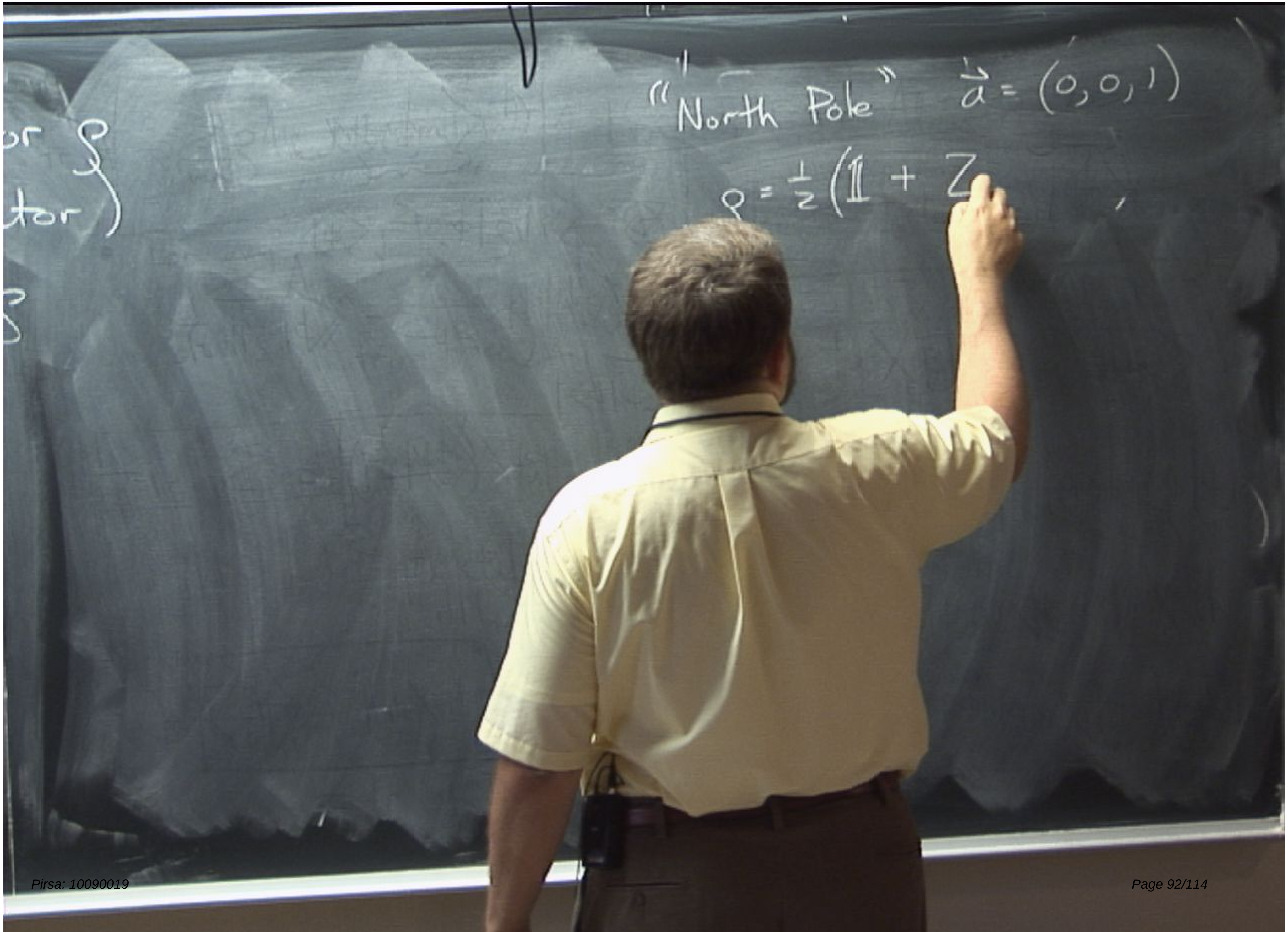
$\text{tr}\, X = $

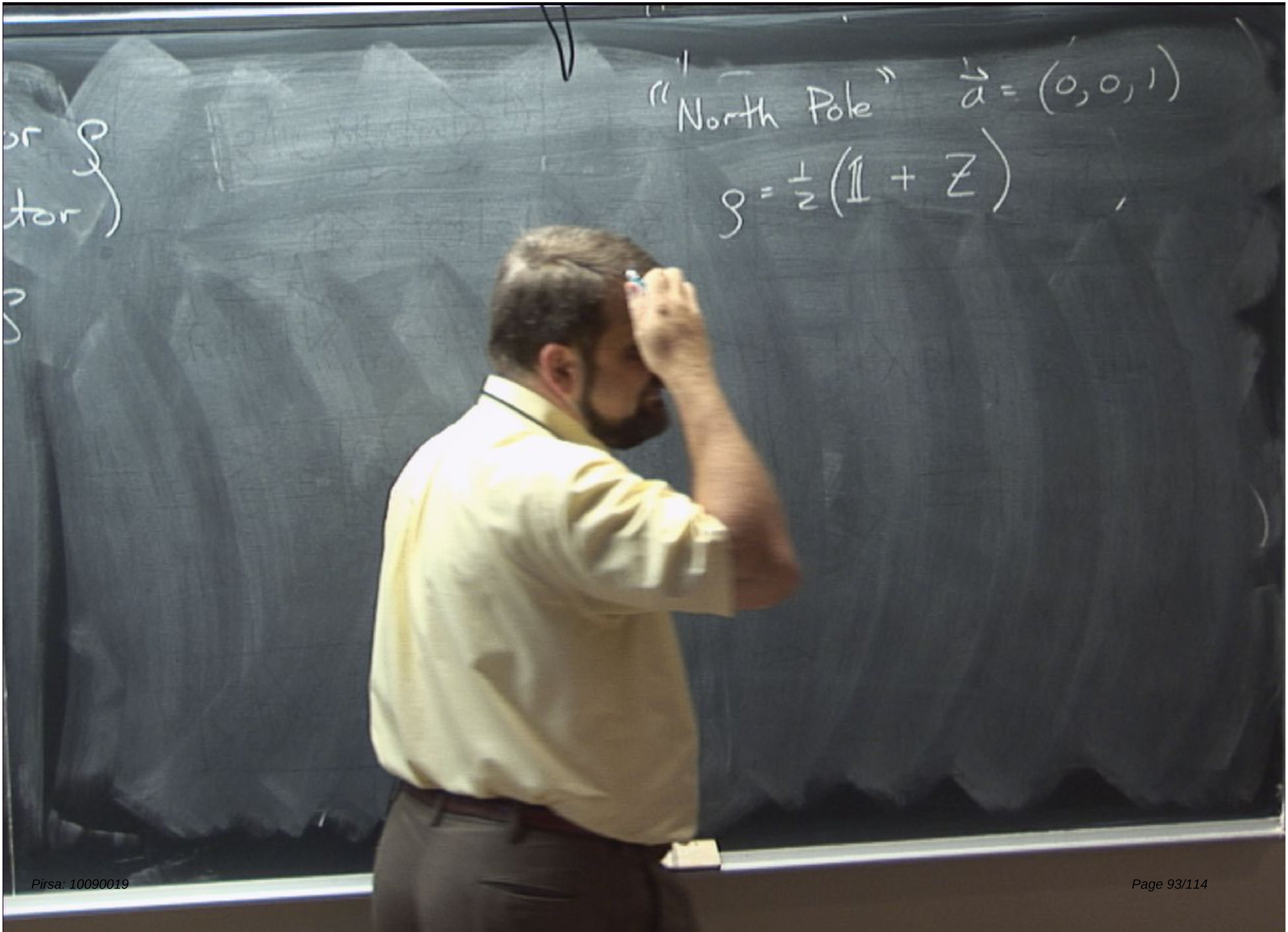
$|X\psi$

$\rho = \frac{1}{2}(\mathbb{1} + a$

rem

$\rho = \frac{1}{2}(\mathbb{1} +$


"North Pole" $\vec{a} = (0,0,1)$
$\rho = \frac{1}{2}(\mathbb{1} + Z$
or ρ
tor)


"North Pole" $\vec{a} = (0, 0, 1)$
$\rho = \frac{1}{2}(\mathbb{1} + Z)$

$\psi|$? (pure)

$\langle\psi| \Rightarrow$ $\mathrm{tr}\,\rho^2 = 1$

$(1 + \vec{a}\cdot\vec{\sigma})$

s $\Longrightarrow$ only 4 terms survive!

