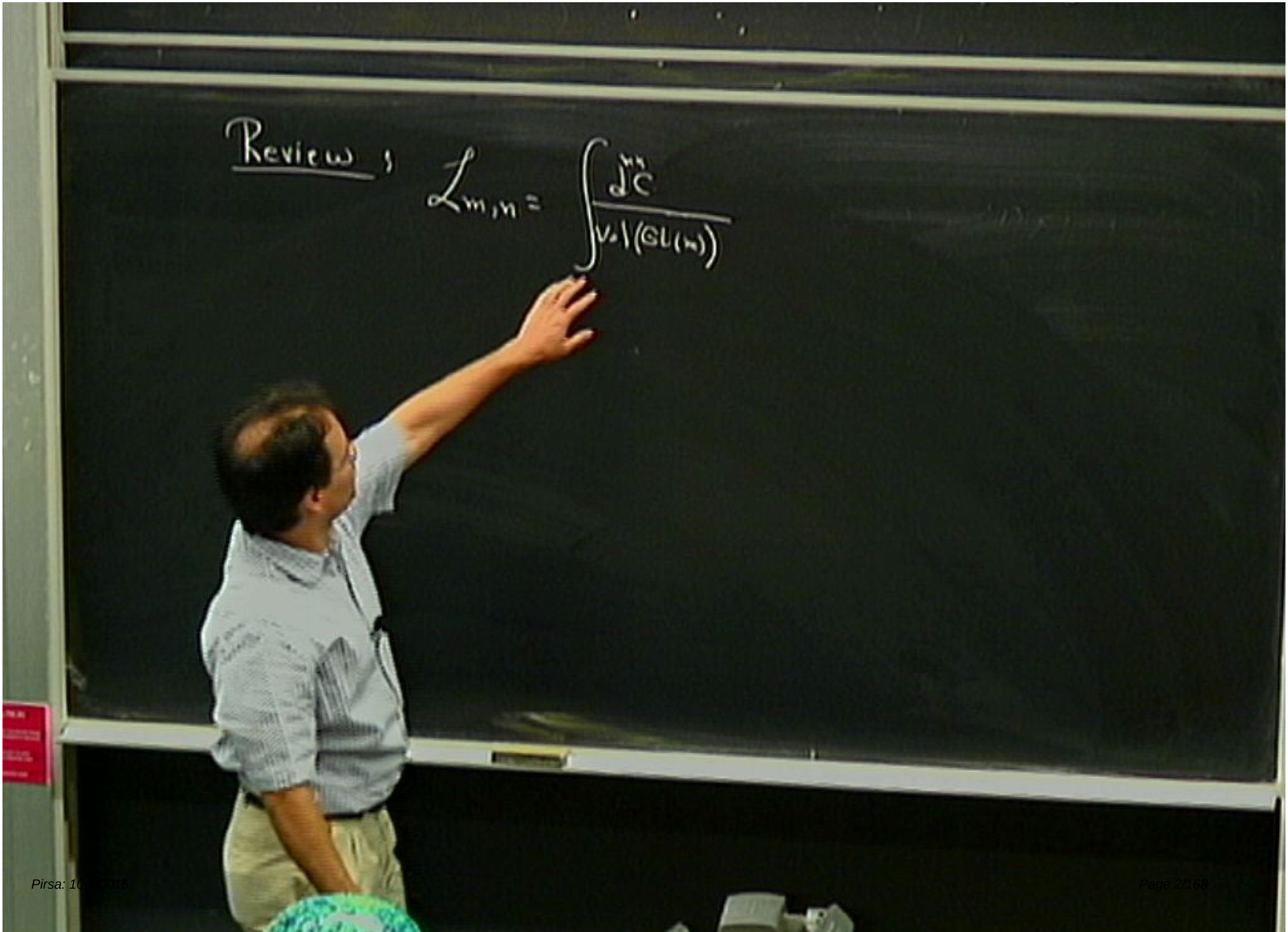
Title: Space-time, Quantum Mechanics and Scattering Amplitudes - 3A

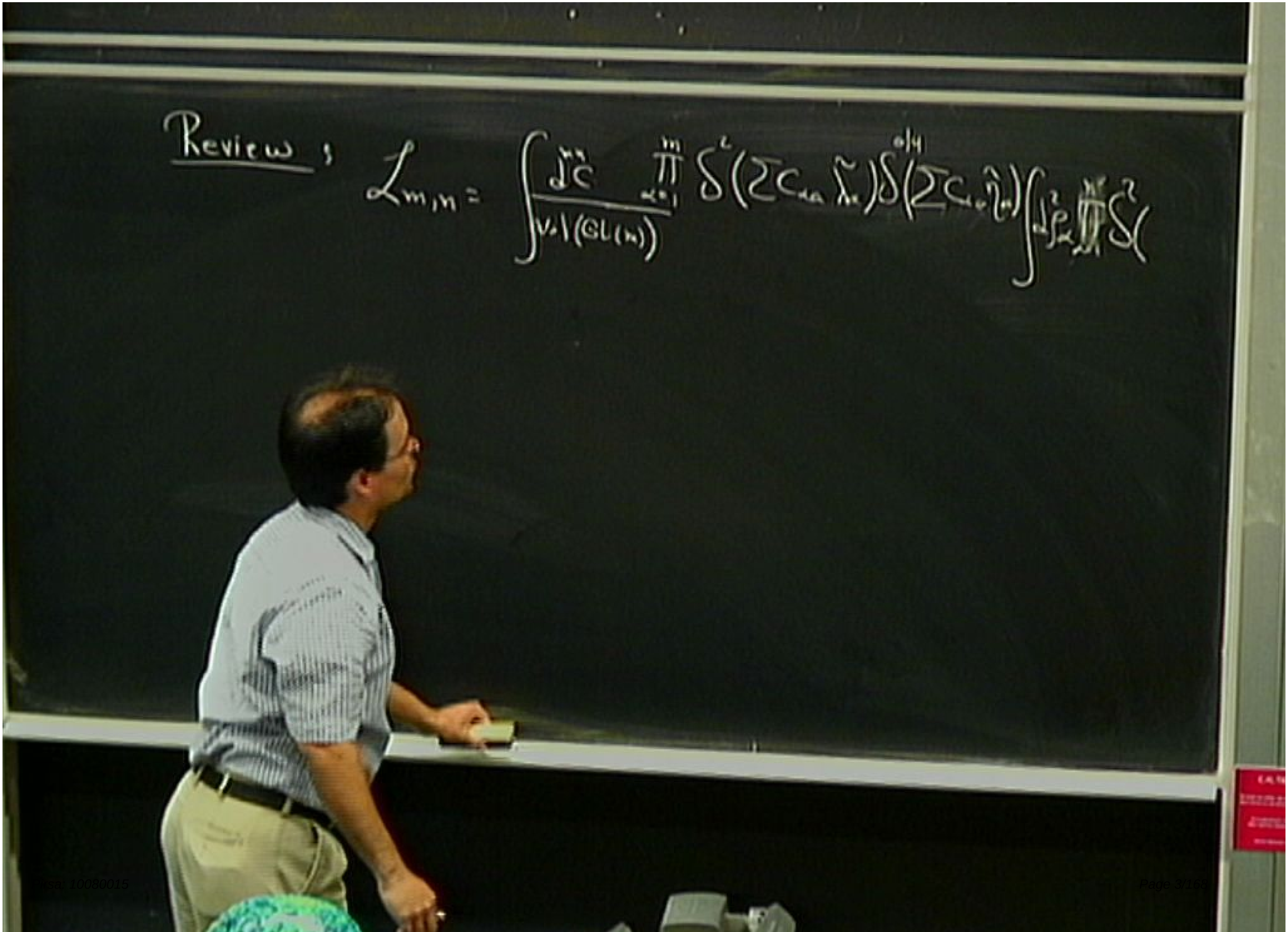
Date: Aug 25, 2010 10:00 AM

URL: http://pirsa.org/10080015

Abstract: This mini-course will review recent developments in our understanding of scattering amplitudes in gauge theories, particularly N=4 SYM in four dimensions. We will discuss a dual theory for these scattering amplitudes associated with contour integrals over Grassmannians, where the remarkable symmetries of the theory are manifest, but space-time and unitarity emerge as secondary concepts.

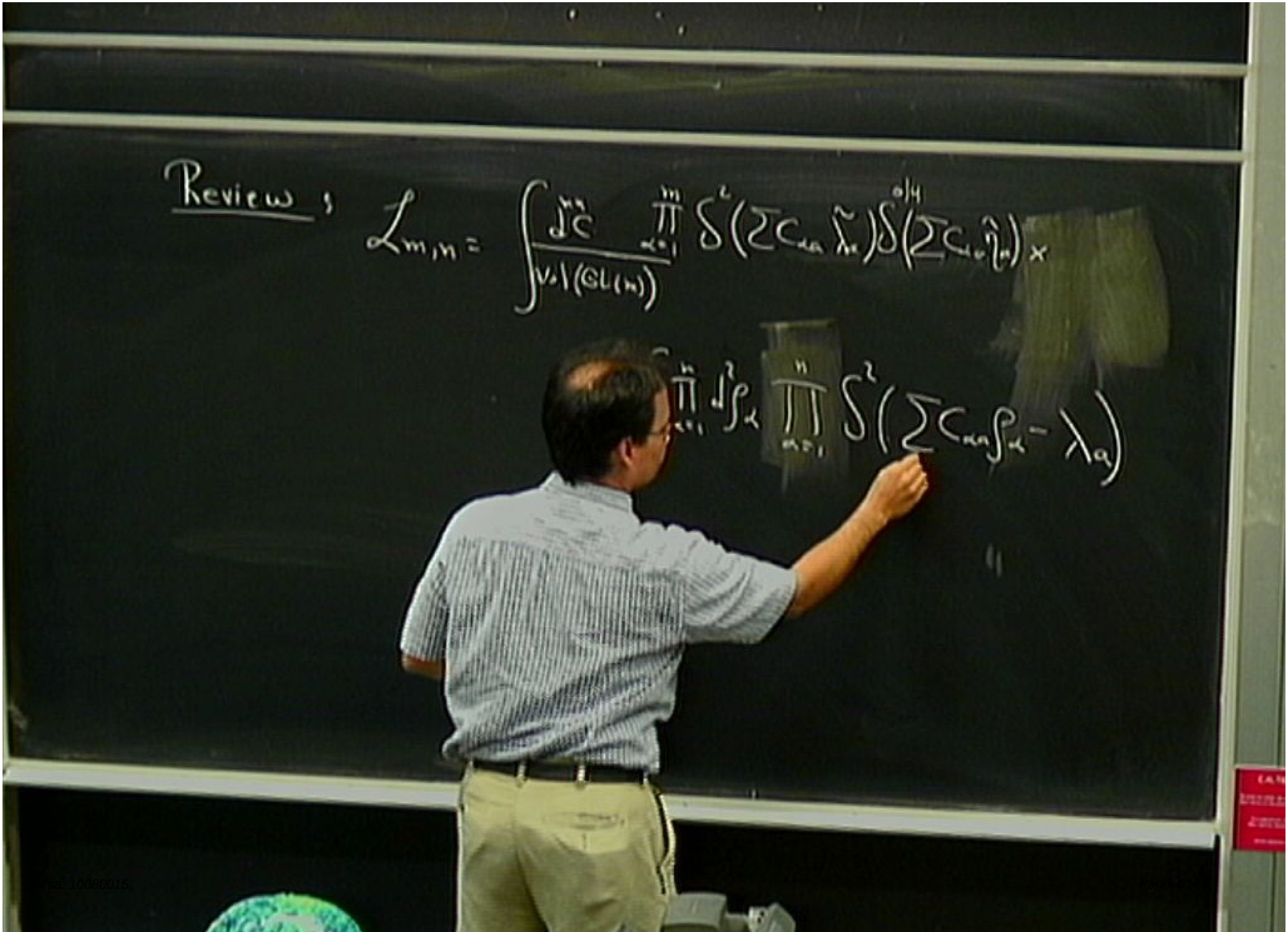


Review: $\mathcal{L}_{m,n} = \int \frac{d^{mn}C}{\text{vol}(GL(n))}$



Review:

$$\mathcal{L}_{m,n} = \int \frac{d^{m\cdot n}C}{\mathrm{vol}(GL(m))} \prod_{\alpha=1}^{m} \delta^2\left(\sum C_{\alpha a}\tilde{\lambda}_a\right)\delta^{0|4}\left(\sum C_{\alpha a}\tilde{\eta}_a\right)\int d^2\rho_\alpha \prod \delta^2($$



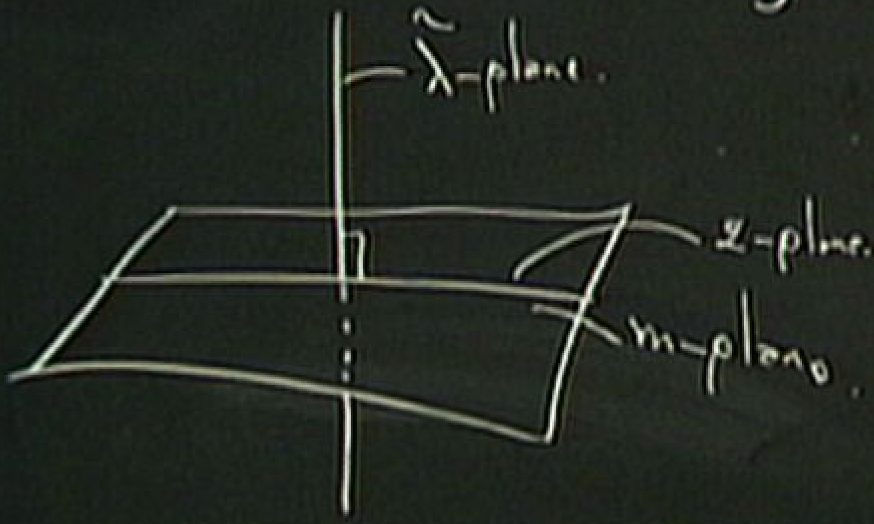
Review:

$$\mathcal{L}_{m,n} = \int \frac{d^{mn}C}{\text{vol}(GL(m))} \prod_{\alpha=1}^{m} \delta^{2}\Big(\sum C_{\alpha a}\tilde{\lambda}_a\Big)\,\delta^{0|4}\Big(\sum C_{\alpha a}\tilde{\eta}_a\Big) \times$$

$$\int \prod_{\alpha=1}^{m} d^2\rho_\alpha \prod_{a=1}^{n} \delta^2\Big(\sum C_{\alpha a}\rho_\alpha - \lambda_a\Big)$$

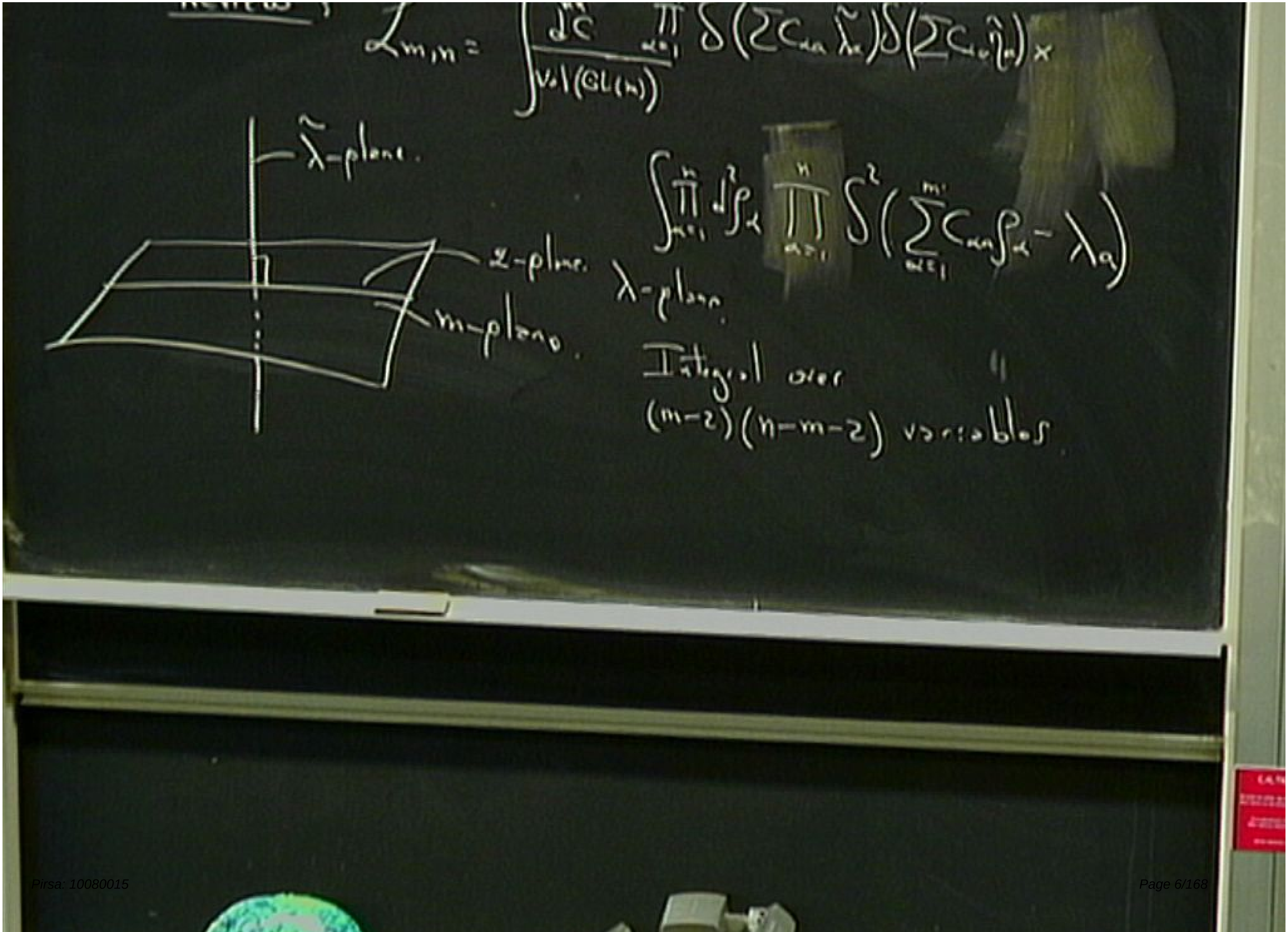
Reviews ;

$$\mathcal{L}_{m,n} = \int \frac{dC}{\text{vol}(GL(k))} \prod_{\alpha=1} \delta\left(\sum C_{\alpha a}\tilde{\lambda}_a\right)\delta\left(\sum C_{\alpha a}\tilde{\eta}_a\right) \times$$



λ-plane.

$$\int \prod_{\alpha=1}^{n} d^2\rho_\alpha \prod_{a=1}^{n} \delta^2\left(\sum_{\alpha=1}^{m'} C_{\alpha a}\rho_\alpha - \lambda_a\right)$$



$$\mathcal{L}_{m,n} = \int \frac{dC}{\text{vol}(GL(m))} \prod_{\alpha=1} \delta\left(\sum C_{\alpha a}\tilde{\lambda}_a\right)\delta\left(\sum C_{\alpha a}\tilde{\eta}_a\right) \times$$

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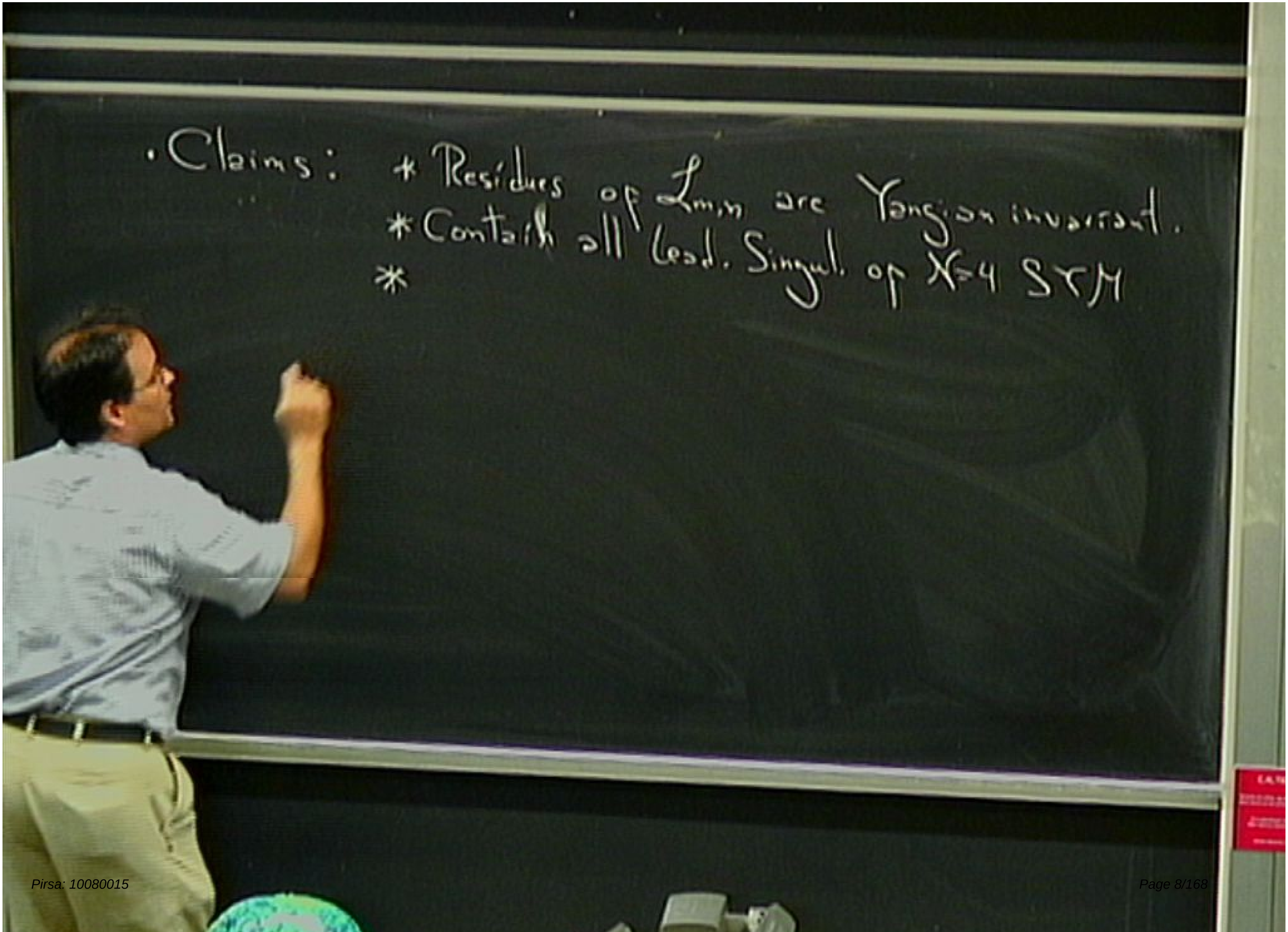
λ-plane.

Integral over
(m−2)(n−m−2) variables

- Claims: * Residues of $\mathcal{L}_{m,n}$ are Yangian invariant.



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* Contain all lead. Singul. of $\mathcal{N}=4$ SYM

*

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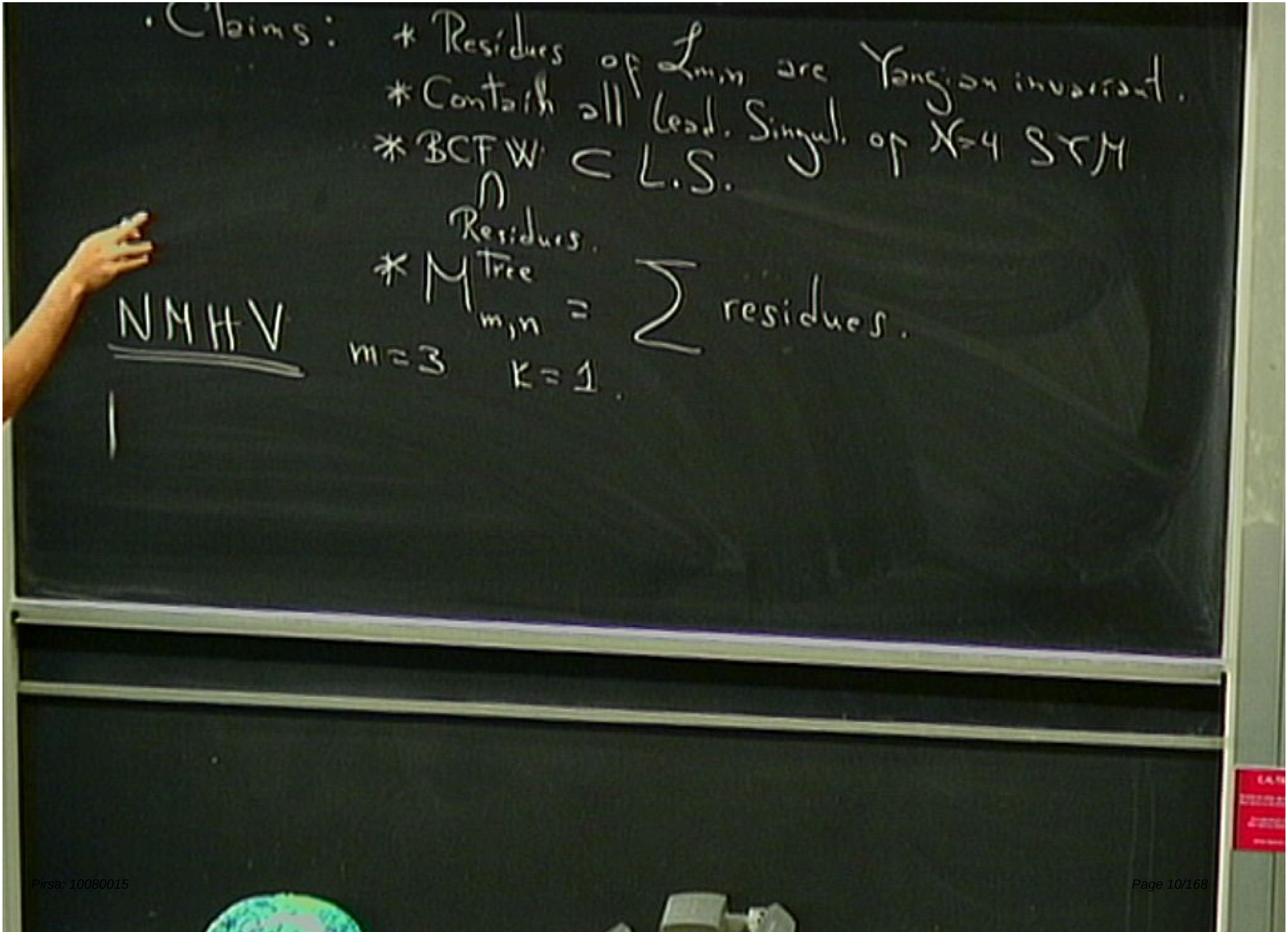
  - * Residues of $\mathcal{L}_{m,n}$ are Yangian invariant.
  - * Contain all lead. Singul. of $\mathcal{N}=4$ SYM
  - * $\begin{array}{c}\text{BCFW} \subset \text{L.S.} \\ \cap \\ \text{Residues.}\end{array}$
  - * $M^{\text{tree}}_{m,n} = \sum$ residues.



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- * Contain all lead. Singul. of $\mathcal{N}=4$ SYM
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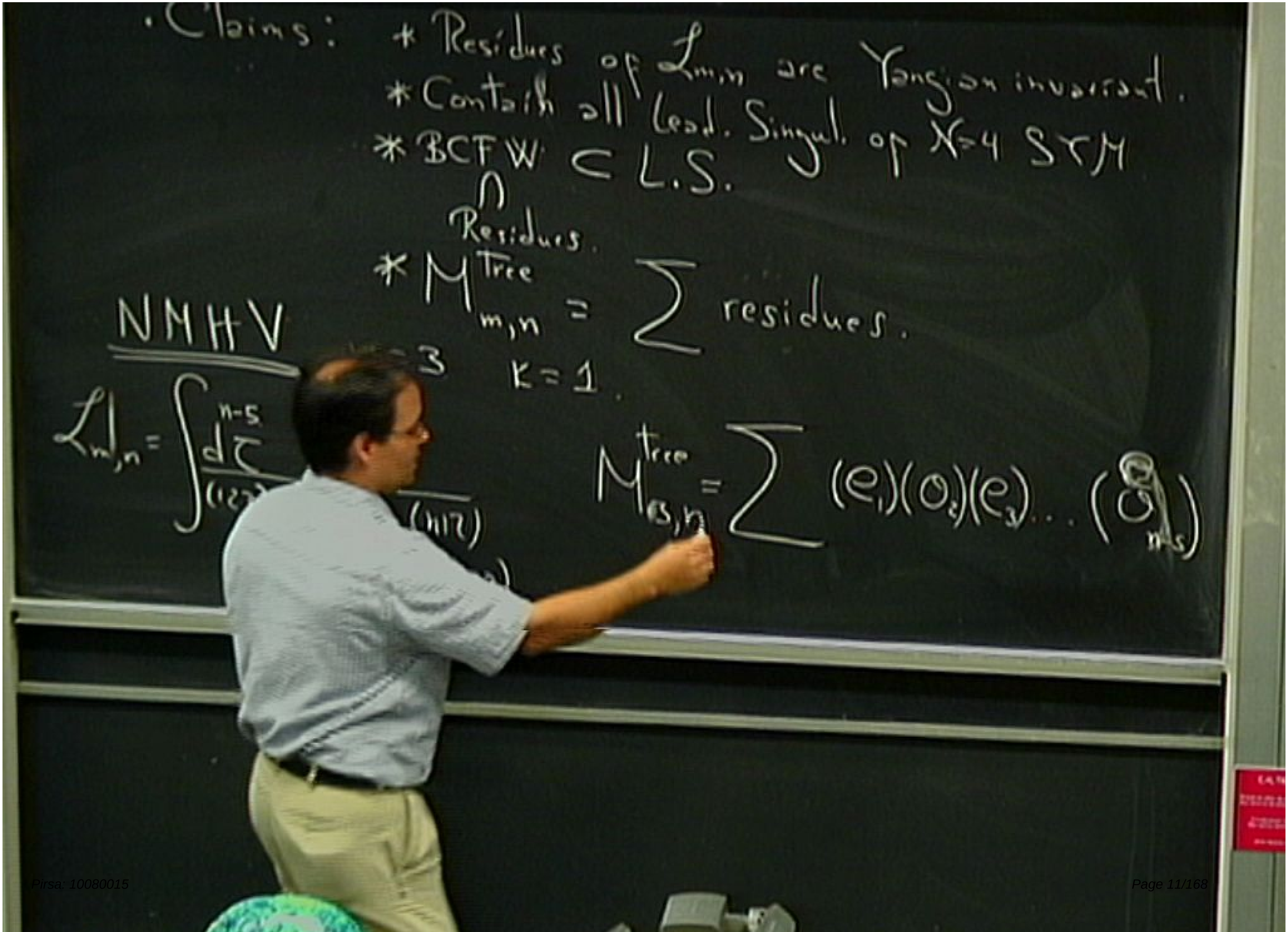
NMHV $m=3$ $k=1$.



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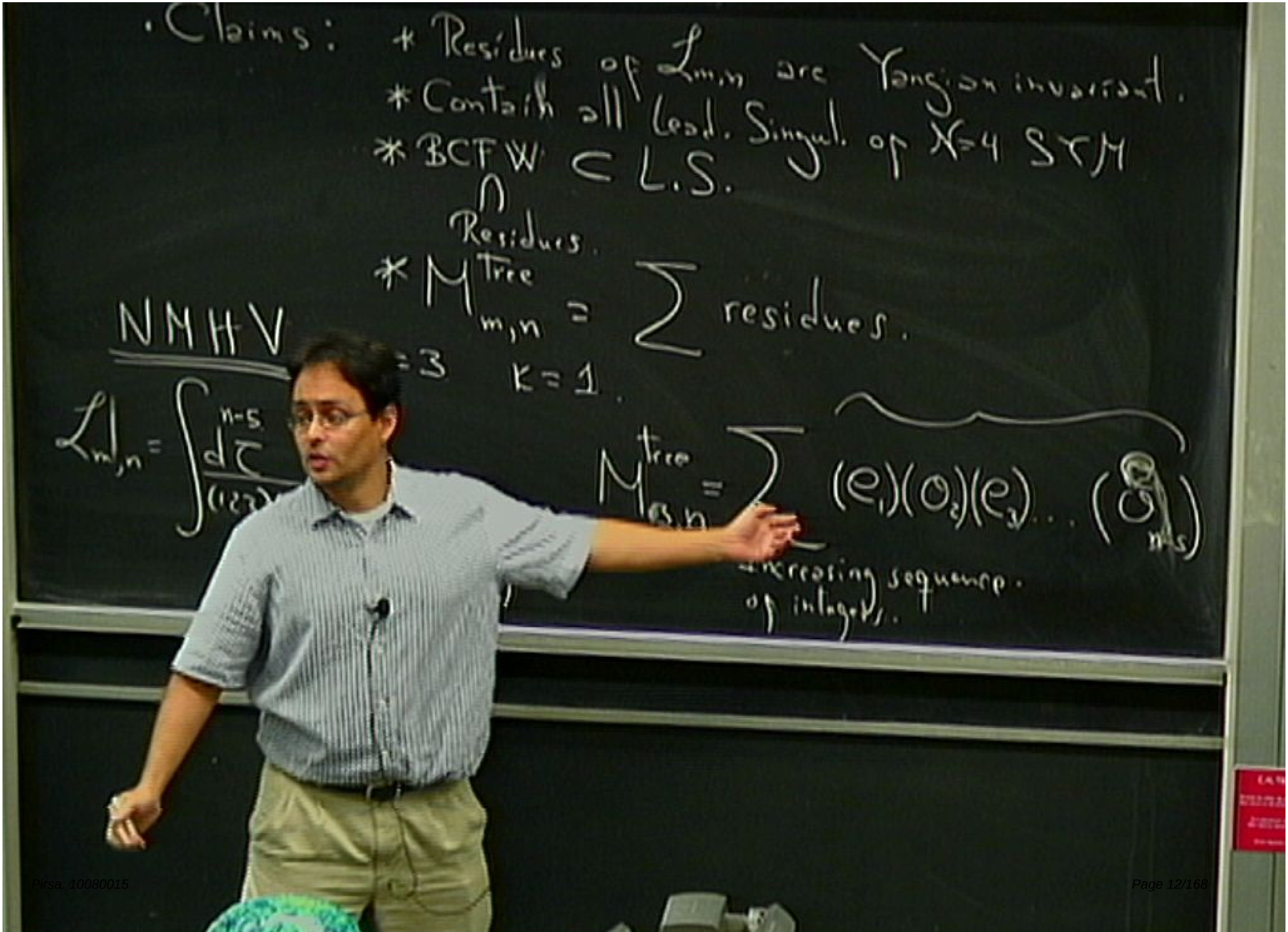
NMHV

$m=3 \quad k=1.$

$$\mathcal{L}_{m,n} = \int \frac{d^{n-5}\tau}{(123)\cdots(n12)}$$

$$M^{\text{tree}}_{3,n} = \sum (e_1)(o_2)(e_3)\ldots(o_{n-5})$$



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* $\mathrm{BCFW} \subset \mathrm{L.S.}$

$\cap$

Residues.

* $M^{\mathrm{tree}}_{m,n} = \sum$ residues.

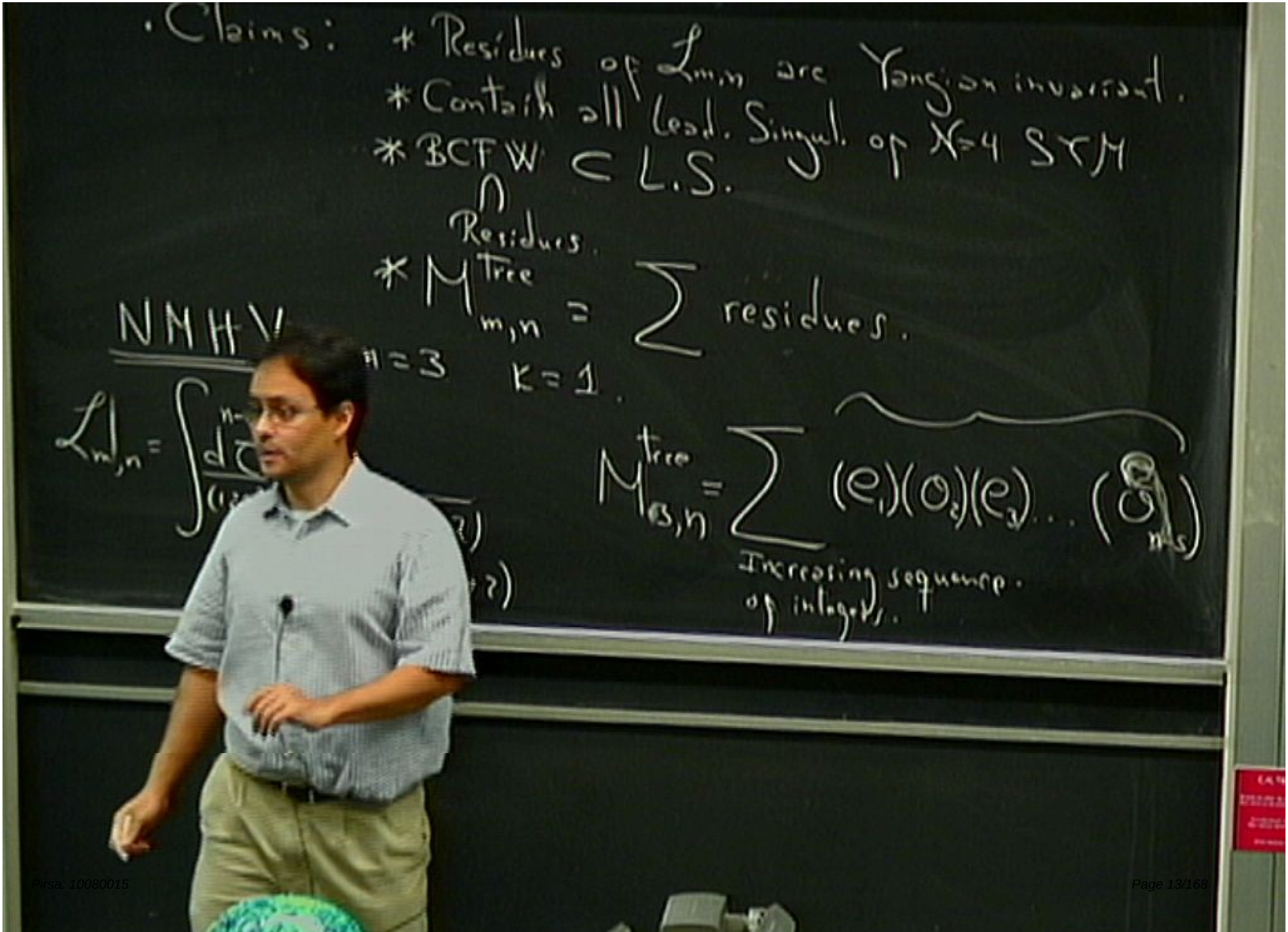
NMHV

$=3 \quad k=1.$

$$\mathcal{L}_{m,n} = \int \frac{d^{n-5}\tau}{(123)\cdots}$$

$$M^{\mathrm{tree}}_{3,n} = \sum \overbrace{(e_1)(o_2)(e_3)\dots(o_{n-5})}$$

increasing sequence of integers.



