

Title: Space-time, Quantum Mechanics and Scattering Amplitudes - Lecture 2B

Date: Aug 24, 2010 03:00 PM

URL: <http://pirsa.org/10080019>

Abstract:

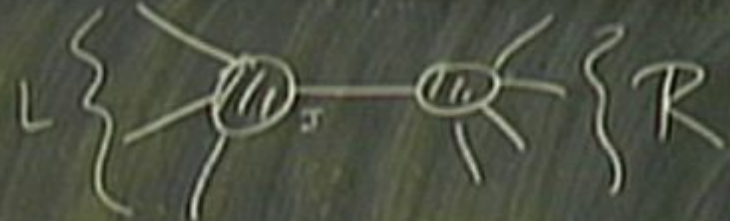
BCFW



BCFW



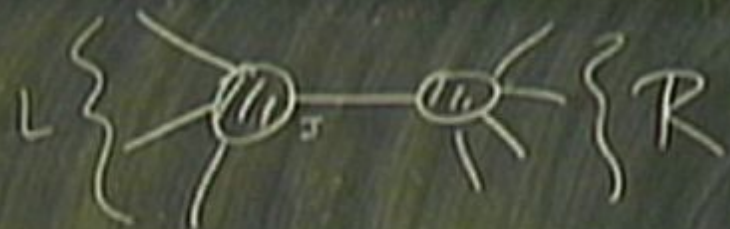
BCFW



$$P_L^2 \rightarrow 0$$

$$M \rightarrow \sum_I M_L(L, I) \frac{1}{P_L^2} M_R(R, -I)$$

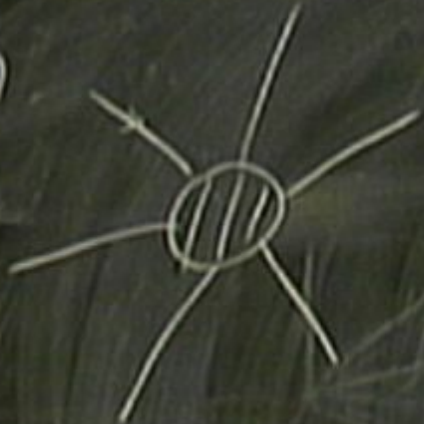
BCFW



$$P_L^2 \rightarrow 0$$

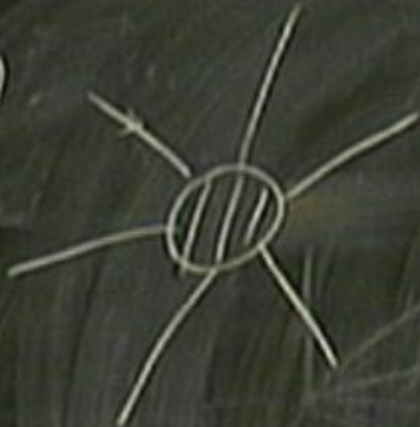
$$M \rightarrow \sum_I M_L(L, I) \frac{1}{P_L^2} M_R(R, -I)$$

$P_1(z)$



P_1

$P_1(z)$



$$P_1'(z) = P_1 + z^L$$

$$P_2'(z) = P_2 - z^L$$

$$P_1^L = P_2^L = 0$$

$$z^L = 0$$

$$z \cdot P_2 = 0$$

$$P_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & 1 & i & 0 & \dots & 0 \end{pmatrix}$$

$$\lambda_1(z) = \lambda_1 + z\lambda_2$$

$$\tilde{\lambda}_2(z) = \tilde{\lambda}_2 - z\tilde{\lambda}_1$$

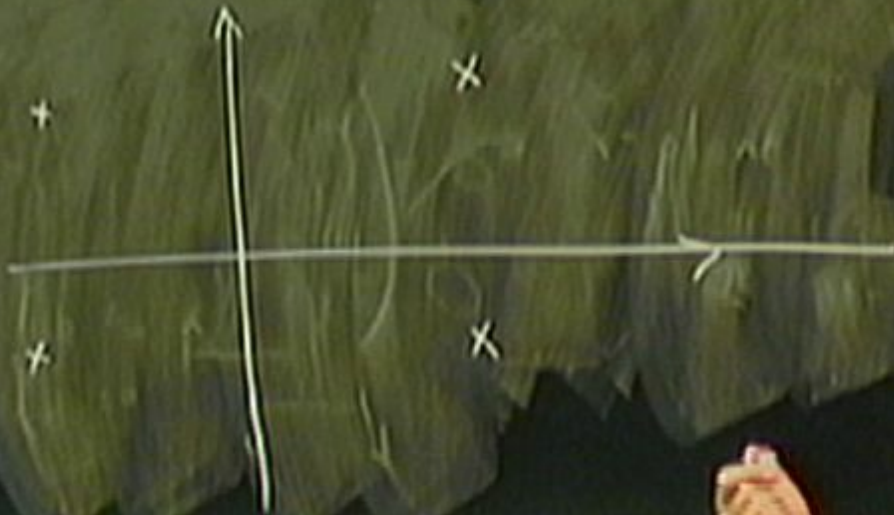
$$\lambda_1(z) = \lambda_1 + z\lambda_2$$

$$\tilde{\lambda}_2(z) = \tilde{\lambda}_2 - z\tilde{\lambda}_1$$

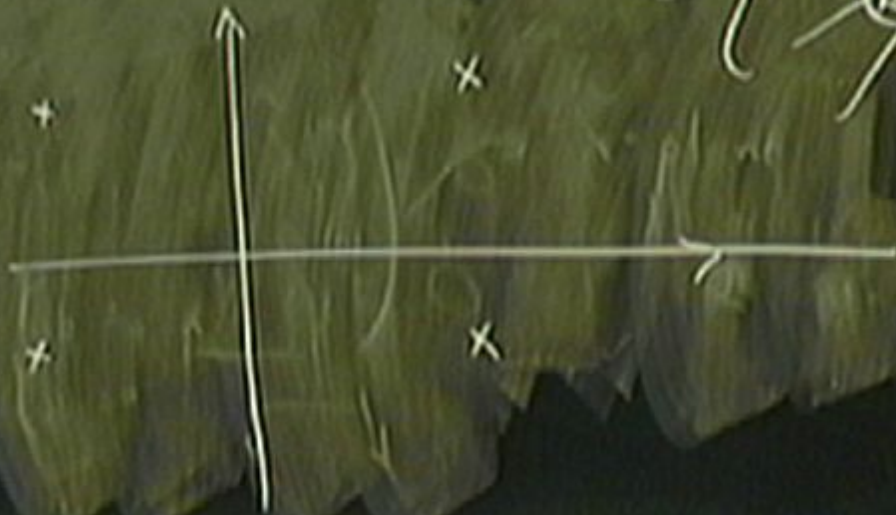
$$\chi_1^{(1)} \tilde{\lambda}_1^{(1)} + \chi_2^{(1)} \tilde{\lambda}_2^{(1)} = \lambda_1 + \lambda_2$$

$$M^{(\lambda, \lambda)} \rightarrow M(z, \lambda, \lambda)$$

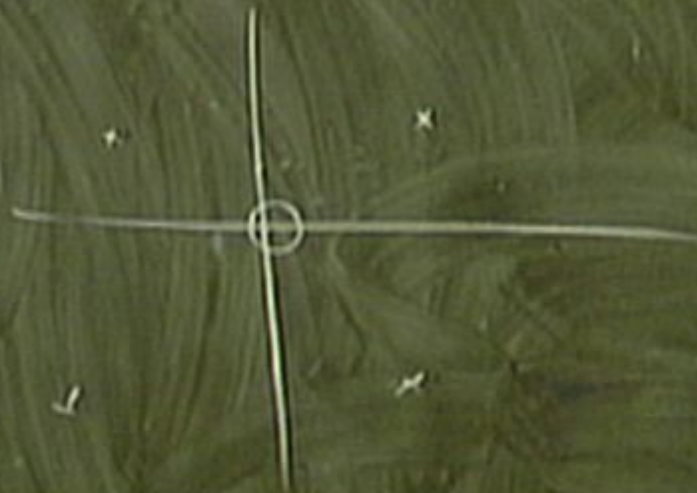
$$M(\lambda, \lambda) \rightarrow M(z, \lambda, \lambda)$$