$a_x^2 + a_y^2 + a_z^2$

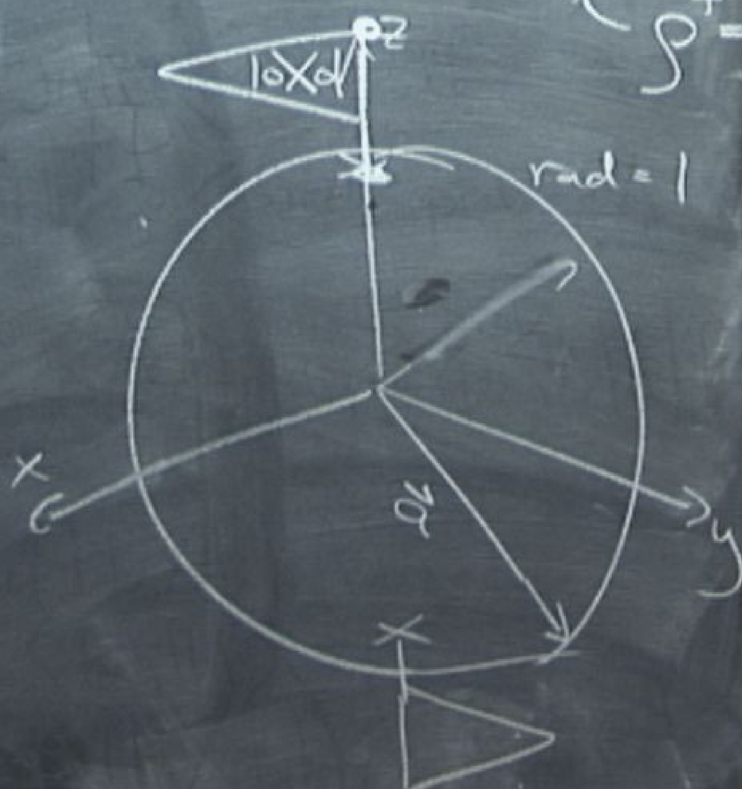
$\vec{a}\cdot\vec{a} = 1$

(for pure state)

$\vec{a}$ = Bloch vector for ρ

(3-D real vector)

$\rho^\dagger = \rho$



or ρ

tor)

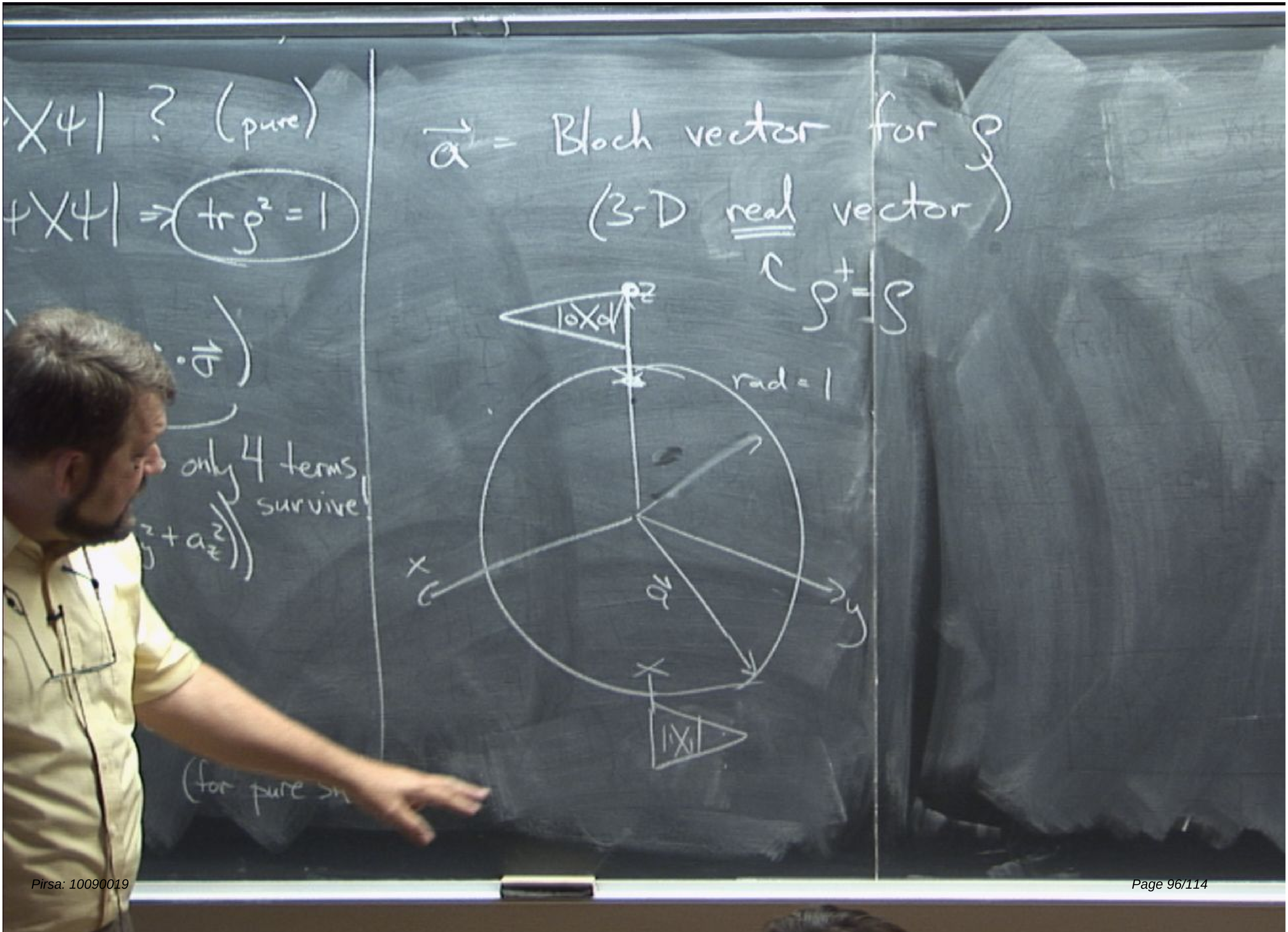
ρ

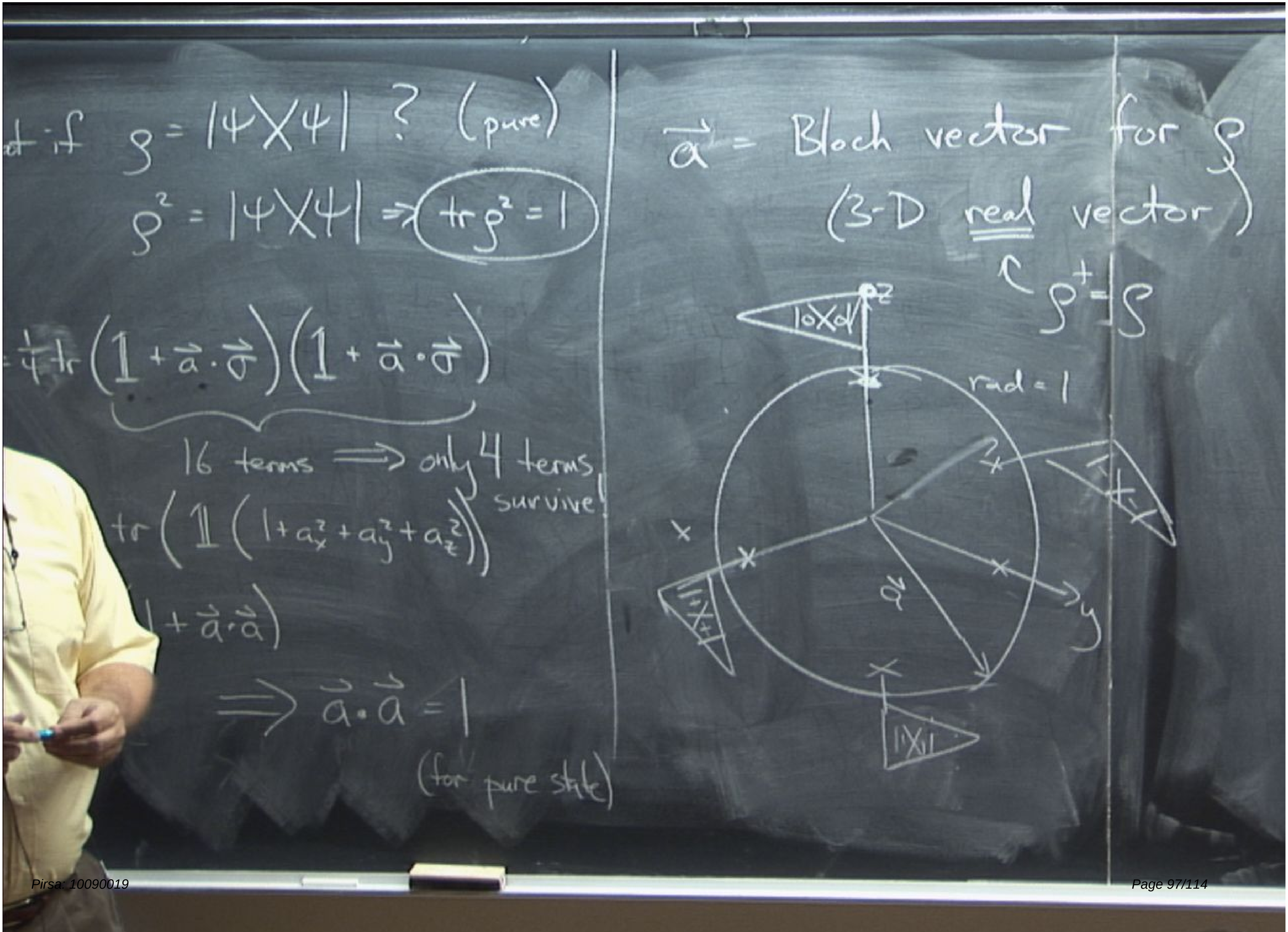
"North Pole" $\vec{a} = (0,0,1)$

$\rho = \frac{1}{2}(\mathbb{1} + Z)$

$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| - |1\rangle\langle 1|)$

$= |0\rangle\langle 0|$


|X4| ? (pure)
tr ρ² = 1
only 4 terms survive!
(for pure st
a = Bloch vector for ρ
(3-D real vector)
ρ† = ρ
|0X0|
z
rad = 1
x
y
a
|1X1|


if ρ = |ψ⟩⟨ψ| ? (pure)
ρ² = |ψ⟩⟨ψ| ⇒ tr ρ² = 1
= ¼ tr (𝟙 + a⃗·σ⃗)(𝟙 + a⃗·σ⃗)
16 terms ⇒ only 4 terms survive!
tr (𝟙 (1 + a_x² + a_y² + a_z²))
1 + a⃗·a⃗
⇒ a⃗·a⃗ = 1
(for pure state)
a⃗ = Bloch vector for ρ
(3-D real vector)
ρ† = ρ
|0⟩⟨0|
z
rad = 1
a⃗
y
|1⟩⟨1|

t if $\rho = |\psi\rangle\langle\psi|$? (pure)

$$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \text{tr}\,\rho^2 = 1$$

$$= \frac{1}{4}\,\text{tr}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$$