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$$\begin{array}{cl} * \text{ BCFW} & \subset \text{L.S.} \\ \cap & \\ \text{Residues.} & \end{array}$$

$$* \; M^{\text{Tree}}_{m,n} = \sum \text{residues.}$$

NNHV

m=3 k=1.

$$\mathcal{L}_{m,n} = \int \frac{d\tau^{n-}}{(1}$$

2)

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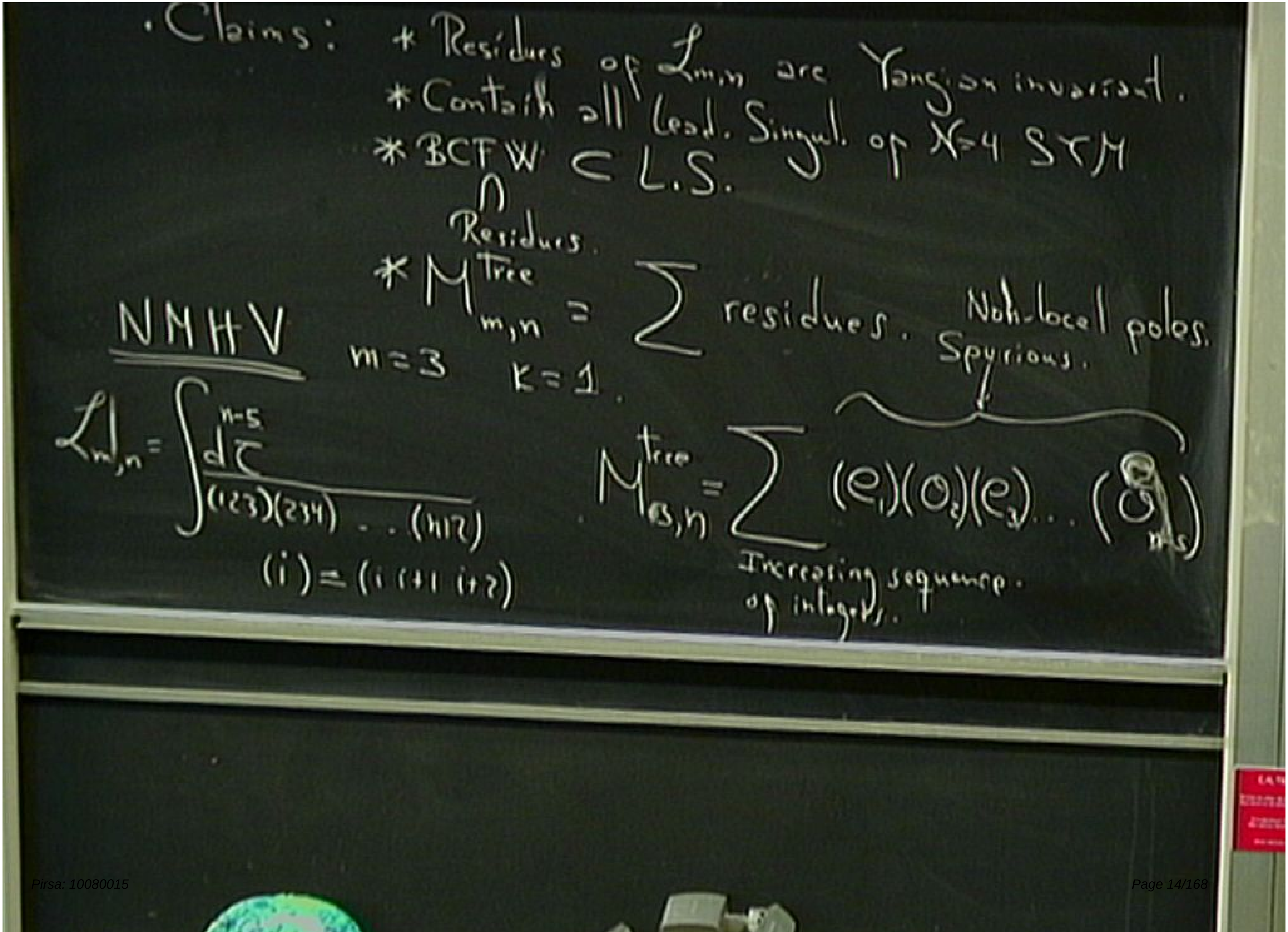
$$M^{\text{tree}}_{3,n} = \sum_{\text{Increasing sequence of integers.}} \overbrace{(e_1)(o_2)(e_3)\dots(o_{n-5})}$$



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* $\underset{\text{Residues.}}{\underset{\cap}{\text{BCFW}}} \subset$ L.S.

* $M^{\text{tree}}_{m,n} = \sum$ residues. Non-local poles. Spurious.

NMHV $m=3$ $k=1$.

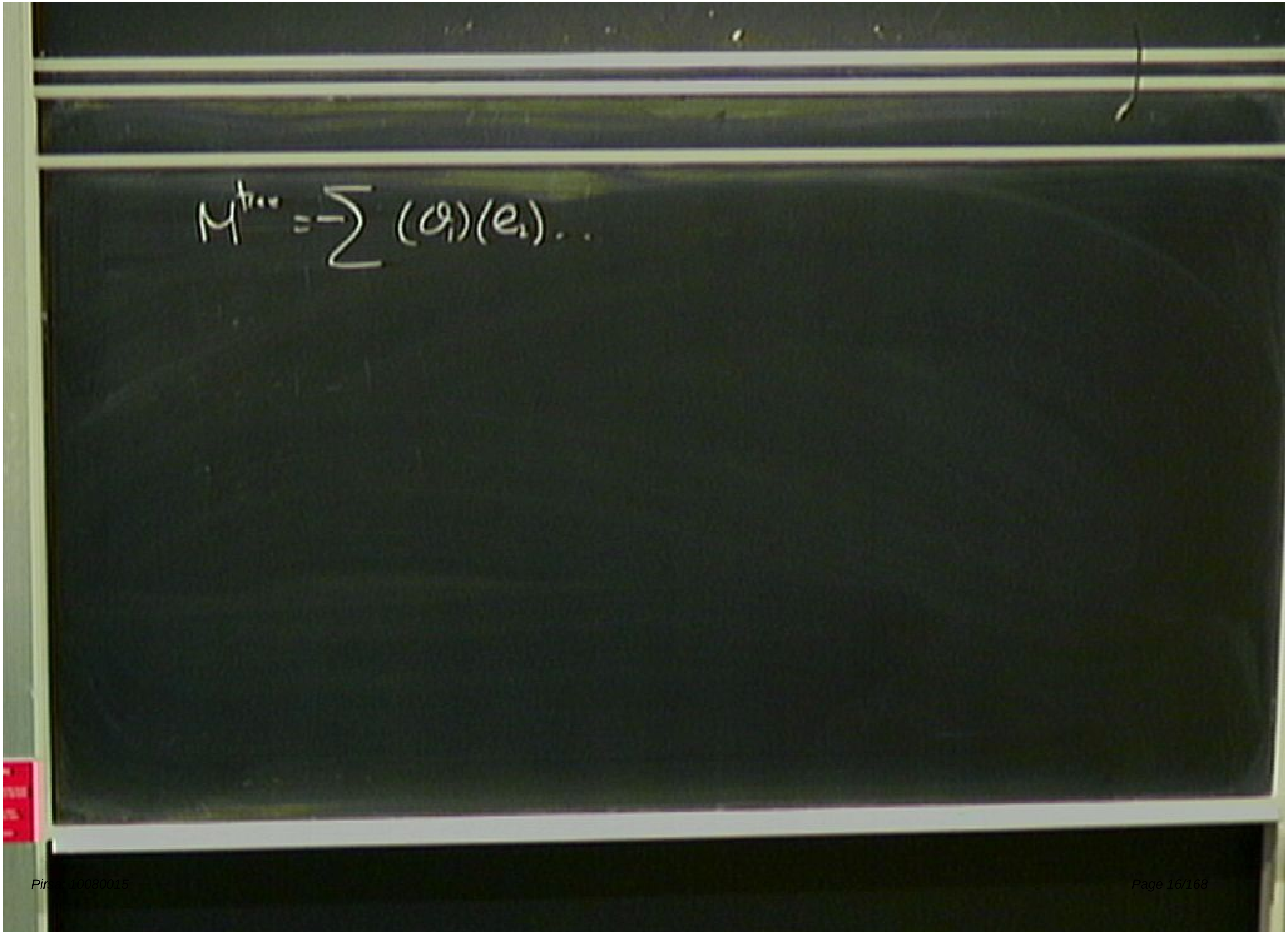
$$\mathcal{L}_{m,n} = \int \frac{d^{n-5}\tau}{(123)(234)\ .\ .\ (n12)}$$

$(i) = (i\ i+1\ i+2)$

$$M^{\text{tree}}_{3,n} = \sum \overbrace{(e_1)(o_2)(e_3)\ldots(o_{n-5})}^{\text{Spurious.}}$$

Increasing sequence of integers.

$$M^{tree} = -\sum (\vartheta_1)(e_2) \ldots$$



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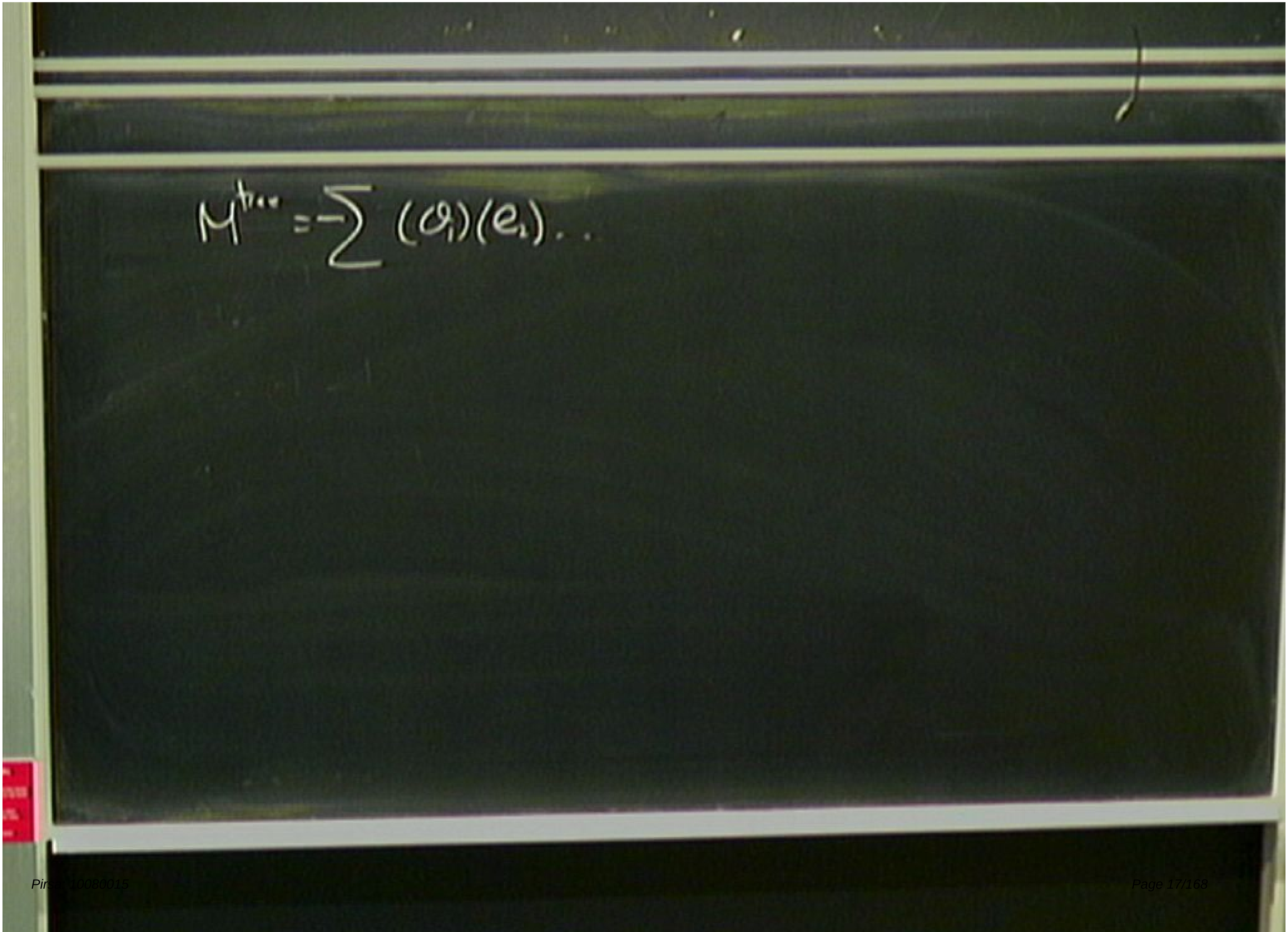

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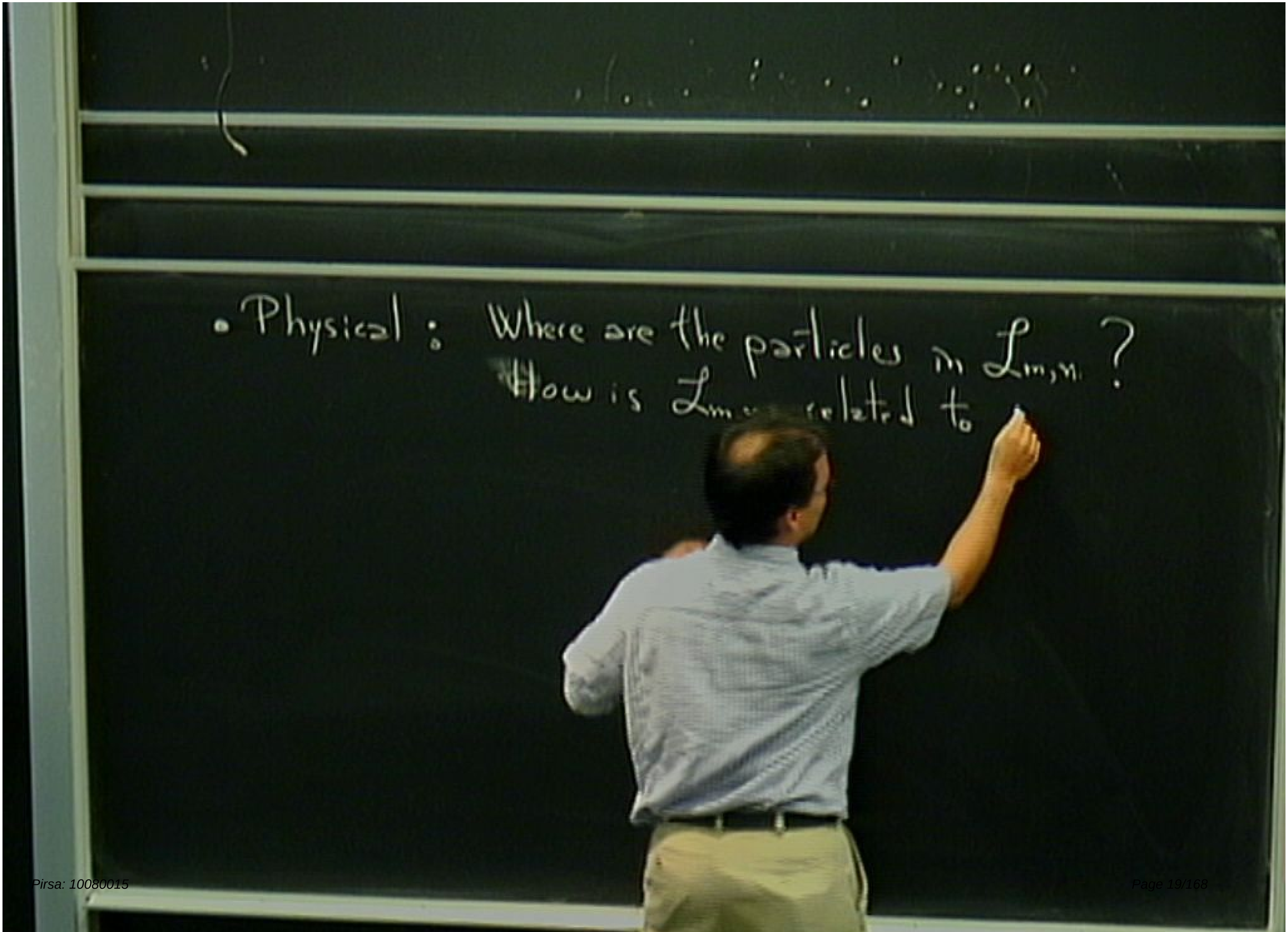
- Physical :



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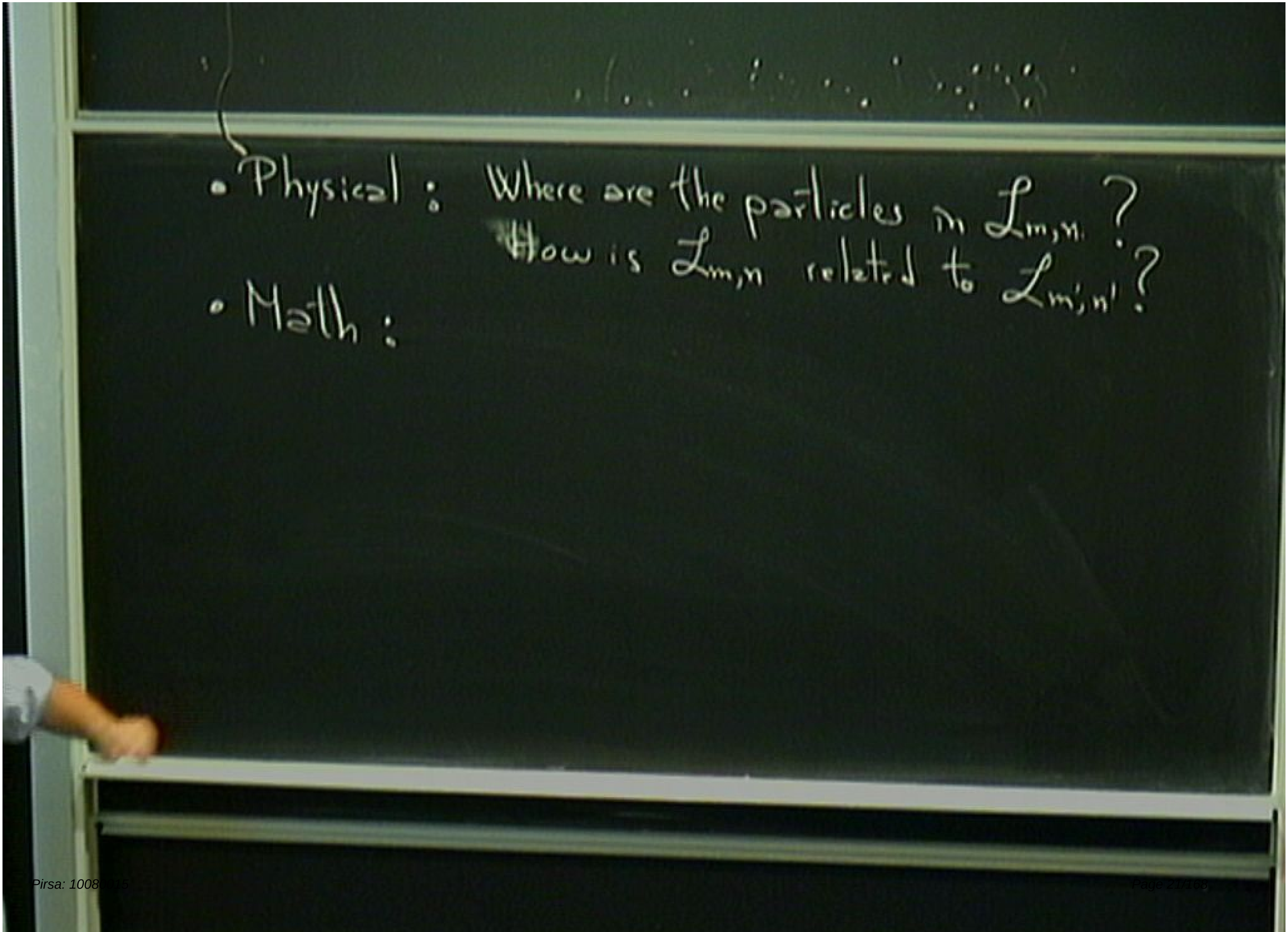

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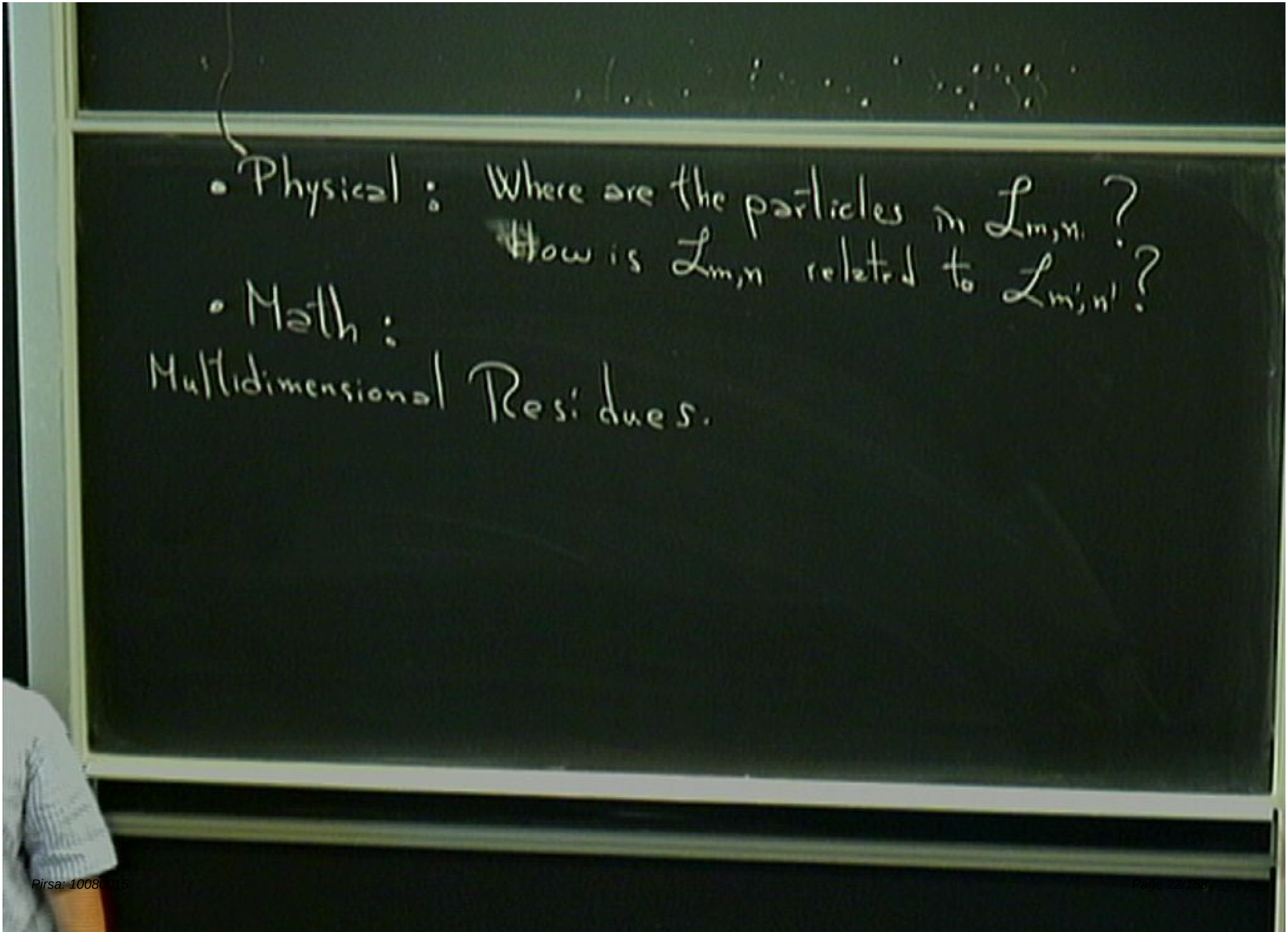

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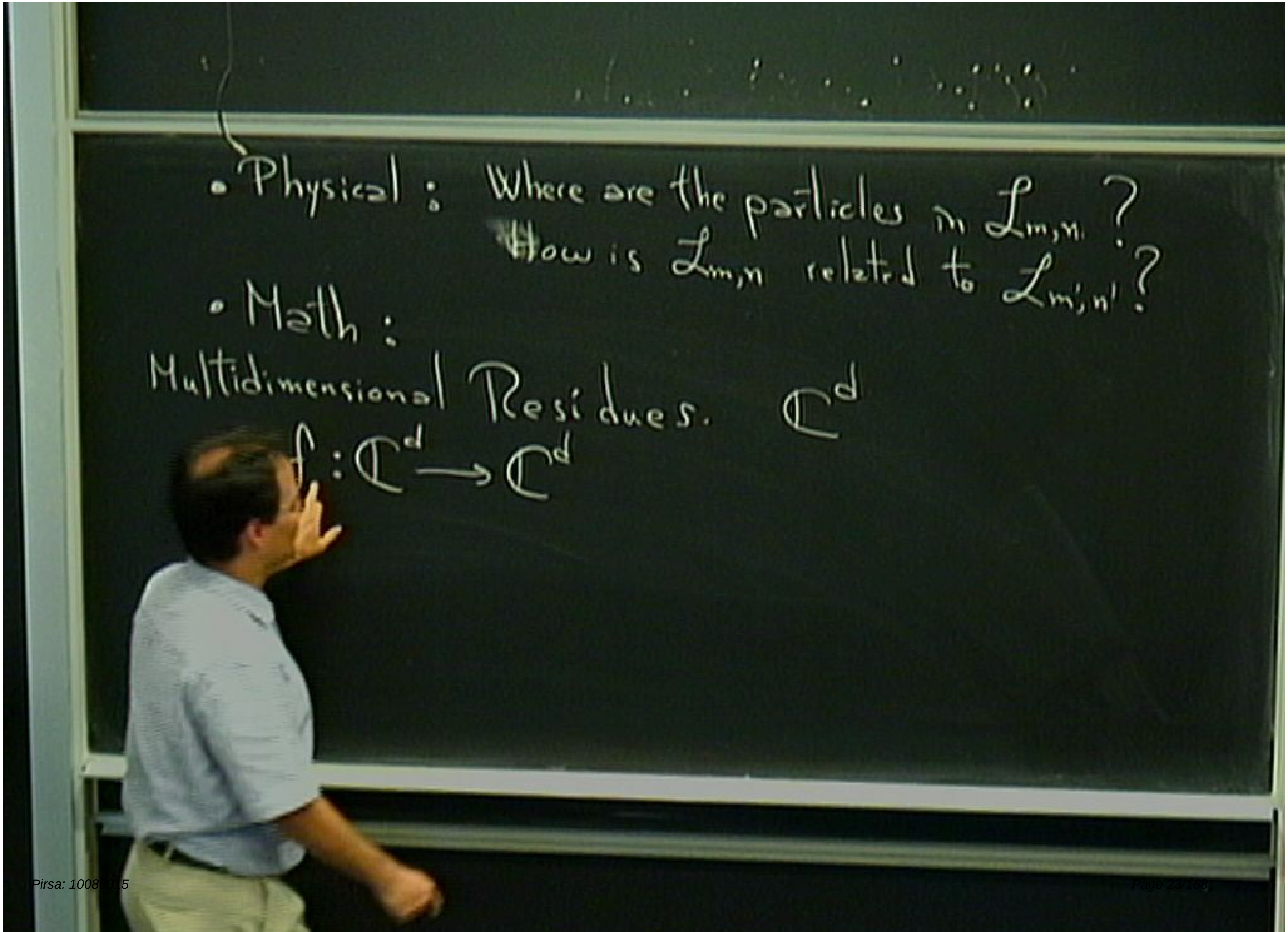
Multidimensional Residues.



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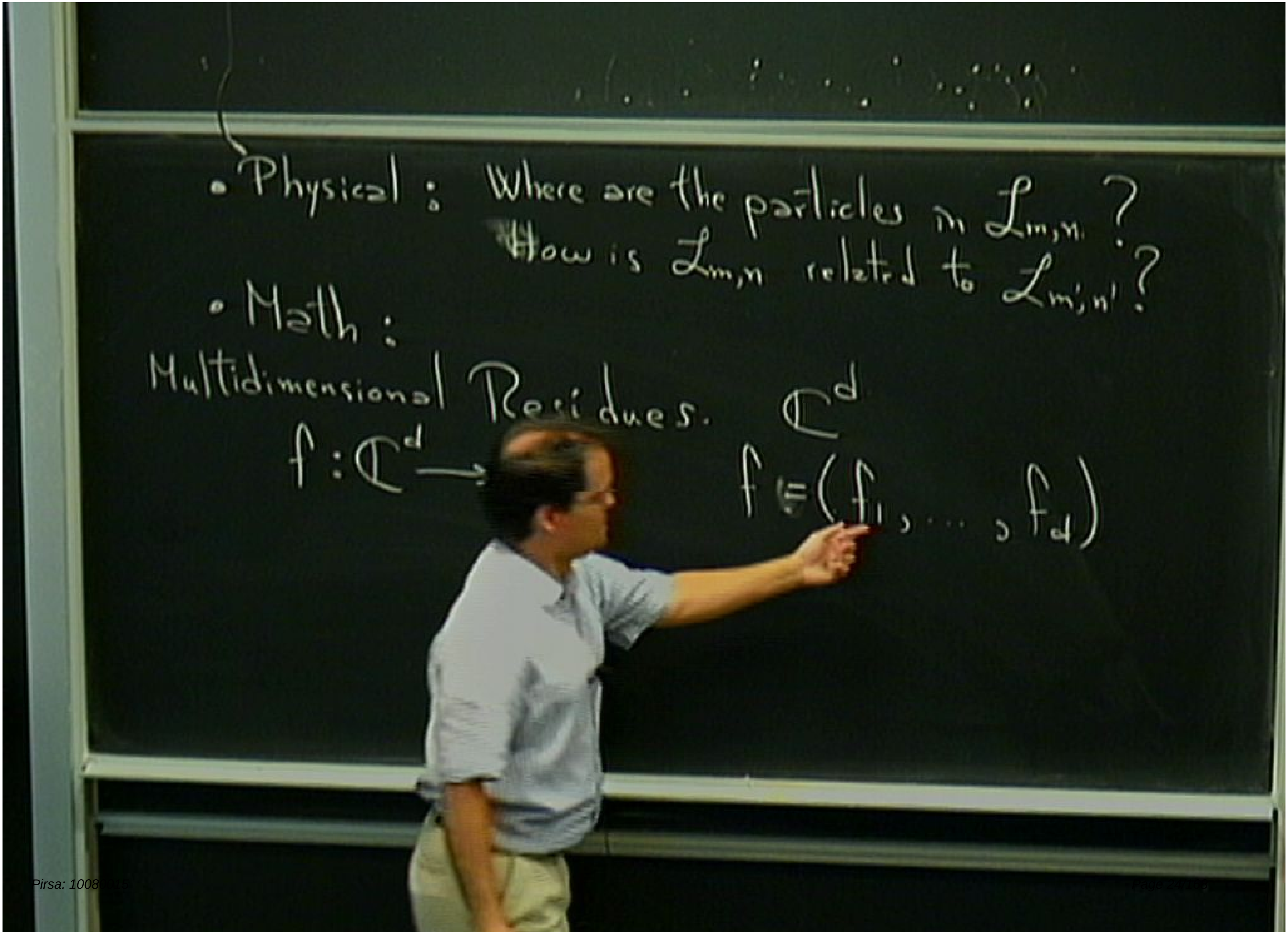
- Math :

Multidimensional Residues. $\mathbb{C}^d$

$f : \mathbb{C}^d \longrightarrow \mathbb{C}^d$



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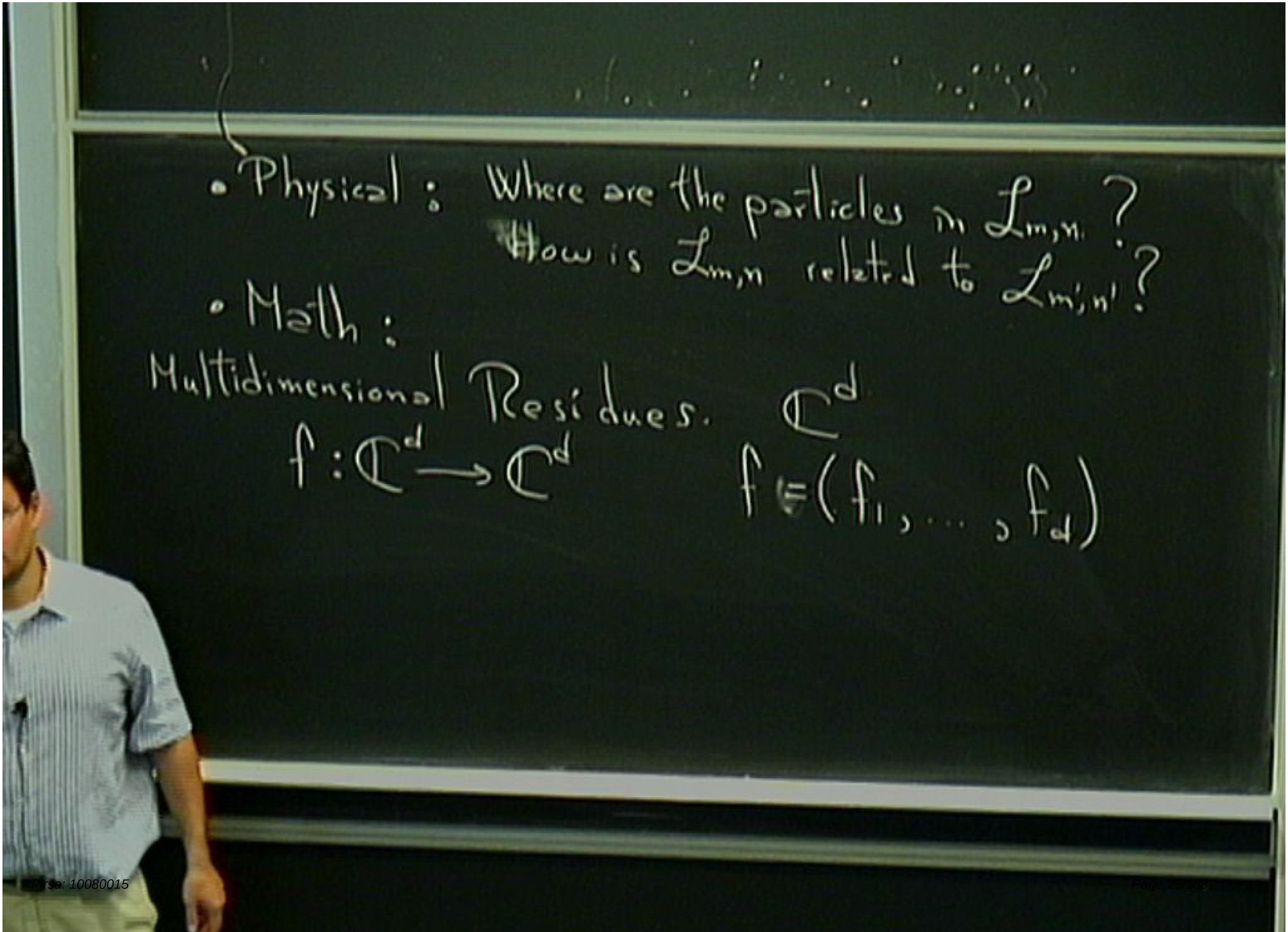
$f:\mathbb{C}^d \longrightarrow$

$f=(f_1, \dots, f_d)$



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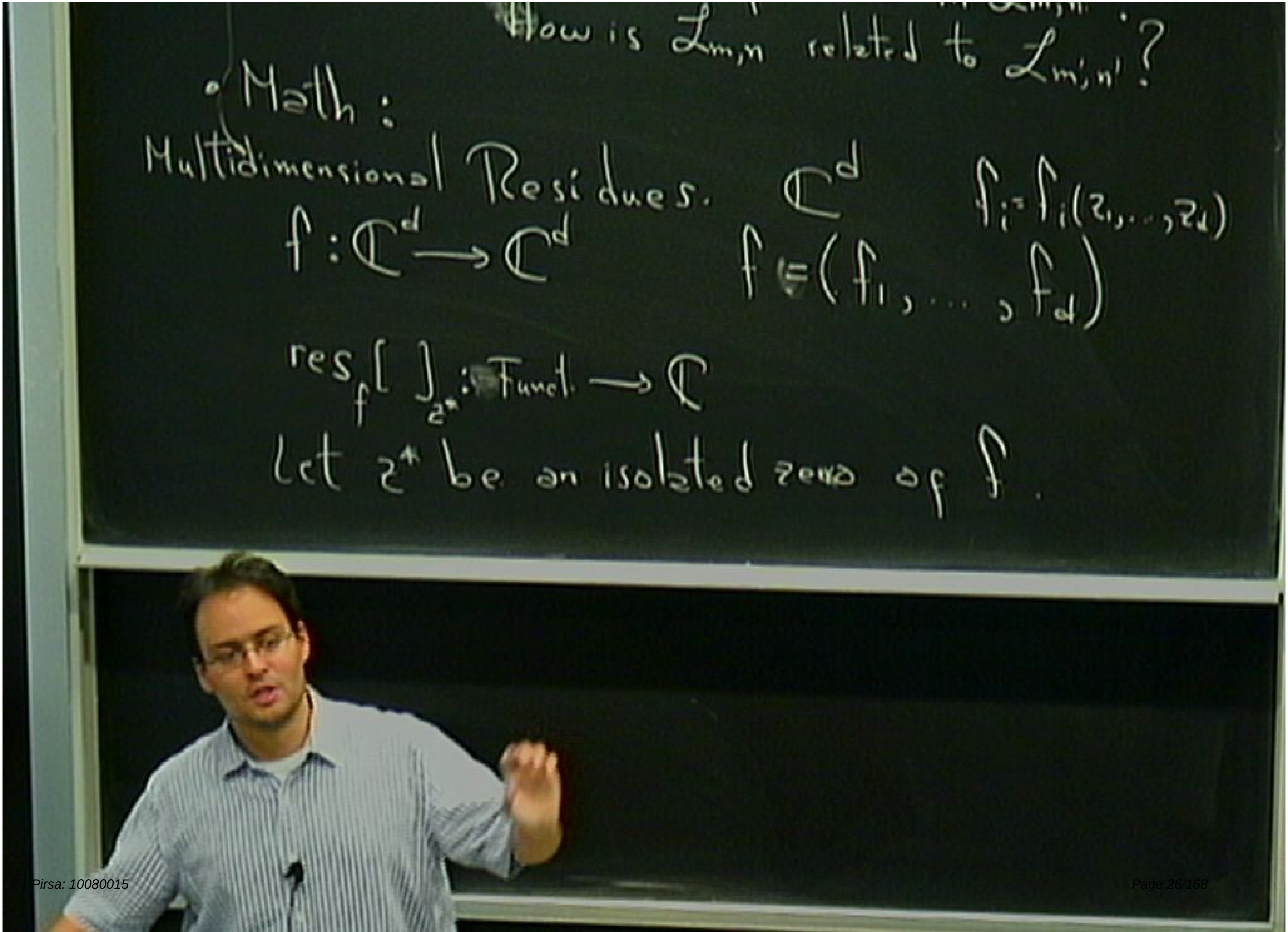


How is $\mathcal{L}_{m,n}$ related to $\mathcal{L}_{m',n'}$?