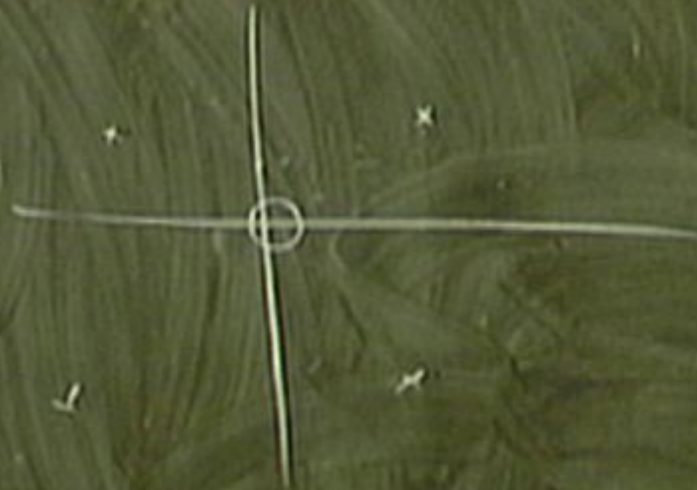
$$M(\lambda, \lambda) \rightarrow M(z, \lambda, \lambda)$$



$$M(0) = \int_{z=0} \frac{dz}{z} M(z)$$

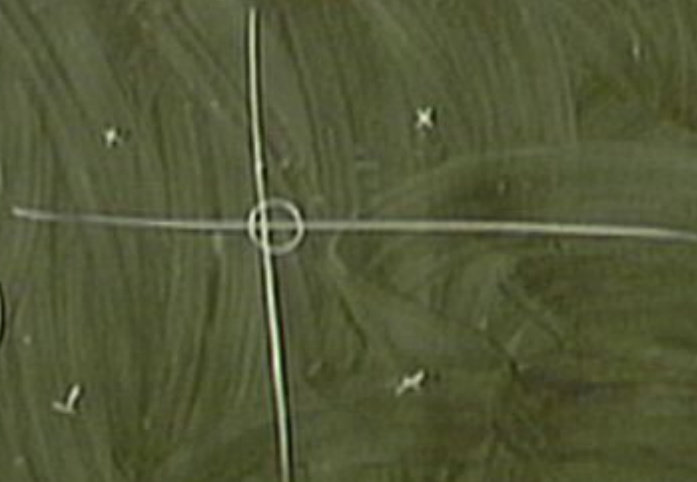


$$M(s) = \int_{z=0}^{\infty} \frac{dz}{z} M(z)$$



$$M(0) = \oint_{z=0} \frac{dz}{z} M(z)$$

$$= \sum \frac{1}{z_k} \operatorname{res} M(z_k) + \text{pole at } \infty$$



$$P_L(z) = P_L + z |z\rangle \langle \tilde{1} |$$

$$P_L^2(z) = P_L^2 + 2z |z\rangle \langle \tilde{1} | P_L |z\rangle$$

$$z_* = - \frac{P_L^2}{2 \langle \tilde{1} | P_L |z\rangle}$$

$H(\omega)$

$$= \sum_{M_L, R} M_L(z_*) \frac{1}{P_L^2}$$

$M_R(z_*)$



$$M(\lambda, \hat{\lambda}, \tilde{\eta})$$

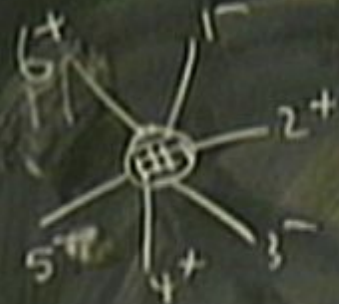
$$M(\lambda, \hat{\lambda}, \tilde{\gamma})$$

$$\lambda(z) = \lambda_1 + z\lambda_2$$

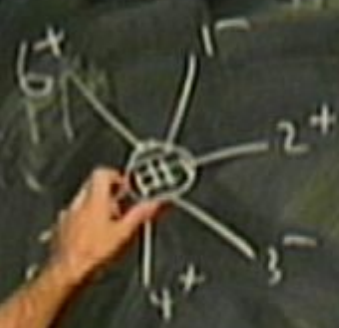
$$\hat{\lambda}(z) = \hat{\lambda}_2 - z\hat{\lambda}_1$$

$$\tilde{\gamma}(z) = \tilde{\gamma}_2 - z\tilde{\gamma}_1$$

$$\delta \left(\underbrace{\lambda_1 \tilde{\gamma}_1 + \lambda_2 \tilde{\gamma}_2 + \dots}_{\text{undefined}} \right)$$



1 2 3 4
5 6 7 8
9 10 11 12
13 14 15 16



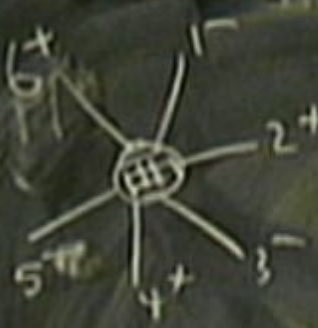
BCFW

$$(1 + g + g^4)$$

$$\left[\frac{[24]^+ \langle 15 \rangle^+}{[23][4] \langle 5 \rangle \langle 1 \rangle (1+g+g^4)^+} \right]$$

$$g: i \rightarrow i+1$$

$$\frac{1}{\langle 5 | (1+g+g^4) | 2 \rangle \langle 1 | (1+g+g^4) | 4 \rangle}$$



BCFW

$$(1 + g + g^2)$$

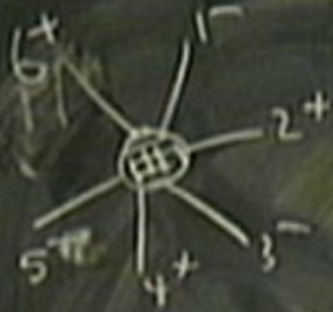
$$\left[\frac{[24]^+ \langle 15 \rangle^+}{[23][4] \langle 5 \rangle \langle 6 \rangle (p_1 + p_4 p_1)^+} \right]$$

$$g: i \rightarrow i+1$$

$$\times \frac{1}{\langle 5 | (6+1) | 2 \rangle \langle 1 | (6+5) | 4 \rangle}$$



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BCFW

$$(1 + g^2 + g^4) \left[\frac{[24]^4 \langle 15 \rangle^4}{[23][4] \langle 56 \rangle \langle 12 \rangle (p_1 + p_4 + p_1)^2} \right] \times \frac{1}{\langle 51(6+1)12 \rangle \langle 1(6+5)14 \rangle}$$

$g: i \rightarrow i+1$

$$\langle 51(6+1)12 \rangle = \langle 51(p_1 + p_1)12 \rangle + \langle 5112 \rangle \langle 12 \rangle$$