16 terms $\Longrightarrow$ only 4 terms survive!

$$= \frac{1}{4}\,\text{tr}\left(\mathbb{1}\left(1 + a_x^2 + a_y^2 + a_z^2\right)\right)$$

$$= \frac{1}{2}\left(1 + \vec{a}\cdot\vec{a}\right)$$

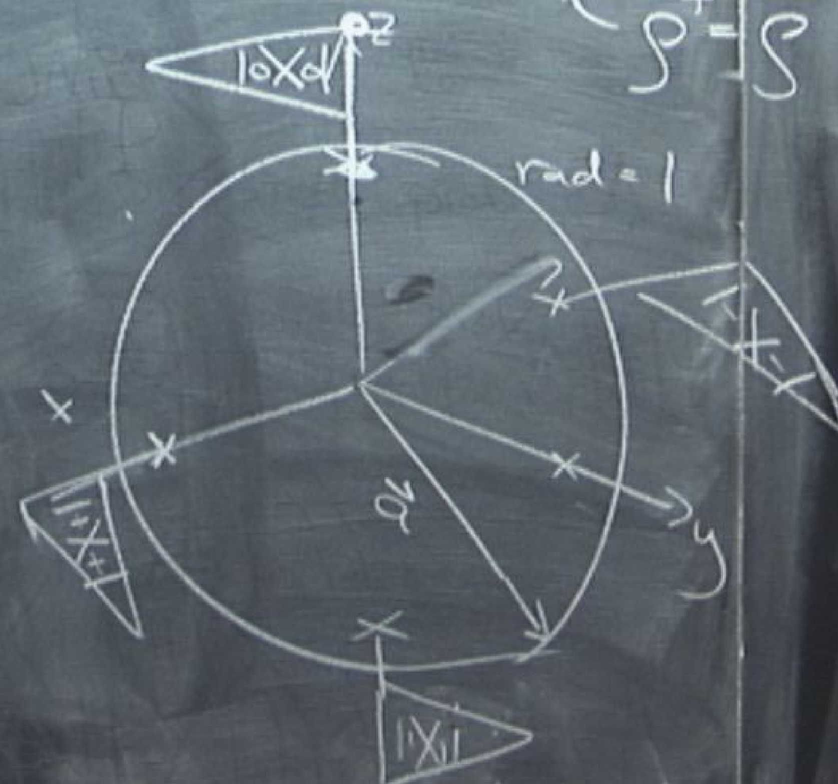
$$\Longrightarrow \vec{a}\cdot\vec{a} = 1$$

(for pure state)

$\vec{a}$ = Bloch vector for ρ

(3-D real vector)

$\rho^\dagger = \rho$



What if $\rho = |\psi\rangle\langle\psi|$? (pure)

$\rho^2 = |\psi\rangle\langle\psi| \Rightarrow \boxed{\mathrm{tr}\,\rho^2 = 1}$

$\mathrm{tr}\,\rho^2 = \frac{1}{4}\mathrm{tr}\underbrace{(\mathbb{1} + \vec{a}\cdot\vec{\sigma})(\mathbb{1} + \vec{a}\cdot\vec{\sigma})}$

16 terms $\Longrightarrow$ only 4 terms survive!

$= \frac{1}{4}\,\mathrm{tr}\left(\mathbb{1}\left(1 + a_x^2 + a_y^2 + a_z^2\right)\right)$

(pure) $1 = \frac{1}{2}\left(1 + \vec{a}\cdot\vec{a}\right)$

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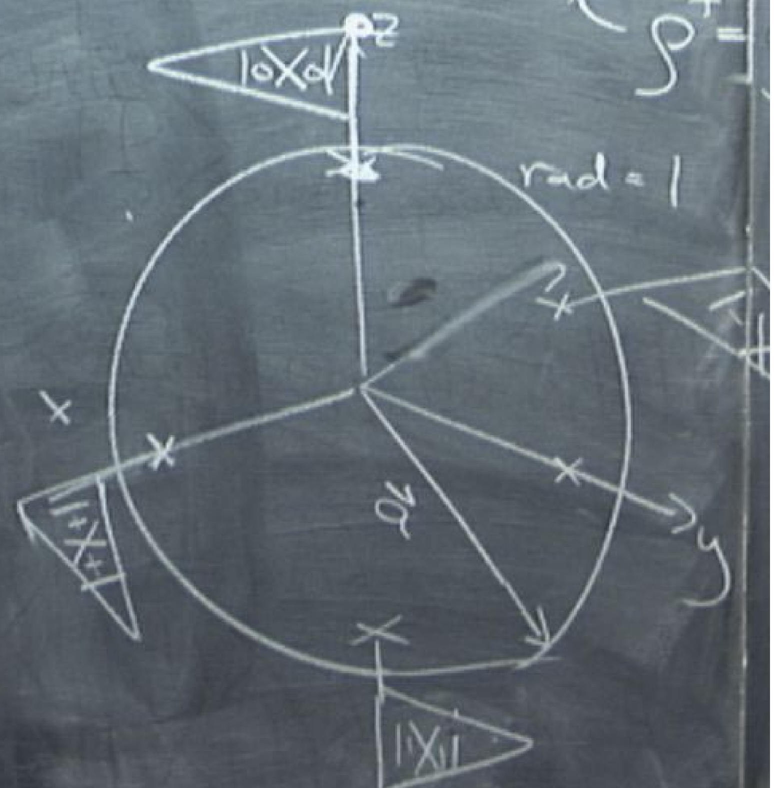
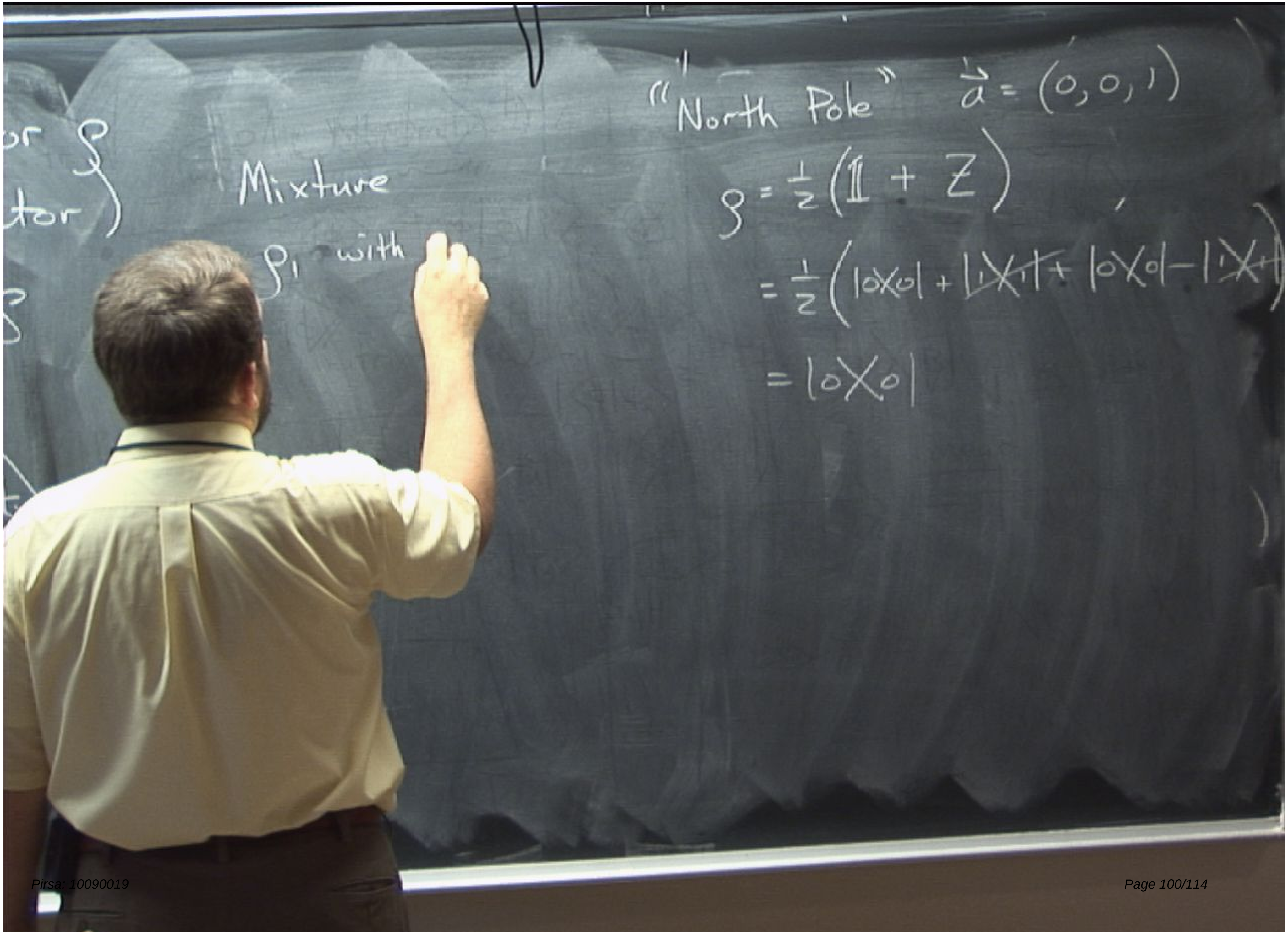
(for pure state)