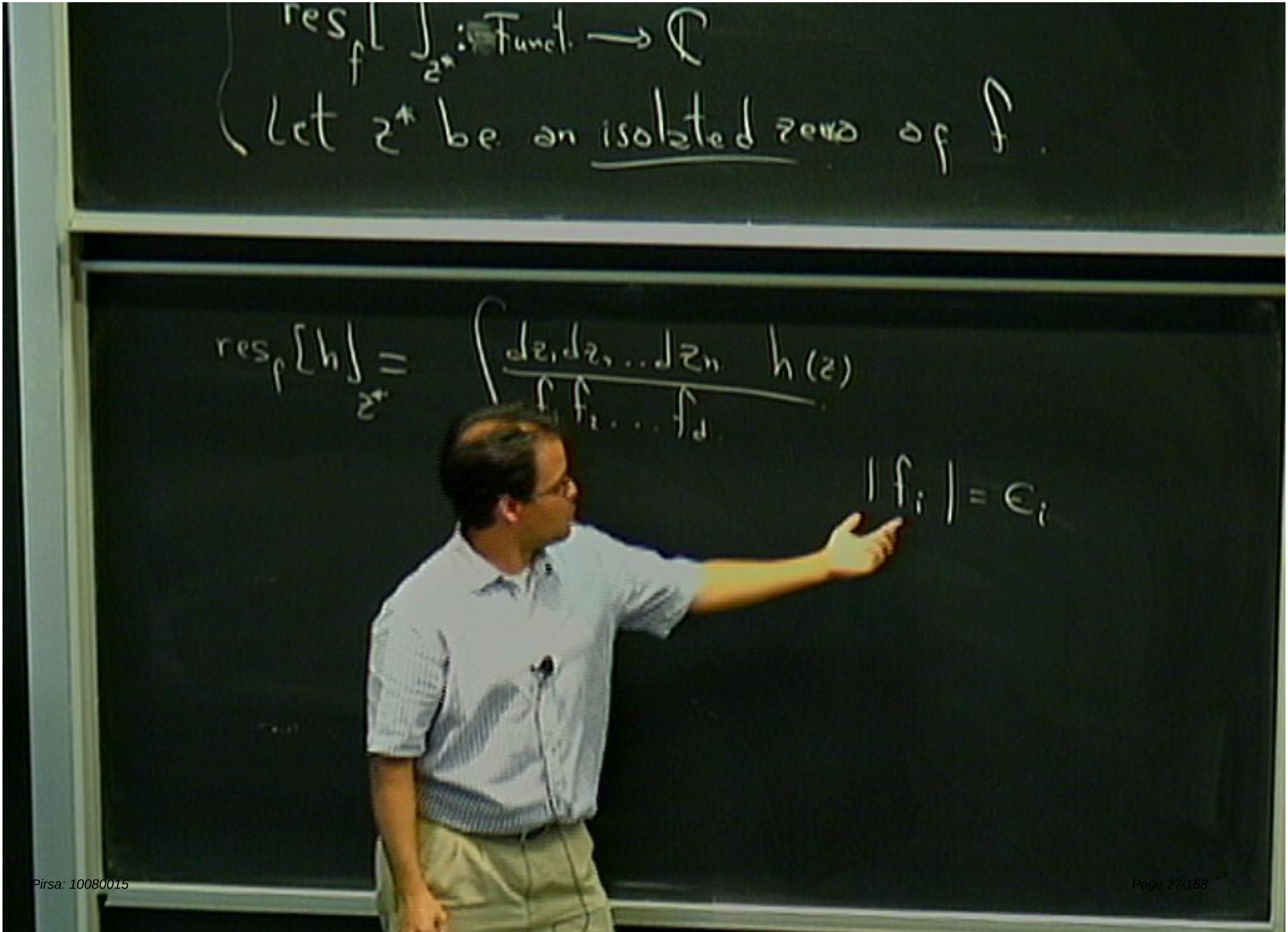
- Math:

Multidimensional Residues. $\mathbb{C}^d$ $f_i = f_i(z_1,\dots,z_d)$

$f:\mathbb{C}^d \longrightarrow \mathbb{C}^d$ $f=(f_1,\dots,f_d)$

$\mathrm{res}_f[\ ]_{z^*}: \text{Funct.} \longrightarrow \mathbb{C}$

Let z^* be an isolated zero of f.



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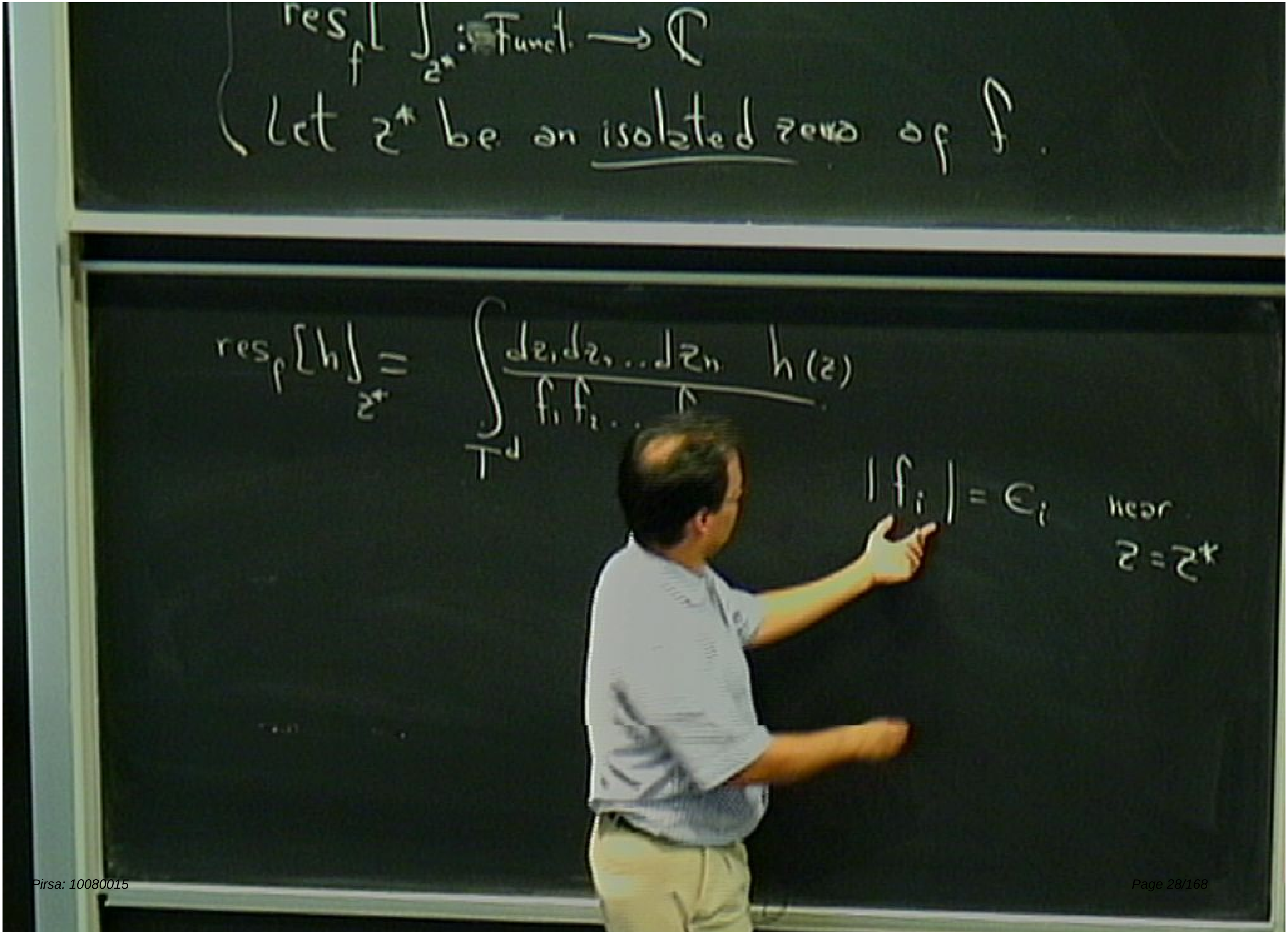
$$\text{res}_f[h]_{z^*} = \int \frac{dz_1 dz_2 \ldots dz_n \quad h(z)}{f_1 f_2 \ldots f_d}$$

$$|f_i| = \epsilon_i$$



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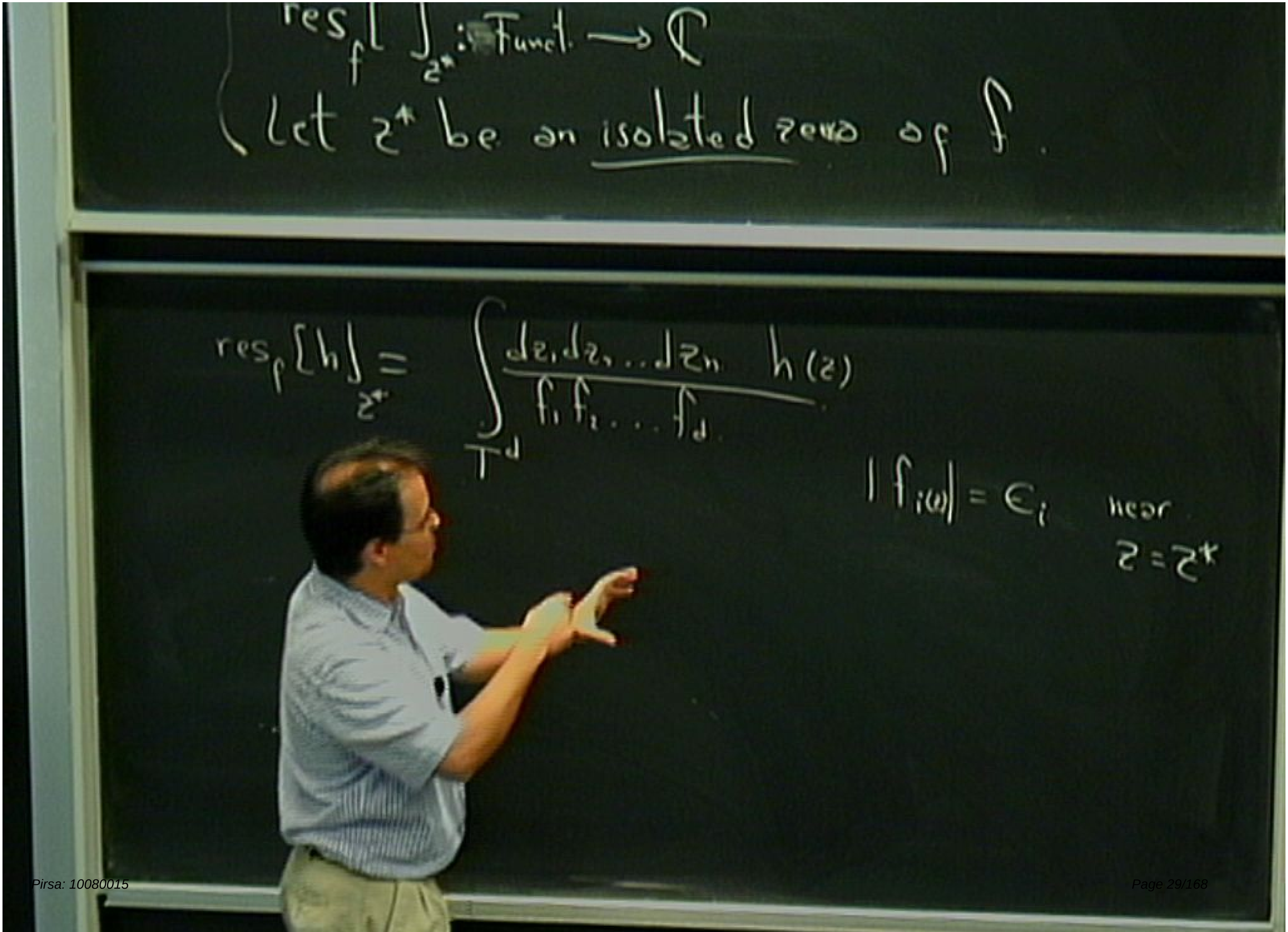

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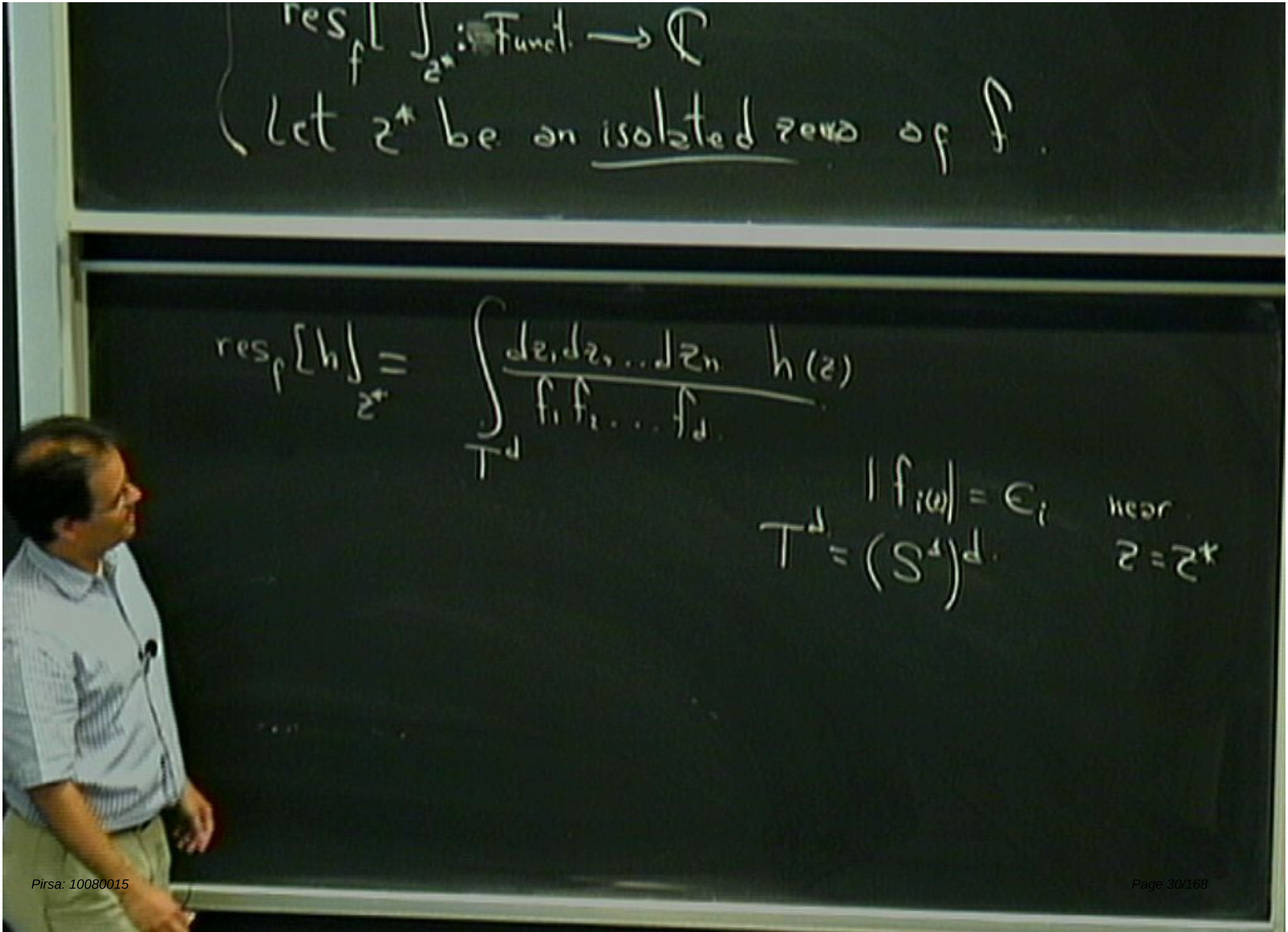

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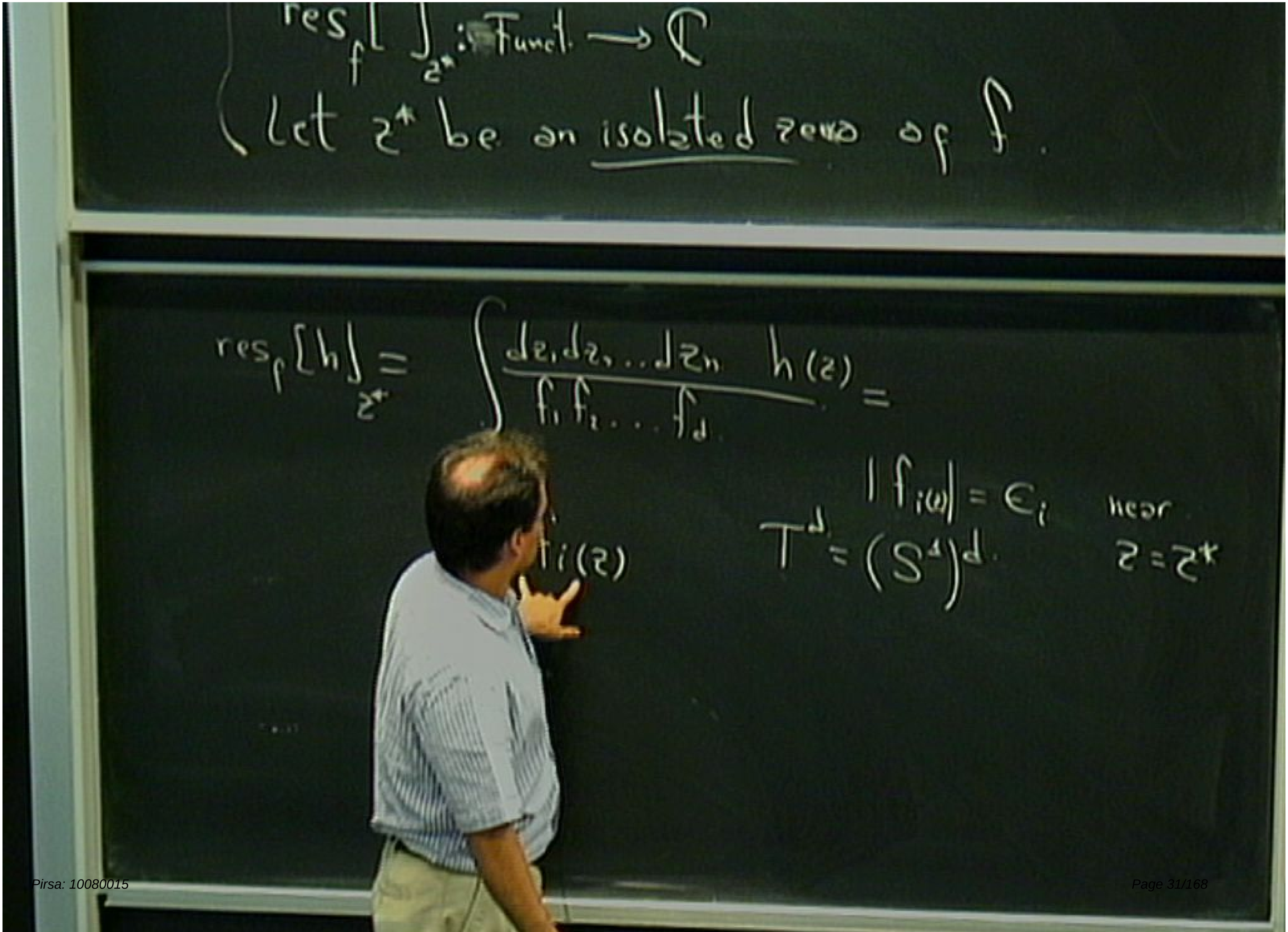
$T^d = (S^1)^d$



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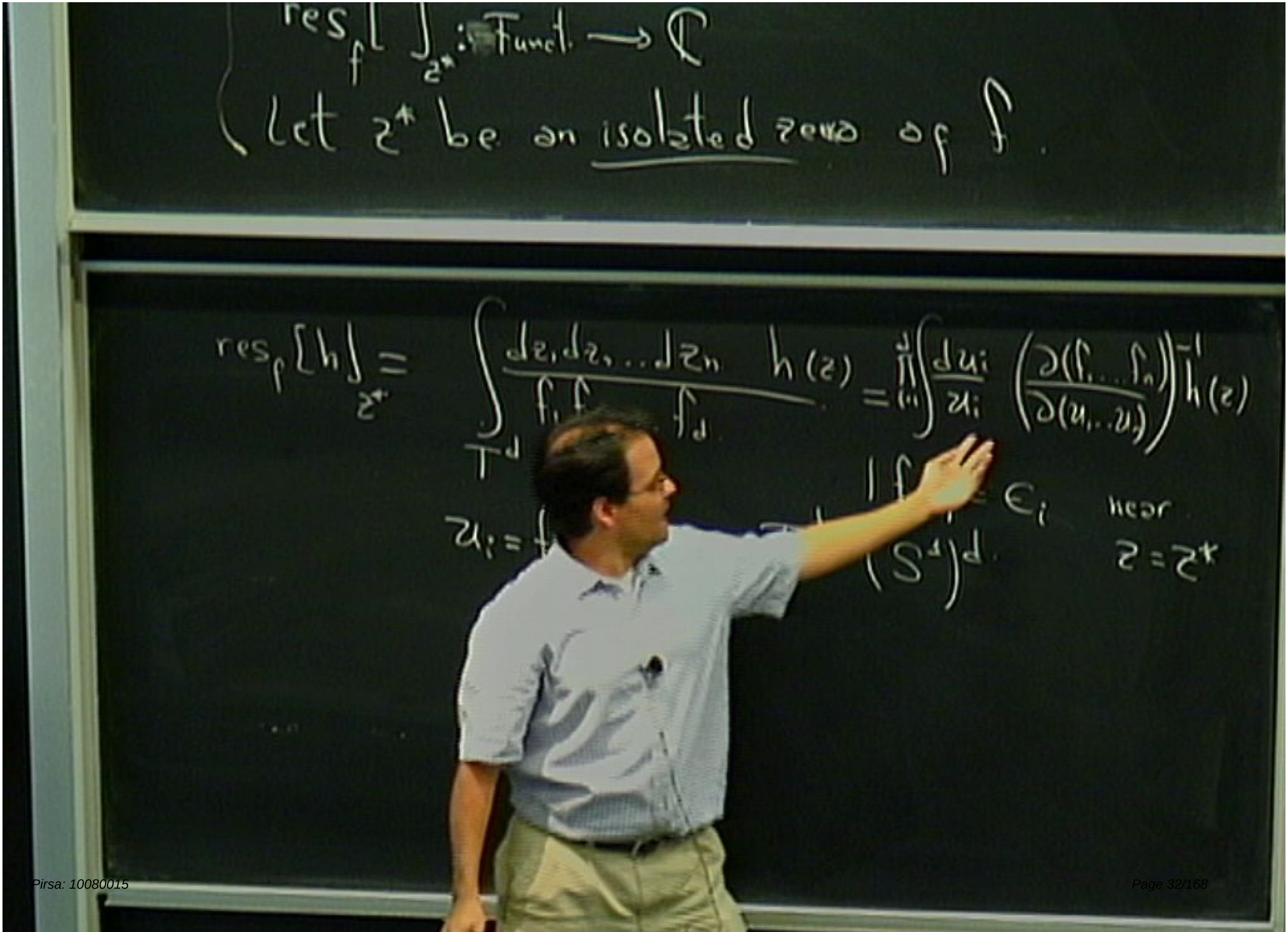
$f_i(z)$ $\quad T^d = (S^1)^d$



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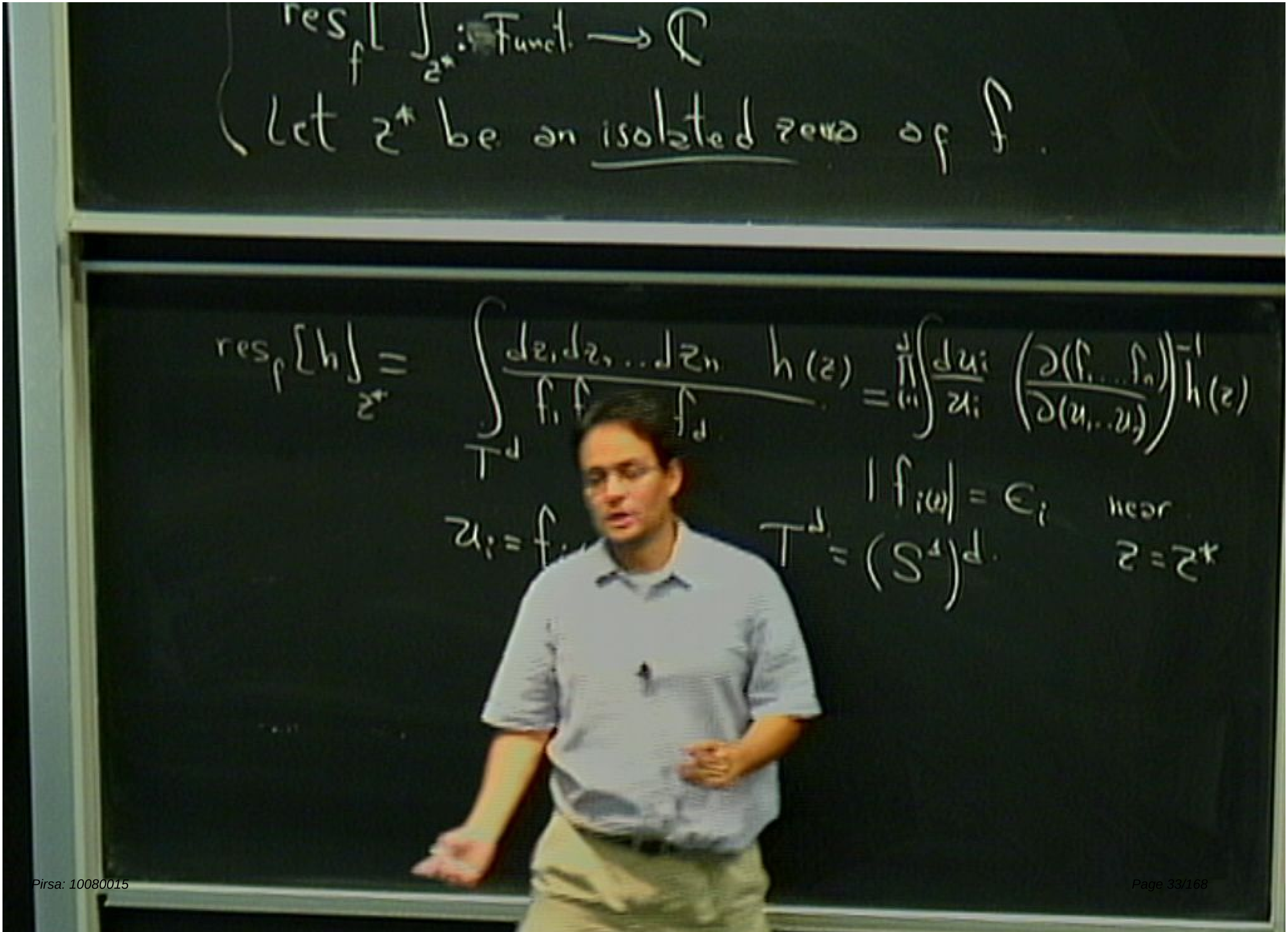
$u_i = f$

$|f \quad \epsilon_i$ near $z = z^*$

$(S^1)^d$



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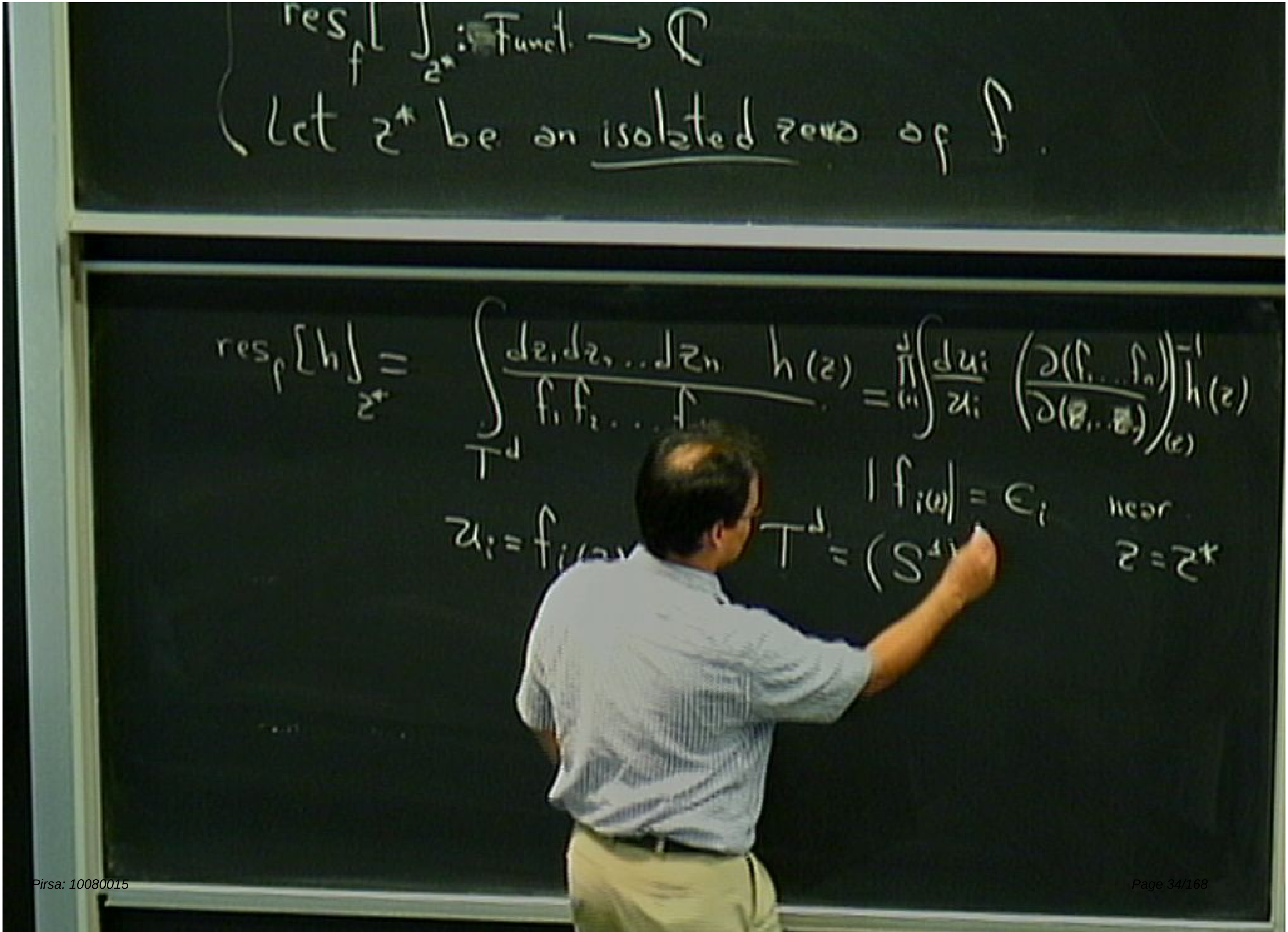
$|f_i(w)| = \epsilon_i$ near $z = z^*$

$u_i = f_i$ $T^d = (S^1)^d$.



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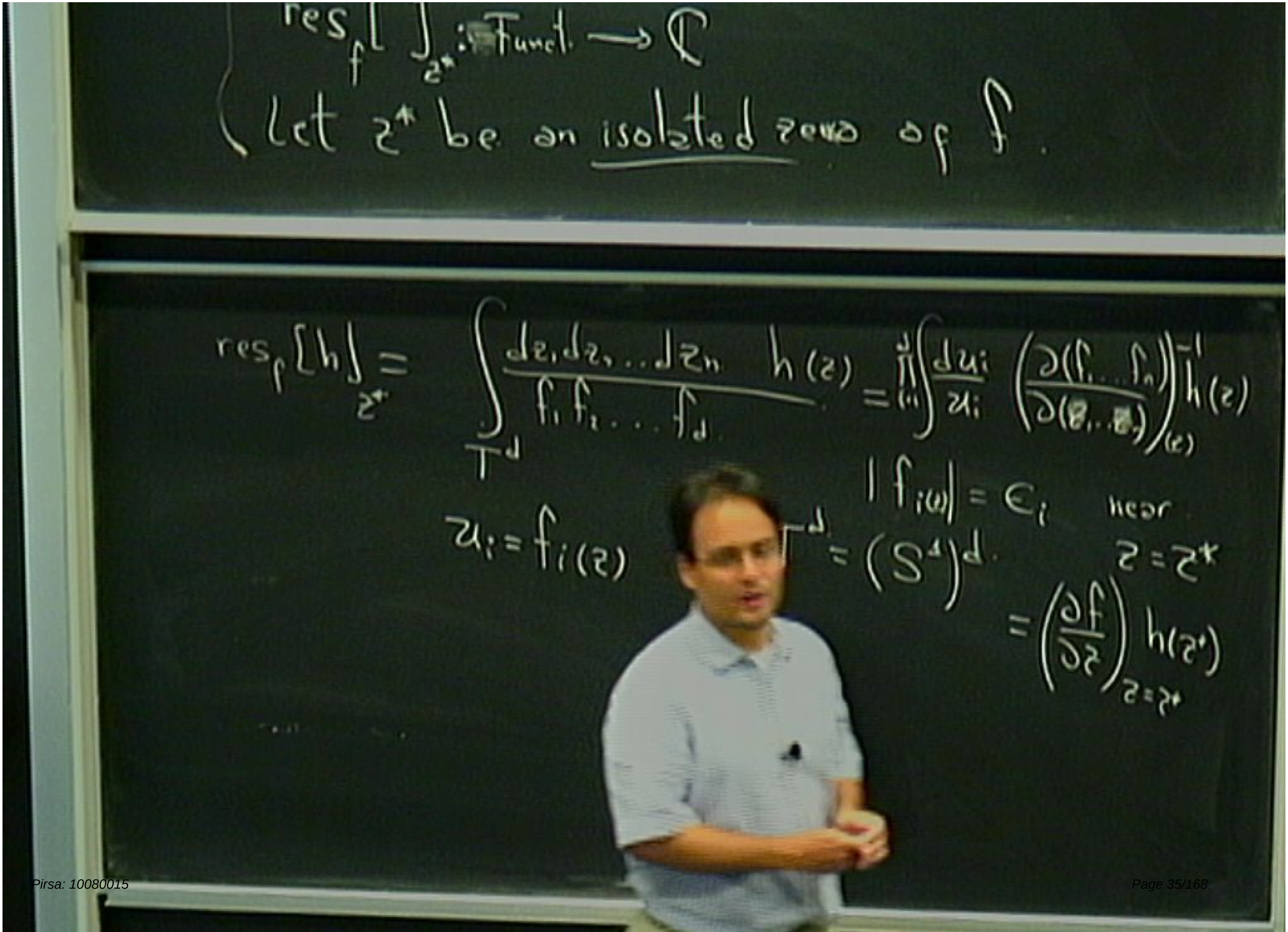
$|f_i(z)| = \epsilon_i$ near $z = z^*$

$u_i = f_i(z)$ $T^d = (S^1)$



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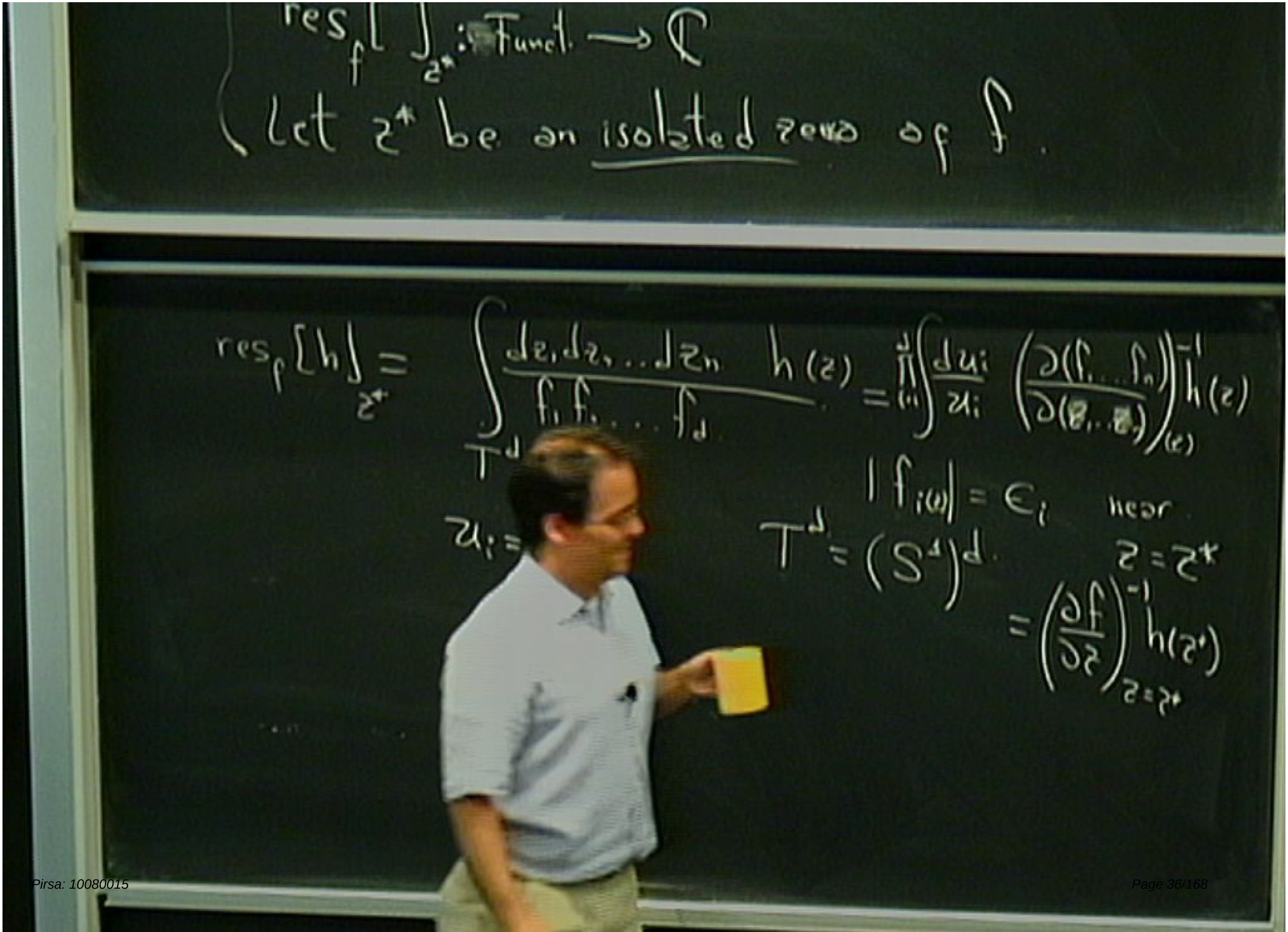
$|f_{i(\omega)}| = \epsilon_i$ near $z = z^*$

$u_i = f_{i(z)}$ $T^d = (S^1)^d$.