B(FW)

$$(1 + g + g^2) \left[\frac{[24]^+ \langle 15 \rangle^+}{[23][4] \langle 5 \rangle \langle 6 \rangle (p_1 + p_2 + p_3)^2} \right]$$

$g: i \rightarrow i+1$

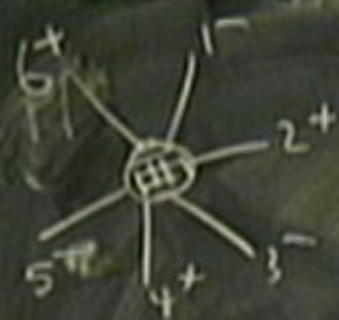
$$\times \frac{1}{\langle 51(6+1)12 \rangle \langle 1(6+5)14 \rangle}$$

$$\langle 51(6+1)12 \rangle = \langle 51(p_1 p_1)12 \rangle$$

$$= (\langle 56 \rangle \langle 62 \rangle + \langle 51 \rangle \langle 12 \rangle)$$

$$\frac{[23]}{[23] \langle 6 \rangle}$$

$$\underbrace{\hspace{2cm}}_{(p_2 + p_3)^2}$$



BCFW

$$(1 + g^2 + g^4)$$

$$\frac{[24]^4 \langle 15 \rangle^4}{[23][44] \langle 56 \rangle \langle 67 \rangle (p_1 + p_4)^2}$$

local / plus

$$\times \frac{1}{\langle 51(6+1)12 \rangle \langle 1(6+5)14 \rangle}$$

non-local

$P(\text{BCFW})$

$(1 + g^2 + g^4)$

$$\left[\frac{\sum_{41} (3+5) |14\rangle^4}{\sum_{34} \sum_{45} \langle 61 \rangle \langle 12 \rangle (p_1 + p_2)^2 \langle 41 + 215 \rangle \langle 21(1+6) | 5 \rangle} \times \frac{1}{\dots} \right]$$

$P(\text{BCFW})$ $(1 + g^2 + g^4)$

$$\left[\frac{\sum_{41} (3+5) 15^4}{\sum_{34} \sum_{45} \langle 41 \rangle \langle 12 \rangle (p_1 + p_2)^2 \langle 41 + 215 \rangle \langle 2(1+6) 15 \rangle} \times \frac{1}{\dots} \right]$$

$$M_{\text{BCFW}} = M_{P(\text{BCFW})} !$$

$$P(\text{B(FW)}) = \frac{\sum_{i=1}^4 (3+5)^{i-1}}{\sum_{i=1}^4 (3+5)^{i-1} (p_1+p_2)^{i-1} (1+2)^{i-1} (1+6)^{i-1}} \times \frac{1}{\dots}$$

$$M_{\text{B(FW)}} = M_{P(\text{B(FW)})} !$$

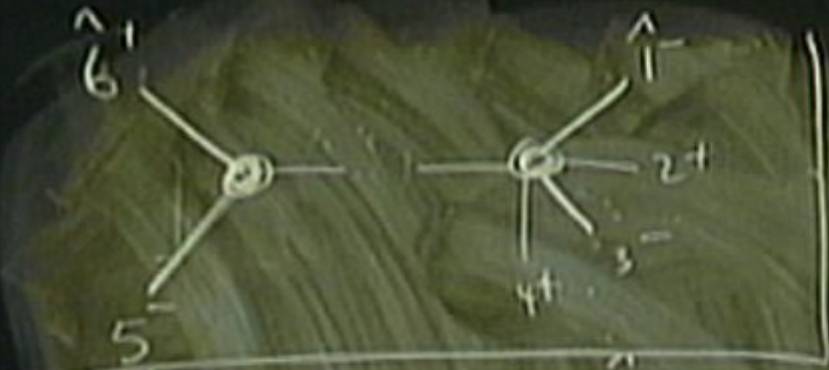
Remarkable
6-term identity.

$\hat{\lambda}_1$
6



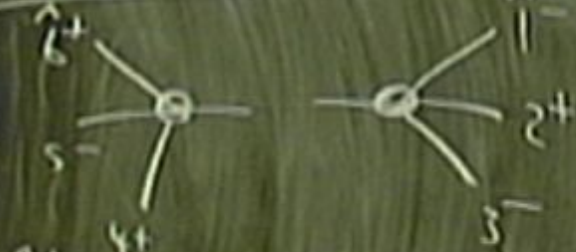
$$\lambda_6(z) = \lambda_6 + z\lambda_1$$

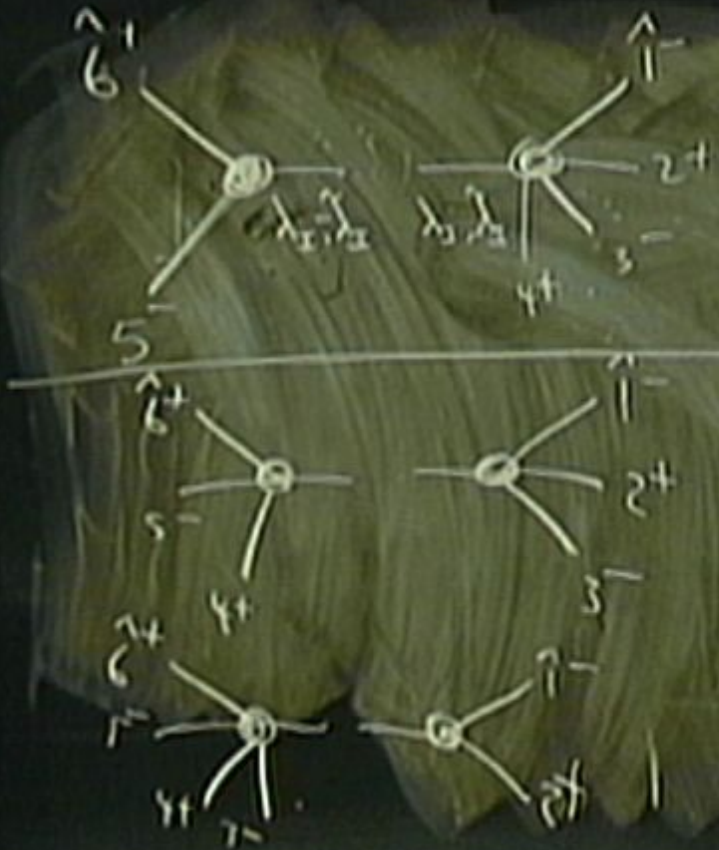
$$\hat{\lambda}_1(z) = \hat{\lambda}_1 - z\lambda_6$$