$\vec{a}$ = Bloch vector f

(3-D real ve

$\rho^\dagger = $




"North Pole" $\vec{a} = (0,0,1)$
$\rho = \frac{1}{2}(\mathbb{1} + Z)$
$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| - |1\rangle\langle 1|)$
$= |0\rangle\langle 0|$
Mixture
ρ_1 with

"North Pole" $\vec{a} = (0,0,1)$

$$\rho = \frac{1}{2}(\mathbb{1} + Z)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + \cancel{|1\rangle\langle 1|} + |0\rangle\langle 0| - \cancel{|1\rangle\langle 1|}\right)$$

$$= |0\rangle\langle 0|$$

Mixture

ρ_1 with prob p_1

ρ_2 with prob. p_2

$$\rho = p_1\rho_1 + p_2\rho_2$$

Mixture

ρ_1 with prob p_1

ρ_2 with prob. p_2

$$\rho = p_1 \rho_1 + p_2 \rho_2$$

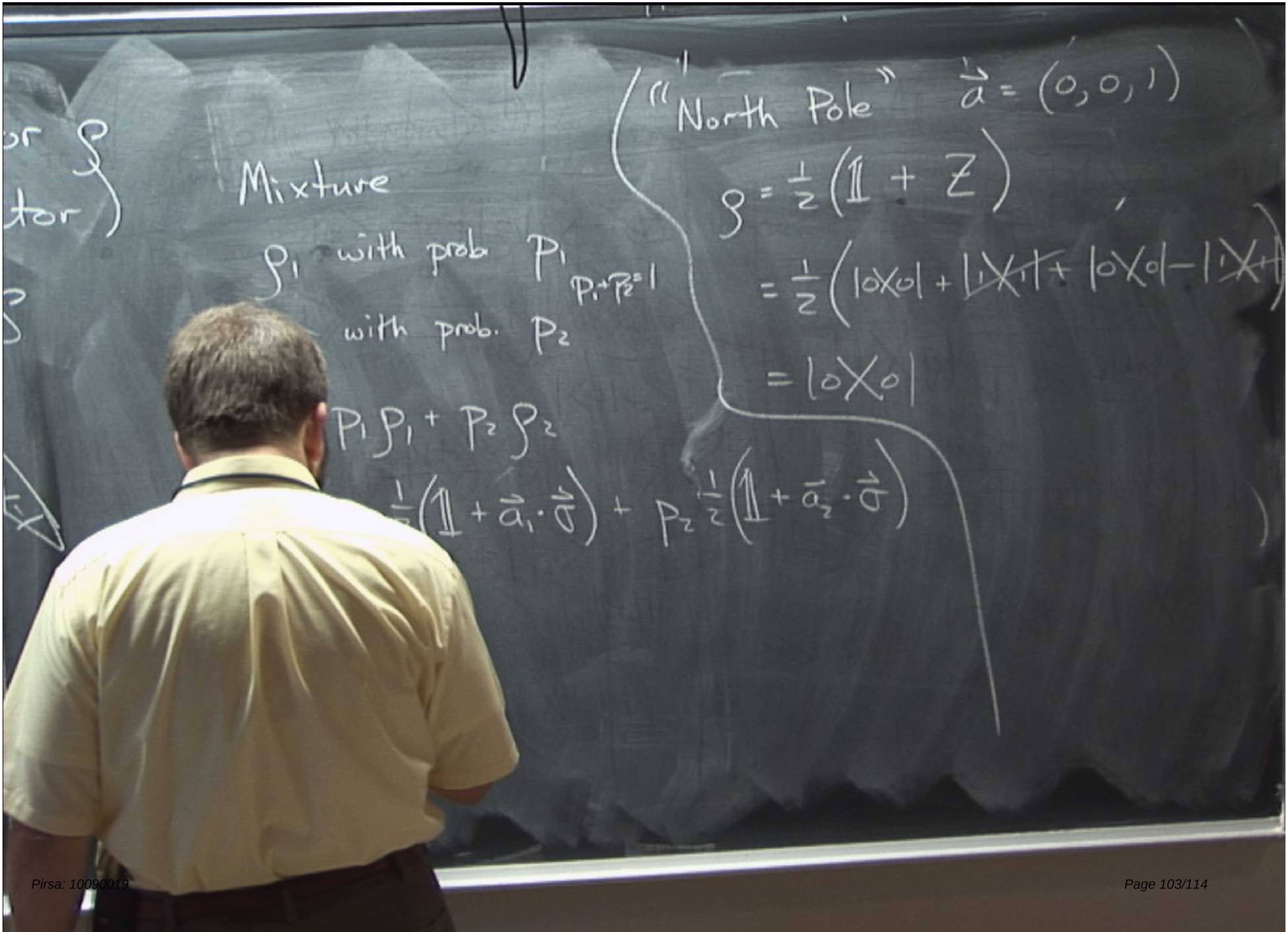
$$= p_1 \frac{1}{2}\left(\mathbb{1} + \vec{a}_1 \cdot \vec{\sigma}\right) + p_2 \frac{1}{2}\left(\mathbb{1} + \vec{a}_2 \cdot \vec{\sigma}\right)$$