$$= \left(\frac{\partial f}{\partial z} \right)_{z=z^*} h(z^*)$$



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$u_i =$

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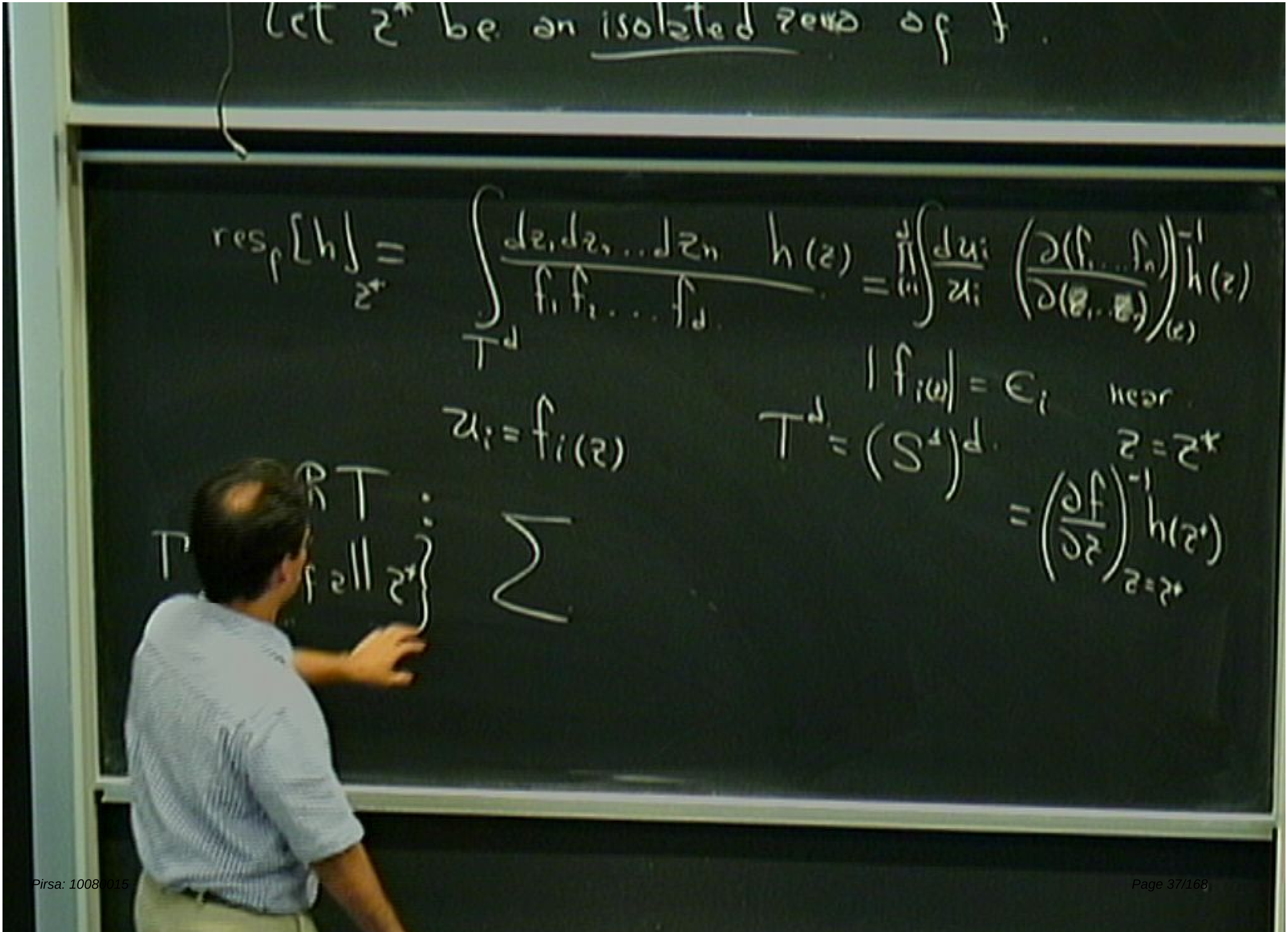

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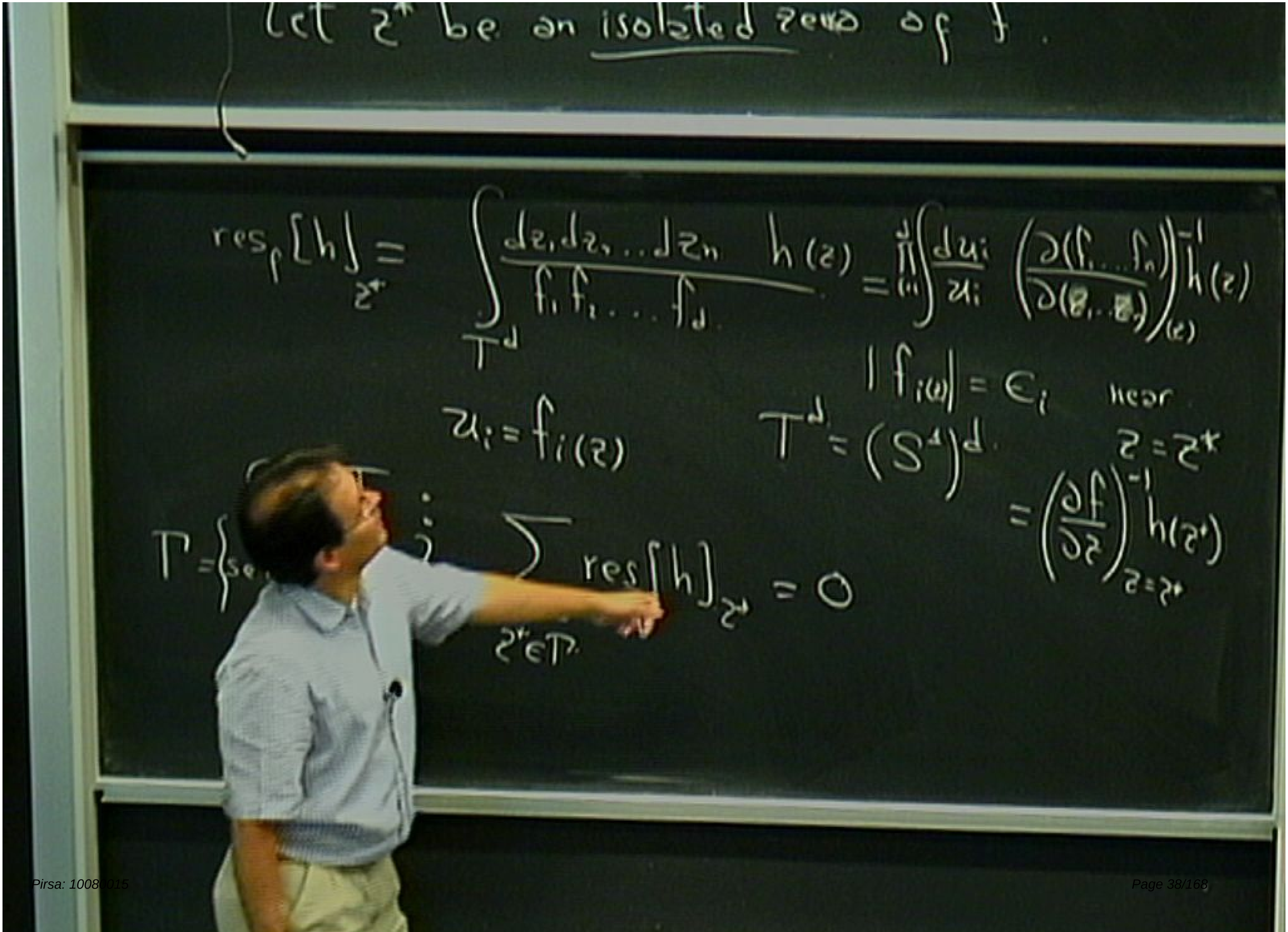
$u_i = f_{i}(z)$ $\qquad T^d = (S^1)^d$

$$= \left(\frac{\partial f}{\partial z}\right)^{-1}_{z=z^*} h(z^*)$$

RT :

T [illegible] f all z^* } $\sum$



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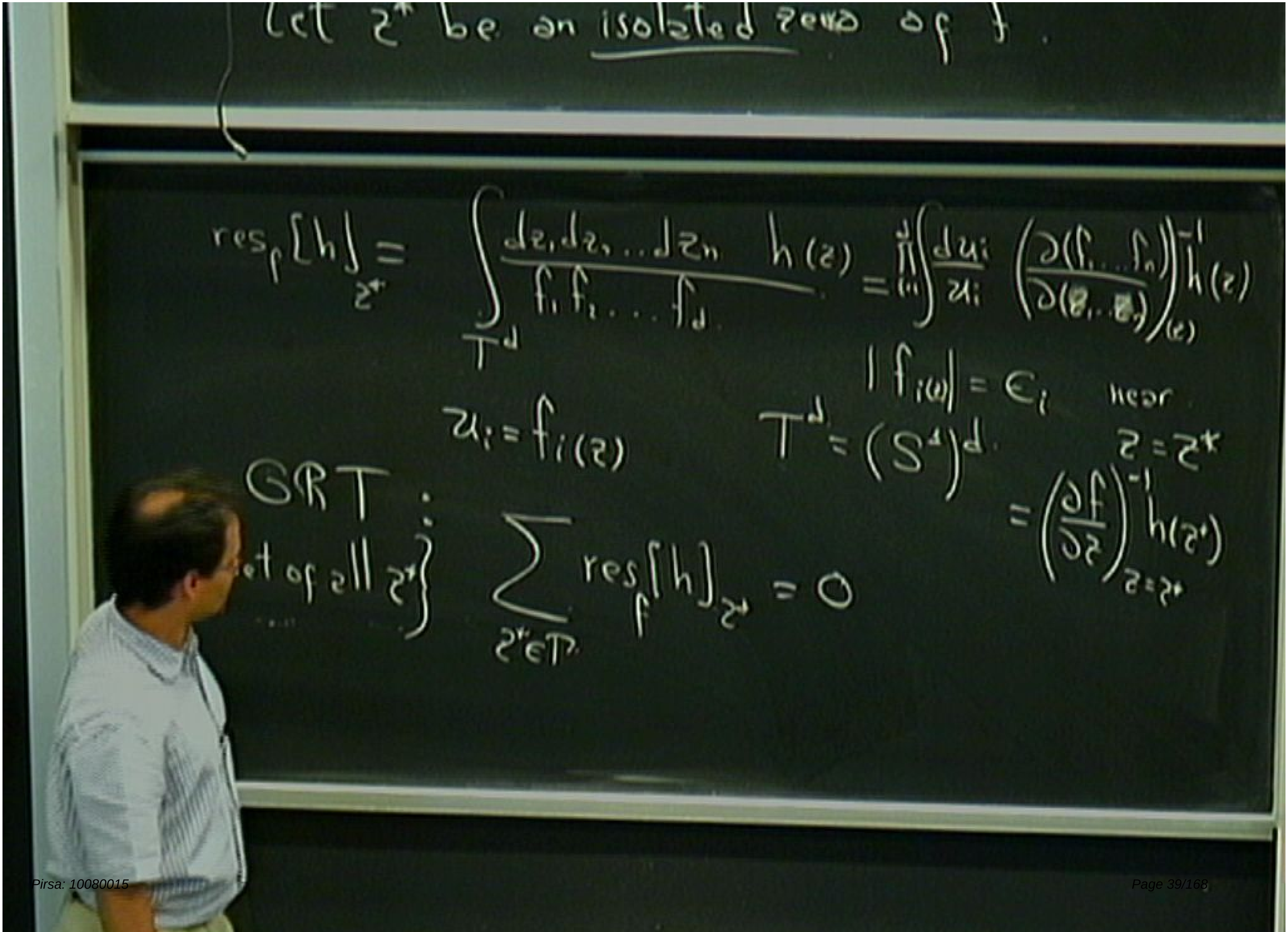
$$= \left(\frac{\partial f}{\partial z}\right)^{-1}_{z=z^*} h(z^*)$$

$\Gamma = \{ se$ $\quad$ $\sum_{z^* \in \Gamma} \operatorname{res}[h]_{z^*} = 0$



Let z^* be an isolated zero of f.

$$\mathrm{res}_f[h]_{z^*} = \int_{T^d} \frac{dz_1 dz_2 \ldots dz_n}{f_1 f_2 \ldots f_d} h(z) = \prod_{i=1}^{d} \int \frac{du_i}{u_i} \left(\frac{\partial(f_1 \ldots f_n)}{\partial([illegible], \ldots [illegible])} \right)^{-1}_{(z)} h(z)$$

$|f_i(\omega)| = \epsilon_i$ near $z = z^*$

$u_i = f_i(z)$ $T^d = (S^1)^d$

$$= \left(\frac{\partial f}{\partial z} \right)^{-1}_{z=z^*} h(z^*)$$

GRT:

et of all z^*} $\displaystyle\sum_{z^* \in P} \mathrm{res}_f[h]_{z^*} = 0$

let z^* be an isolated zero of f.

$$\operatorname{res}_f[h]_{z^*} = \int_{T^d} \frac{dz_1 dz_2 \dots dz_n \; h(z)}{f_1 f_2 \dots f_d} = \prod_{i=1}^{d} \int \frac{du_i}{u_i} \left(\frac{\partial(f_1 \dots f_n)}{\partial(z_1 \dots z_n)}\right)^{-1}_{(z)} h(z)$$

$|f_{i(\omega)}| = \epsilon_i$ near $z = z^*$

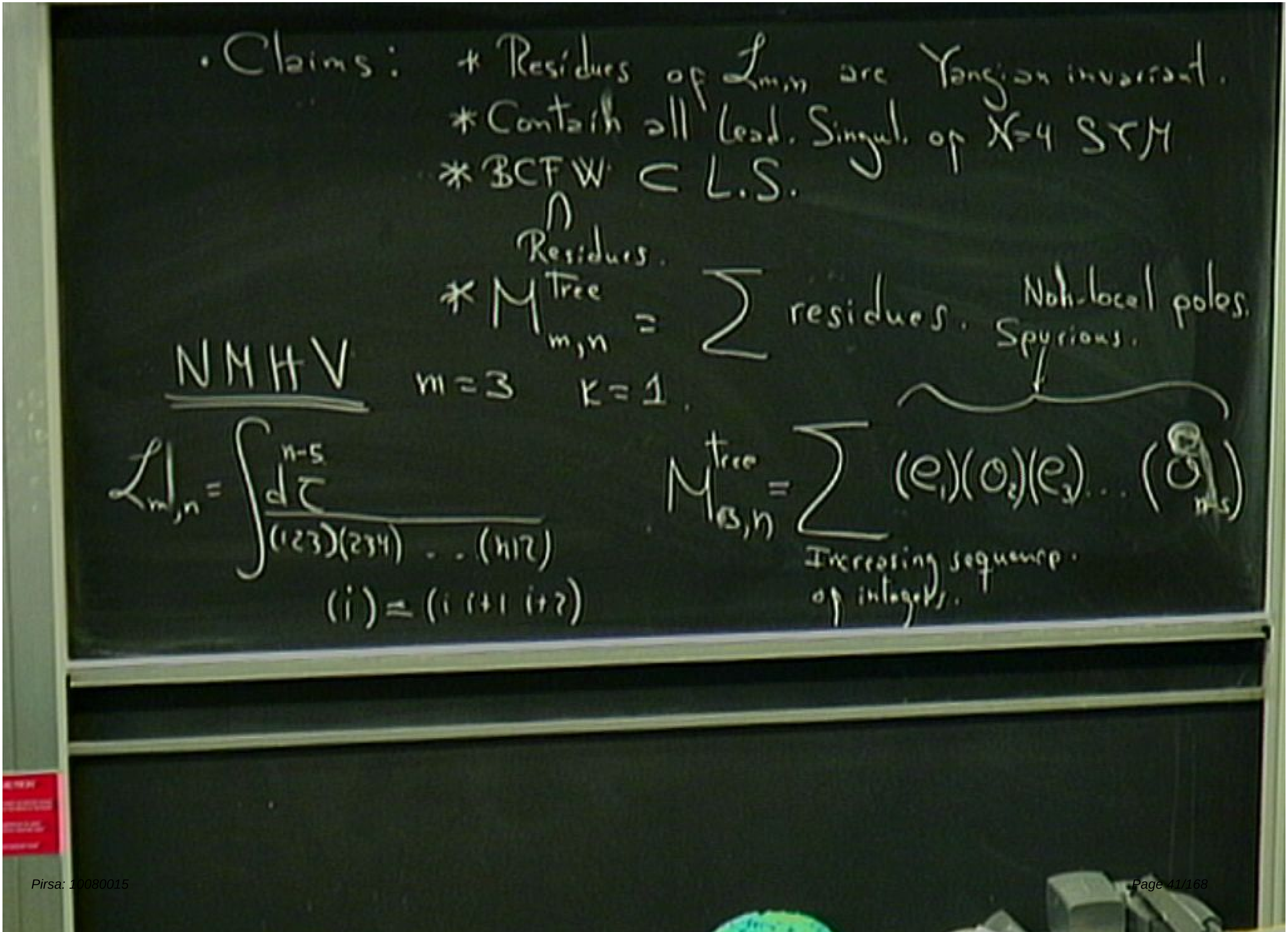
$u_i = f_{i(z)}$ $\qquad T^d = (S^1)^d$

$$= \left(\frac{\partial f}{\partial z}\right)^{-1}_{z=z^*} h(z^*)$$

GRT:

$\Gamma = \{\text{set of all } z^*\}$

$$\sum_{z^* \in \Gamma} \operatorname{res}_f[h]_{z^*} = 0$$



- Claims: * Residues of $\mathcal{L}_{m,n}$ are Yangian invariant.

  * Contain all lead. Singul. of $\mathcal{N}=4$ SYM

  * BCFW $\subset$ L.S.

  $\cap$

  Residues.

  * $M^{\text{Tree}}_{m,n} = \sum$ residues.

Non-local poles.

Spurions.

NMHV $m=3 \quad k=1.$

$$\mathcal{L}_{m,n} = \int \frac{d^{n-5}\tau}{(123)(234)\ldots(n12)}$$

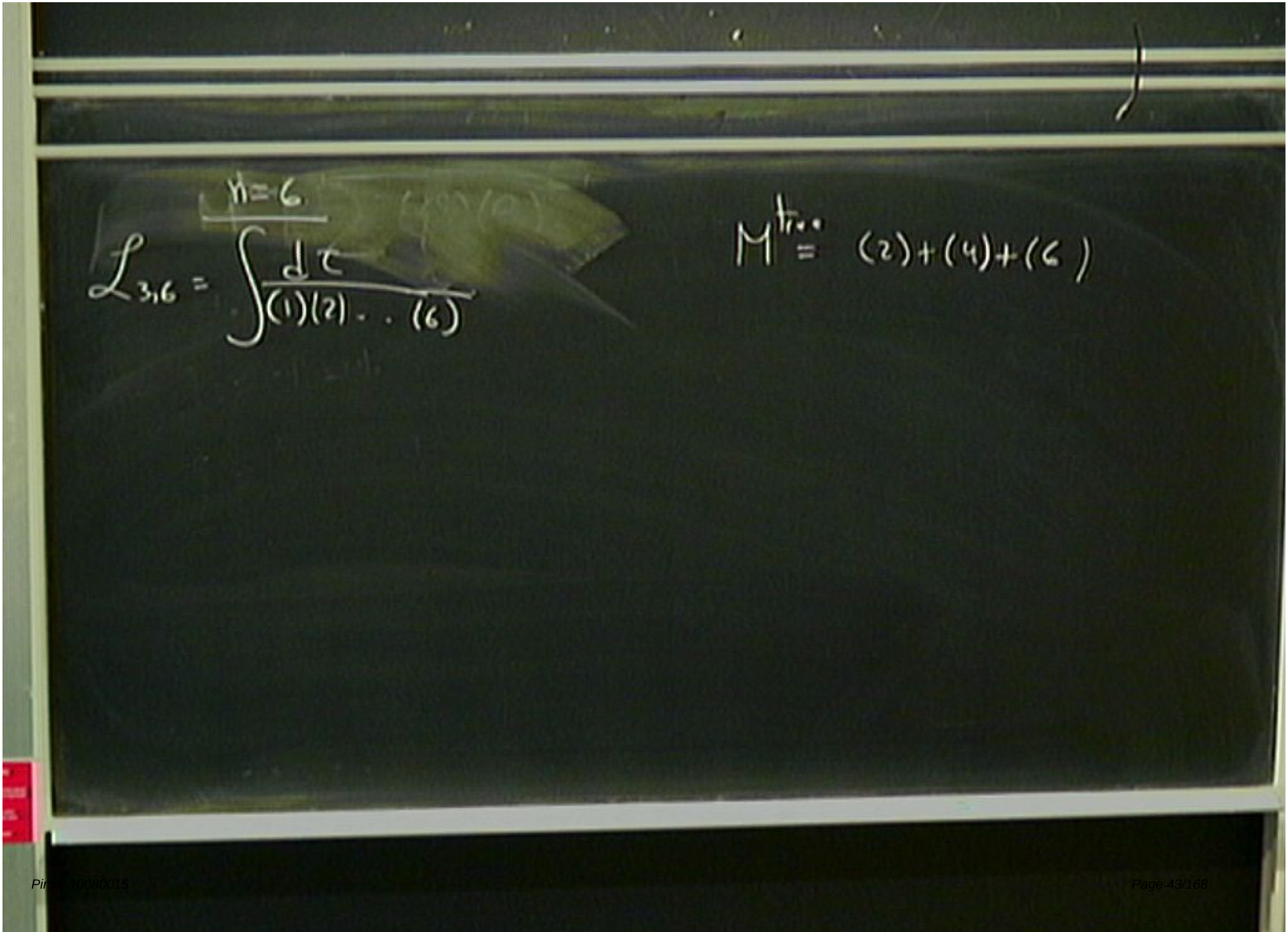
$$(i) = (i\ i{+}1\ i{+}2)$$

$$M^{\text{tree}}_{3,n} = \sum \underbrace{(e_1)(o_2)(e_3)\ldots(o_{n-5})}$$

Increasing sequence of integers.

$$k=6$$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\cdots(6)}$$



$n=6$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

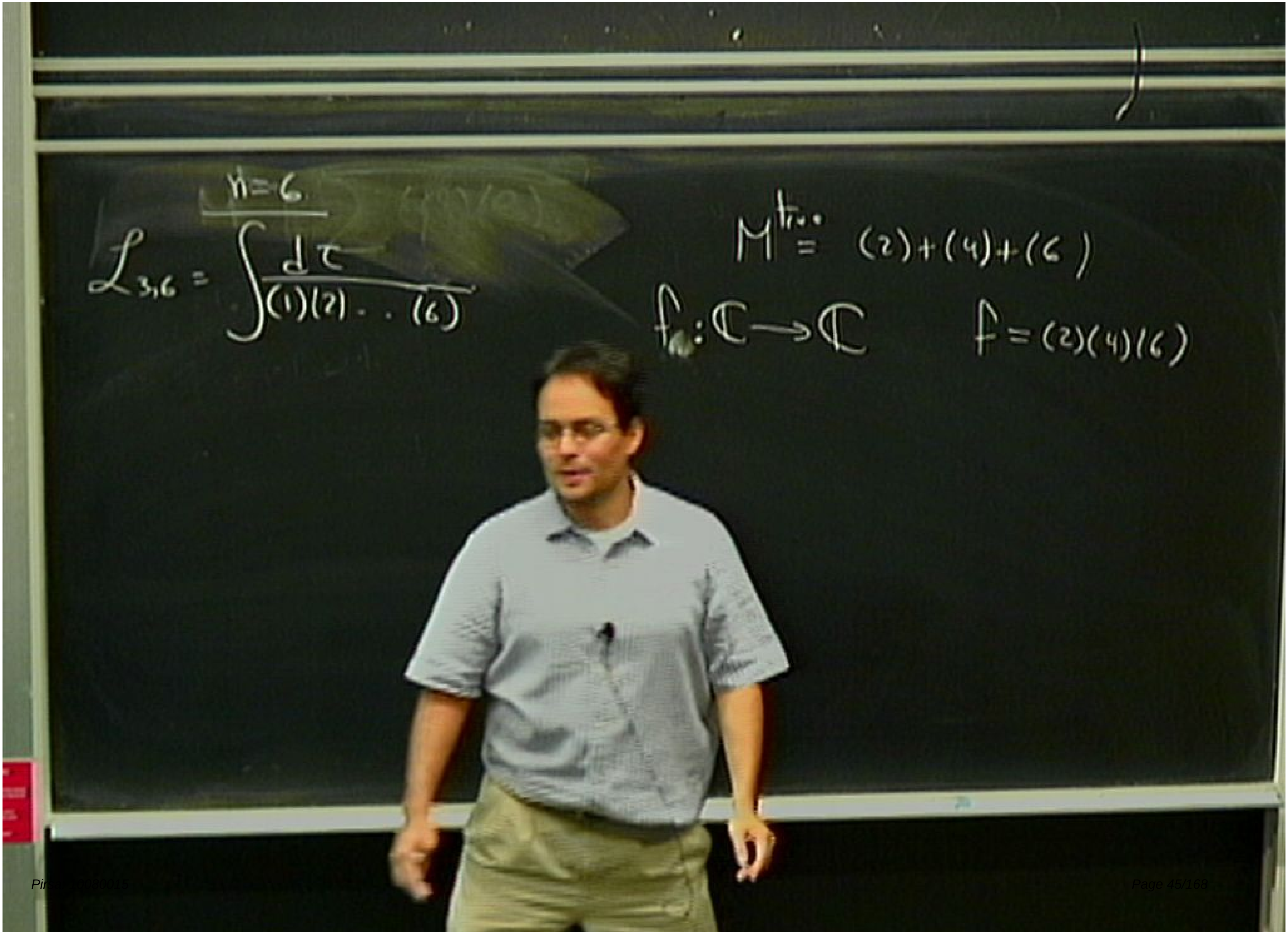
$$M^{tree} = (2)+(4)+(6)$$

$\underline{n=6}$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

$$M^{tree} = (2)+(4)+(6)$$

$$f_{\text{[illegible]}} : \mathbb{C} \longrightarrow \mathbb{C}$$



$n=6$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\cdots(6)}$$

$$M^{\text{tree}} = (2)+(4)+(6)$$

$$f: \mathbb{C} \longrightarrow \mathbb{C} \qquad f = (2)(4)(6)$$

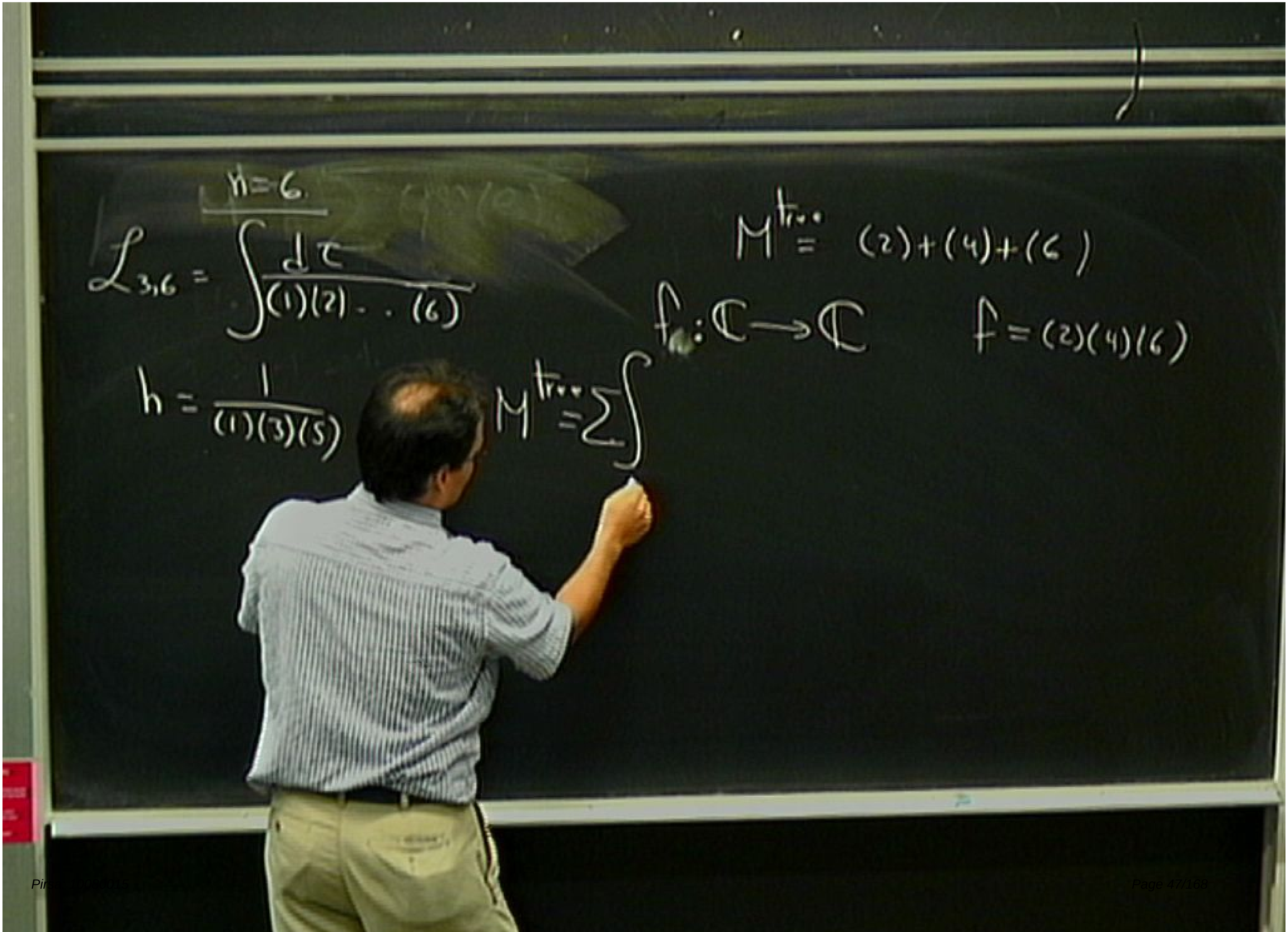
$\underline{n=6}$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

$$h = \frac{1}{(1)(3)(5)}$$

$$M^{\text{tree}} = (2)+(4)+(6)$$

$$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad f = (2)(4)(6)$$



$n=6$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

$$h = \frac{1}{(1)(3)(5)}$$

$$M^{tree} = \sum \int$$

$$M^{tree} = (2)+(4)+(6)$$

$$f_*: \mathbb{C} \longrightarrow \mathbb{C}$$

$$f = (2)(4)(6)$$

$\underline{n=6}$

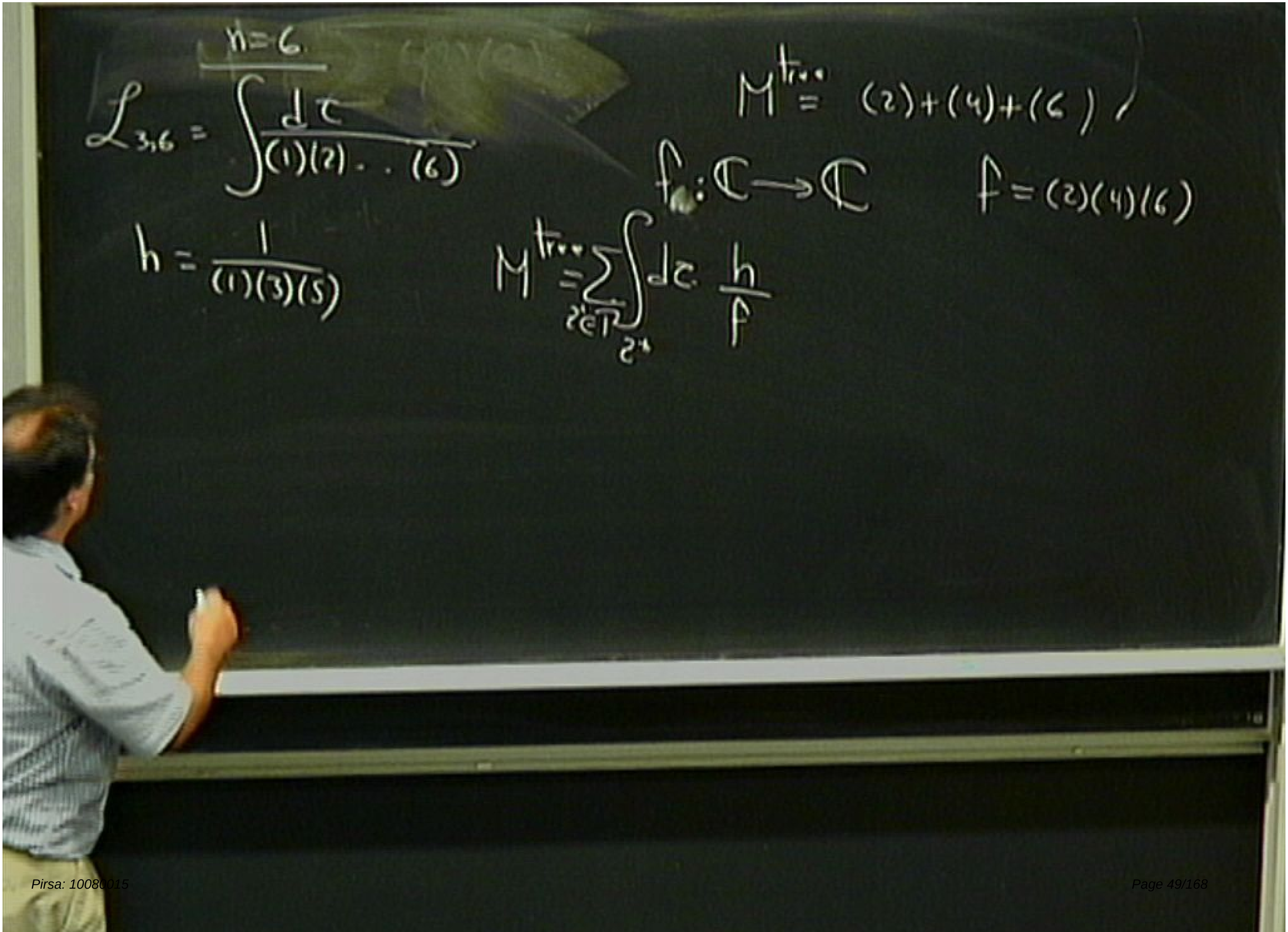
$$\mathcal{L}_{3,6} = \int \frac{dz}{(1)(2)\,.\,.\,(6)}$$

$$M^{\text{tree}} = (2)+(4)+(6)$$

$$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad f = (2)(4)(6)$$

$$h = \frac{1}{(1)(3)(5)}$$

$$M^{\text{tree}} = \sum_{z^*\in T} \int_{z^*} dz\, \frac{h}{f}$$



$n=6$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

$$h = \frac{1}{(1)(3)(5)}$$

$$M^{tree} = \sum_{\tau^* \in T} \int_{\tau^*} d\tau \, \frac{h}{f}$$

$$M^{tree} = (2)+(4)+(6)\,) \checkmark$$

$$f : \mathbb{C} \to \mathbb{C} \qquad f = (2)(4)(6)$$

$\underline{n=6}$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\,.\,.\,(6)}$$