$$\lambda_6(z) = \lambda_6 + z\lambda_1$$

$$\lambda_1(z) = \lambda_1 - z\lambda_6$$





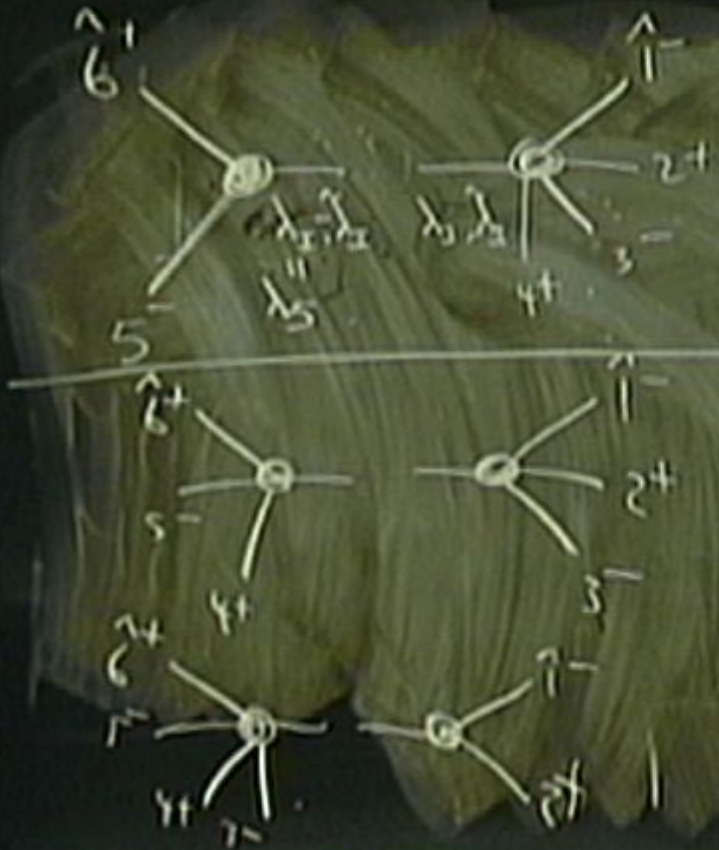
$$\lambda_6(z) = \lambda_6 + z\lambda_1$$

$$\lambda_1(z) = \lambda_1 - z\lambda_6$$

$$\langle 56(z) \rangle = 6$$

$$\langle 56 \rangle + z^x \langle 16 \rangle = 0$$





$$\hat{\lambda}_6(z) = \hat{\lambda}_6 + z \hat{\lambda}_1$$

$$\hat{\lambda}_1(z) = \hat{\lambda}_1 - z \hat{\lambda}_6$$

$$\langle 56(z) \rangle = 6$$

$$\langle 56 \rangle + z^* \langle 16 \rangle = 0$$

$$-\lambda_5 \hat{\lambda}_I + \lambda_5 \hat{\lambda}_S + \lambda_6^{(2x)} \hat{\lambda}_6 = 0$$

$$-\lambda_5 \tilde{\lambda}_I + \lambda_5 \tilde{\lambda}_S + \lambda_6^{(2)} \tilde{\lambda}_6 = 0$$

$$-\langle 56 \rangle \tilde{\lambda}_I = \langle 56 \rangle \tilde{\lambda}_S + z_* \langle 16 \rangle$$

$$-\lambda_5 \tilde{\lambda}_I + \lambda_5 \tilde{\lambda}_5 + \lambda_6^{(z_*)} \tilde{\lambda}_6 = 0$$

$$-\langle 56 \rangle \tilde{\lambda}_I = \langle 56 \rangle \tilde{\lambda}_5 + z_* \langle 16 \rangle \tilde{\lambda}_6$$

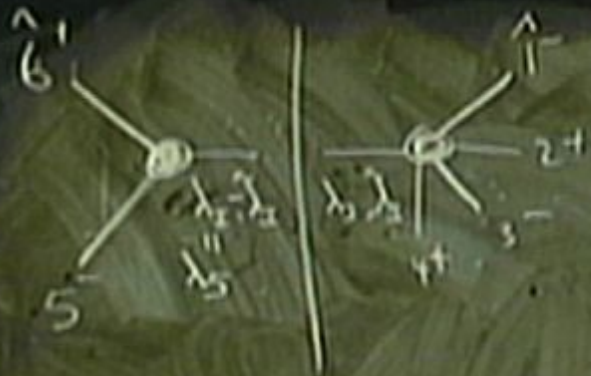
$$= \langle 56 \rangle \tilde{\lambda}_5 - \langle 56 \rangle \tilde{\lambda}_6$$

$$-\lambda_5 \hat{\lambda}_I + \lambda_5 \hat{\lambda}_S + \lambda_6^{(z_x)} \hat{\lambda}_6 = 0$$

$$-\langle 56 \rangle \hat{\lambda}_I = \langle 56 \rangle \hat{\lambda}_S + z_x \langle 16 \rangle \hat{\lambda}_6$$

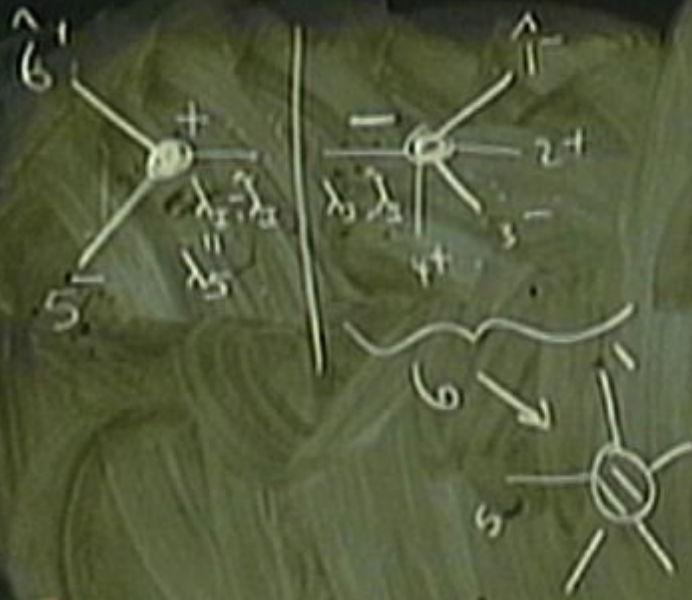
$$= \langle 56 \rangle \hat{\lambda}_S + \frac{\langle 56 \rangle \langle 16 \rangle}{\langle 51 \rangle} \hat{\lambda}_6$$

$$\hat{\lambda}_I = \hat{\lambda}_S + \frac{\langle 61 \rangle \hat{\lambda}_6}{\langle 51 \rangle} = \frac{(P_5 + P_6) |1\rangle}{\langle 51 \rangle}$$



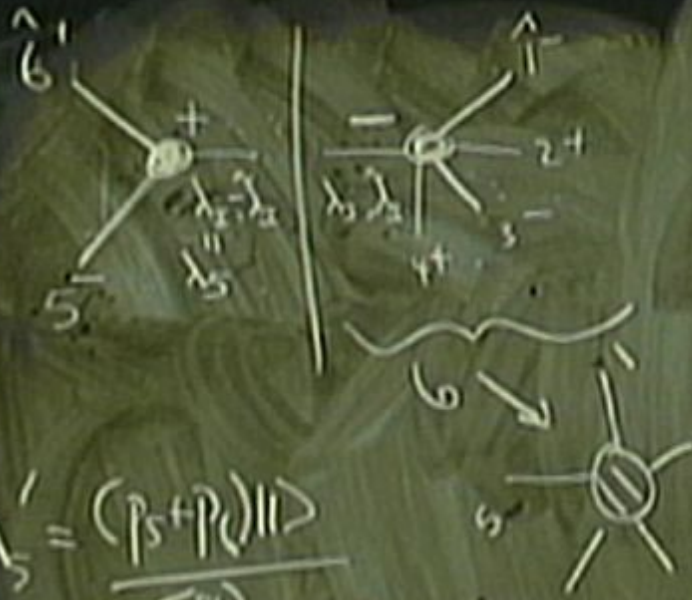
$$\hat{\lambda}_1 = \frac{(P_1 + P_2) | \psi \rangle}{\langle \psi |}$$