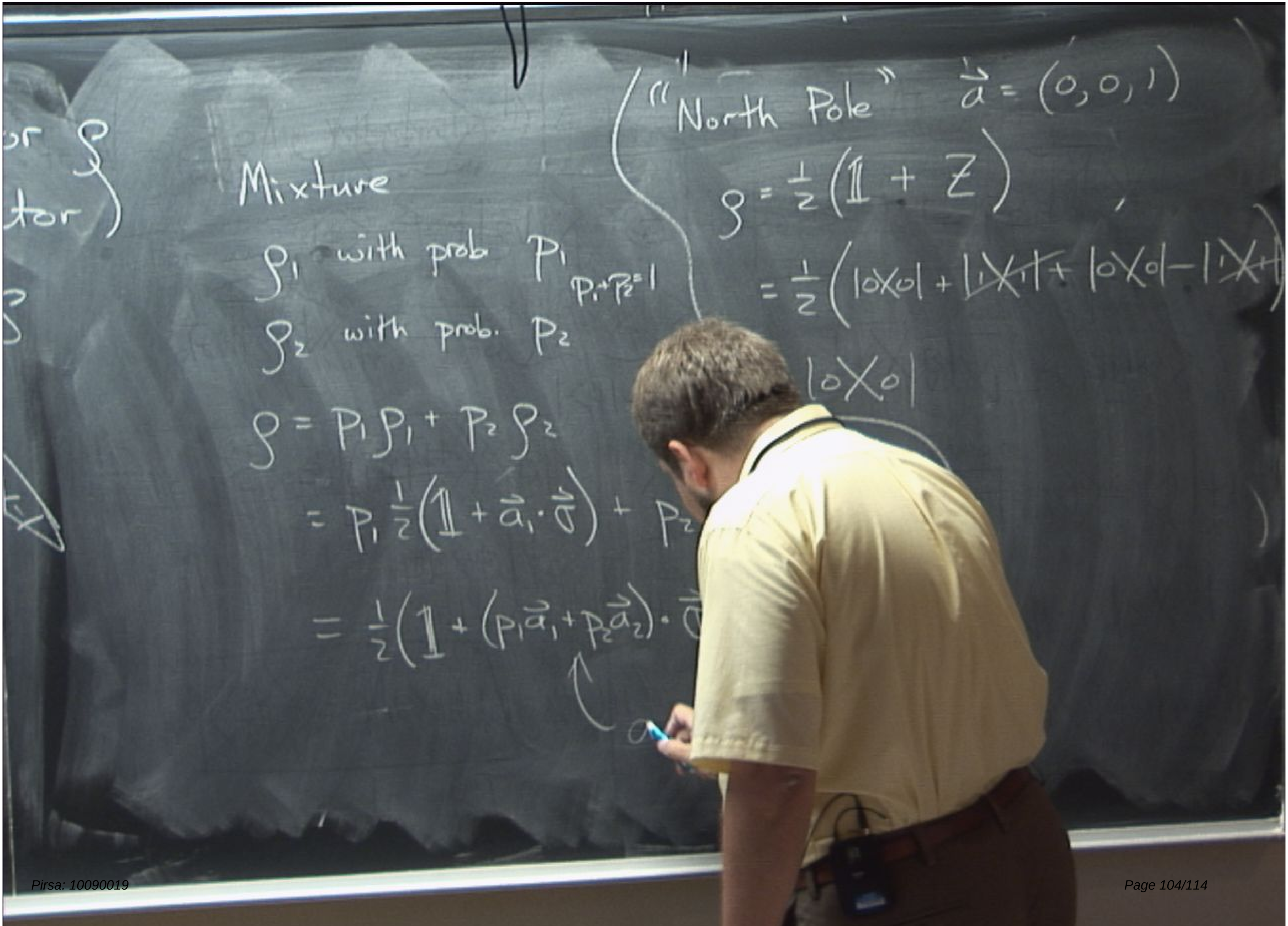
"North Pole" $\vec{a} = (0,0,1)$

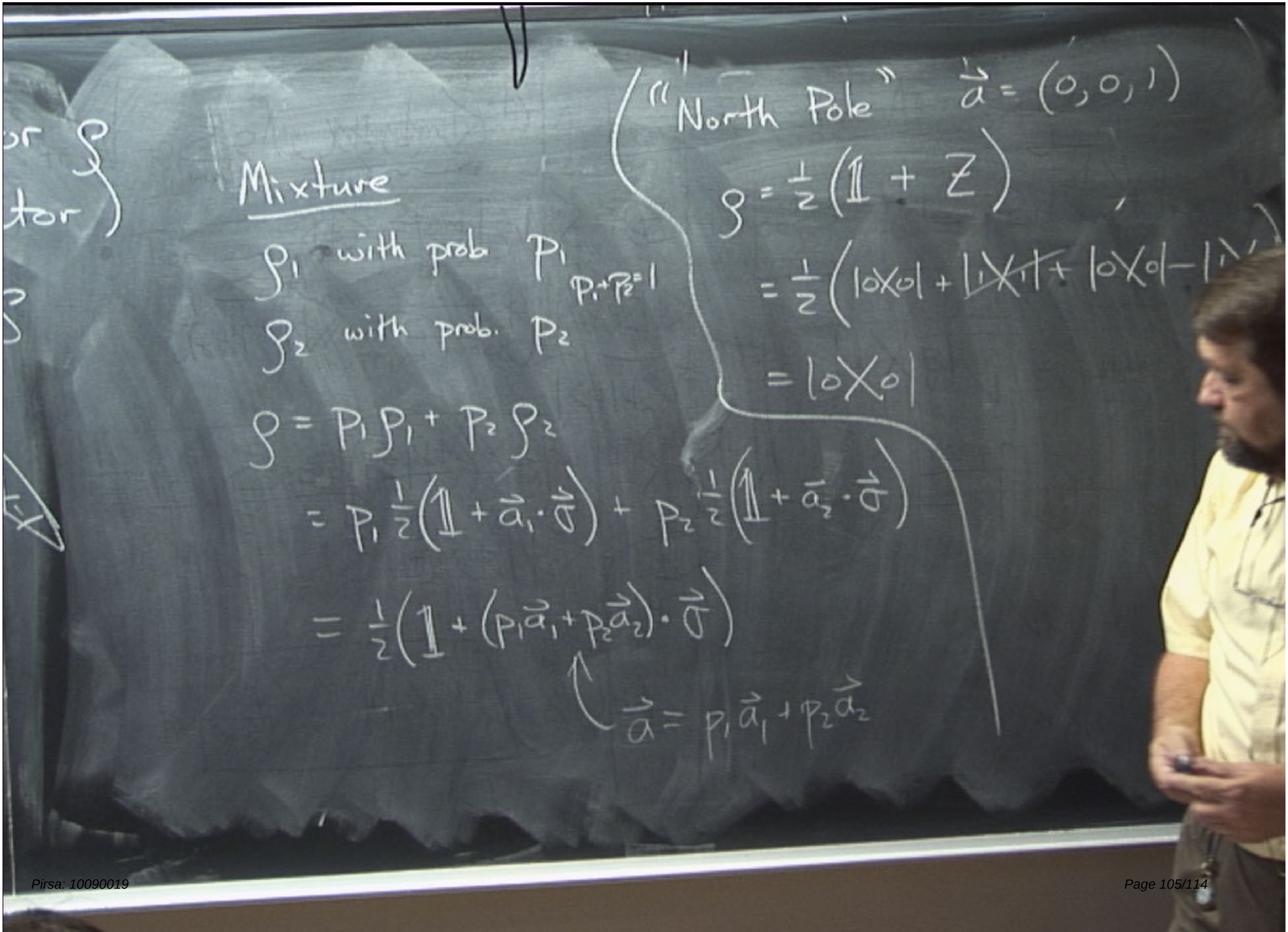
$$\rho = \frac{1}{2}\left(\mathbb{1} + Z\right)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$= |0\rangle\langle 0|$$


Mixture
ρ1 with prob p1
p1+p2=1
with prob. p2
p1ρ1 + p2ρ2
"North Pole" a = (0,0,1)
ρ = 1/2 (𝟙 + Z)
= 1/2 (|0⟩⟨0| + |1⟩⟨1| + |0⟩⟨0| − |1⟩⟨1|)
= |0⟩⟨0|
1/2 (𝟙 + a1 · σ) + p2 1/2 (𝟙 + a2 · σ)


or ρ
tor)
Mixture
ρ1 with prob p1
p1+p2=1
ρ2 with prob. p2
ρ = p1ρ1 + p2ρ2
= p1 ½(𝟙 + a1·σ) + p2
= ½(𝟙 + (p1a1 + p2a2)·σ
"North Pole" a = (0,0,1)
ρ = ½(𝟙 + Z)
= ½(|0⟩⟨0| + |1⟩⟨1| + |0⟩⟨0| − |1⟩⟨1|)
|0⟩⟨0|


Mixture
ρ1 with prob p1
p1+p2=1
ρ2 with prob. p2
ρ = p1 ρ1 + p2 ρ2
= p1 ½(𝟙 + a⃗1 · σ⃗) + p2 ½(𝟙 + a⃗2 · σ⃗)
= ½(𝟙 + (p1 a⃗1 + p2 a⃗2) · σ⃗)
a⃗ = p1 a⃗1 + p2 a⃗2
"North Pole" a⃗ = (0, 0, 1)
ρ = ½(𝟙 + Z)
= ½(|0⟩⟨0| + |1⟩⟨1| + |0⟩⟨0| − |1⟩
= |0⟩⟨0|

or ρ

tor)

Mixture

ρ_1 with prob p_1 $p_1 + p_2 = 1$

ρ_2 with prob. p_2

$$\rho = p_1 \rho_1 + p_2 \rho_2$$

$$= p_1 \frac{1}{2}\left(\mathbb{1} + \vec{a}_1 \cdot \vec{\sigma}\right) + p_2 \frac{1}{2}\left(\mathbb{1} + \vec{a}_2 \cdot \vec{\sigma}\right)$$

$$= \frac{1}{2}\left(\mathbb{1} + (p_1 \vec{a}_1 + p_2 \vec{a}_2) \cdot \vec{\sigma}\right)$$

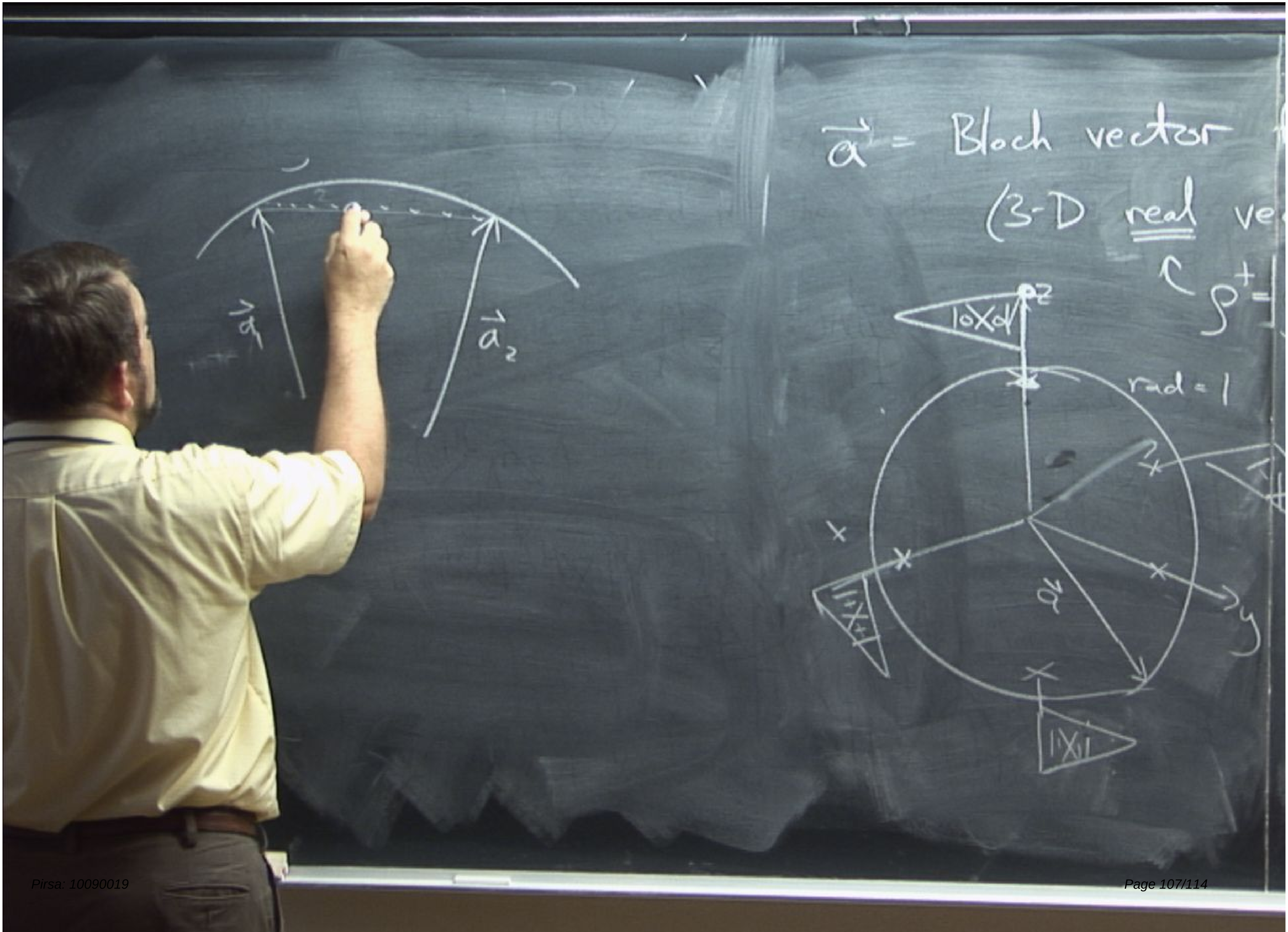
$$\vec{a} = p_1 \vec{a}_1 + p_2 \vec{a}_2$$