$$h = \frac{1}{(1)(3)(5)}$$

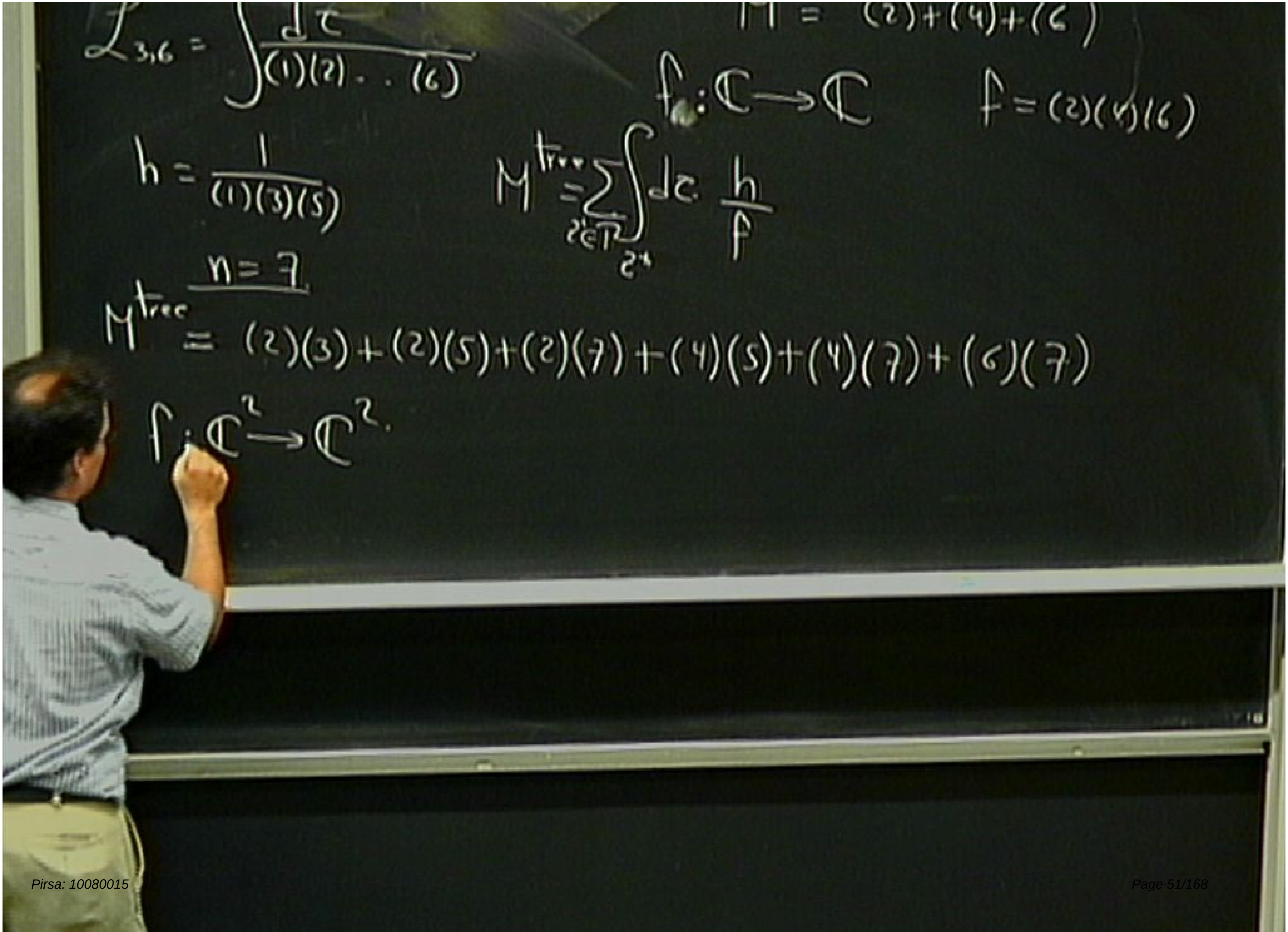
$\underline{n=7}$

$M^{tree} =$

$M^{tree} = (2)+(4)+(6)$,

$f : \mathbb{C} \to \mathbb{C}$ $\qquad f = (2)(4)(6)$

$$M^{tree} = \sum_{2' \in T} \int_{2^*} dc.\ \frac{h}{f}$$



$$\mathcal{L}_{3,6} = \int \frac{dz}{(1)(2)\ldots(6)}$$

$$M = (2)+(4)+(6)$$

$$f_{\cdot} : \mathbb{C} \longrightarrow \mathbb{C} \qquad f = (2)(4)(6)$$

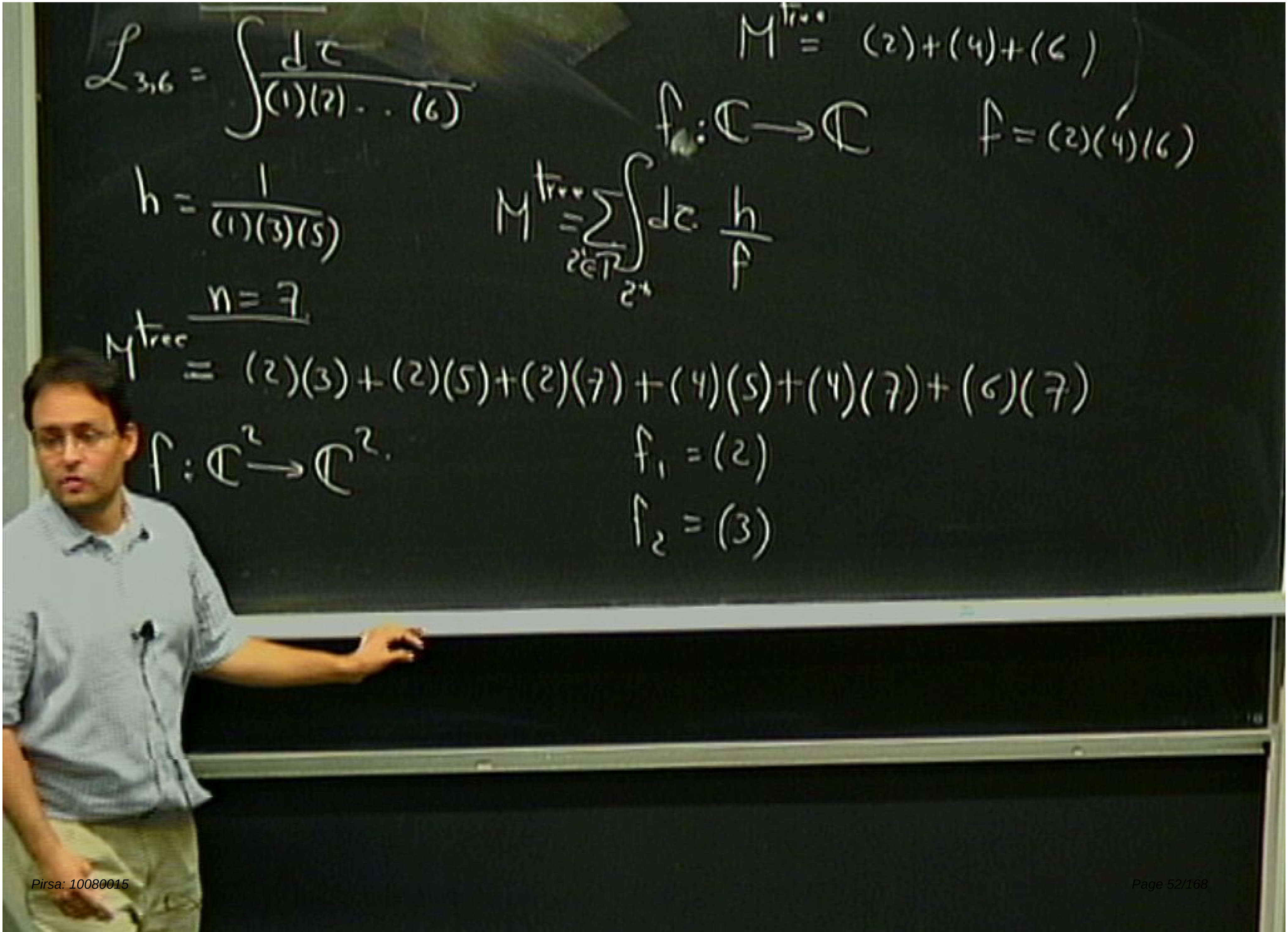
$$h = \frac{1}{(1)(3)(5)}$$

$$M^{tree} = \sum_{z^* \in T} \int_{z^*} dz \, \frac{h}{f}$$

$$\underline{n = 7}$$

$$M^{tree} = (2)(3) + (2)(5) + (2)(7) + (4)(5) + (4)(7) + (6)(7)$$

$$f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2.$$



$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(1)(2)\cdots(6)}$$

$$M^{tree} = (2)+(4)+(6)$$

$$f : \mathbb{C} \to \mathbb{C} \qquad f = (2)(4)(6)$$

$$h = \frac{1}{(1)(3)(5)}$$

$$M^{tree} = \sum_{\tau^* \in T} \int_{\tau^*} d\tau \, \frac{h}{f}$$

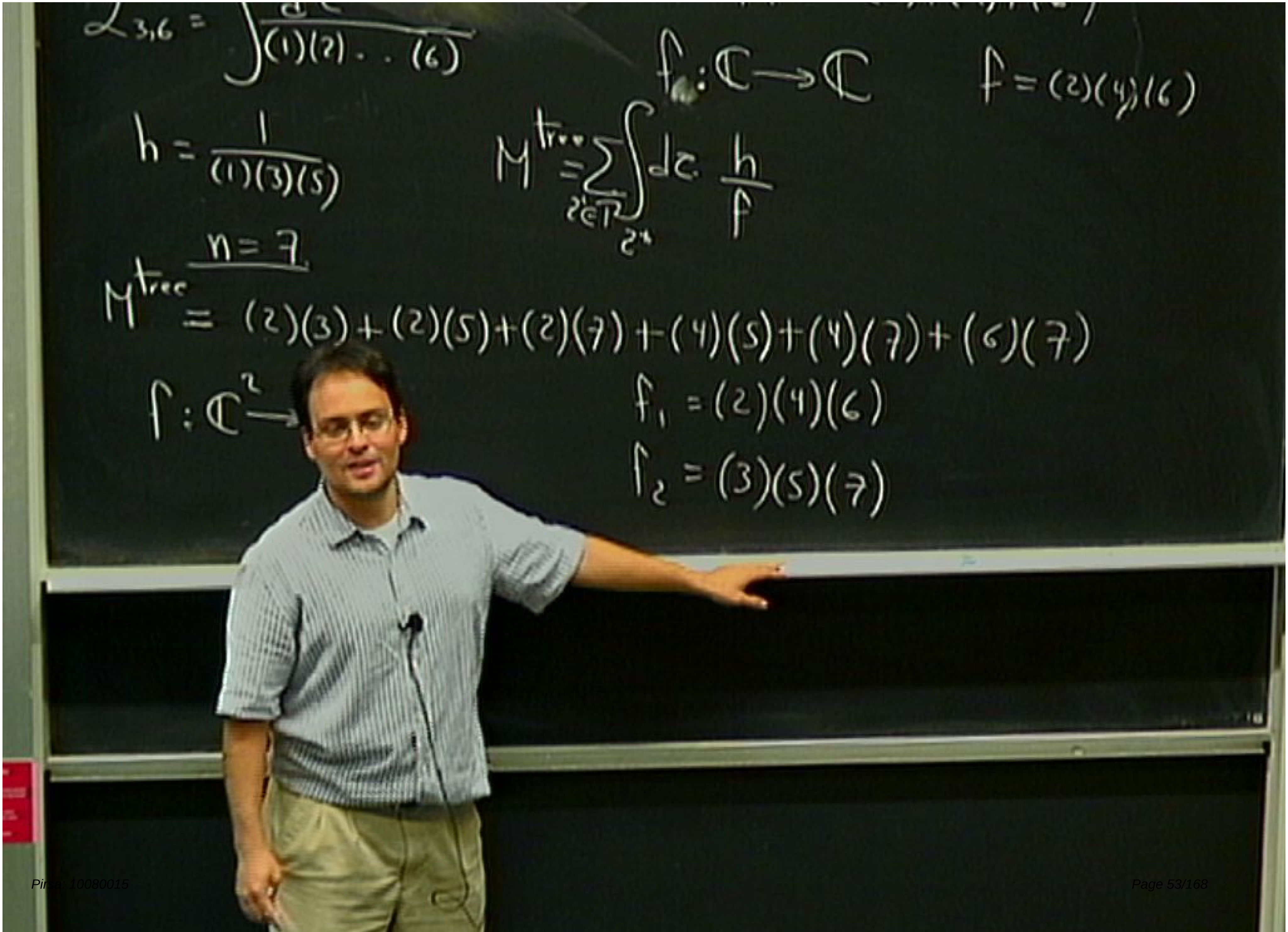
$$\underline{n = 7}$$

$$M^{tree} = (2)(3)+(2)(5)+(2)(7)+(4)(5)+(4)(7)+(6)(7)$$

$$f : \mathbb{C}^2 \to \mathbb{C}^2.$$

$$f_1 = (2)$$

$$f_2 = (3)$$



$$\mathcal{L}_{3,6} = \int \frac{dc}{(1)(2)\ldots(6)}$$

$$f_0 : \mathbb{C} \to \mathbb{C} \qquad f = (2)(4)(6)$$

$$h = \frac{1}{(1)(3)(5)} \qquad M^{tree} = \sum_{c^* \in T} \int_{c^*} dc\, \frac{h}{f}$$

$\underline{n = 7}$

$$M^{tree} = (2)(3) + (2)(5) + (2)(7) + (4)(5) + (4)(7) + (6)(7)$$

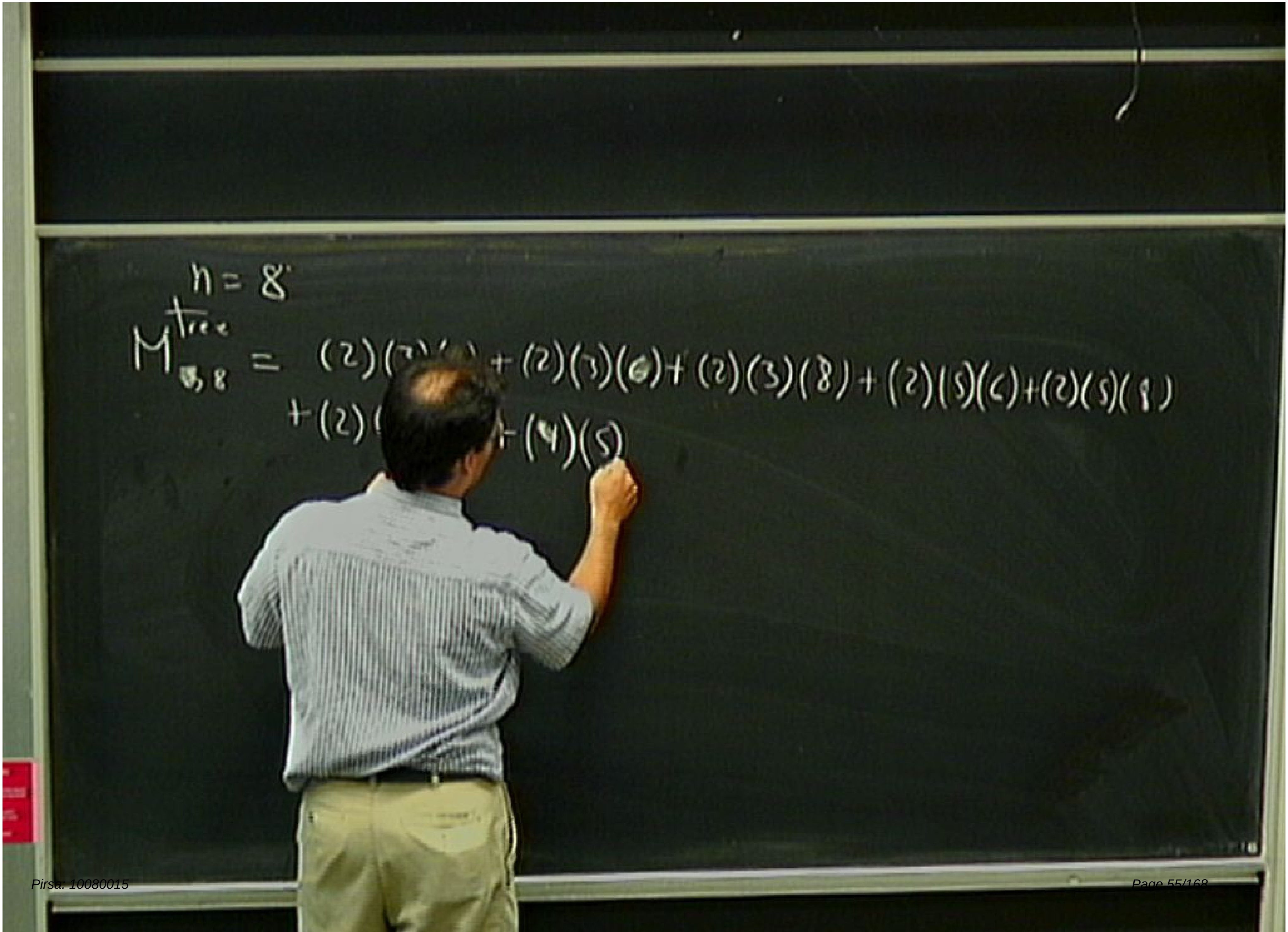
$$f : \mathbb{C}^2 \to$$

$$f_1 = (2)(4)(6)$$

$$f_2 = (3)(5)(7)$$

$n = 8$

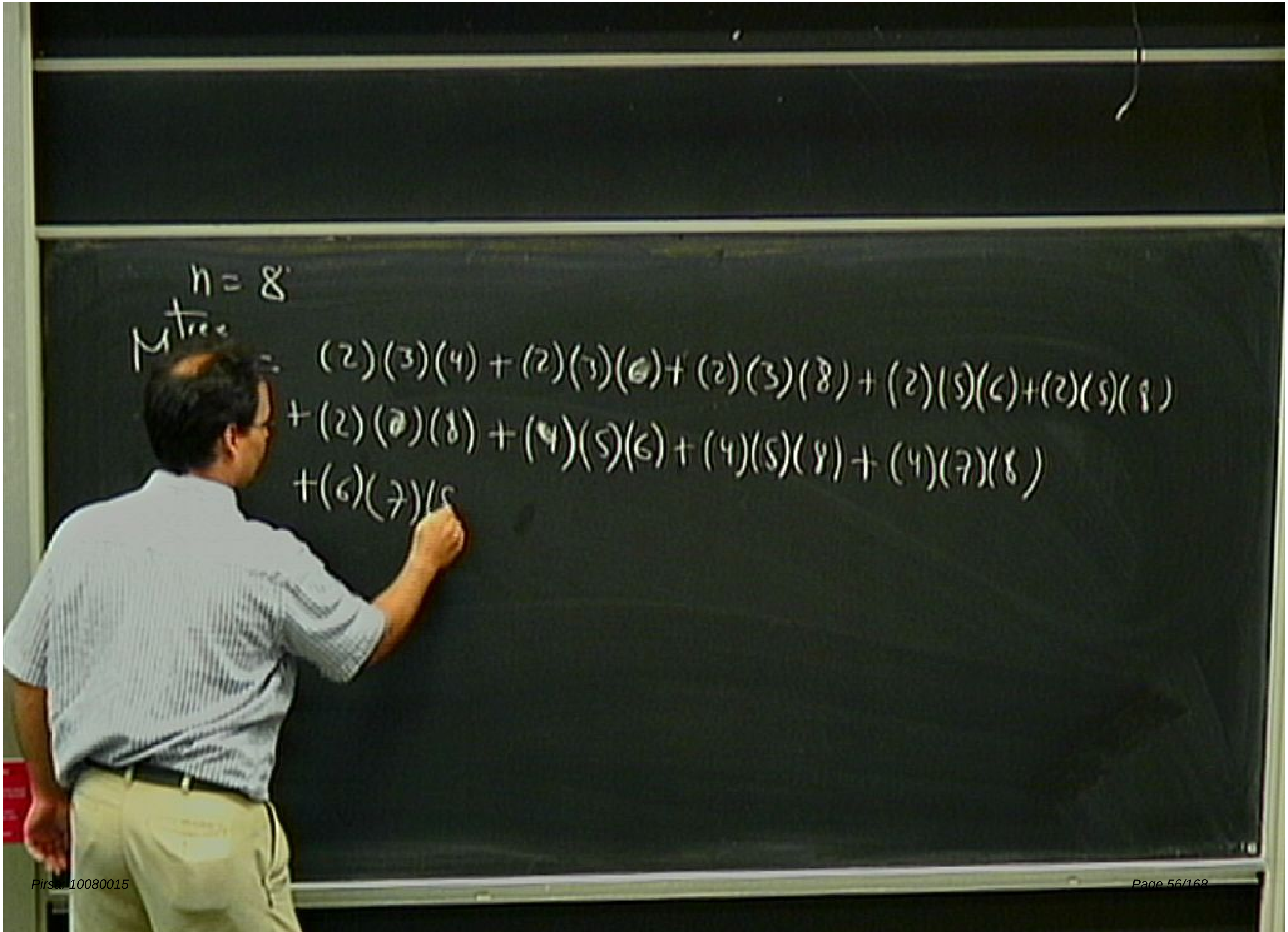
$M^{\text{tree}}_{\text{[illegible]},8} =$



$n = 8$

$$M^{\text{tree}}_{3,8} = (2)(3)(\quad + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$

$$+ (2)(\quad + (4)(5)$$

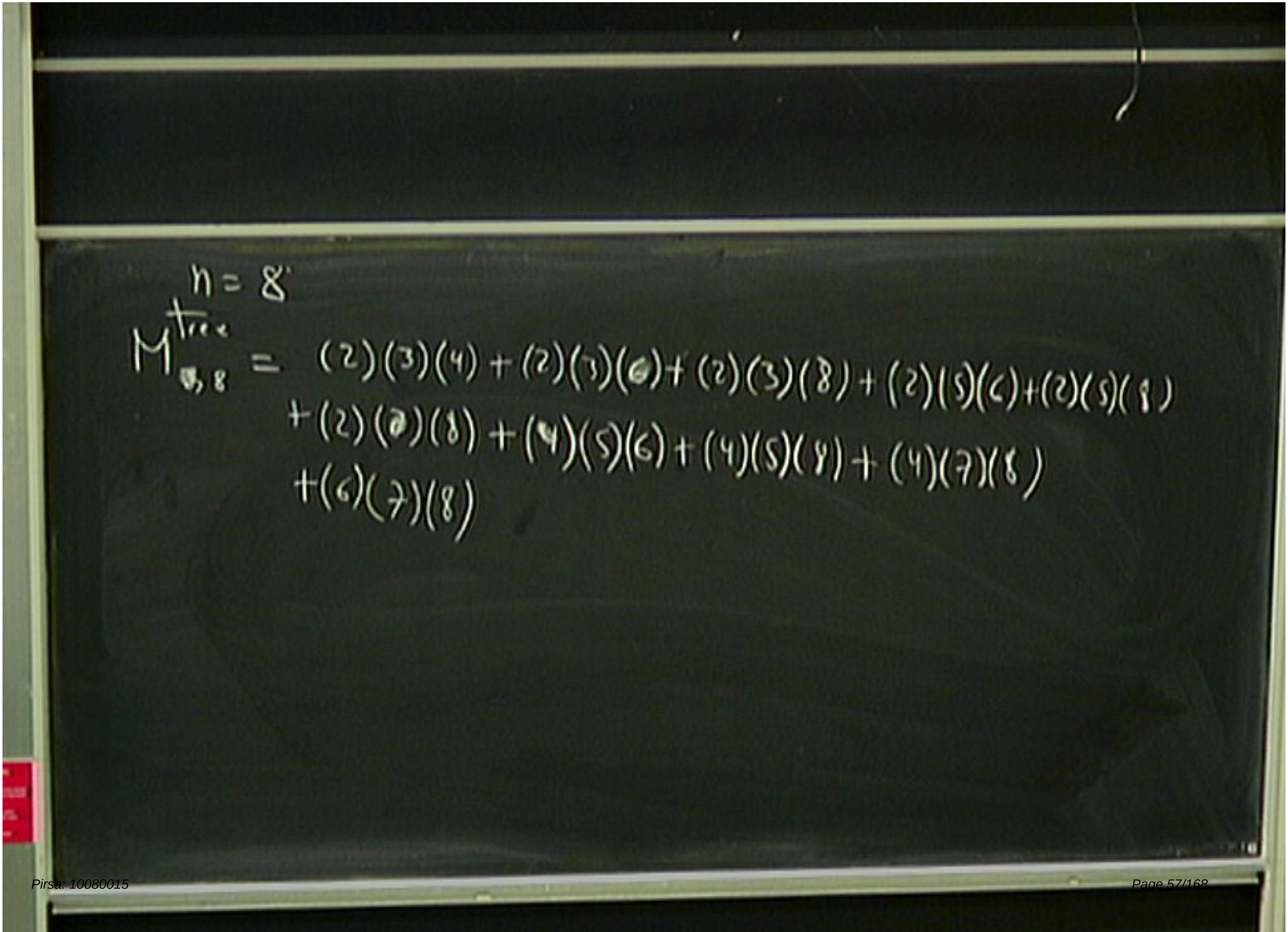


$n = 8$

$$M^{tree} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$

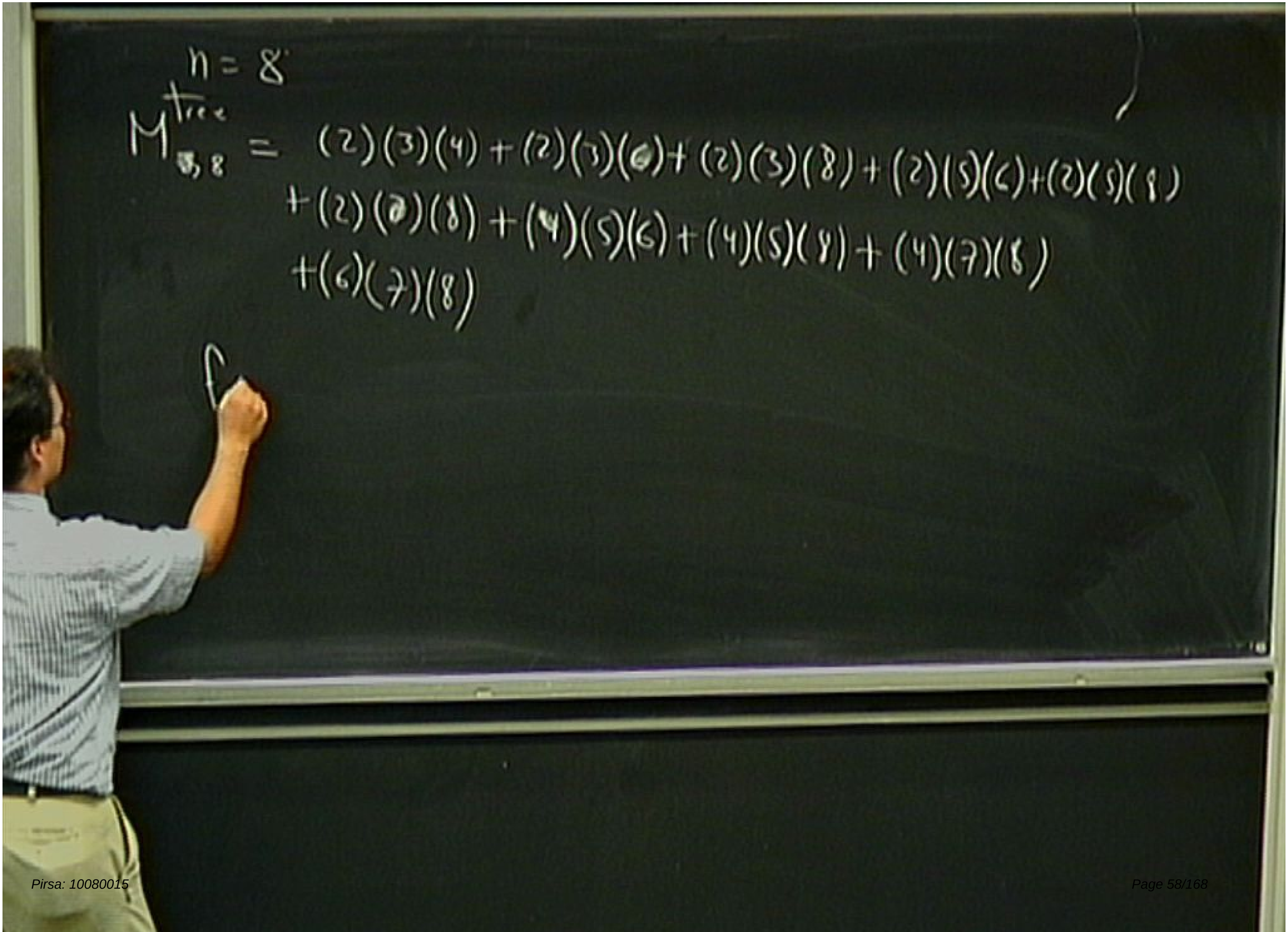
$$+ (2)(7)(8) + (4)(5)(6) + (4)(5)(8) + (4)(7)(8)$$

$$+ (6)(7)(8$$



$n = 8$

$$M^{\text{tree}}_{3,8} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$
$$+ (2)(7)(8) + (4)(5)(6) + (4)(5)(8) + (4)(7)(8)$$
$$+ (6)(7)(8)$$



$n = 8$

$$M^{tree}_{3,8} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$
$$+ (2)(7)(8) + (4)(5)(6) + (4)(5)(8) + (4)(7)(8)$$
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$n = 8$

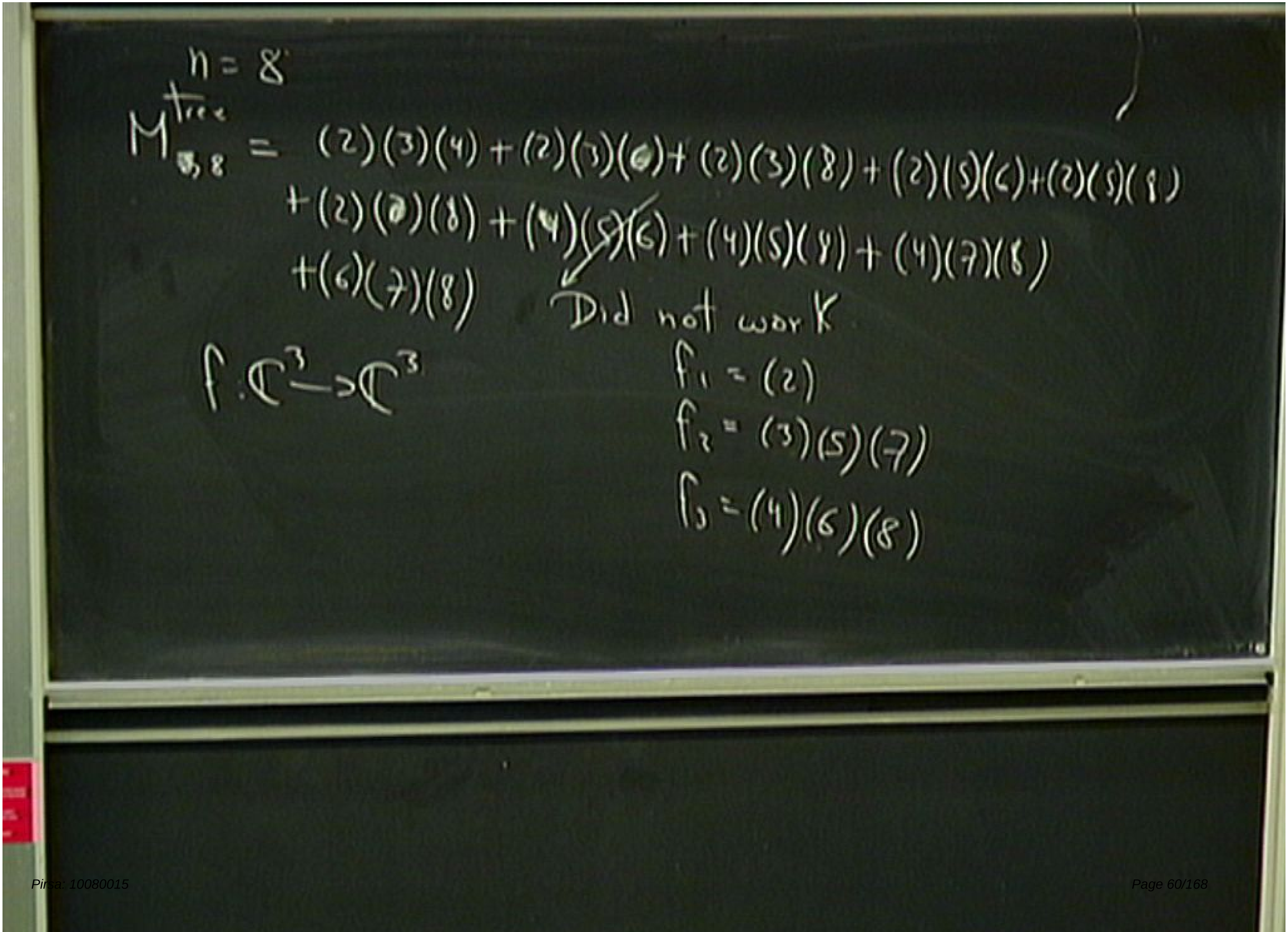
$$M^{\text{tree}}_{3,8} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$
$$+ (2)(7)(8) + (4)(5)(6) + (4)(5)(8) + (4)(7)(8)$$
$$+ (6)(7)(8)$$

$f : \mathbb{C}^3 \to \mathbb{C}^3$

$f_1 = (2)$

$f_2 = (3)(5)(7)$

$f_3 = (4)(6)(8)$



$n = 8$

$$M^{\text{tree}}_{3,8} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$
$$+ (2)(7)(8) + (4)\cancel{(5)}(6) + (4)(5)(8) + (4)(7)(8)$$
$$+ (6)(7)(8)$$

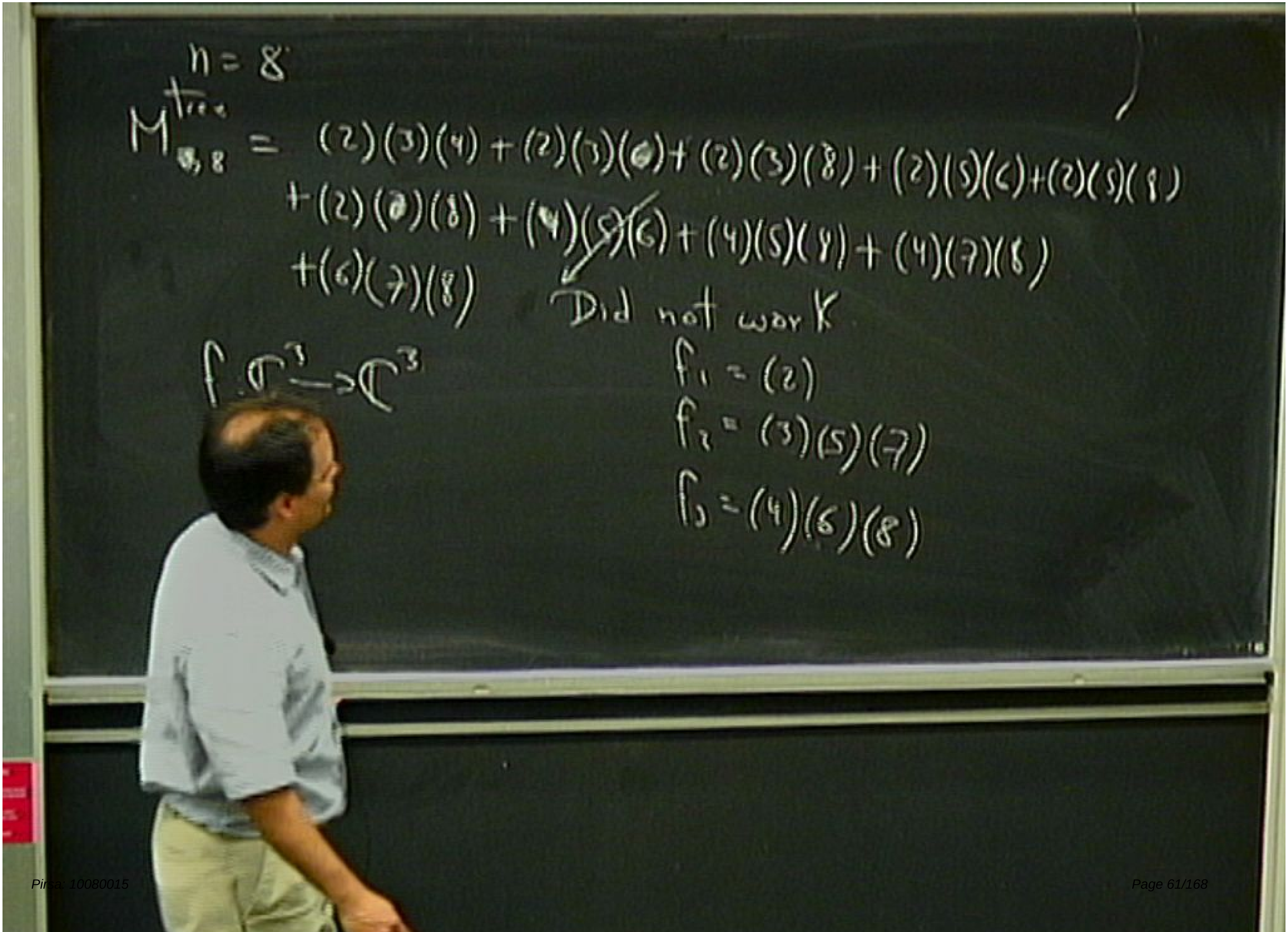
Did not work

$f : \mathbb{C}^3 \to \mathbb{C}^3$

$f_1 = (2)$

$f_2 = (3)(5)(7)$

$f_3 = (4)(6)(8)$



$n = 8$

$$M^{\text{tree}}_{3,8} = (2)(3)(4) + (2)(3)(6) + (2)(3)(8) + (2)(5)(6) + (2)(5)(8)$$
$$+ (2)(7)(8) + (4)(\cancel{5})(6) + (4)(5)(8) + (4)(7)(8)$$
$$+ (6)(7)(8)$$

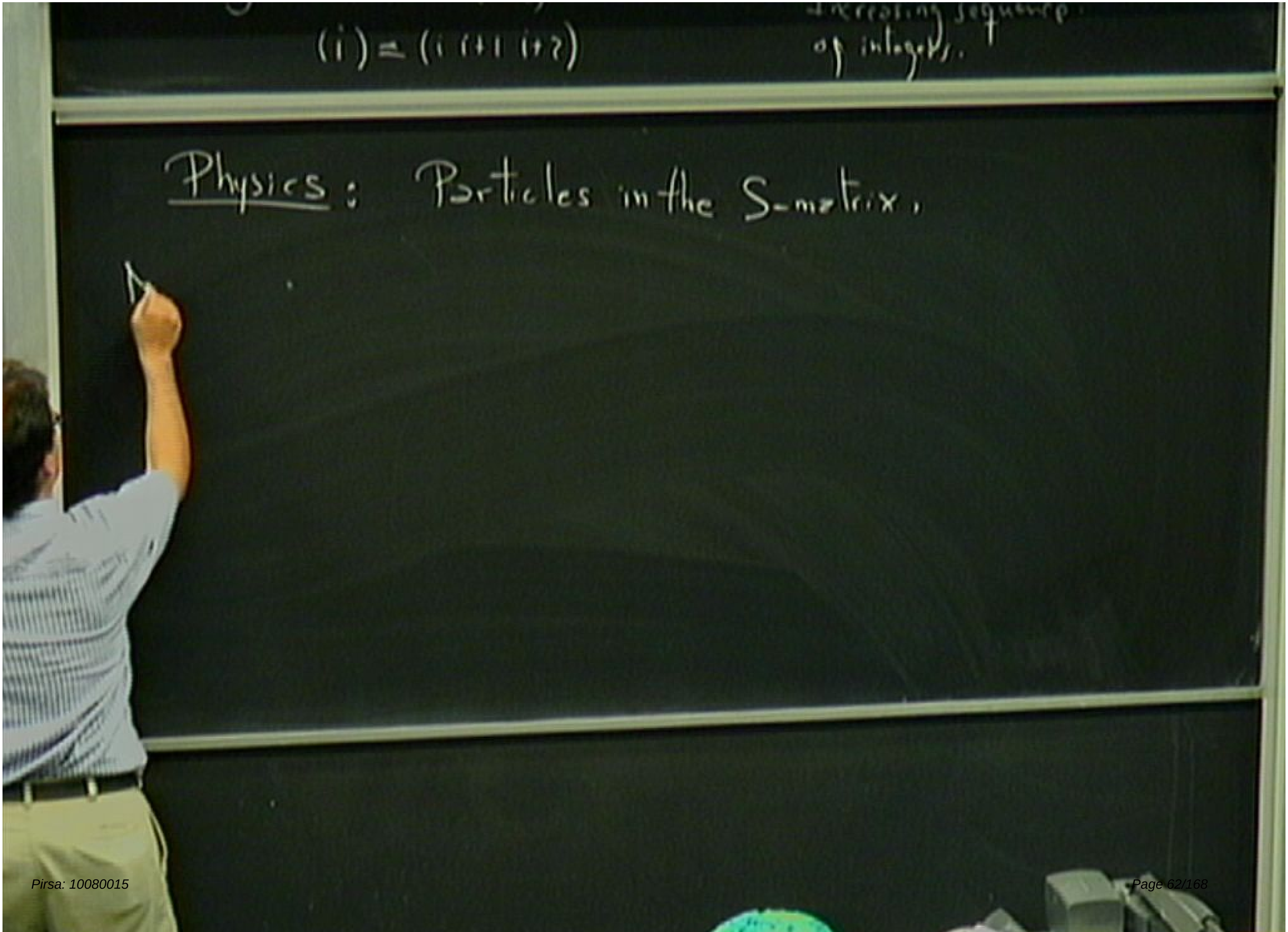
Did not work

$f : \mathbb{C}^3 \to \mathbb{C}^3$

$f_1 = (2)$

$f_2 = (3)(5)(7)$

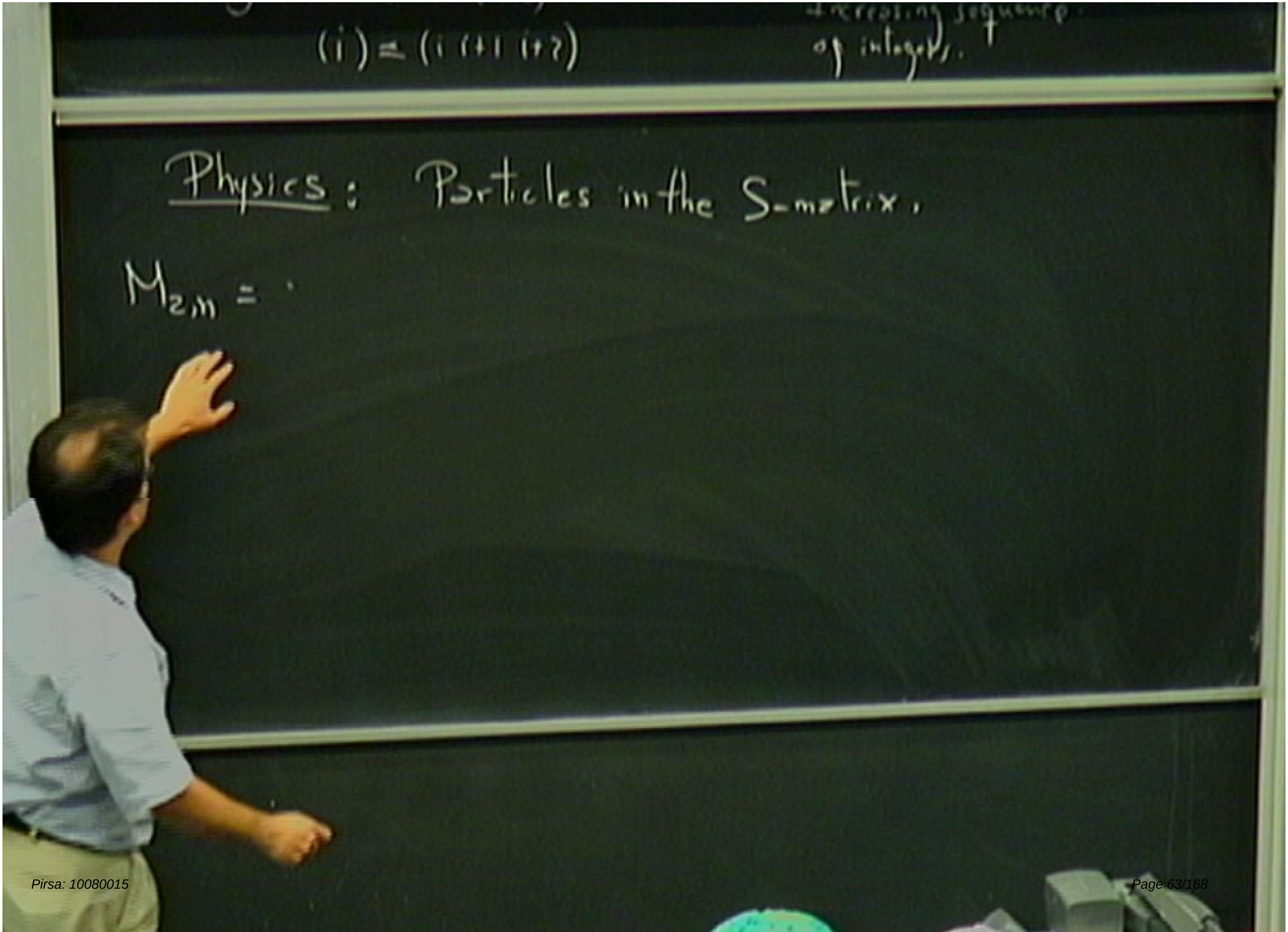
$f_3 = (4)(6)(8)$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence
of integers.