$$\langle \psi | + \langle \psi | = 0$$



$$\hat{\lambda}_1 = \frac{(P_1 + P_2)|5\rangle}{\langle 15 \rangle}$$

$$\langle 56 \rangle + z^* \langle 51 \rangle = 0$$



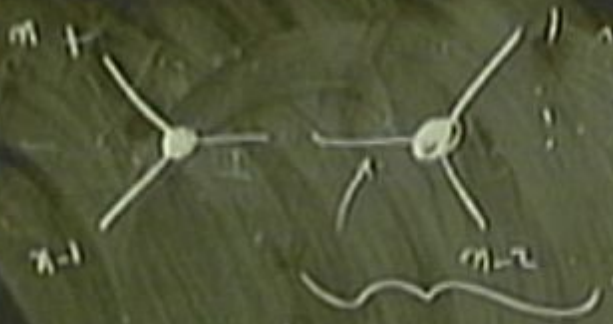
$$\hat{\lambda}_1 = \frac{(P_1 + P_2) | 5 \rangle}{\langle 15 \rangle}$$

$$\hat{\lambda}'_5 = \frac{(P_5 + P_6) | 1 \rangle}{\langle 51 \rangle}$$

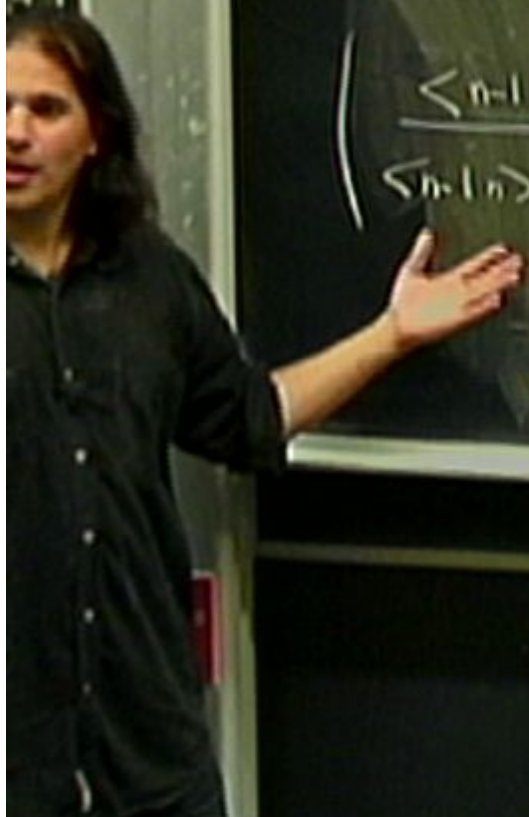
$$\hat{\lambda} = \frac{(P_5 + P_1) | 5 \rangle}{\langle 15 \rangle}$$

$$\langle 56 \rangle + 3 \quad 0$$





$$\left(\frac{\langle n-1 \ 1 \rangle}{\langle n-1 \ n \rangle \langle n \ 1 \rangle} \right) \times M_R \left[\sum_{i=1}^n \frac{(P_i + P_{i+1}) \cdot D}{(s_{i-1,i})} , \sum_{i=1}^n \frac{(P_i + P_{i+1}) \cdot D}{(s_{i,i+1})} , \dots \right]$$





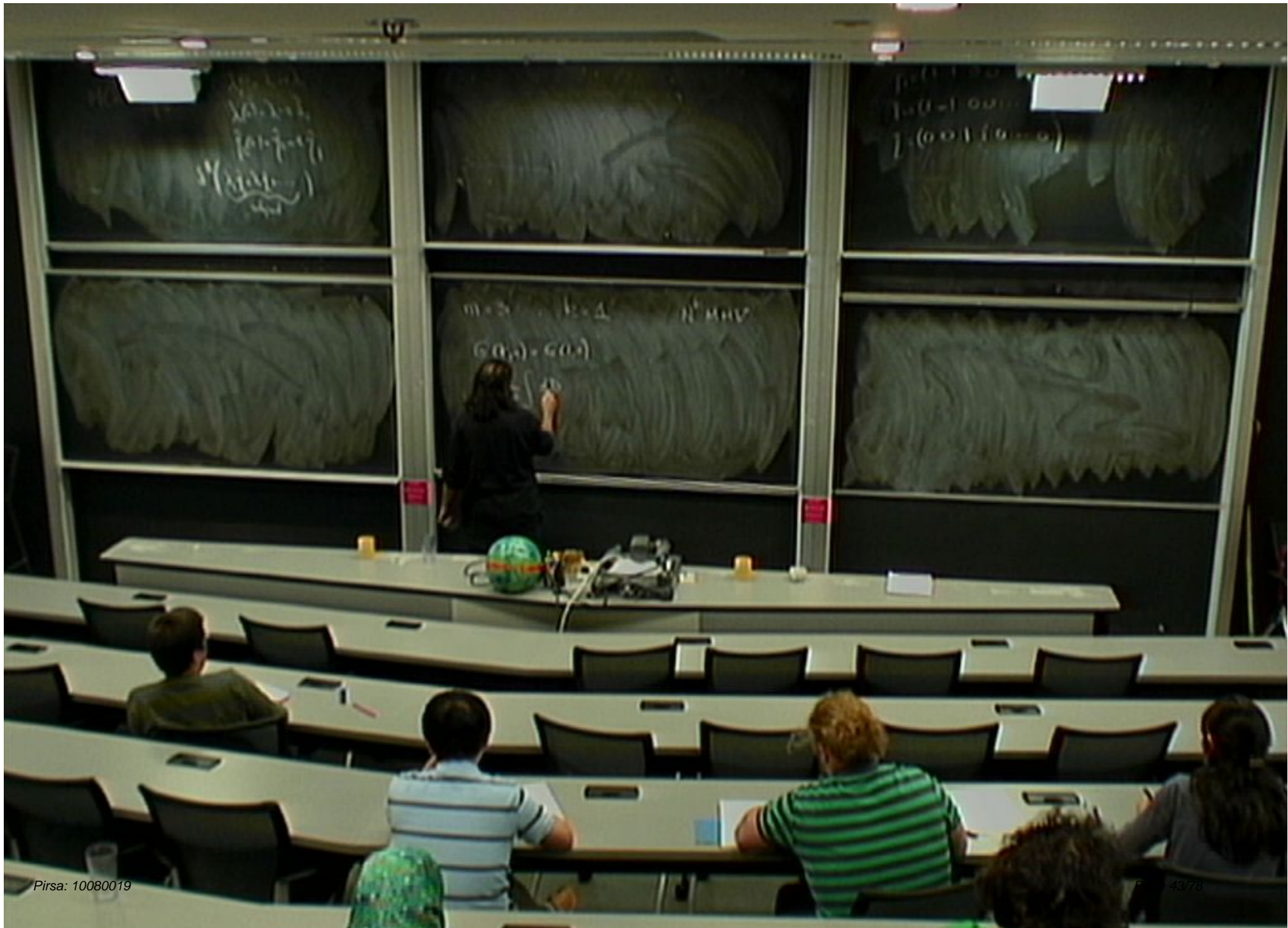
$$\lambda = \frac{(p_1 + p_2) | 5 \rangle}{\langle 15 \rangle}$$

$$\frac{\langle 51 \rangle}{\langle 56 \rangle \langle 61 \rangle} \times \frac{[24]^4 \langle 15 \rangle^3}{[23][34] \underbrace{\langle 51(p_2 + p_1) | 2 \rangle}_{[12]} \langle 1 | [6+5] | 4 \rangle \frac{1}{(p_1 + p_2 + p_3)^2}}$$



$$\lambda = \frac{(p_1 + p_2) / 5}{5}$$

$$\frac{\langle 51 \rangle}{\langle 56 \rangle \langle 61 \rangle} \times \frac{[24]^4 \langle 15 \rangle^3}{\underbrace{[23][34]}_{[12]} \underbrace{\langle 51(p_2 + p_1) / 2 \rangle}_{\langle 11 \ 6+5 \ 14 \rangle} \frac{1}{(p_1 + p_2 + p_3)^2}}$$