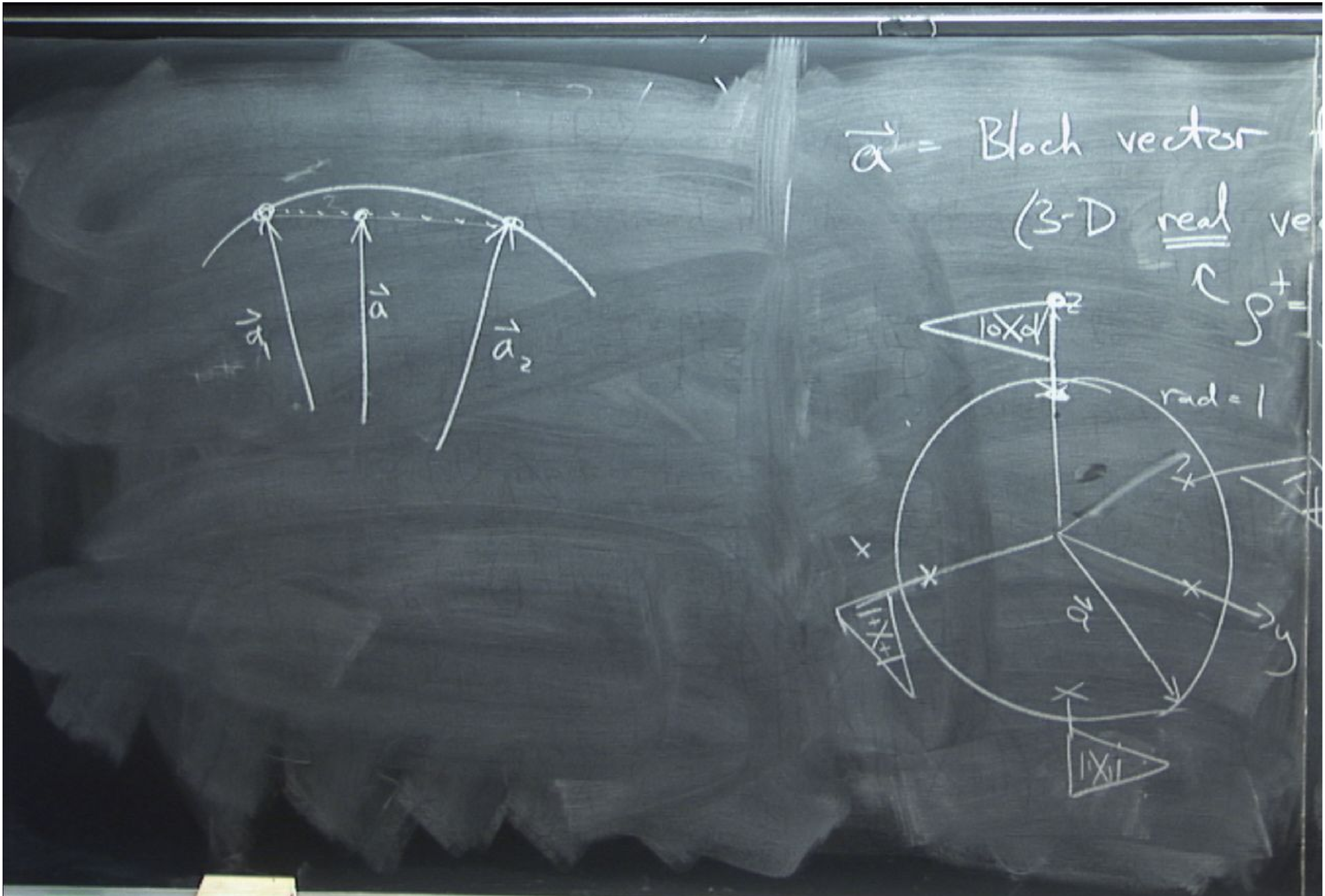
"North Pole" $\vec{a} = (0, 0, 1)$

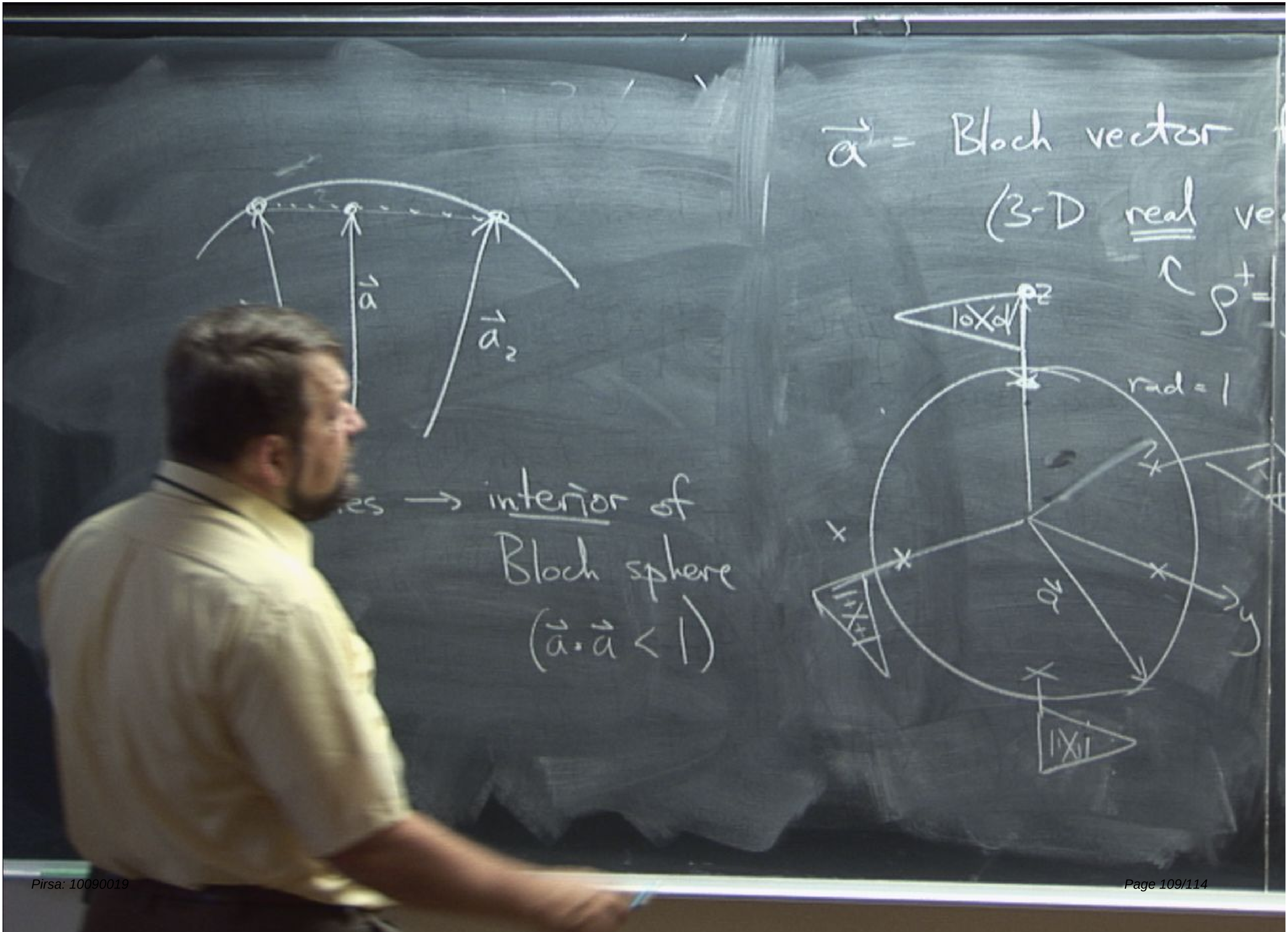
$$\rho = \frac{1}{2}\left(\mathbb{1} + Z\right)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$= |0\rangle\langle 0|$$


a = Bloch vector
(3-D real ve
rad = 1
|0⟩⟨0|
|1⟩⟨1|
|+⟩⟨+|
z
y
a1
a2

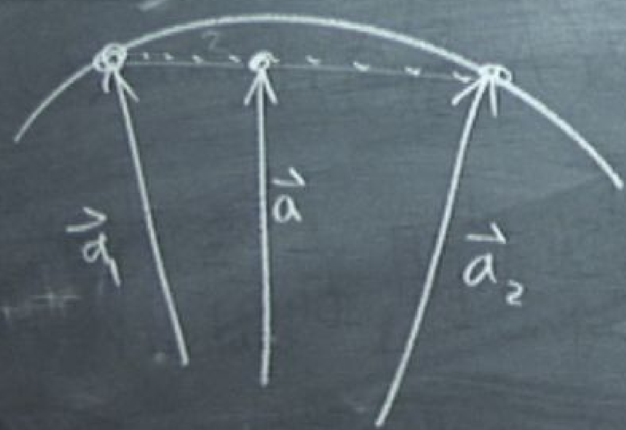

a = Bloch vector
(3-D real ve
rad = 1
|0⟩⟨0|
|1⟩⟨1|
z
y


a = Bloch vector
(3-D real ve
a
a₂
rad = 1
|0⟩⟨0|
|1⟩⟨1|
|+⟩⟨+|
z
y
a
interior of
Bloch sphere
(a·a < 1)