Physics: Particles in the S-matrix.

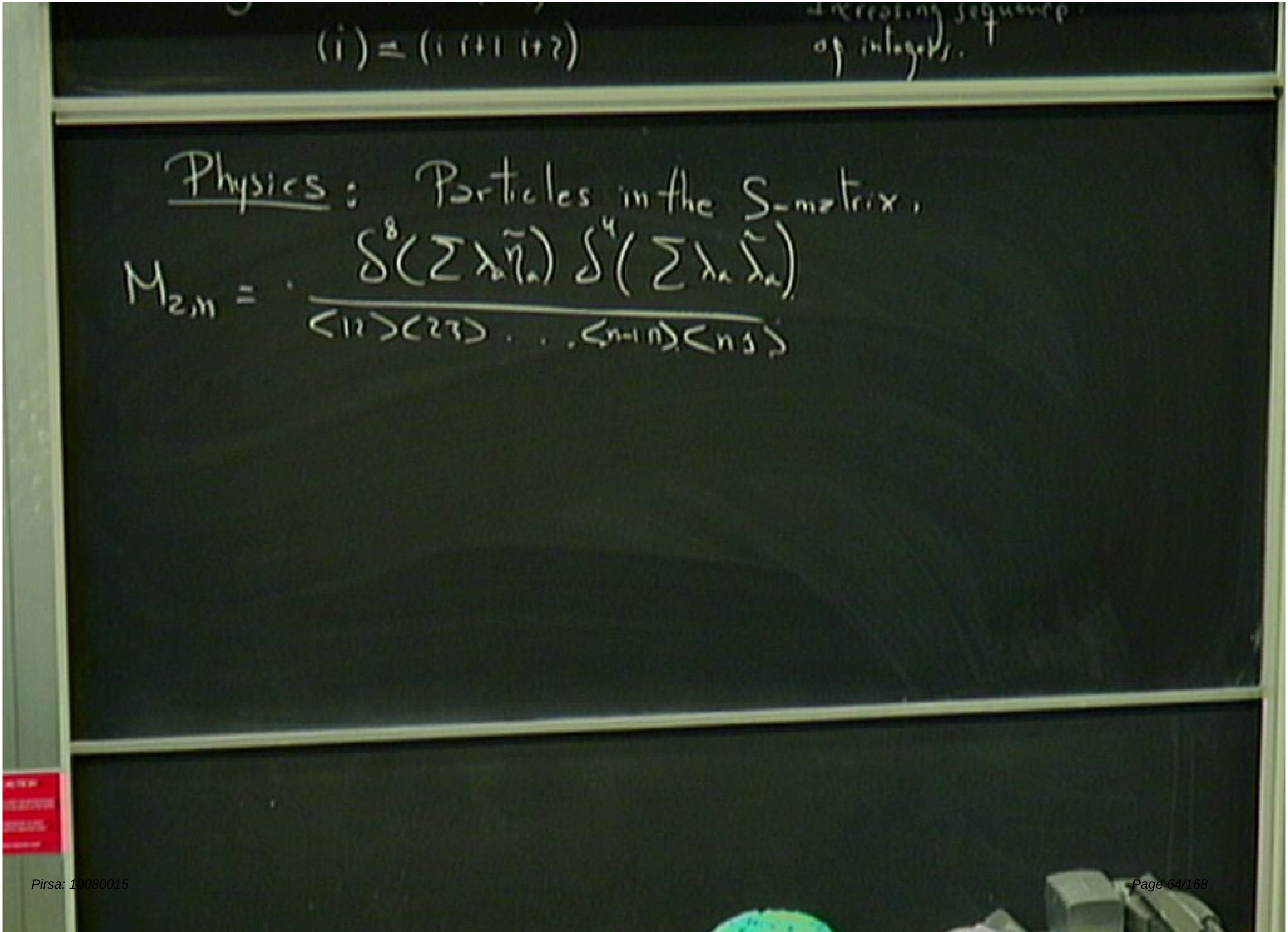


$(i) = (i \; i+1 \; i+2)$

increasing sequence of integers.

Physics: Particles in the S-matrix.

$M_{2,n} =$

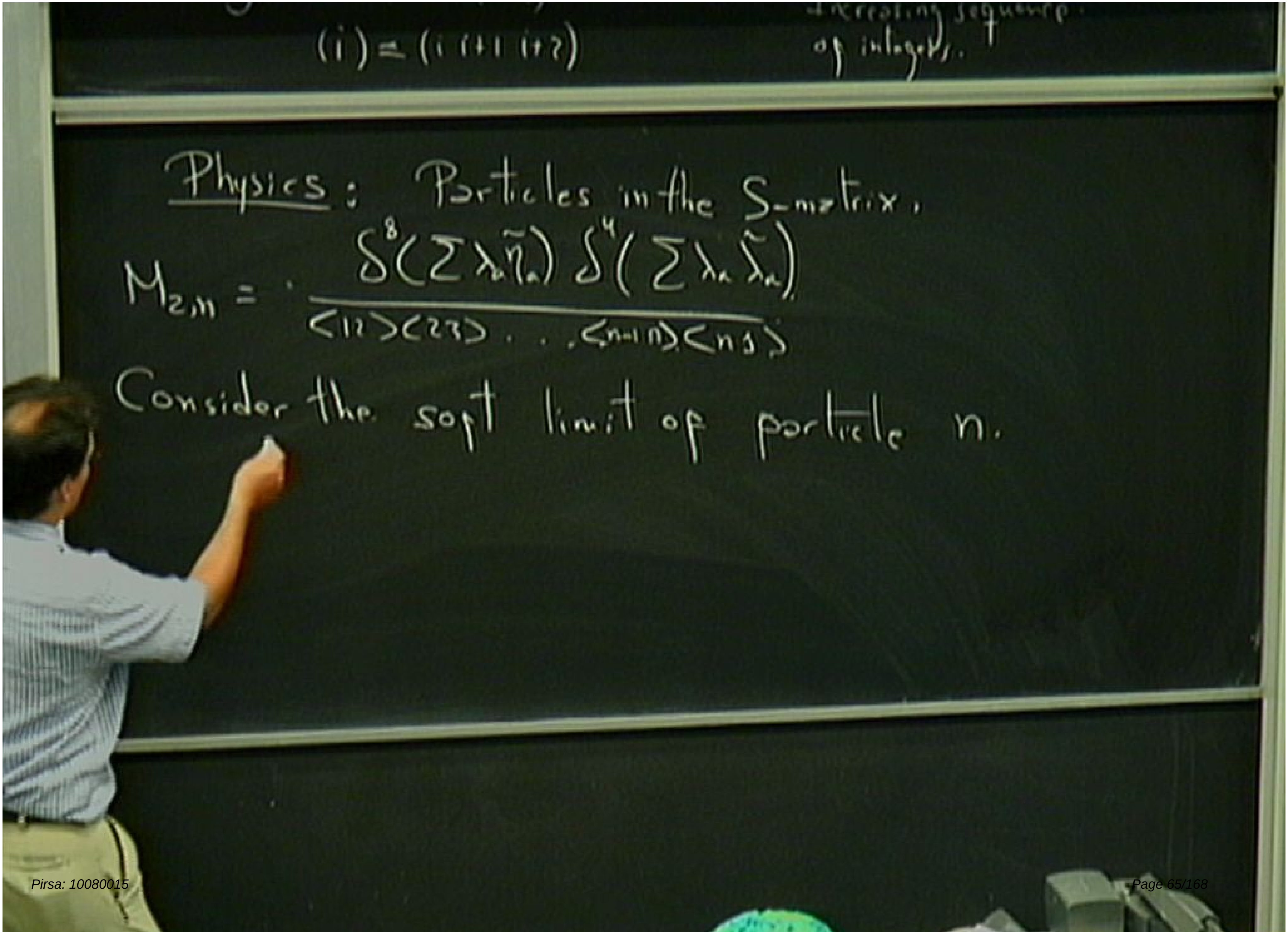


$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8\left(\sum \lambda_a \tilde{\eta}_a\right)\delta^4\left(\sum \lambda_a \tilde{\lambda}_a\right)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n 1\rangle}$$



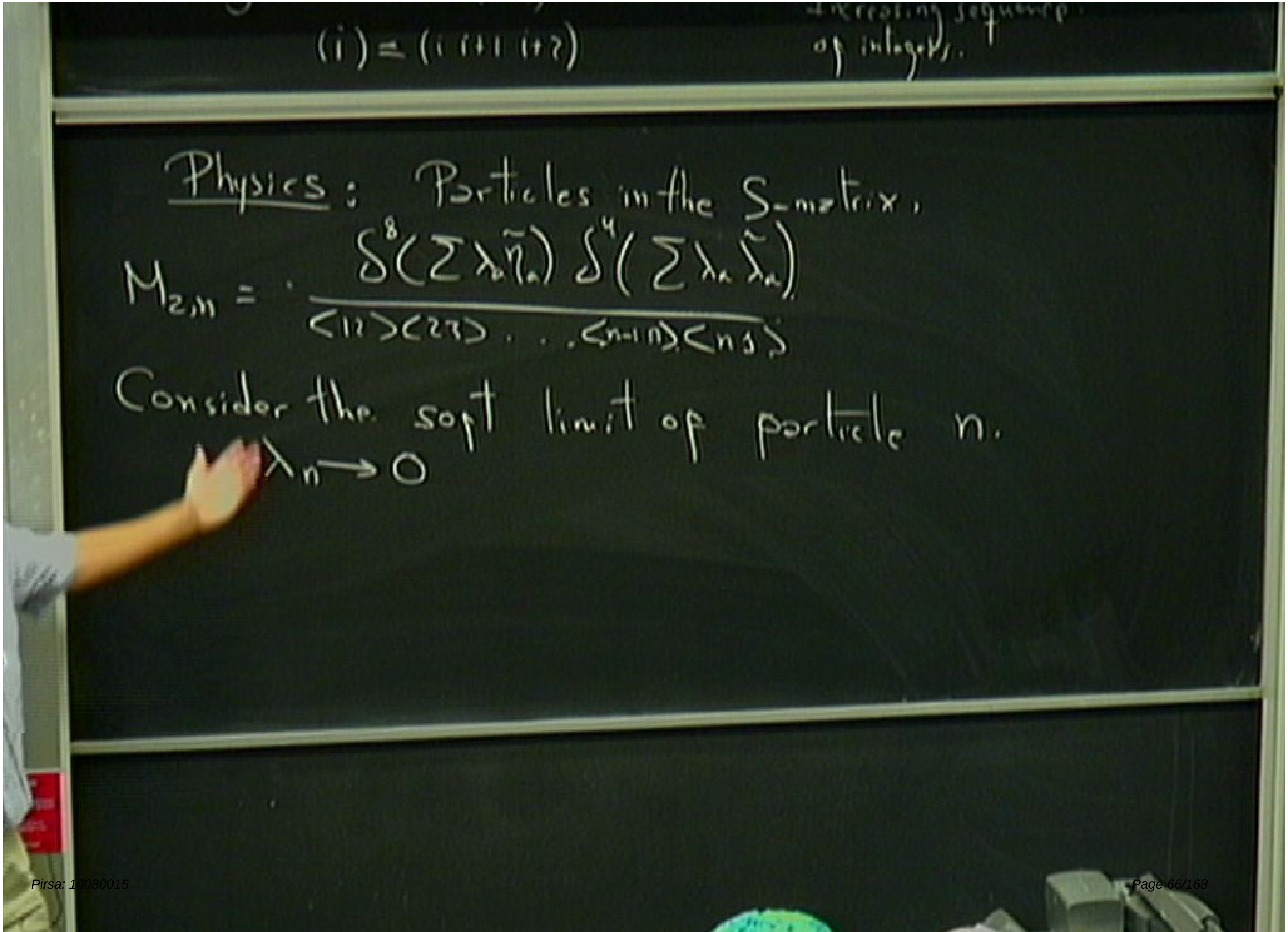
$(i) \equiv (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n 1\rangle}$$

Consider the soft limit of particle n.



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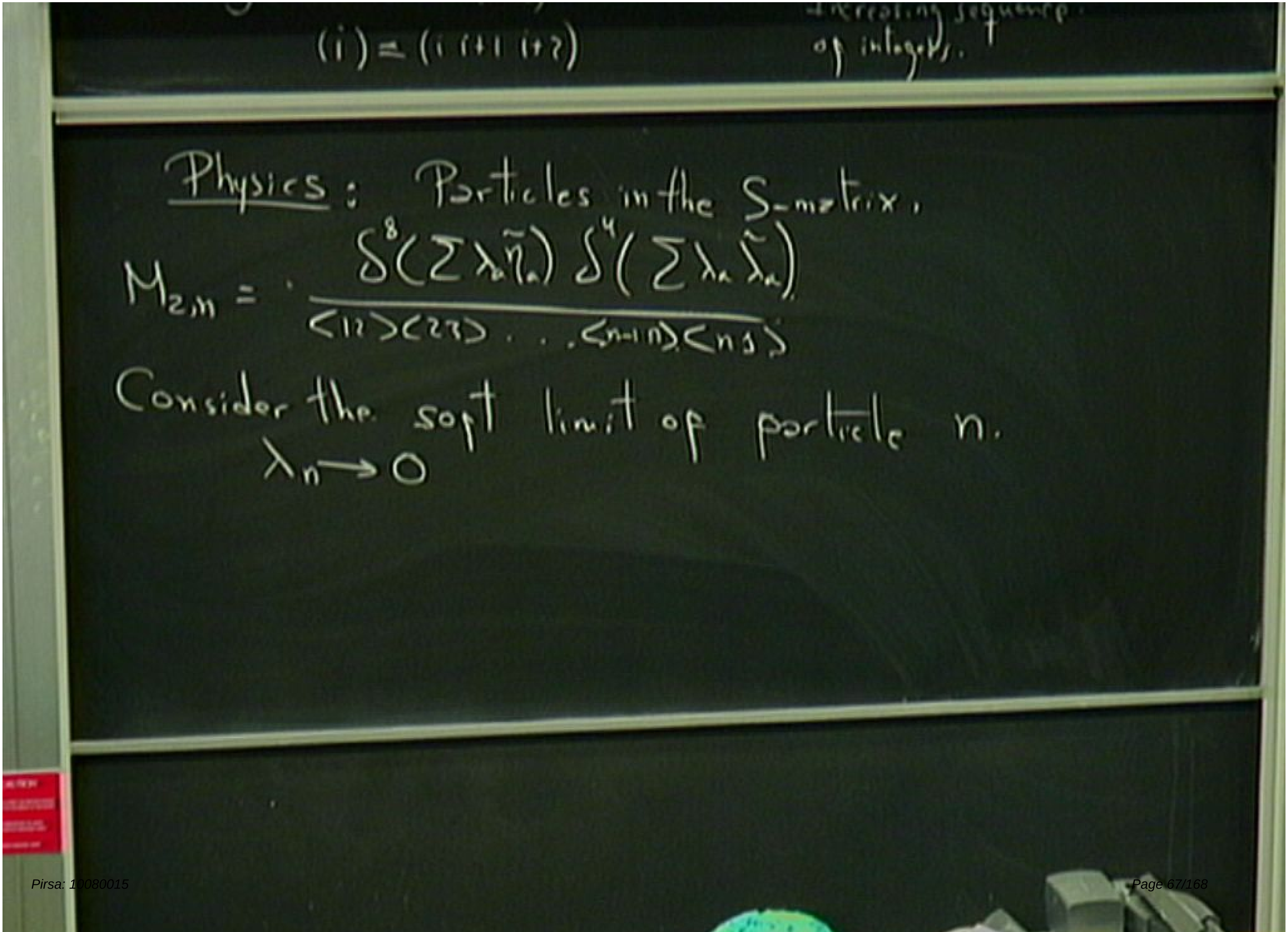
Consider the soft limit of particle n.

$\lambda_n \rightarrow 0$



$(i) = (i\ i{+}1\ i{+}2)$

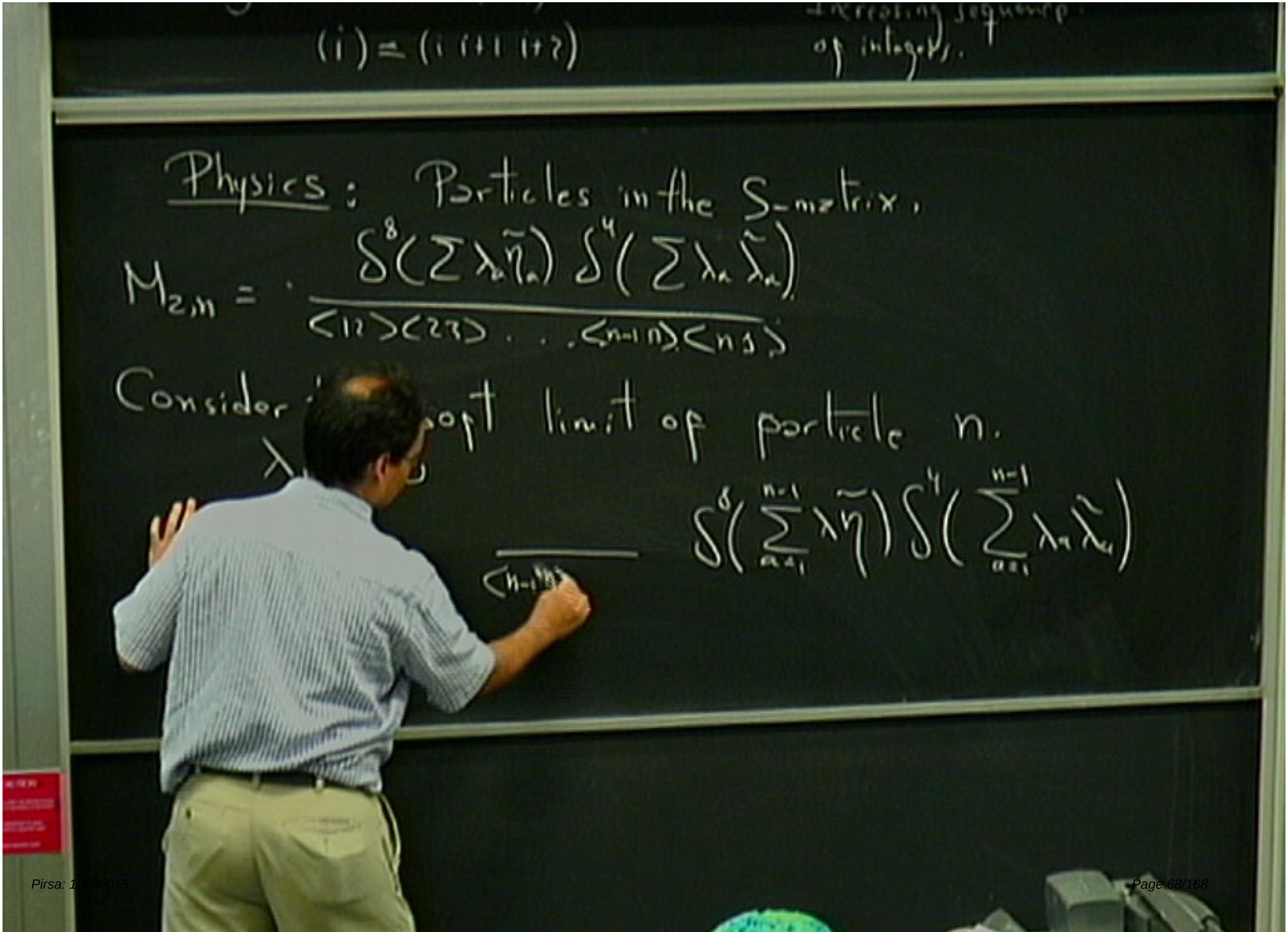
increasing sequence of integers.

<u>Physics</u>: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle\ldots\langle n{-}1\,n\rangle\langle n 1\rangle}$$

Consider the soft limit of particle n.

$\lambda_n \to 0$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

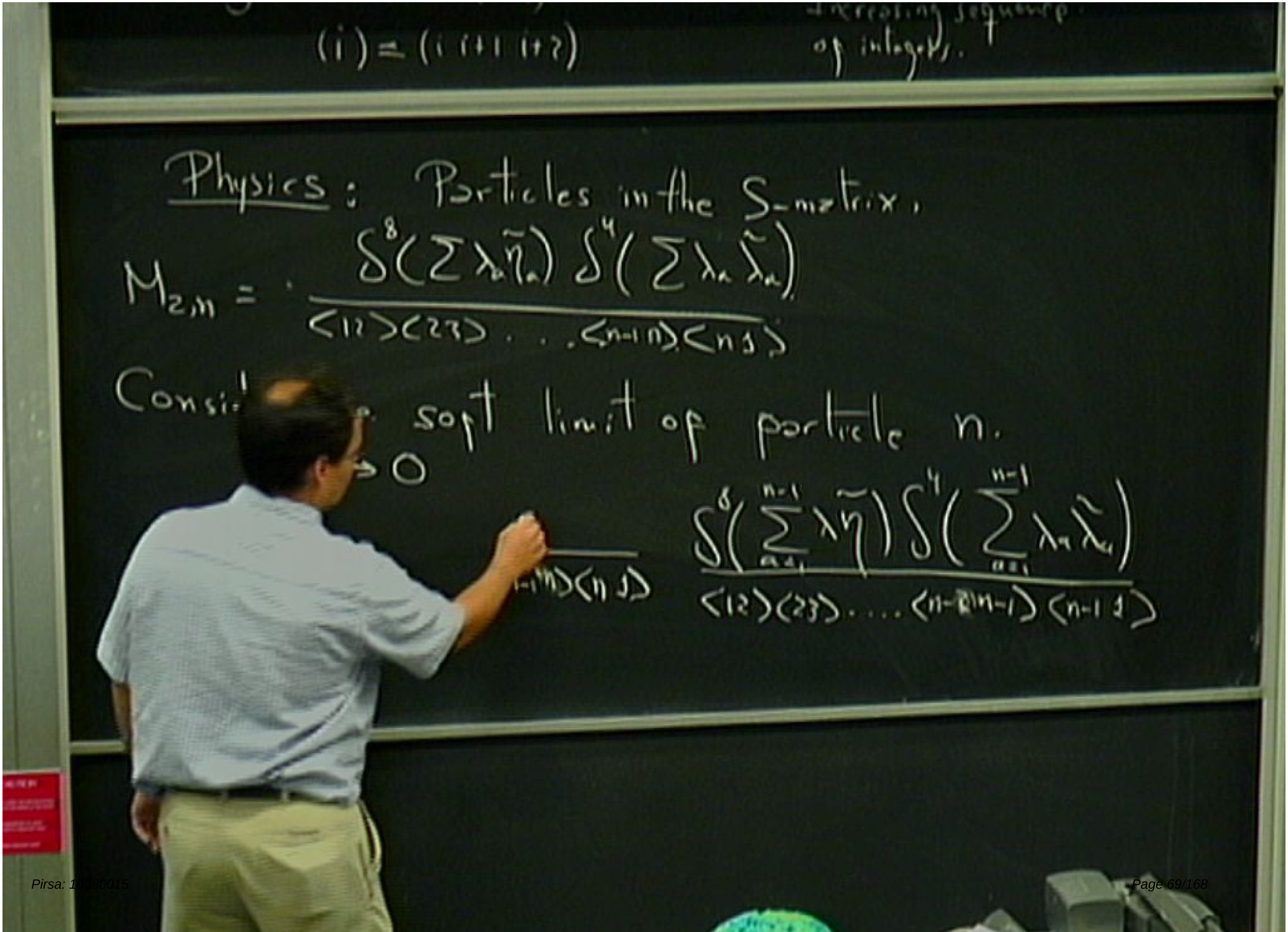
Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\, \delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n 1\rangle}$$

Consider [illegible] oft limit of particle n.

λ_n

$$\frac{\quad}{\langle n{-}1\, n\rangle} \quad \delta^8\Big(\sum_{a=1}^{n-1} \lambda \tilde{\eta}\Big)\, \delta^4\Big(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a\Big)$$



$(i) = (i\ i{+}1\ i{+}2)$

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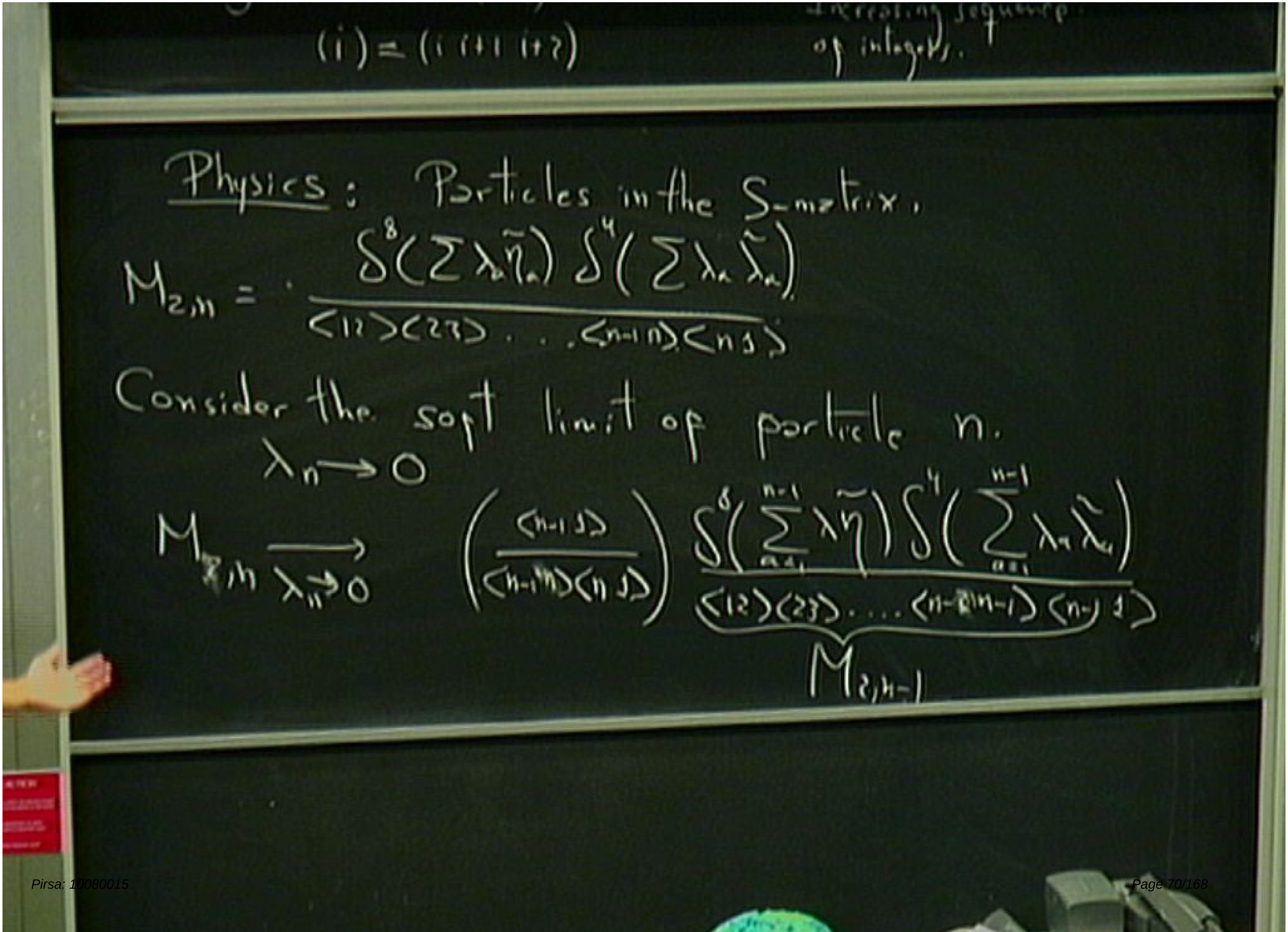
<u>Physics</u>: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\, \delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$

Consi[illegible] soft limit of particle n.

$[illegible] \to 0$

$$\frac{[illegible]}{[illegible]\, n{-}1\, n\rangle\langle n\, 1\rangle} \qquad \frac{\delta^8(\sum_{a=1}^{n-1} \lambda \tilde{\eta})\, \delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}2\, n{-}1\rangle\langle n{-}1\, 1\rangle}$$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence
of integers.

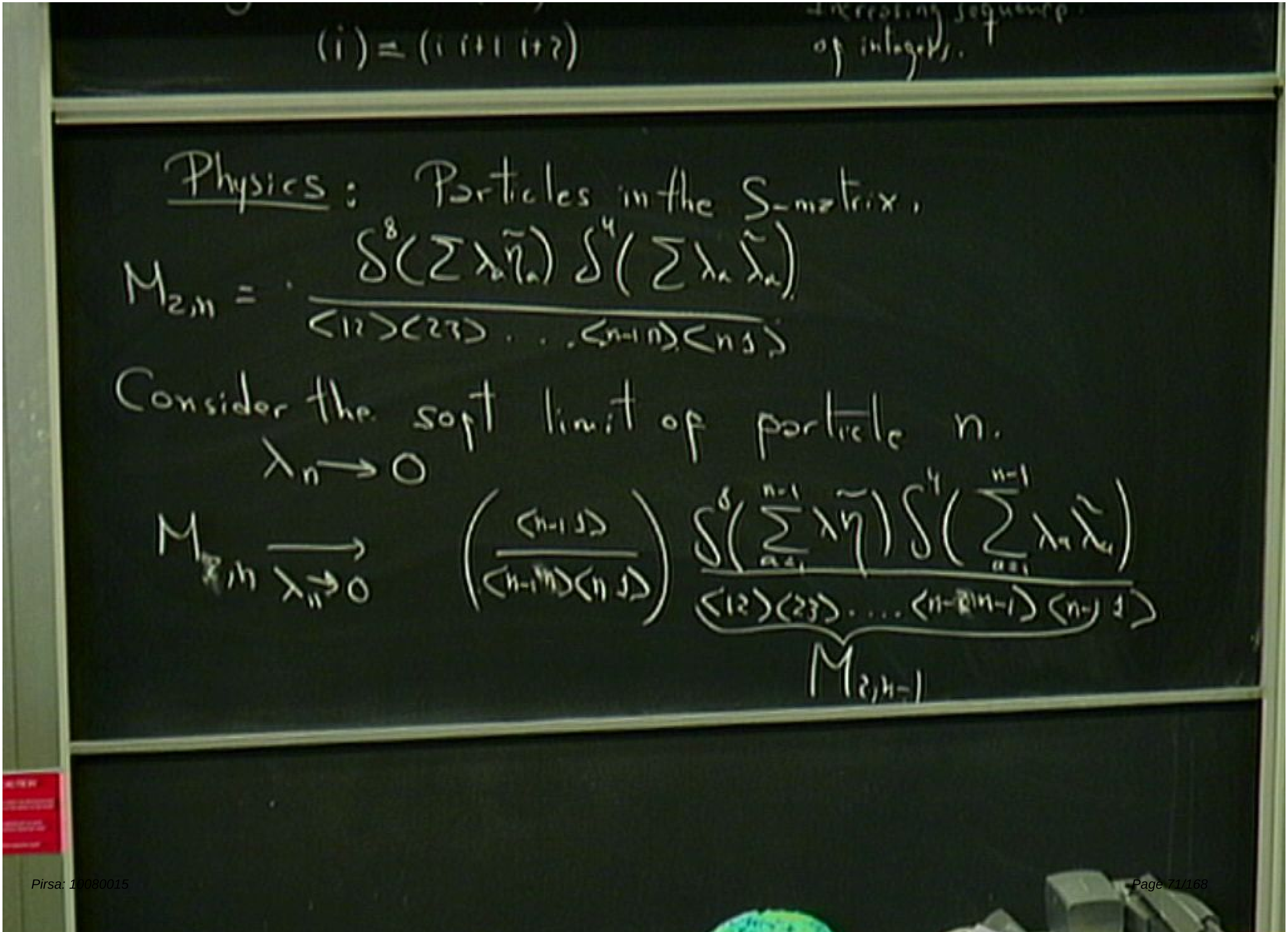
<u>Physics</u>: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$

Consider the soft limit of particle n.

$$\lambda_n \to 0$$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\ 1\rangle}{\langle n{-}1\ n\rangle\langle n\ 1\rangle}\right) \underbrace{\frac{\delta^8(\sum_{a=1}^{n-1} \lambda \tilde{\eta})\,\delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}2\ n{-}1\rangle\langle n{-}1\ 1\rangle}}_{M_{2,n-1}}$$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

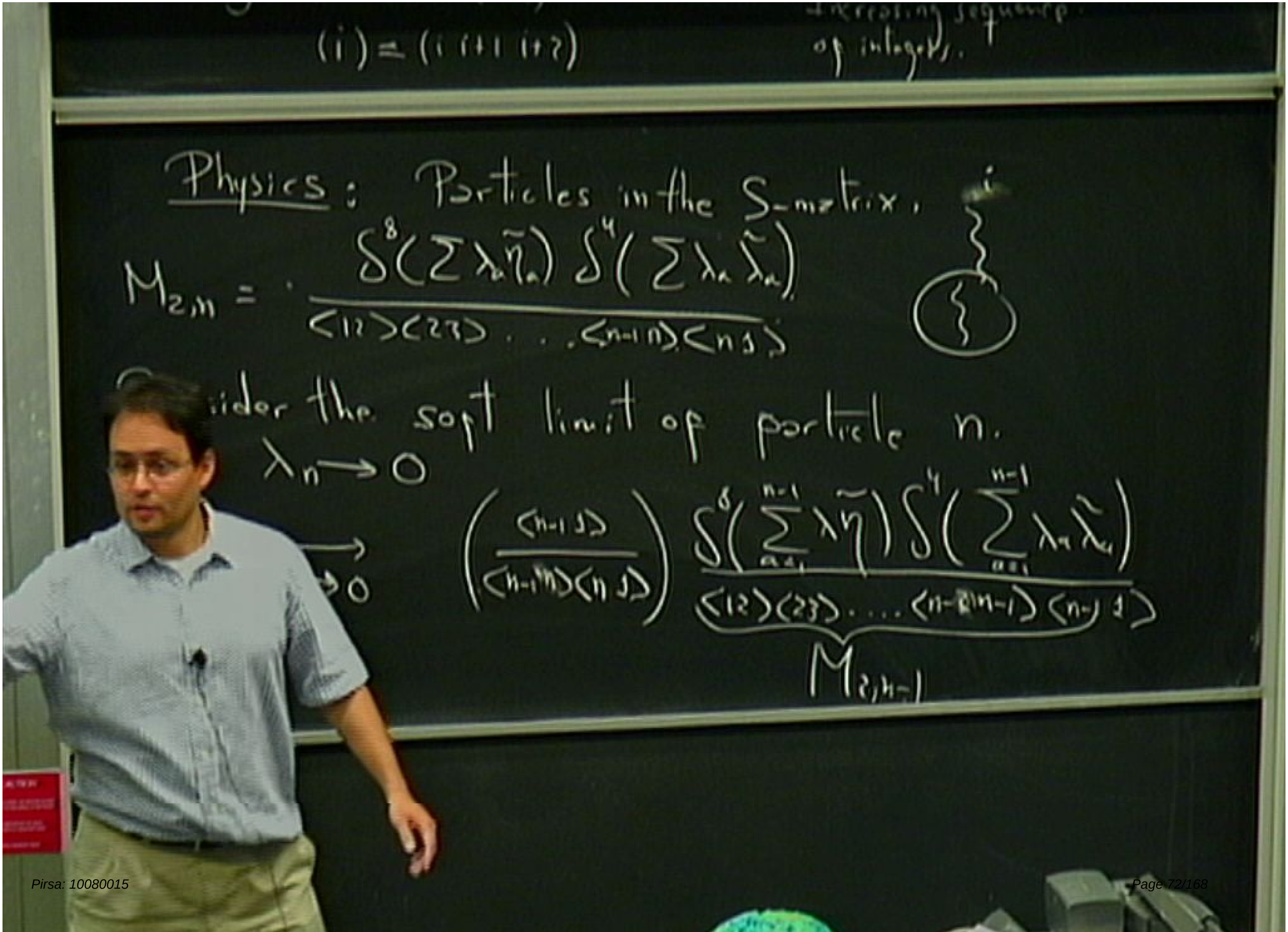
Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$

Consider the soft limit of particle n.

$$\lambda_n \to 0$$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\, 1\rangle}{\langle n{-}1\, n\rangle\langle n\, 1\rangle}\right) \frac{\delta^8(\sum_{a=1}^{n-1} \lambda \tilde{\eta})\,\delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\underbrace{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}2\, n{-}1\rangle\langle n{-}1\, 1\rangle}_{M_{2,n-1}}}$$



$(i) = (i\; i+1\; i+2)$

increasing sequence of integers.

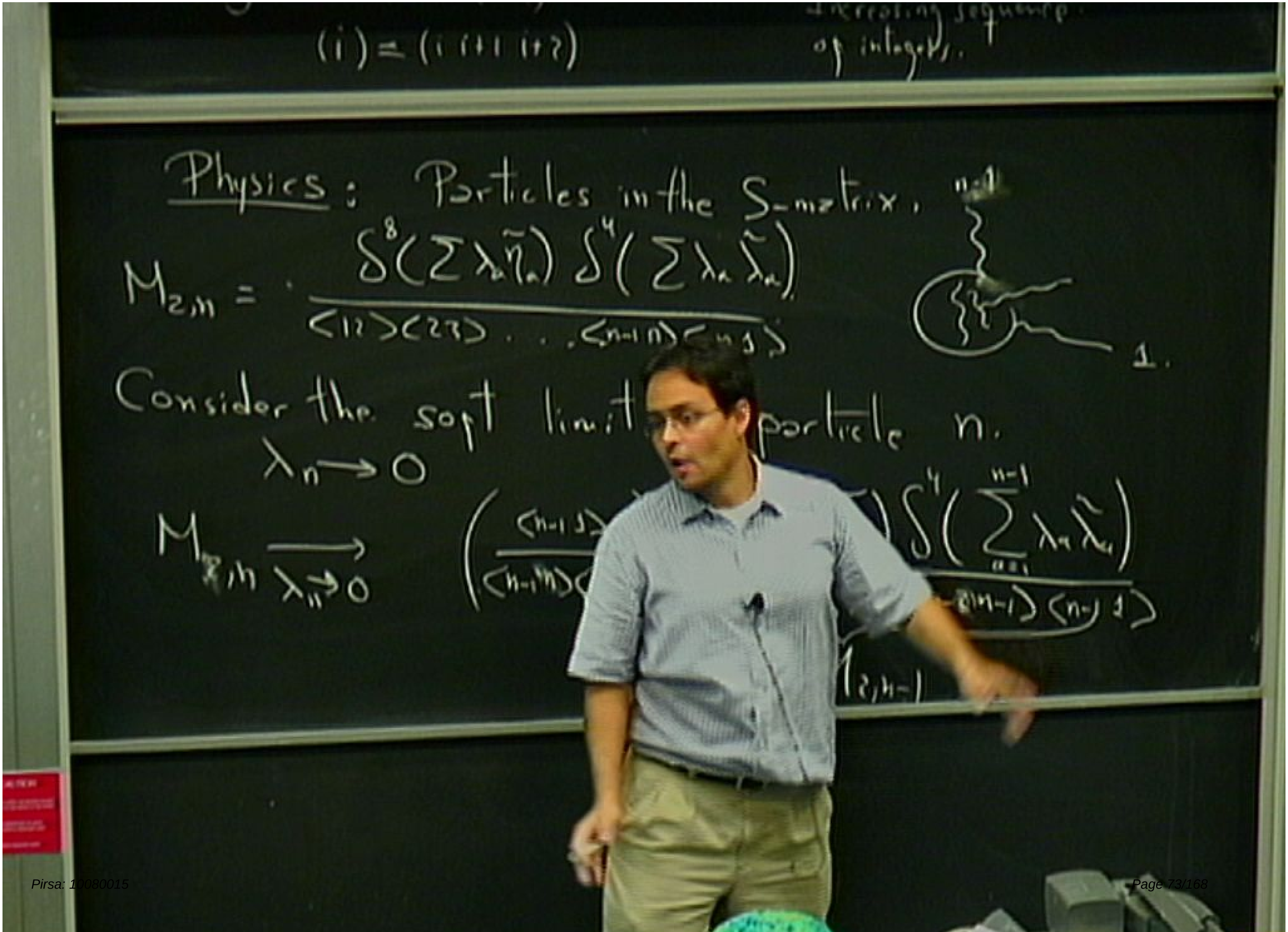
Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n\, 1\rangle}$$

sider the soft limit of particle n.