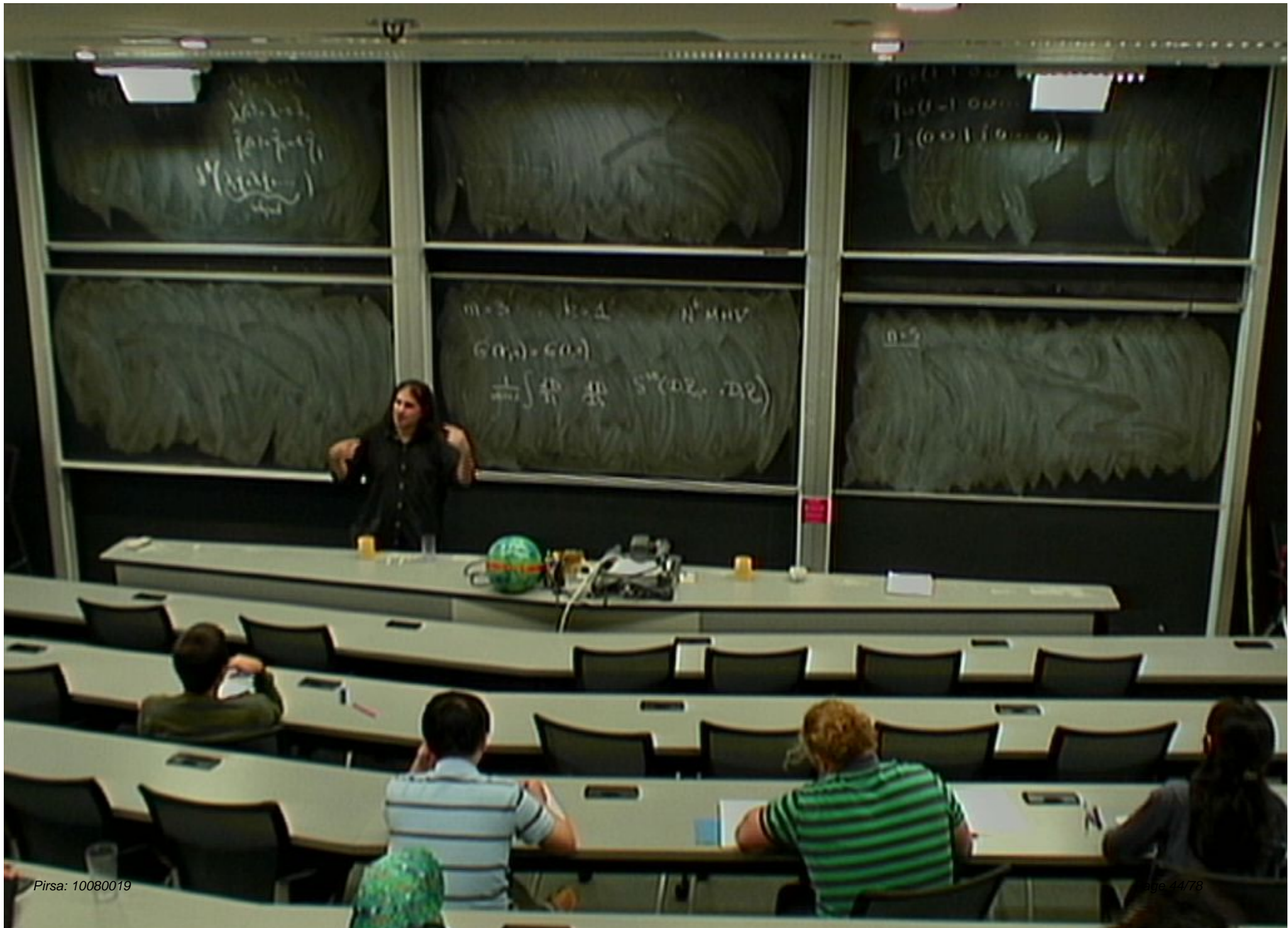


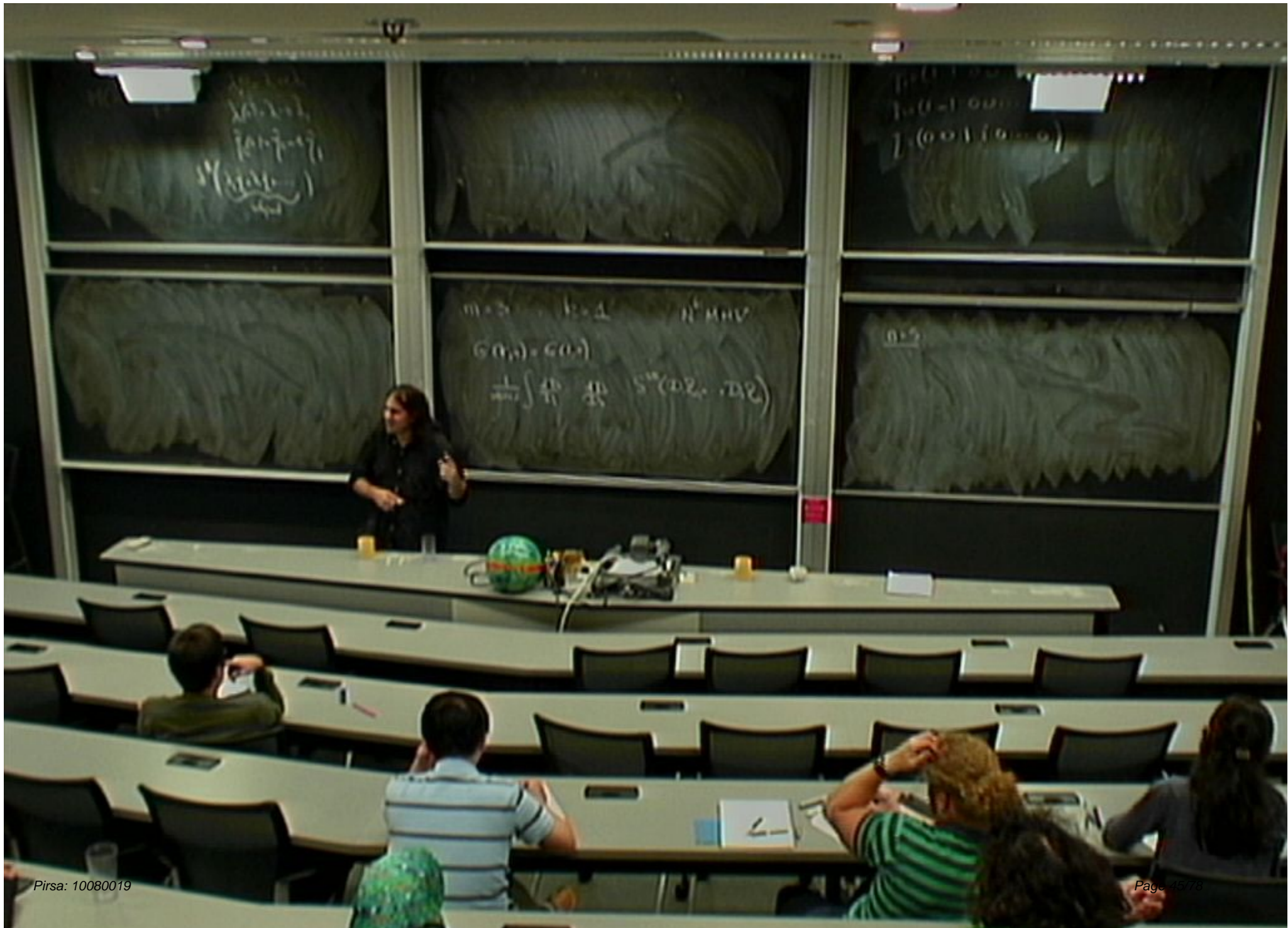
$$\lambda_1 = 2, \lambda_2 = 1$$
$$p_1(x) = (x-2)^2$$
$$p_2(x) = (x-1)$$