$\vec{a}$ = Bloch vector f

(3-D real ve

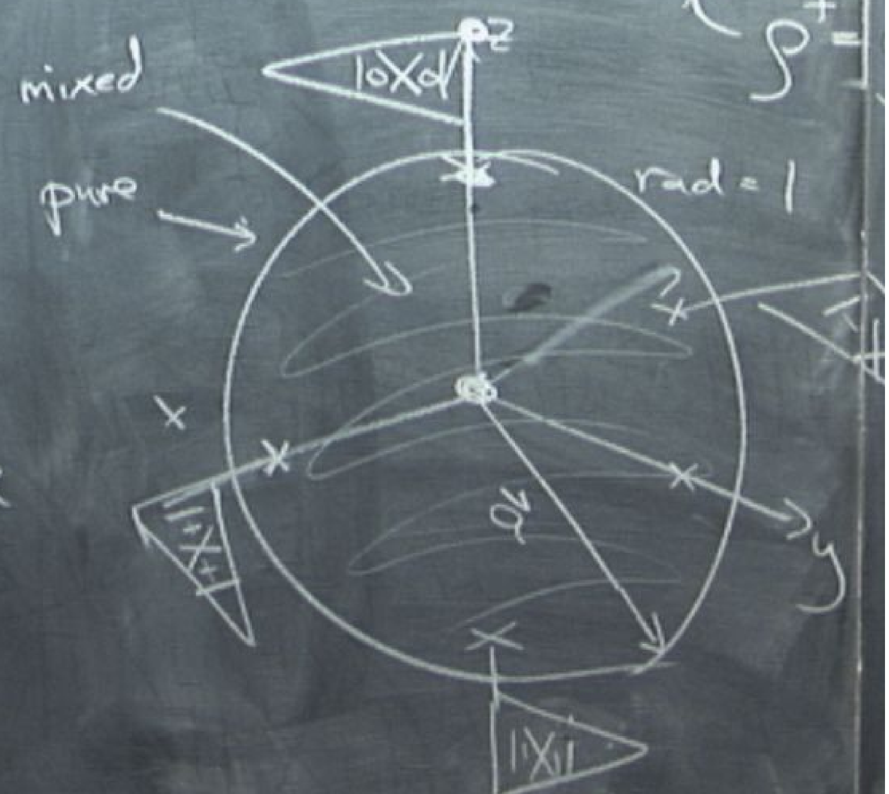
$\rho^\dagger =$

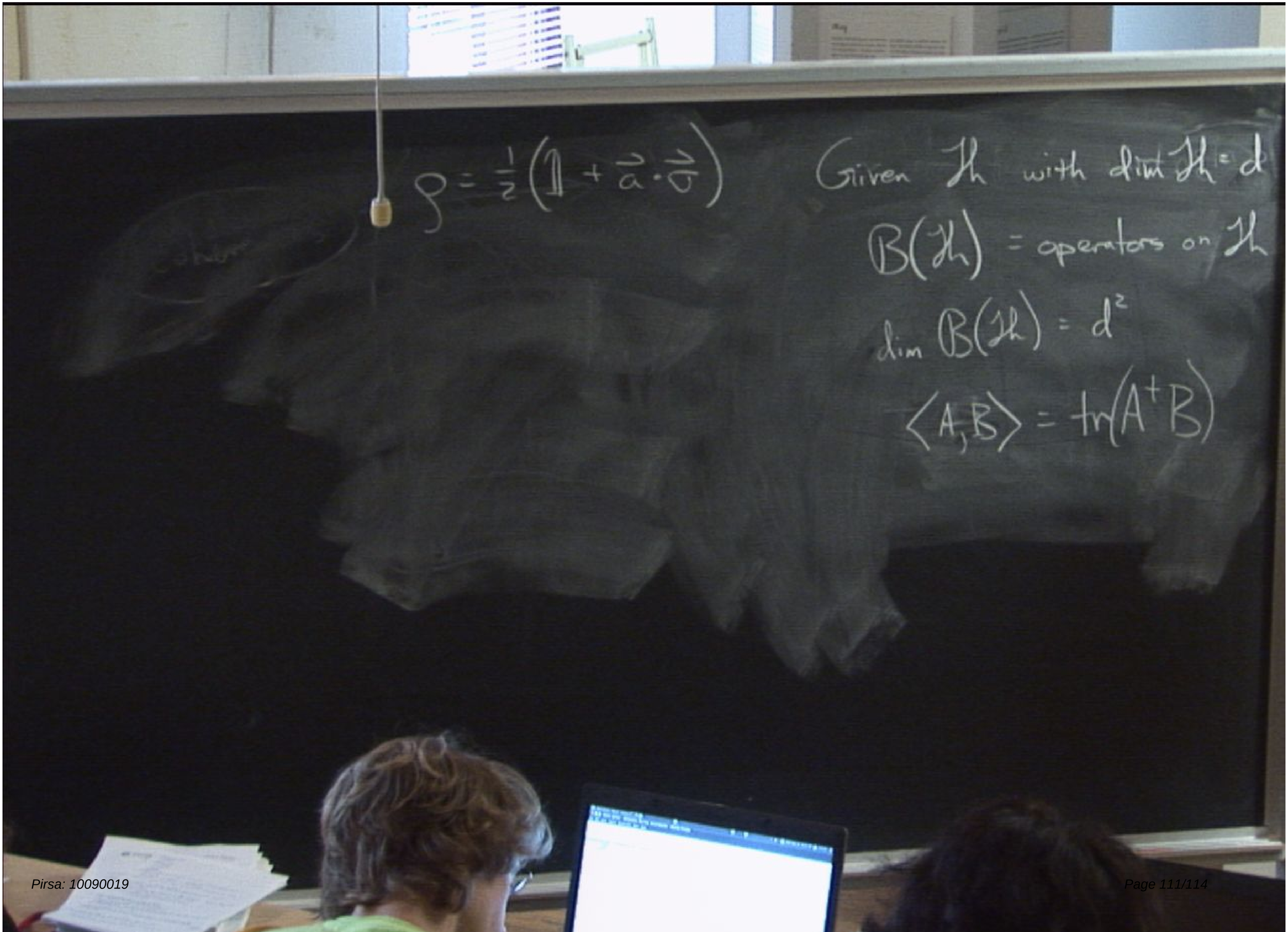


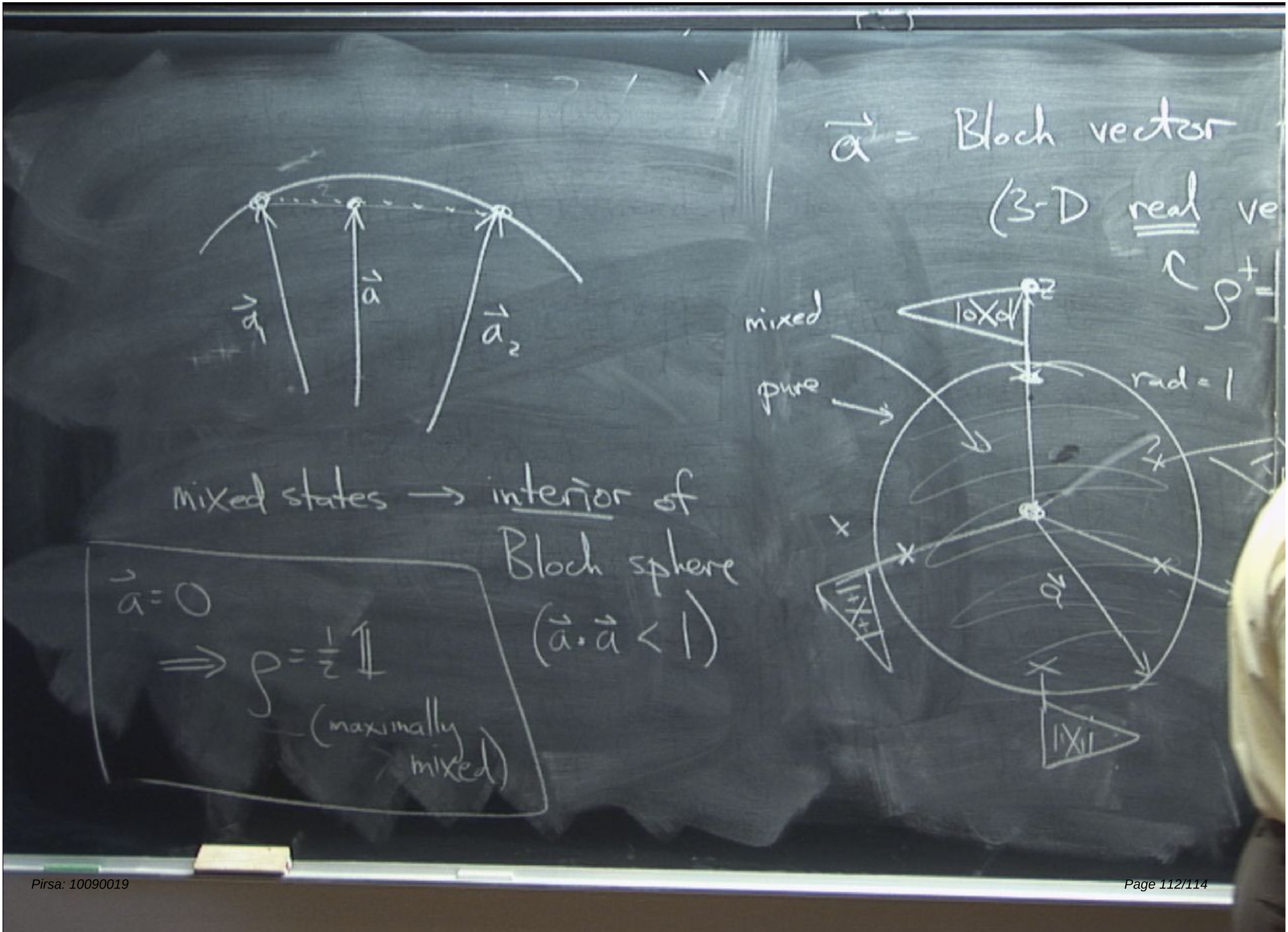
mixed states → interior of Bloch sphere ($\vec{a}\cdot\vec{a} < 1$)