$\lambda_n \to 0$

$$\xrightarrow[\;\to 0]{} \left(\frac{\langle n-1\, 1\rangle}{\langle n-1\, n\rangle\langle n\, 1\rangle}\right) \underbrace{\frac{\delta^8(\sum_{a=1}^{n-1} \lambda\tilde{\eta})\,\delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \cdots \langle n-2\; n-1\rangle\langle n-1\; 1\rangle}}_{M_{2,n-1}}$$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

<u>Physics</u>: Particles in the S-matrix.

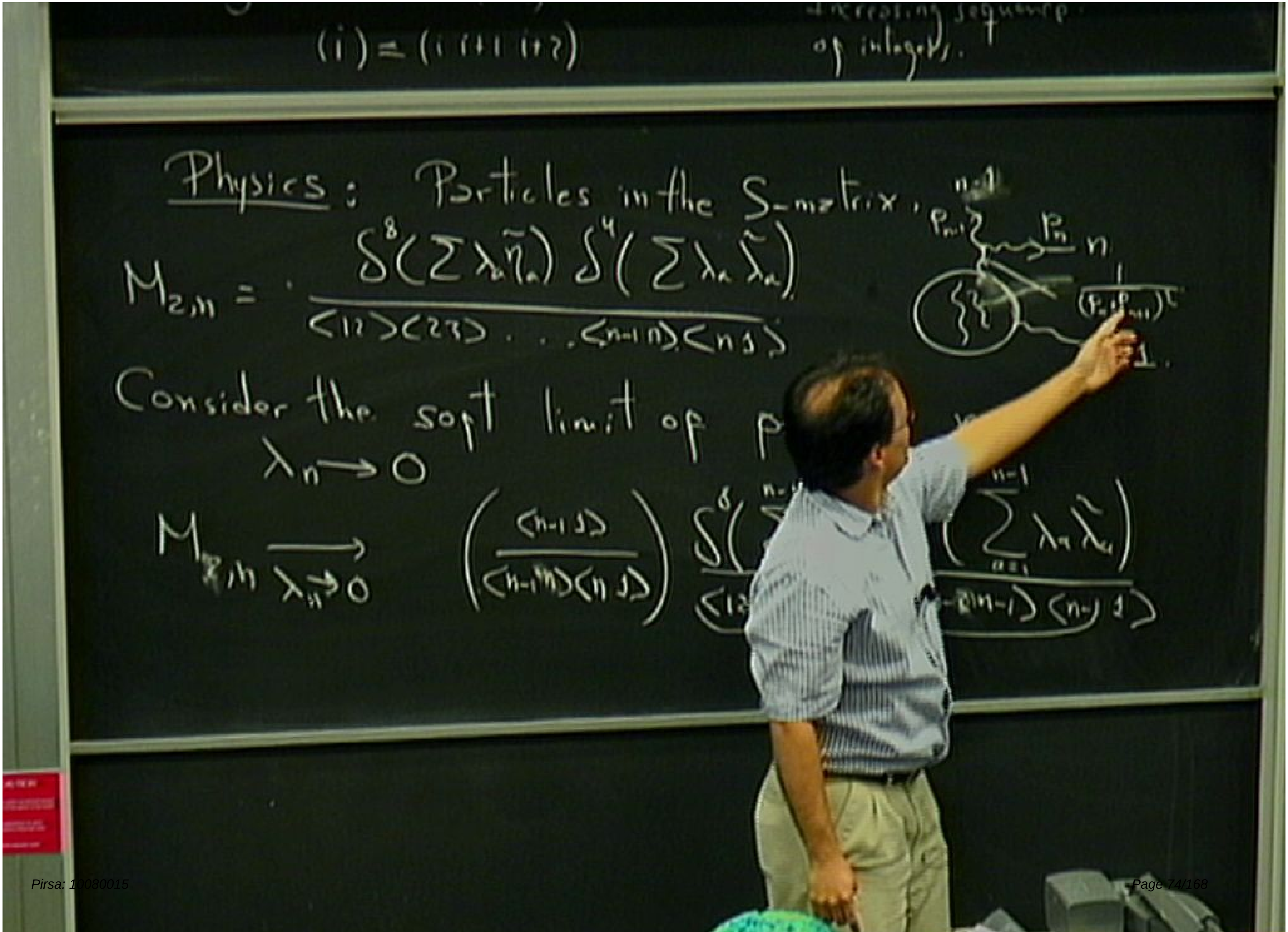
$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$

Consider the soft limit of particle n.

$\lambda_n \to 0$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\ 1\rangle}{\langle n{-}1\, n\rangle\langle}\right) \delta^4\Big(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a\Big) \ldots \frac{}{\ldots n{-}1\rangle\,\langle n{-}1\ 1\rangle}$$

$M_{2,n-1}$



$(i) = (i\ i{+}1\ i{+}2)$

increasing sequence of integers.

Physics: Particles in the S-matrix,

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$

Consider the soft limit of p

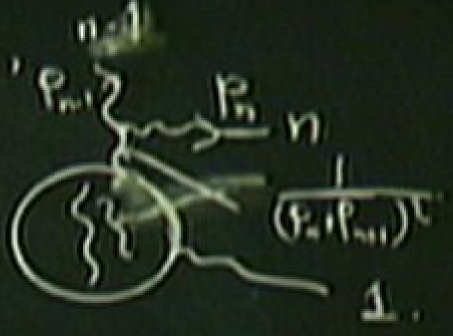
$$\lambda_n \to 0$$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\ 1\rangle}{\langle n{-}1\ n\rangle\langle n\ 1\rangle}\right) \frac{\delta^8(\sum^{n-1} \ldots)\ \ldots(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle \ldots \langle n{-}2\ n{-}1\rangle\langle n{-}1\ 1\rangle}$$

$(i) = (i\ i+1\ i+2)$ increasing sequence of integers.

Physics: Particles in the S-matrix,

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\, \delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n-1\, n\rangle\langle n\, 1\rangle}$$



Consider the soft limit of particle n.

$\lambda_n \to 0$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n-1\, 1\rangle}{\langle n-1\, n\rangle\langle n\, 1\rangle}\right) \underbrace{\frac{\delta^8(\sum_{a=1}^{n-1} \lambda \tilde{\eta})\, \delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n-2\, n-1\rangle\langle n-1\, 1\rangle}}_{M_{2,n-1}}$$

$(i) = (i\ i{+}1\ i{+}2)$

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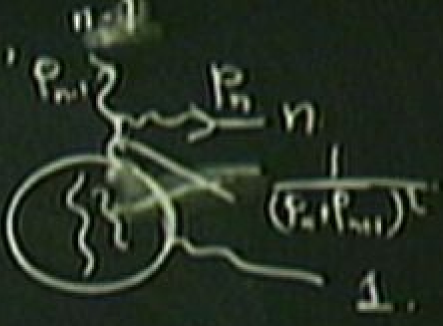
Physics: Particles in the S-matrix,

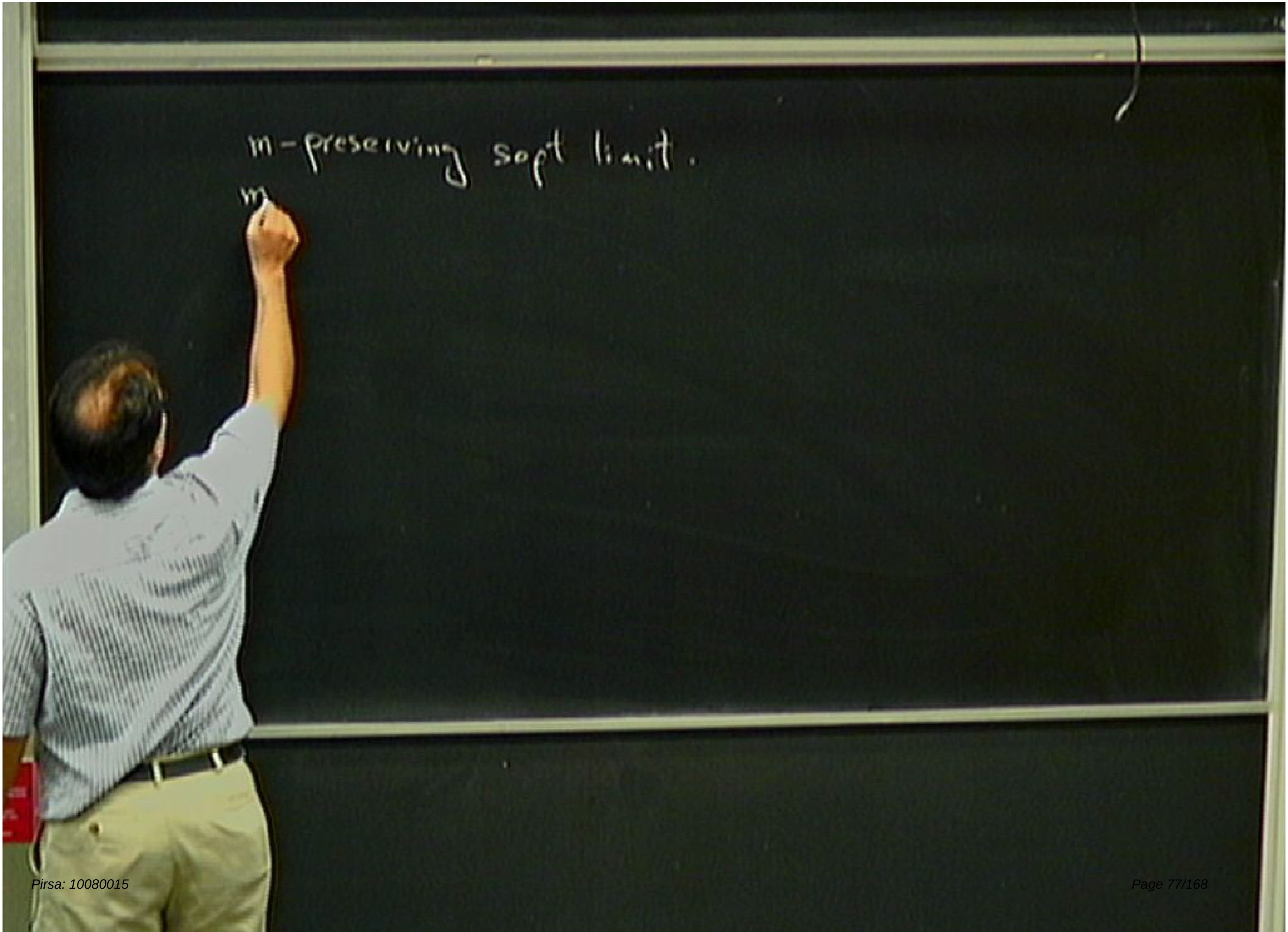
$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \dots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$



Consider the soft limit of particle n.

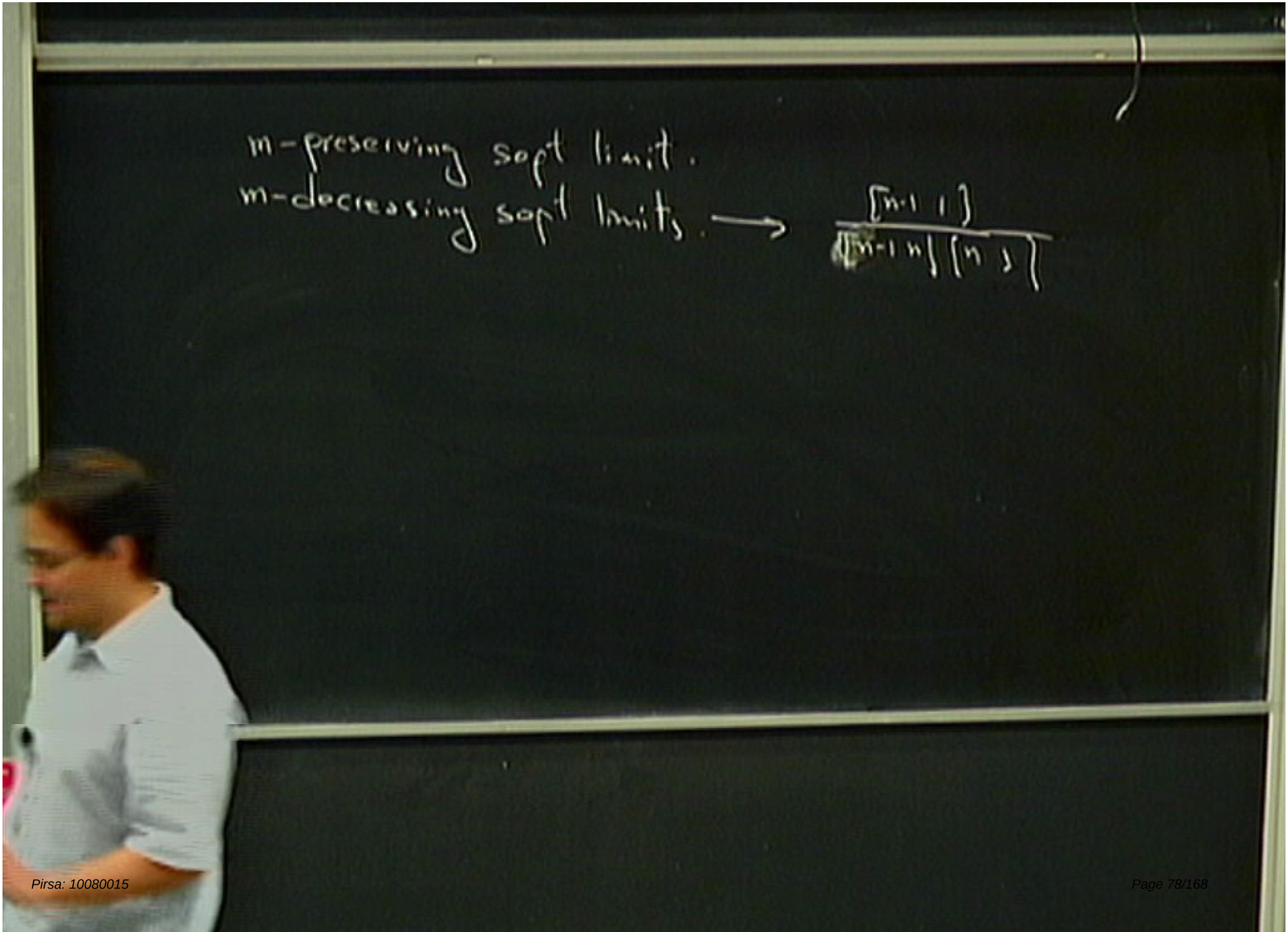
$$\lambda_n \to 0$$

$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\, 1\rangle}{\langle n{-}1\, n\rangle\langle n\, 1\rangle}\right) \underbrace{\frac{\delta^8(\sum_{a=1}^{n-1} \lambda \tilde{\eta})\,\delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \dots \langle n{-}2\, n{-}1\rangle\langle n{-}1\, 1\rangle}}_{M_{2,n-1}}$$



m-preserving sept limit.

m

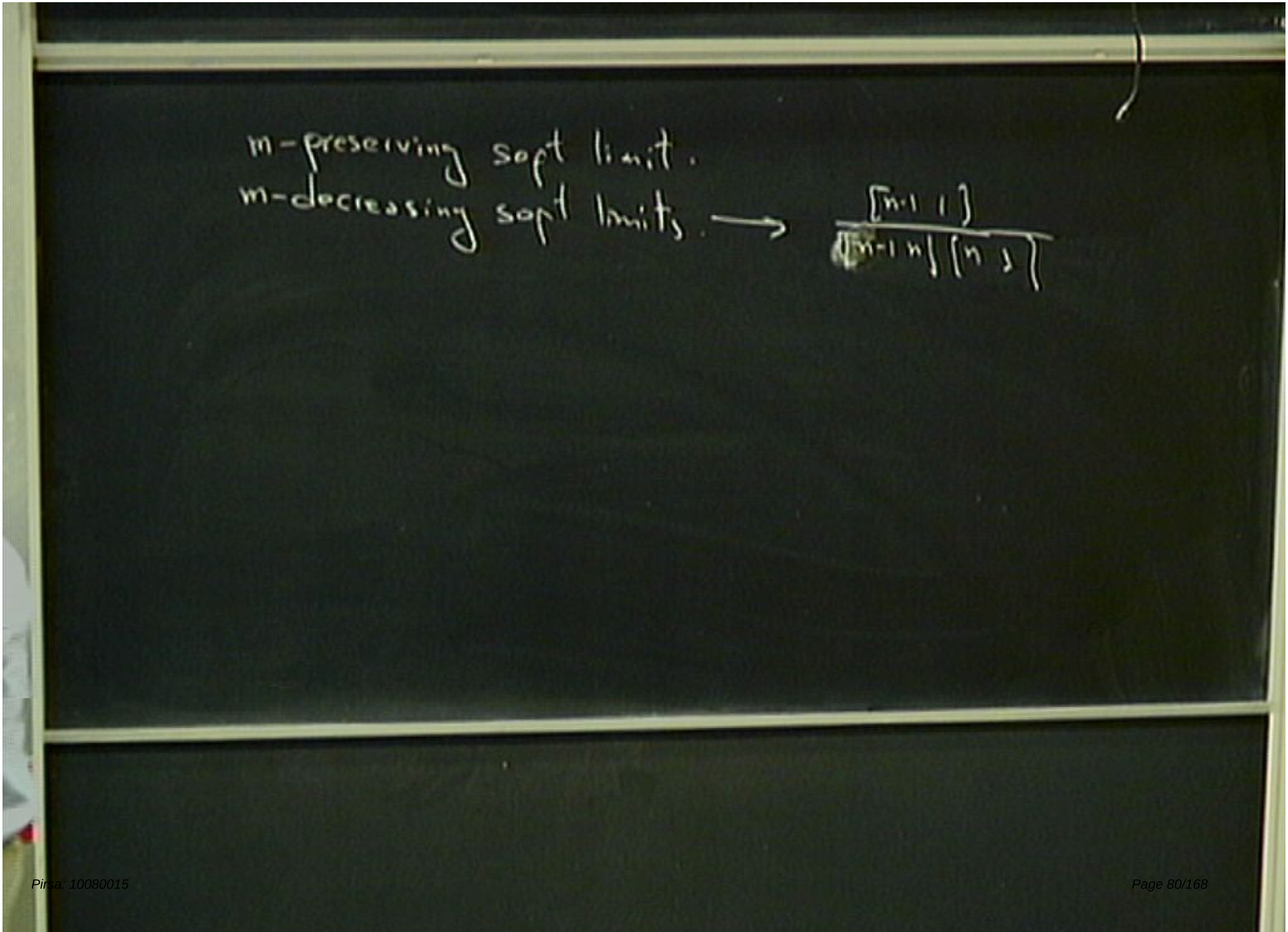


m-preserving soft limit.

m-decreasing soft limits. $\longrightarrow$ $\frac{\{n-1 \;\; 1\}}{\{n-1 \;\; n\}\{n \;\; 1\}}$

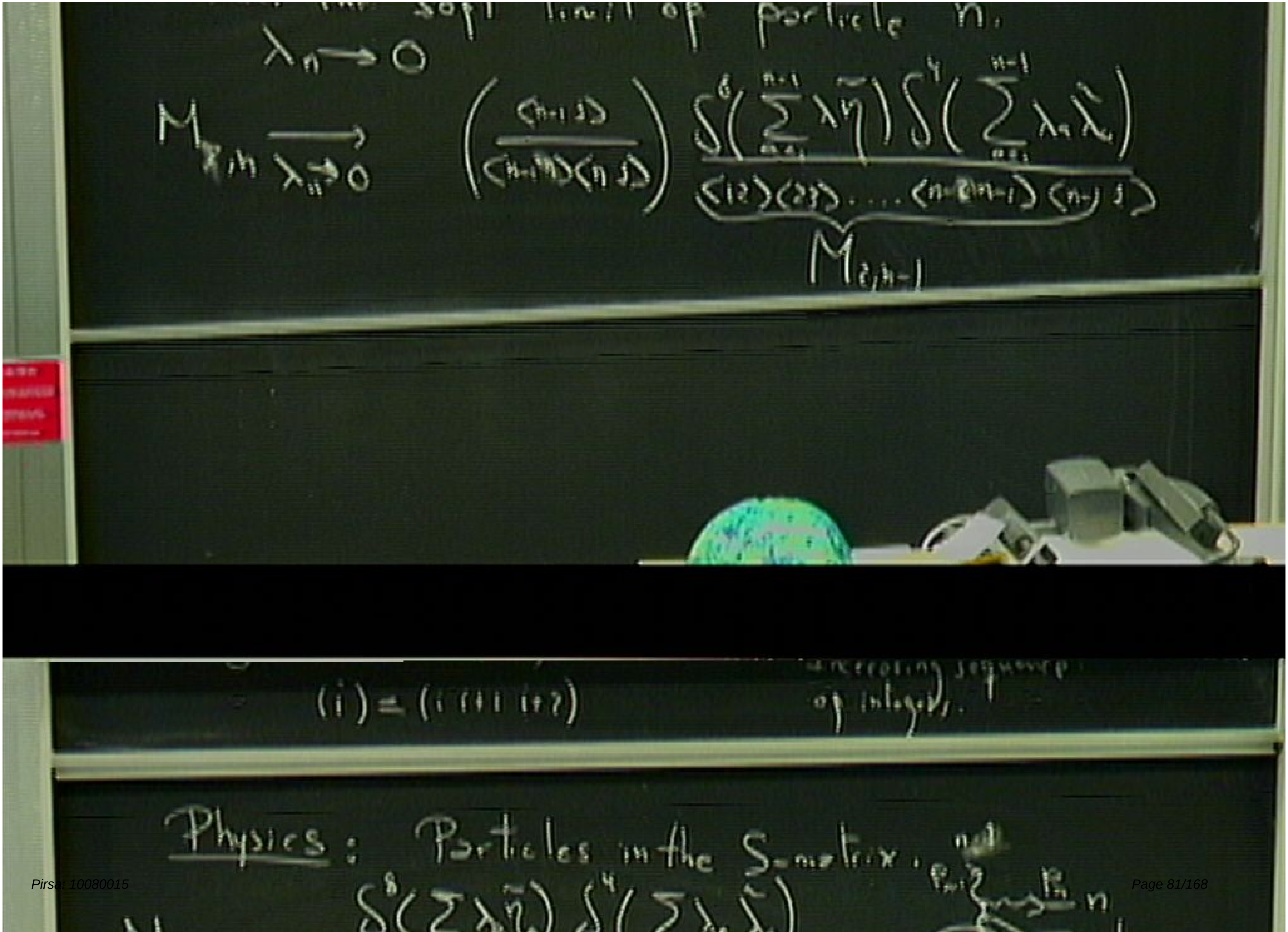
m-preserving soft limit.

m-decreasing soft limits. $\longrightarrow$ $\dfrac{\{n\text{-}1\ \ 1\}}{\{n\text{-}1\ \ n\}\{n\ \ 1\}}$



m-preserving sopt limit.

m-decreasing sopt limits $\longrightarrow$ $$\frac{\left[n-1 \;\; 1 \right]}{\left\{ n-1 \;\; n \right\} \left\{ n \;\; 2 \right\}}$$



soft limit of particle n.

$\lambda_n \to 0$

$$M_{2,n} \xrightarrow[\lambda_n\to 0]{} \left(\frac{\langle n-1\, 1\rangle}{\langle n-1\, n\rangle\langle n\, 1\rangle}\right) \underbrace{\frac{\delta^8\left(\sum_{i=1}^{n-1}\lambda\tilde\eta\right)\delta^4\left(\sum_{i=1}^{n-1}\lambda_i\tilde\lambda_i\right)}{\langle 12\rangle\langle 23\rangle\cdots\langle n-2\, n-1\rangle\langle n-1\, 1\rangle}}_{M_{2,n-1}}$$

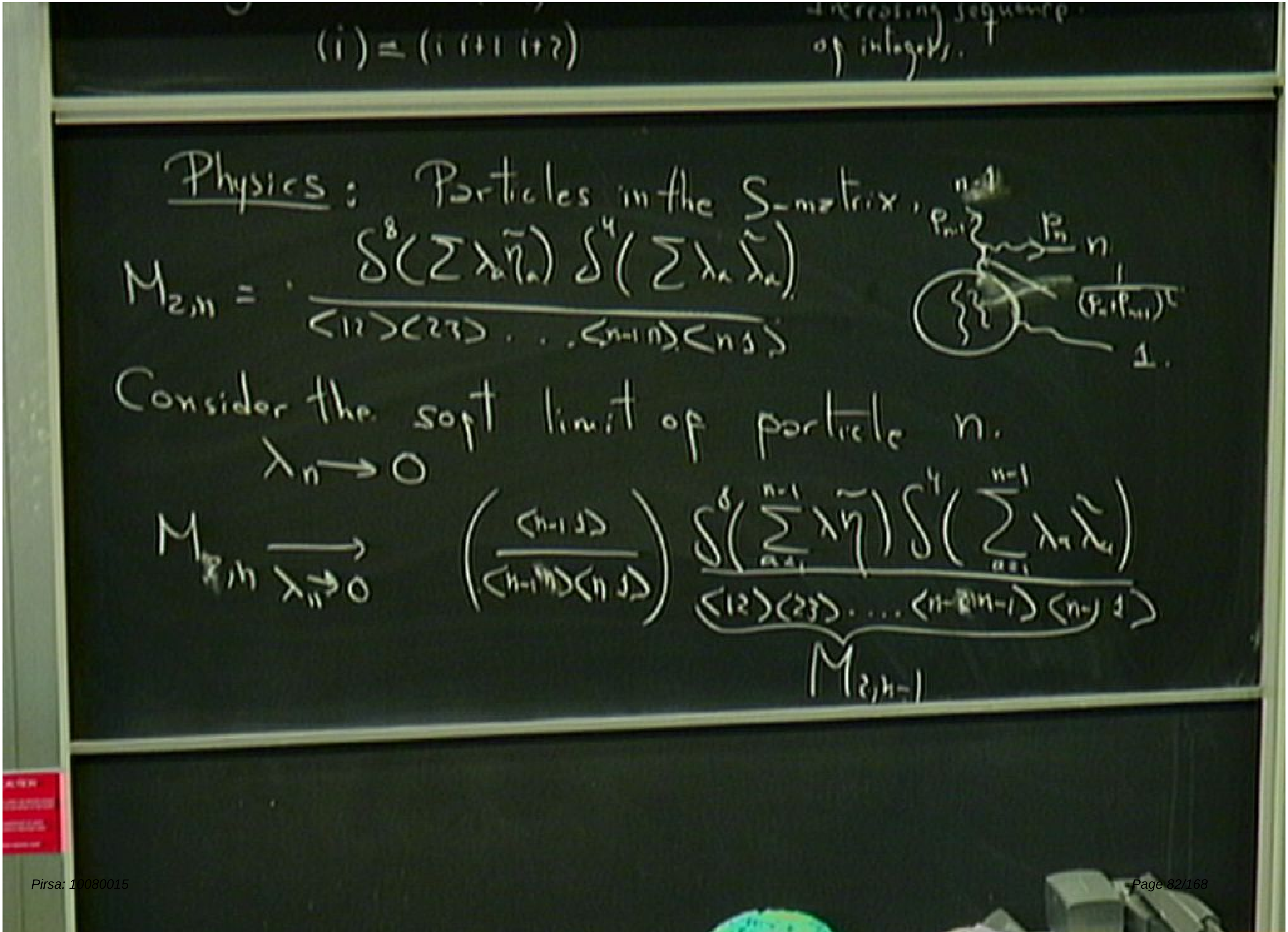
[redacted]

$(i) = (i\ i+1\ i+2)$

increasing sequence of integers.

Physics: Particles in the S-matrix,

$\delta^8(\sum\lambda\tilde\eta)\,\delta^4(\sum\lambda\tilde\lambda)$



$(i) = (i\ i{+}1\ i{+}2)$

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Physics: Particles in the S-matrix.

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde{\eta}_a)\,\delta^4(\sum \lambda_a \tilde{\lambda}_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\,n\rangle\langle n\,1\rangle}$$

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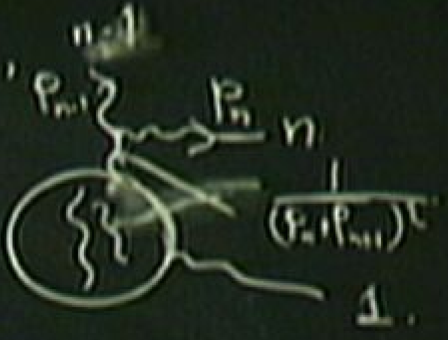
$(i) = (i\ i{+}1\ i{+}2)$ increasing sequence of integers.

Physics: Particles in the S-matrix,

$$M_{2,n} = \frac{\delta^8(\sum \lambda_a \tilde\eta_a)\,\delta^4(\sum \lambda_a \tilde\lambda_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}1\, n\rangle\langle n\, 1\rangle}$$



Consider the soft limit of particle n.

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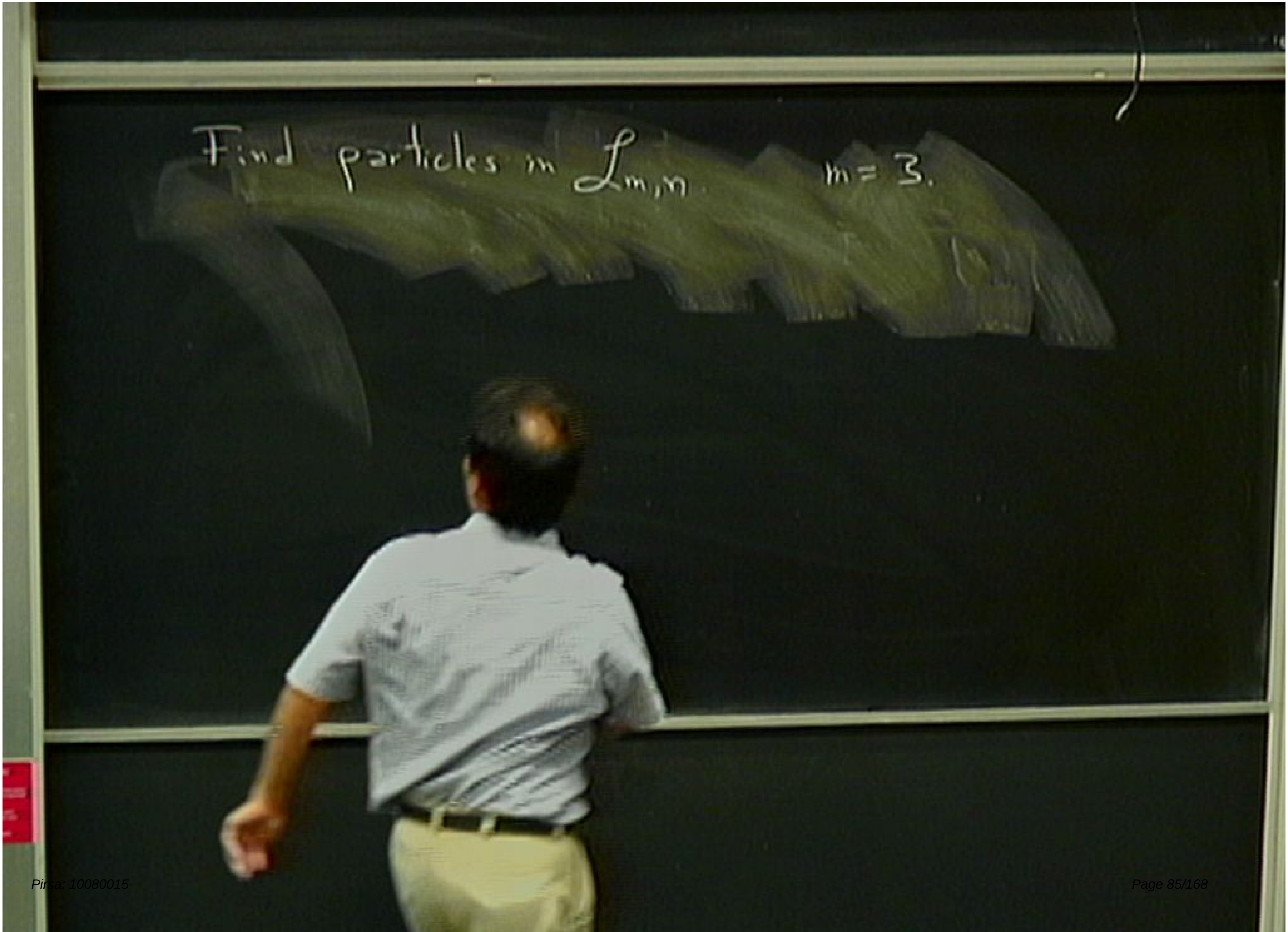
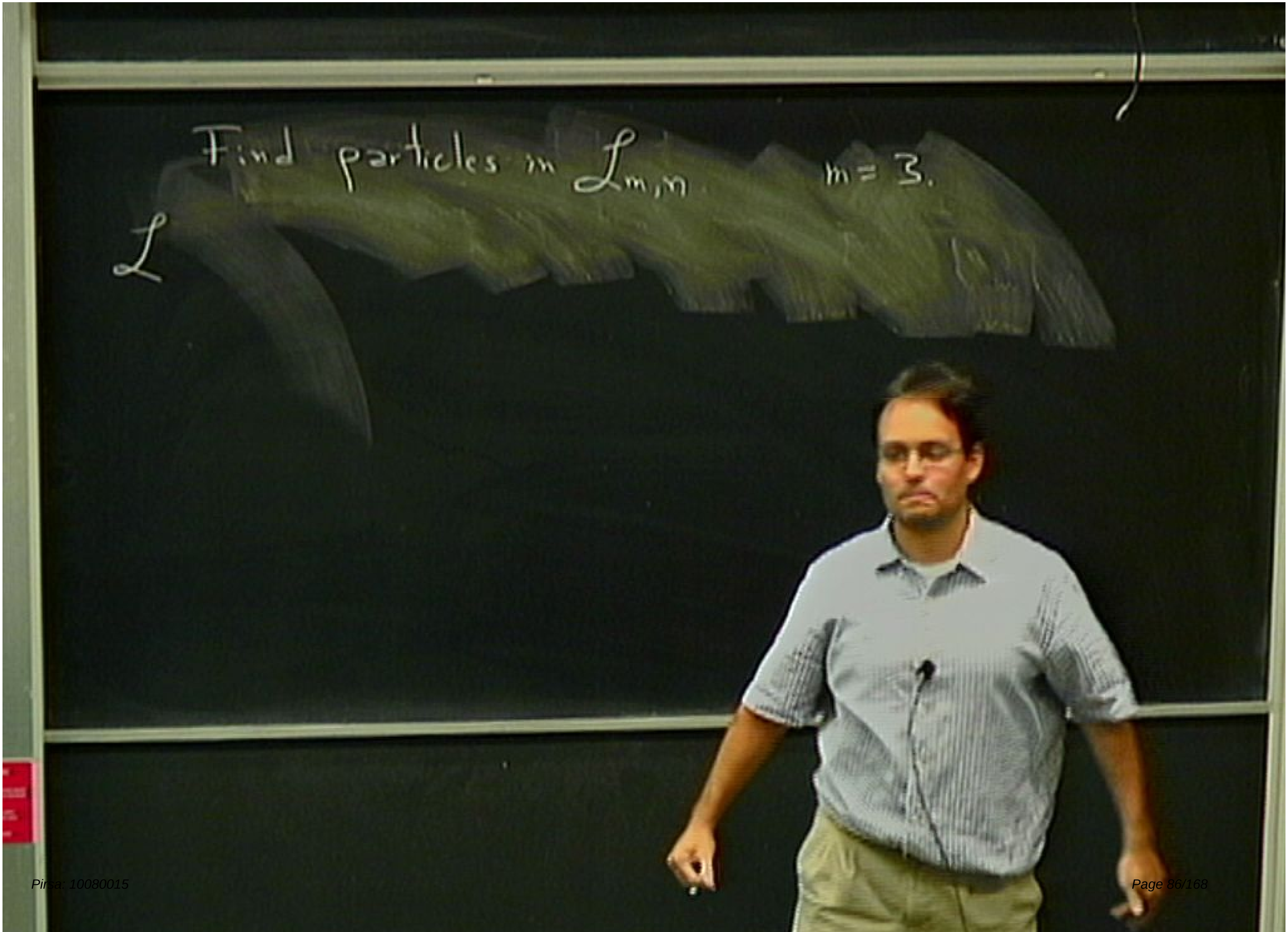
$$M_{2,n} \xrightarrow[\lambda_n \to 0]{} \left(\frac{\langle n{-}1\, 1\rangle}{\langle n{-}1\, n\rangle\langle n\, 1\rangle}\right) \underbrace{\frac{\delta^8(\sum_{a=1}^{n-1} \lambda\tilde\eta)\,\delta^4(\sum_{a=1}^{n-1} \lambda_a \tilde\lambda_a)}{\langle 12\rangle\langle 23\rangle \ldots \langle n{-}2\, n{-}1\rangle\langle n{-}1\, 1\rangle}}_{M_{2,n-1}}$$

m-preserving sep^t limit.

m-decreasing sep^t limits. $\longrightarrow$ $\dfrac{\{n\text{-}1\ \ 1\}}{\{n\text{-}1\ n\}\{n\ \ 2\}}$

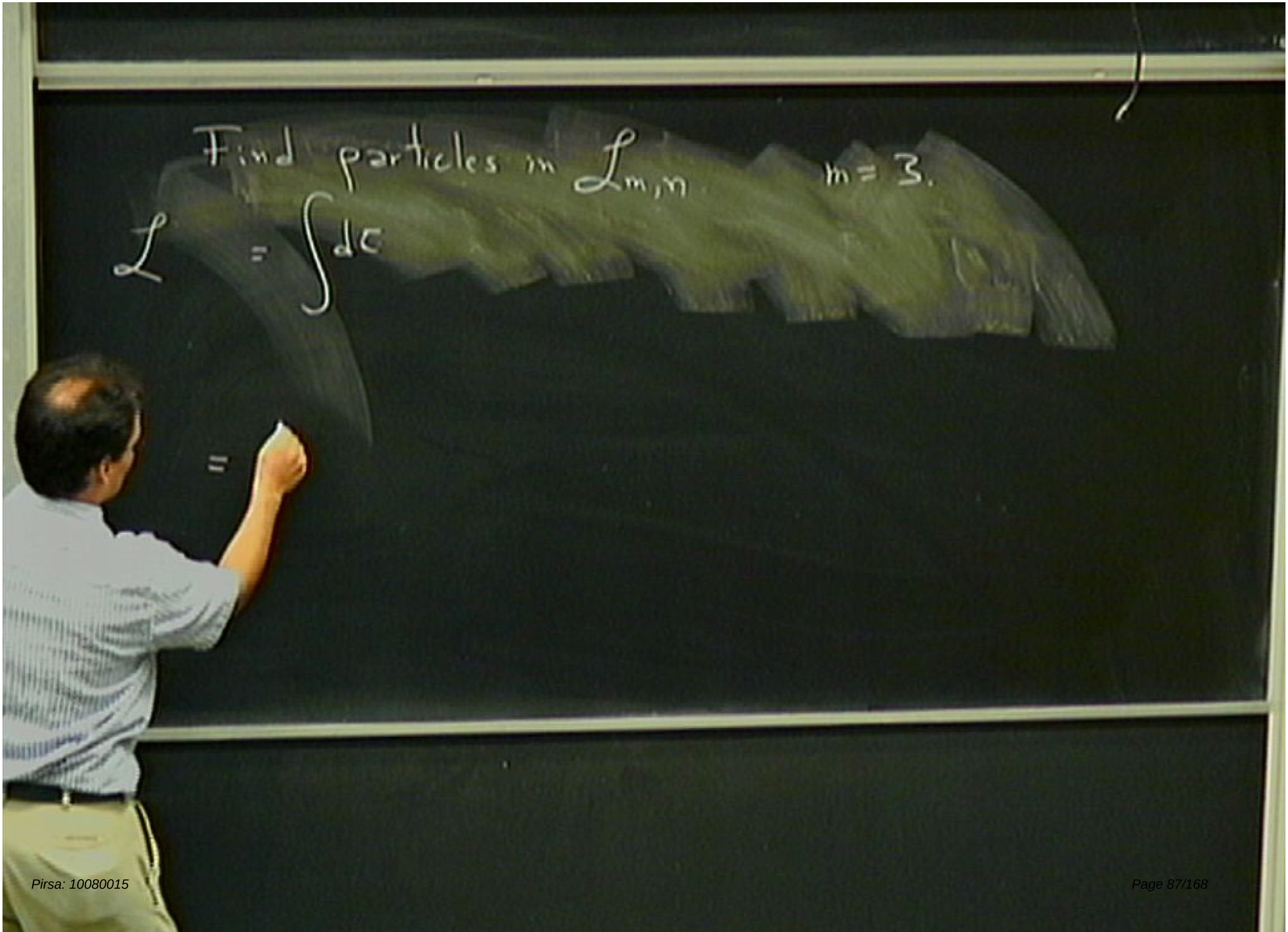


Find particles in $\mathcal{L}_{m,n}$. $m = 3$.