$$p_1(x) = (x-2)^2$$
$$p_2(x) = (x-1)$$
$$p_3(x) = (x-1)(x-2)$$

$$m=3, k=1, n^{\text{th}} \text{ row}$$
$$C(3,0) = C(3,3)$$







$$m=3, \quad k=1 \quad N^k \text{ MHV}$$

$$G(k, n) = G(1, n)$$

$$\frac{1}{\text{vol}GL(3)} \int \frac{dD_1}{D_1} \dots \frac{dD_n}{D_n} \int^{411} (\mathbb{D}_1 Z_1 + \dots + \mathbb{D}_n Z_n)$$

$$m=3, \quad k=1 \quad N^k \text{ MHV}$$

$$G(k, n) = G(1, n)$$

$$\frac{1}{\text{vol}GL(3)} \int \frac{dD_1}{D_1} \dots \frac{dD_n}{D_n} \int^{4|4} (\mathbb{D}_1 Z_1 + \dots + \mathbb{D}_n Z_n)$$

$$\underline{n=5}$$

$$\int \frac{dD_1}{D_1} \dots \frac{dD_4}{D_4}$$

$$\delta^{4|4} (D_1 z_1 + \dots + D_4 z_4 + z_5)$$

$$D_5 = 1$$

$$\underline{n=5}$$

$$D_5 = 1$$

$$\int \frac{dD_1}{D_1} \dots \frac{dD_4}{D_4}$$

$$\delta^{4|4}(D_1 z_1 + \dots + D_4 z_4 + z_5)$$

$$-z_5^I = D_1 z_1^I + D_2 z_2^I + \dots + D_4 z_4^I$$

$$\underline{n=5}$$

$$D_5 = 1$$

$$\int \frac{dD_1}{D_1} \dots \frac{dD_4}{D_4}$$

$$\delta^{4|4}(D_1 z_1 + \dots + D_4 z_4 + z_5)$$

$$-z_5^I = D_1 z_1^I + D_2 z_2^I + \dots + D_4 z_4^I$$

$$\langle 5234 \rangle = D_1 \langle 1234 \rangle$$

$$D_1 = \frac{\langle 2345 \rangle}{\langle 1234 \rangle}, \quad D_2 = \dots$$

$$\underline{n=5}$$

$$D_5 = 1$$

$$\int \frac{dD_1}{D_1} \dots \frac{dD_4}{D_4}$$

$$\delta^{4|4}(D_1 z_1 + \dots + D_4 z_4 + z_5)$$

$$-z_5^I = D_1 z_1^I + D_2 z_2^I + \dots + D_4 z_4^I$$

$$-(5234) = D_1 (1234)$$

$$D_1 = \frac{(2345)}{(1234)}$$

$$D_2 = \dots$$

$$\underline{n=5}$$

$$\int \frac{dD_1}{D_1} \dots \frac{dD_4}{D_4}$$

$$D_5 = 1$$
$$\delta^{4|4}(D_1 z_1 + \dots + D_4 z_4 + z_5)$$

$$-z_5^I = D_1 z_1^I + D_2 z_2^I + \dots + D_4 z_4^I$$

$$- \langle 5234 \rangle = D_1 \langle 1234 \rangle$$

$$D_1 = \frac{\langle 2345 \rangle}{\langle 1234 \rangle}, \quad D_2 = \dots$$

$$\frac{\langle 1234 \rangle^4}{\langle 2345 \rangle \langle 1345 \rangle \langle 1245 \rangle \langle 1235 \rangle \langle 1234 \rangle} = \frac{1}{\int^{\text{ol4}} \left[\eta_5 + \frac{\langle 2345 \rangle \eta_1}{\langle 1247 \rangle} + \dots \right]}$$

$$= \frac{\int^{\text{ol4}} \left[\langle 1234 \rangle \eta_5 + \text{cyclic} \right]}{\langle 1234 \rangle \langle 2345 \rangle \langle 1431 \rangle \langle 4512 \rangle \langle 5123 \rangle}$$

$$\langle 1234 \rangle \langle 2345 \rangle \langle 1431 \rangle \langle 4512 \rangle \langle 5123 \rangle$$

$$\frac{\langle 1234 \rangle^4}{\langle 2345 \rangle \langle 1345 \rangle \langle 1245 \rangle \langle 1235 \rangle \langle 1234 \rangle}$$

$$\int_0^1 \left[1_5 + \frac{\langle 2345 \rangle}{\langle 1247 \rangle} 1_1 + \dots \right]$$

$$= \int_0^1 \left[\langle 1234 \rangle 1_5 + \text{cyclic} \right]$$

$$\langle 1234 \rangle \langle 2145 \rangle \langle 1431 \rangle \langle 4512 \rangle \langle 5123 \rangle$$

$$\int \frac{dD_1}{D_1}$$

$$\frac{dD_n}{D_n}$$

$$\delta^{4|4} (D_1 Z_1 + \dots + D_n Z_n)$$

$$K(n-k-4)$$

$$(n-5)$$

$$(m-2) (n-m-2)$$

$$\int \frac{dD_1}{D_1}$$

$$\frac{dD_6}{D_6}$$

$$S^{414} (D_1 Z_1 + \dots + D_6 Z_6)$$

$$\int \frac{dD_1}{D_1} \int \frac{dD_3}{D_3} \frac{dD_4}{D_4} \delta^{4|4} (D_1 z_1 + \dots + D_6 z_6)$$

\downarrow

$D_2 = 0$

$$\int \frac{dD_1}{D_1} \frac{dD_2}{D_2} \frac{dD_3}{D_3} \frac{dD_4}{D_4} \frac{dD_5}{D_5} \frac{dD_6}{D_6} \delta^{4|4} (D_1 z_1 + \dots + D_6 z_6)$$

$$\int \frac{dD_1}{D_1} \int \frac{dD_3}{D_3} \left[\int \frac{dD_4}{D_4} \delta^{4|4} (D_1 z_1 + \dots + D_6 z_6) \right]$$