$\vec{a} = 0$

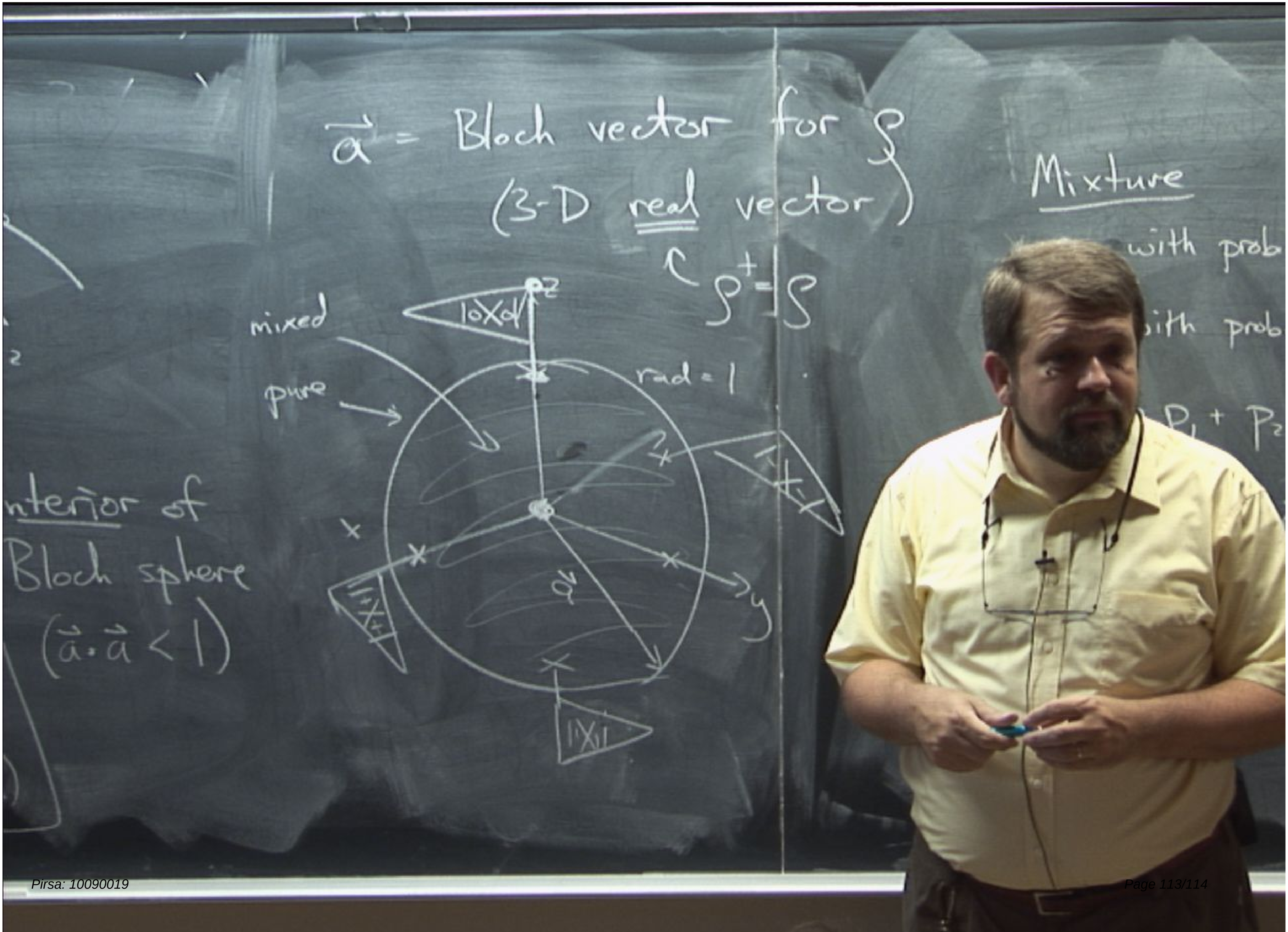
$\Longrightarrow$

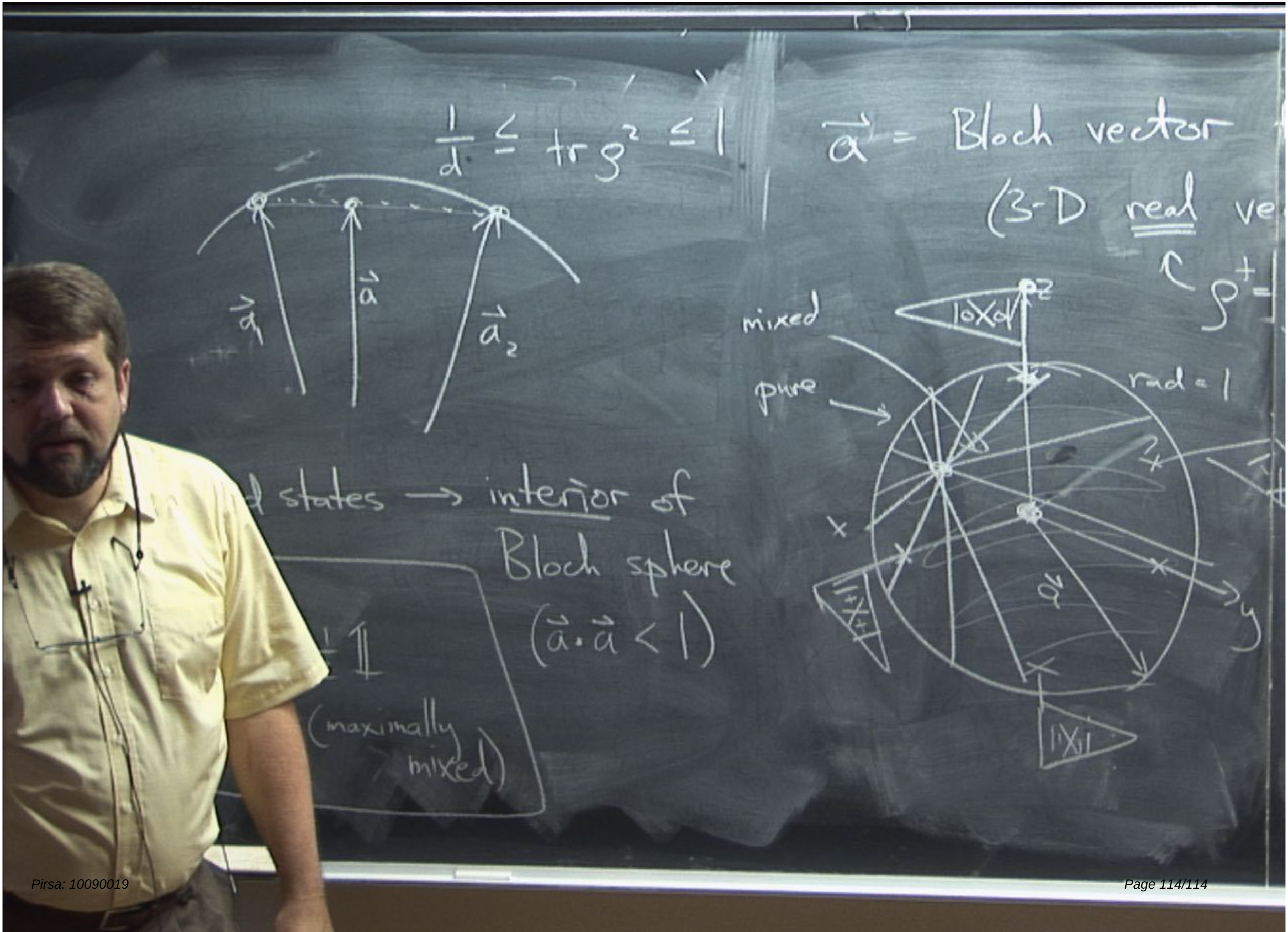



$\rho = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})$
Given $\mathcal{H}$ with $\dim \mathcal{H} = d$
$\mathbb{B}(\mathcal{H})$ = operators on $\mathcal{H}$
$\dim \mathbb{B}(\mathcal{H}) = d^2$
$\langle A, B\rangle = \mathrm{tr}(A^{\dagger}B)$



a⃗ = Bloch vector
(3-D real ve
ρ† =
a⃗
a⃗₁
a⃗₂
mixed
pure
|0⟩⟨0|
z
rad = 1
|1⟩⟨1|
mixed states → interior of Bloch sphere
(a⃗ · a⃗ < 1)
a⃗ = 0
⇒ ρ = ½ 𝟙
(maximally mixed)


a = Bloch vector for ρ
(3-D real vector)
ρ† = ρ
mixed
pure
|0⟩⟨0|
z
rad = 1
y
|1⟩⟨1|
nterior of
Bloch sphere
(a·a < 1)
Mixture
with prob
ith prob



1/d ≤ tr ρ² ≤ 1
a = Bloch vector
(3-D real ve
a₁
a
a₂
mixed
|0⟩⟨0|
z
pure
rad = 1
states → interior of
Bloch sphere
(a·a < 1)
ρ
y
|1⟩⟨1|
(maximally
mixed)