$D_3 = 0$

$$\int \frac{dD_1}{D_1} \frac{dD_2}{D_2} \frac{dD_3}{D_3} \frac{dD_4}{D_4} \frac{dD_5}{D_5} \frac{dD_6}{D_6} \delta^{4|4} (D_1 z_1 + \dots + D_6 z_6)$$

$$[abcde] = \frac{\delta^{d4} [\uparrow_a \langle bcde \rangle + \text{cycle}]}{\langle abcd \rangle \langle bcde \rangle \langle cabc \rangle}$$

$$[D_3=0] \\ (3) = [12456]$$

$$(5) = [12346]$$

$$[abcde] = \frac{\delta^{d4} [\uparrow_r \langle bcde \rangle + \text{cyclic}]}{\langle abcd \rangle \langle bcde \rangle \langle eabc \rangle}$$

$$[D_3=0] \\ (3) = [12456]$$

$$(5) = [12346]$$

$$(6) [12345]$$



$$[abcde] = \frac{\delta^{du} [n]_{ir} \langle bcde \rangle + \text{cyclic}}{\langle abcd \rangle \langle bcde \rangle \langle ca bc \rangle}$$

$$[D_3=0] \\ (3) = [12456]$$

$$(5) = [12346]$$

$$(6) \leftrightarrow [12345]$$

$$B(FW) = (6) + (4) + (2)$$

$$P(B(FW)) = (5) - (3) - (1)$$

$$[abcde] = \frac{\delta^{du} [\uparrow_{ir} \langle bcde \rangle + \text{cyclic}]}{\langle abcd \rangle \langle bcde \rangle \langle eabc \rangle}$$

$[D_3=0]$

$$(3) = [12456]$$

$$(5) = [12346]$$

$$(6) \leftrightarrow [12345]$$

$$B(FW) = (6) + (4) + (2)$$

$$P(B(FW)) = (5) - (3) - (1)$$

$$[abcde] = \frac{\delta^{du} [\uparrow_r \langle bcde \rangle + \text{cyclic}]}{\langle abcd \rangle \langle bcde \rangle \langle caeb \rangle}$$

$$[D_1=0] \\ (3) = [12456]$$

$$(5) = [12346]$$

$$(6) \leftrightarrow [12345]$$

$$B(FW) = (6) + (4) + (2)$$

$$P(B(FW)) = (5) - (3) - (1)$$

$$(1)(2) + (1)(1)$$

$$(1)(2) + (1)(4) + (1)(6)$$

$$\dots + (3)(4) + (3)(6)$$

$$+ (5)(4)$$

$$\begin{aligned} & \left| \begin{aligned} & (2)(3) + (2)(5) + (2)(7) \\ & + (4)(5) + (4)(7) \\ & + (6)(7) \end{aligned} \right. \end{aligned}$$

$=$ Residue thm

$$\sum_{\text{increasing}} \binom{n}{e} \binom{n}{a} \binom{n}{e} \binom{n}{o}$$

$(n-5)$



$$\sum_{\text{increasing}} \binom{n}{e} \binom{n}{a} \binom{n}{e} \binom{n}{o} = (-1)^{n-5} \sum \underbrace{\binom{n}{a} \binom{n}{e} \binom{n}{o}}_{n-5}$$

$$\int \frac{d\tau_1 d\tau_2}{D_1(\tau_1, \tau_2) D_2(\tau_1, \tau_2)}$$



$$\int \frac{d\tau_1 d\tau_2}{D_1(\tau_1, \tau_2) D_2(\tau_1, \tau_2)}$$

$$\binom{7}{2} = 21$$

$$\underbrace{(D_1, D_2)}_{P_1} \underbrace{(D_1, D_4, D_5, D_6, D_7)}_{P_2}$$

$$\int \frac{d\tau_1 d\tau_2}{D_1(\tau_1, \tau_2) D_2(\tau_1, \tau_2)}$$

$$\binom{7}{2} = 21$$

$$\underbrace{(D_1, D_2)}_{P_1} \quad \underbrace{(D_1, D_4, D_5, D_6, D_7)}_{P_2}$$



$$\int \frac{d\tau_1 d\tau_2}{D_1(\tau_1, \tau_2) D_7(\tau_1, \tau_2)}$$

$$\frac{1}{\underbrace{(D_1, D_2)}_{P_1} \cdot \underbrace{(D_3, D_4, D_5, D_6, D_7)}_{P_2}}$$

$$\binom{7}{2} = 21$$

$$\begin{aligned} & (1)(3) + (1)(4) + (1)(5) + (1)(6) + (1)(7) \\ & + (2)(4) + (2)(5) + (2)(6) + (2)(7) \\ & + (3)(7) \end{aligned}$$

$$= 0$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

$$\sum_i (1)(i) = 0$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

For
all n

$$\sum_i (i)(i) = 0$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

$$\sum_i (j)(i) = 0$$

$$(2)(1) + (2)(3) + (2)(4) + (2)(5) + (2)(6) + (2)(7) = 0$$

$$(4)(1) + (4)(2) + (4)(3) + (4)(5) + (4)(6) + (4)(7) = 0$$

$$(6)(1) + (6)(2) + (6)(3) + (6)(4) + (6)(5) + (6)(7) = 0$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

$$\sum_i (j)(i) = 0$$

$$\begin{aligned} & (2)(1) + (2)(3) + (2)(4) + (2)(5) + (2)(6) + (2)(7) = 0 \\ & + (4)(1) + (4)(2) + (4)(3) + (4)(5) + (4)(6) + (4)(7) = 0 \\ & (6)(1) + (6)(2) + (6)(3) + (6)(4) + (6)(5) + (6)(7) = 0 \end{aligned}$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

$$\sum_i (j)(i) = 0$$

$$\begin{aligned} & [(1)(2) + (1)(4) + (1)(6) \\ & \quad + (3)(4) + (3)(6) \\ & \quad + (5)(6)] \\ & + [(2)(2) + (2)(5) + (2)(7) \\ & \quad + (4)(5) + (4)(7) + (6)(7)] = 0 \end{aligned}$$

$$\begin{aligned} & [(2)(1)] + (2)(3) + (2)(4) + (2)(5) + (2)(6) + (2)(7) = 0 \\ & + [(4)(1)] + (4)(2) + [(4)(3)] + (4)(5) + (4)(6) + (4)(7) = 0 \\ & + [(6)(1)] + (6)(2) + [(6)(3)] + (6)(4) + [(6)(5)] + (6)(7) = 0 \end{aligned}$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(7) = 0$$

$$\sum_i (j)(i) = 0$$

$$\begin{aligned} & [(1)(2) + (1)(4) + (1)(6) \\ & \quad + (3)(4) + (3)(6) \\ & \quad + (5)(6)] \\ & + [(2)(2) + (2)(5) + (2)(7) \\ & \quad + (4)(5) + (4)(7) + (6)(7)] = 0 \end{aligned}$$

$$\begin{aligned} & [(2)(1)] + (2)(3) + (2)(4) + (2)(5) + (2)(6) + (2)(7) = 0 \\ & + [(4)(1)] + (4)(2) + [(4)(3)] + (4)(5) + (4)(6) + (4)(7) = 0 \\ & + [(6)(1)] + (6)(2) + [(6)(3)] + (6)(4) + [(6)(5)] + (6)(7) = 0 \end{aligned}$$

$$(1)(2) + (1)(3) + (1)(4) + \dots + (1)(n) = 0$$

$$\sum_{i=1}^n (i)(i) = 0$$

$$\begin{aligned} & [1(2) + 1(4) + 1(6) \\ & + 2(4) + 2(6) \\ & + 3(6) \\ & + \dots + (n-2)(n) \\ & + (n-1)(n) + (n)(n) \end{aligned} = 0$$

$$\begin{aligned} & [2(1)] + (2)(2) + (2)(4) + (2)(6) + (2)(8) + (2)(10) + \dots \\ & + [4(1)] + (4)(2) + [4(4)] + (4)(6) + (4)(8) + (4)(10) + \dots \\ & + [6(1)] + (6)(2) + [6(4)] + (6)(6) + [6(8)] + (6)(10) + \dots \end{aligned}$$