Find particles in $\mathcal{L}_{m,n}$ m = 3.

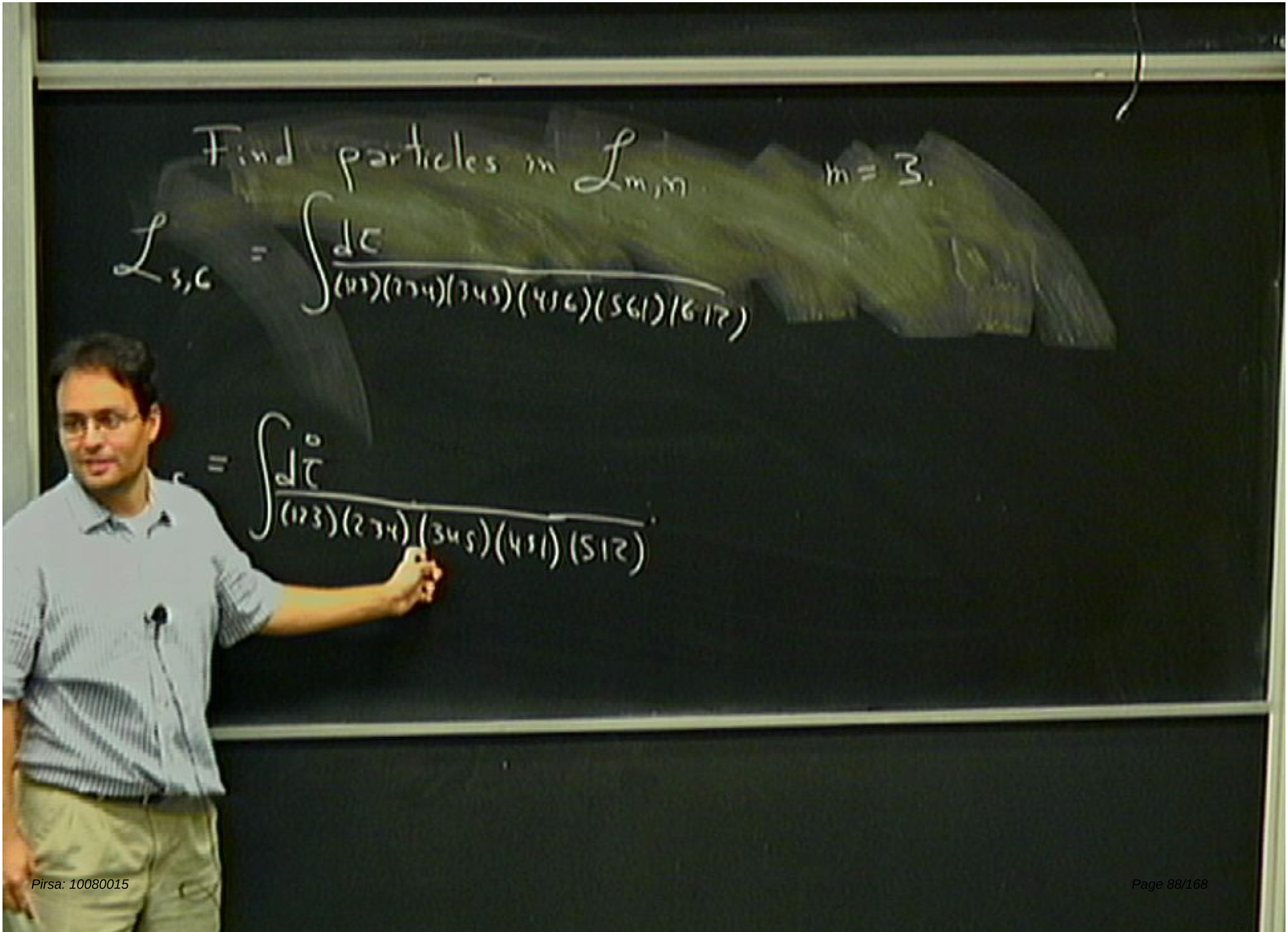
$\mathcal{L}$



Find particles in $\mathcal{L}_{m,n}$ $m=3.$

$\mathcal{L} = \int dc$

$=$



Find particles in $\mathcal{L}_{m,n}$ $m = 3.$

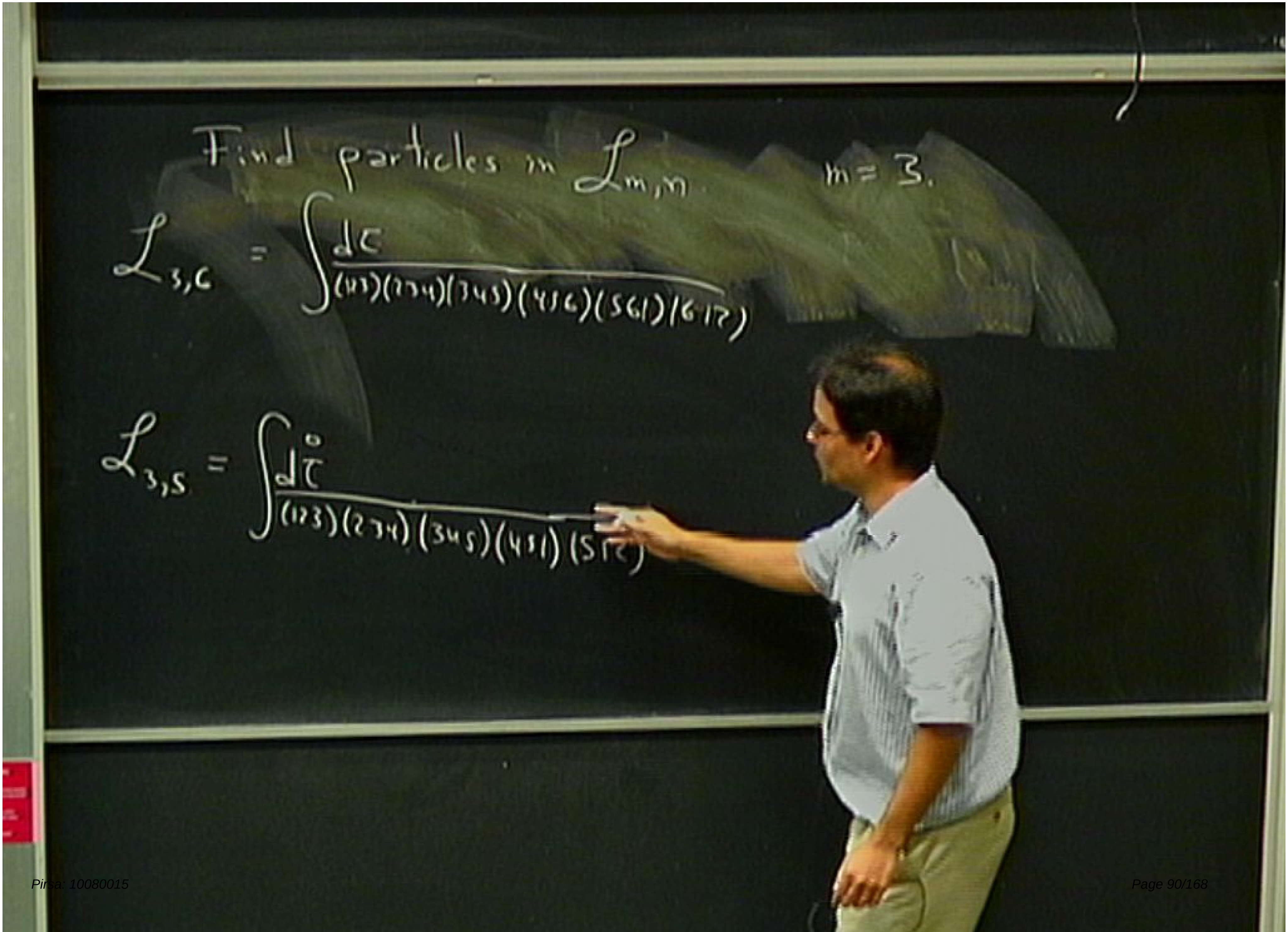
$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)}$$

$$= \int \frac{d\mathring{C}}{(123)(234)(345)(451)(512)}$$

Find particles in $\mathcal{L}_{m,n}$ $m = 3.$

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)}$$

$$\mathcal{L}_{3,5} = \int \frac{d\overset{\circ}{C}}{(123)(234)(345)(451)(512)}$$



Find particles in $\mathcal{L}_{m,n}$. $m=3$.

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)}$$

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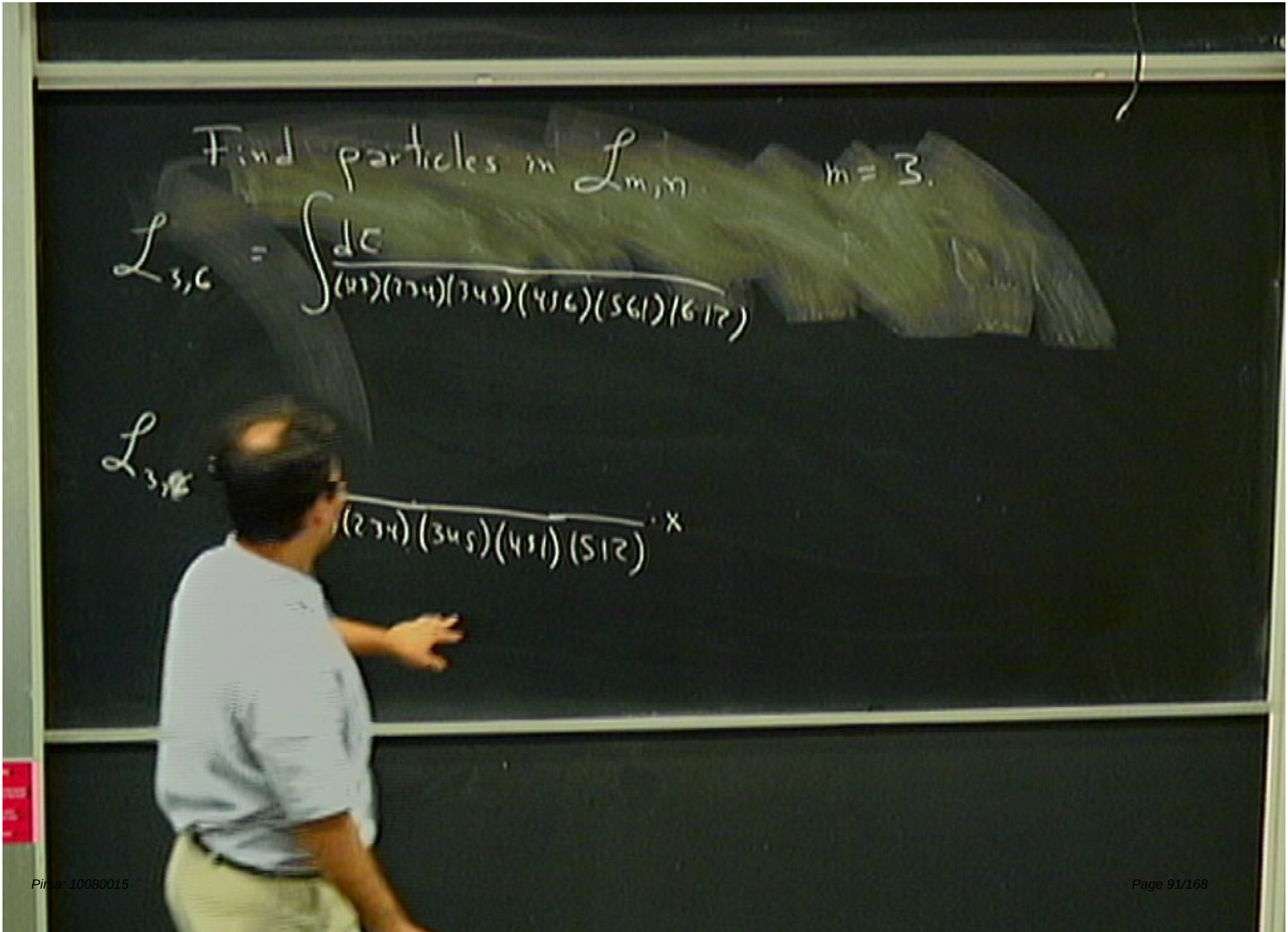
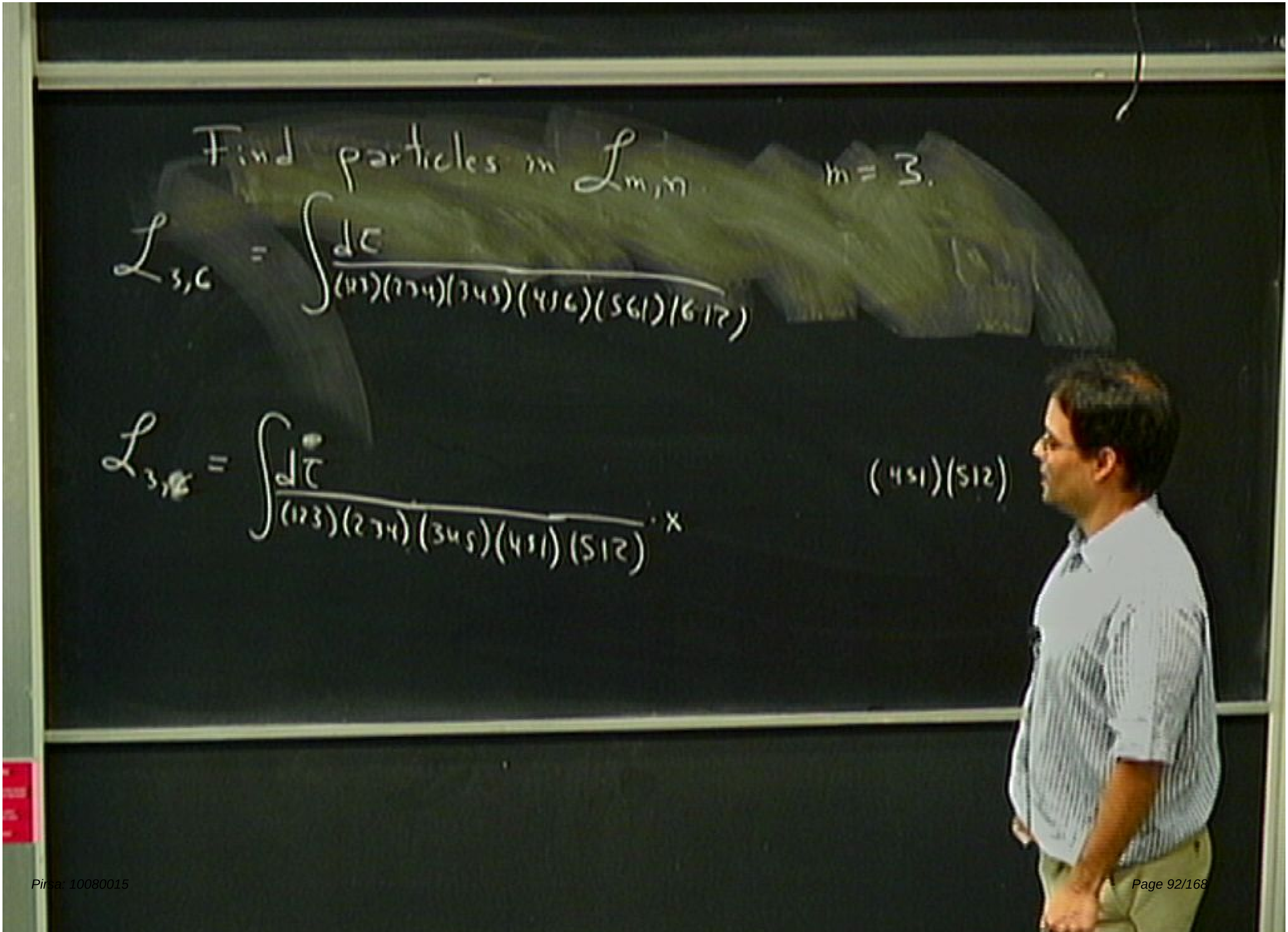


Find particles in $\mathcal{L}_{m,n}$ $m = 3.$

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)}$$

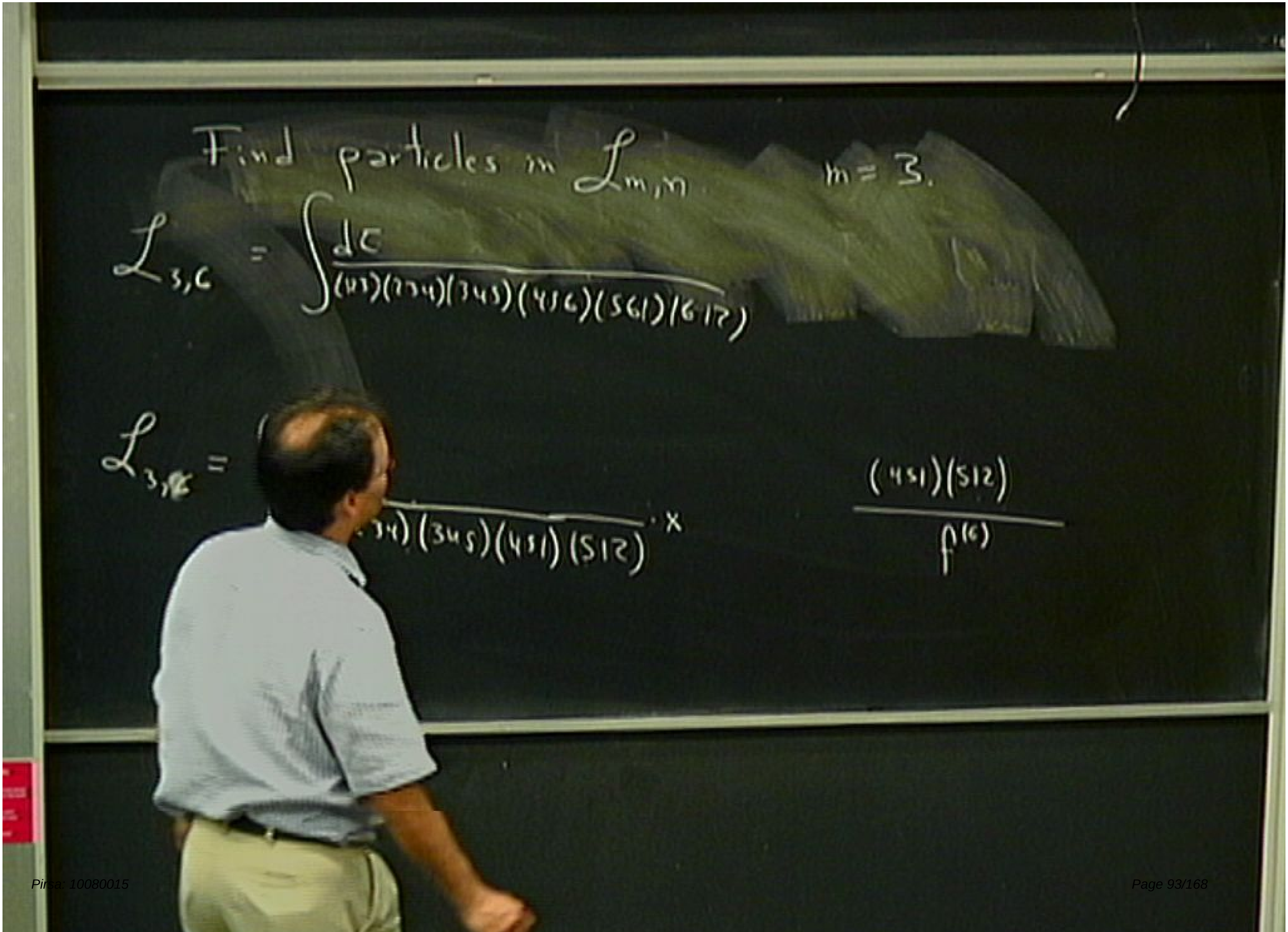
$$\mathcal{L}_{3,5} \quad \frac{}{(234)(345)(451)(512)} \cdot x$$



Find particles in $\mathcal{L}_{m,n}$ $\quad m=3.$

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)}$$

$$\mathcal{L}_{3,5} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \cdot x \qquad (451)(512)$$



Find particles in $\mathcal{L}_{m,n}$ m = 3.

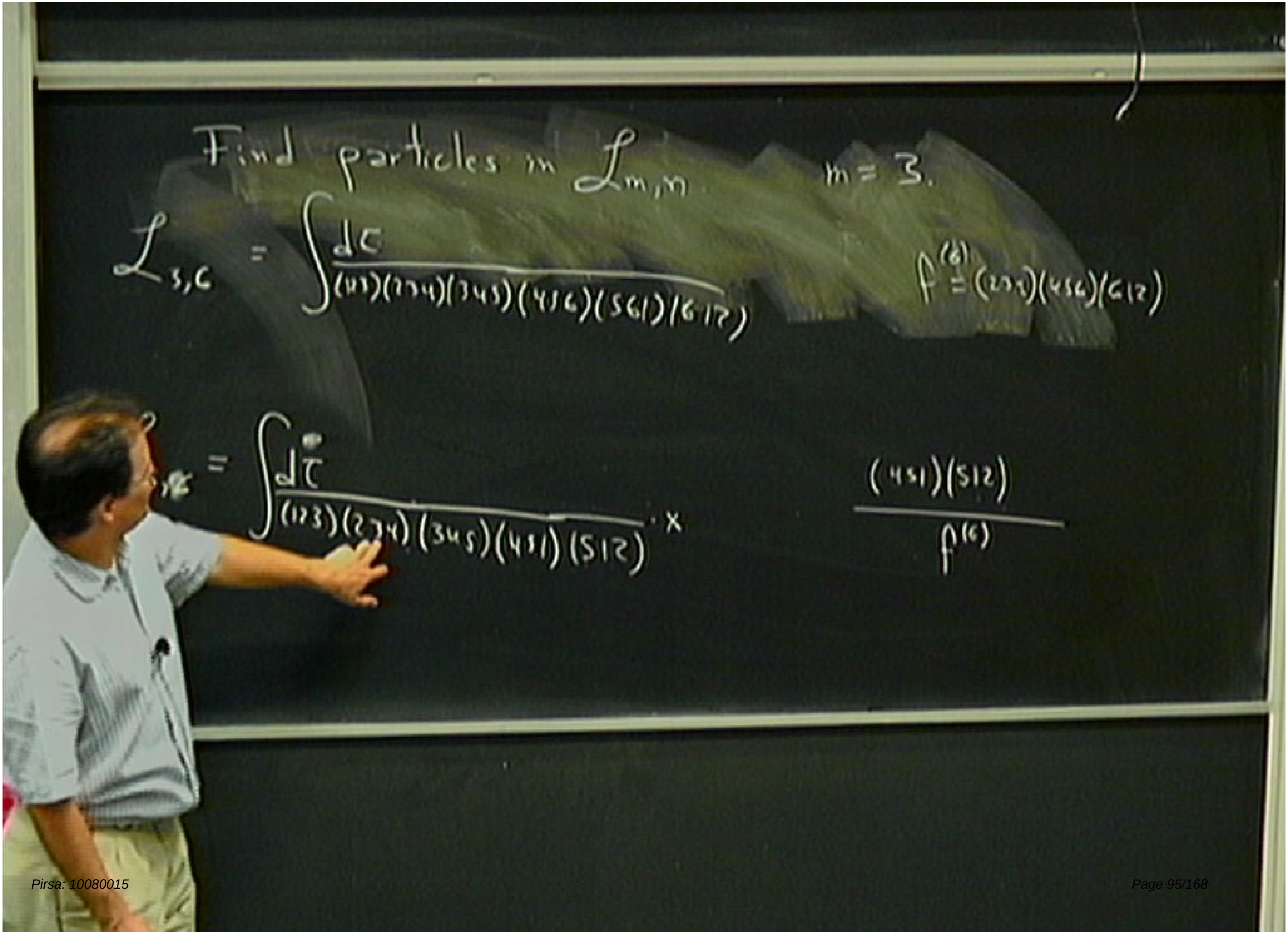
$$\mathcal{L}_{3,6} = \int \frac{dc}{(123)(234)(345)(456)(561)(612)}$$

$$\mathcal{L}_{3,5} = \quad \frac{}{\;34)(345)(451)(512)} \times \qquad \frac{(451)(512)}{f^{(6)}}$$

Find particles in $\mathcal{L}_{m,n}$ $m=3.$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} = (234)(456)(612)$$

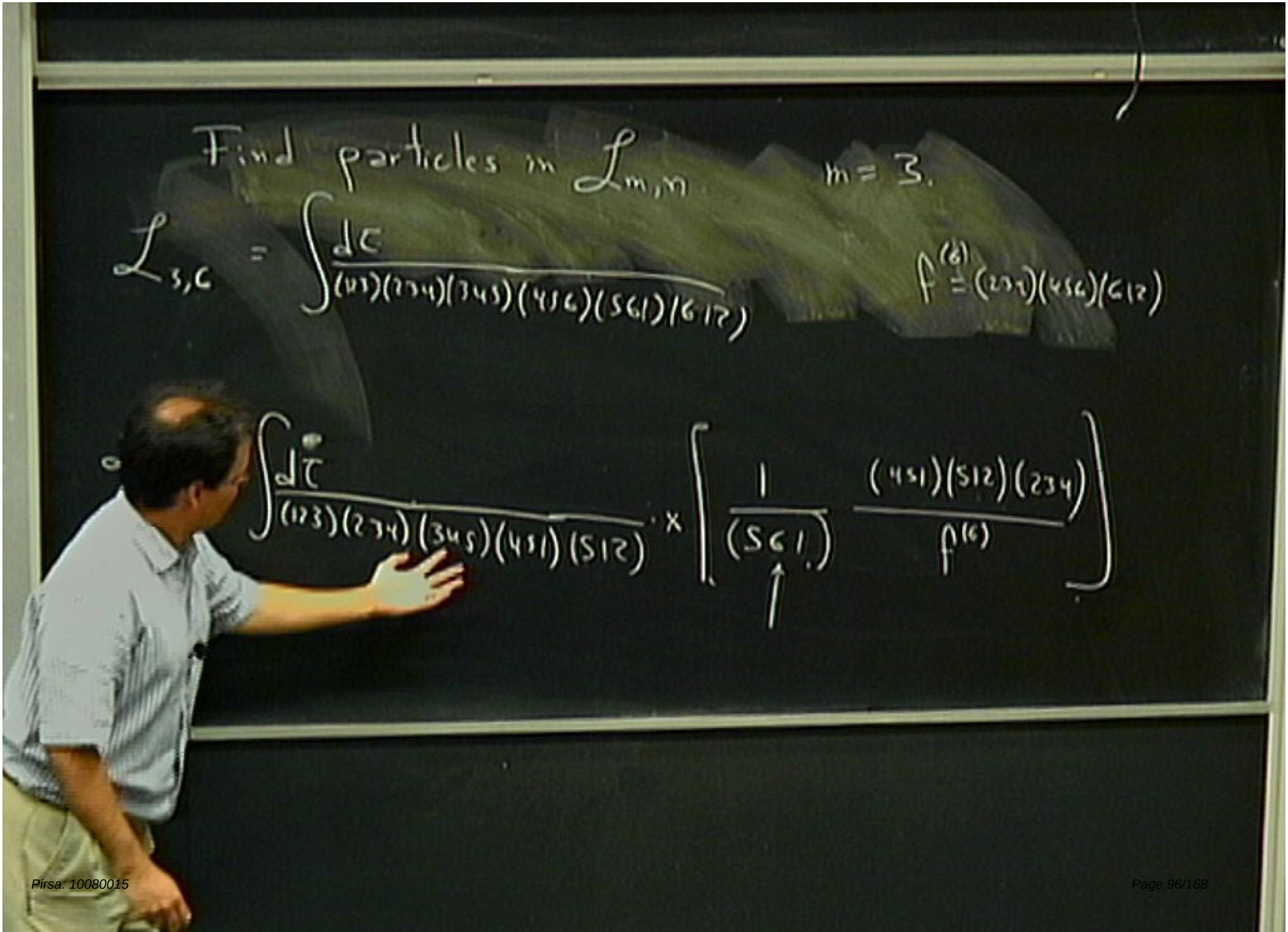
$$\mathcal{L}_{3,5} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \qquad \frac{(451)(512)}{f^{(6)}}$$



Find particles in $\mathcal{L}_{m,n}$ $\quad m = 3.$

$$\mathcal{L}_{3,6} = \int \frac{dc}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} = (234)(456)(612)$$

$$= \int \frac{d\tilde{c}}{(123)(234)(345)(451)(512)} \cdot \times \qquad \frac{(451)(512)}{f^{(6)}}$$

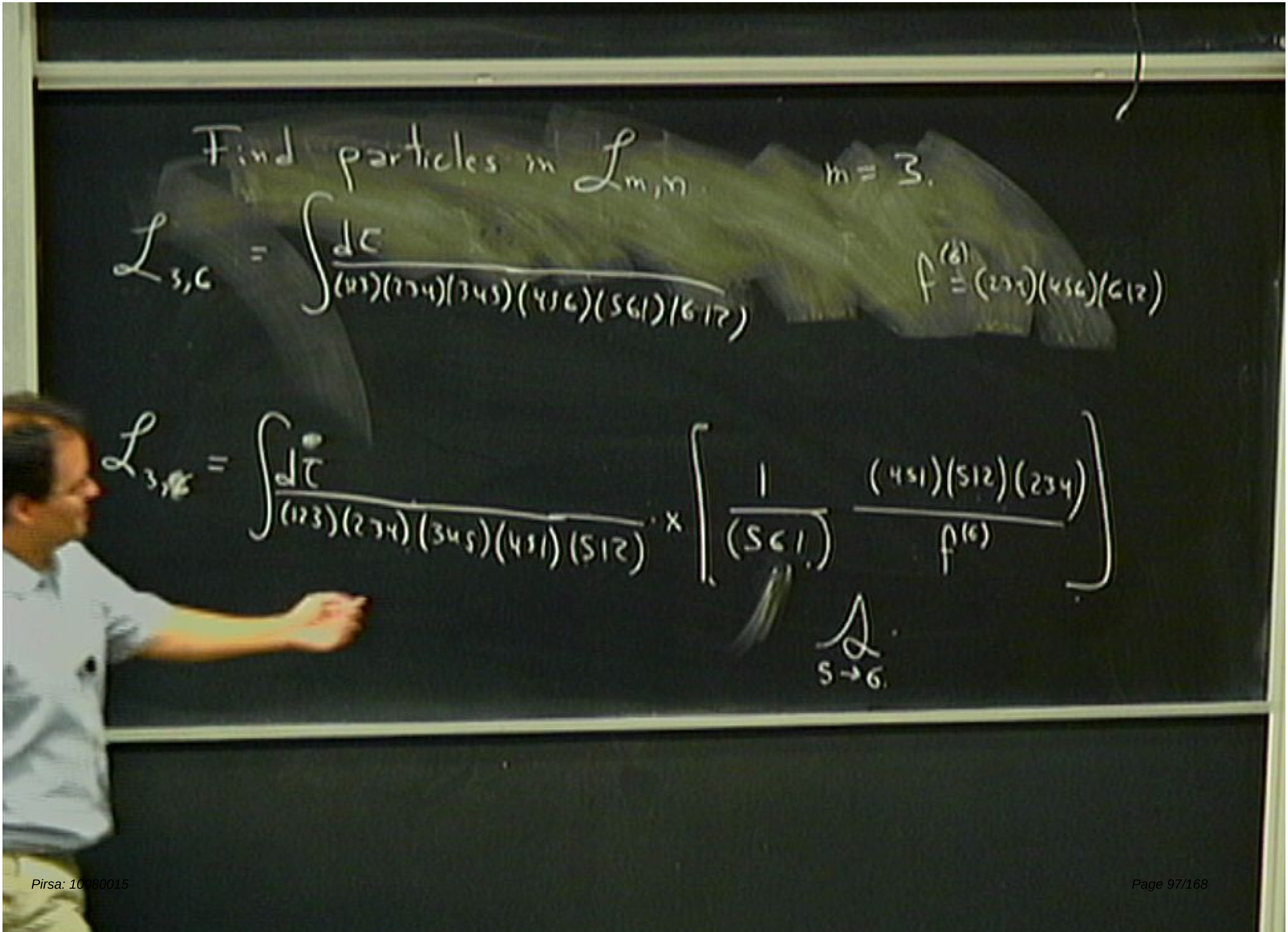


Find particles in $\mathcal{L}_{m,n}$ $m = 3$.

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(456)(561)(612)}$$

$$f^{(6)} = (234)(456)(612)$$

$$= \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \left[\frac{1}{\underset{\uparrow}{(561)}} \frac{(451)(512)(234)}{f^{(6)}} \right]$$



Find particles in $\mathcal{L}_{m,n}$ $m = 3.$

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} \doteq (234)(456)(612)$$

$$\mathcal{L}_{3,5} = \int \frac{d\hat{C}}{(123)(234)(345)(451)(512)} \times \left[\frac{1}{(561)} \; \frac{(451)(512)(234)}{f^{(6)}} \right]$$

$$\underset{5\to 6}{\mathcal{Q}}.$$

Find particles in $\mathcal{L}_{m,n}$. $m = 3$.

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(456)(561)(612)}$$

$$f^{(6)} \doteq (234)(456)(612)$$

$$\mathcal{L}_{3,5} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \left[\frac{1}{(561)} \, \frac{(451)(512)(234)}{f^{(6)}} \right]$$

$$f = \mathbb{C} \to \mathbb{C} \qquad f = f^{(6)} \qquad \underset{5\to 6}{\mathcal{A}}.$$

Find particles in $\mathcal{L}_{m,n}$ $m = 3$.

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} = (234)(456)(612)$$

$$\mathcal{L}_{3,5} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \left[\frac{1}{(561)} \; \frac{(451)(512)(234)}{f^{(6)}} \right]$$

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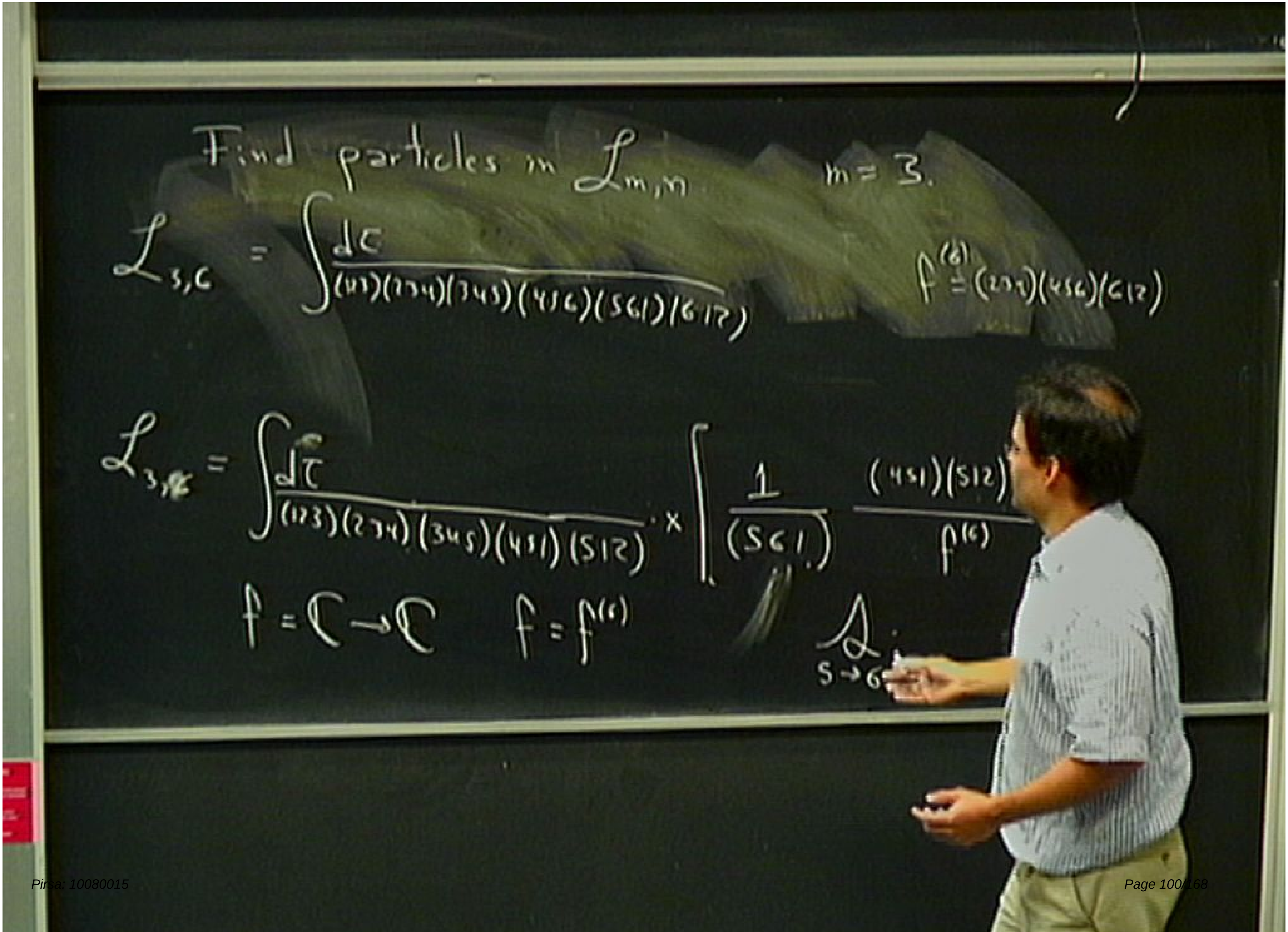


Find particles in $\mathcal{L}_{m,n}$ $m = 3$.

$$\mathcal{L}_{3,6} = \int \frac{dC}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} \doteq (234)(456)(612)$$

$$\mathcal{L}_{3,5} = \int \frac{dC}{(123)(234)(345)(451)(512)} \times \int \left(\frac{1}{(561)} \frac{(451)(512)}{f^{(6)}} \right.$$

$f = \mathbb{C} \to \mathbb{C}$ $\quad f = f^{(6)}$ $\quad \mathcal{A}_{5\to 6}$

Find particles in $\mathcal{L}_{m,n}$ $m = 3.$

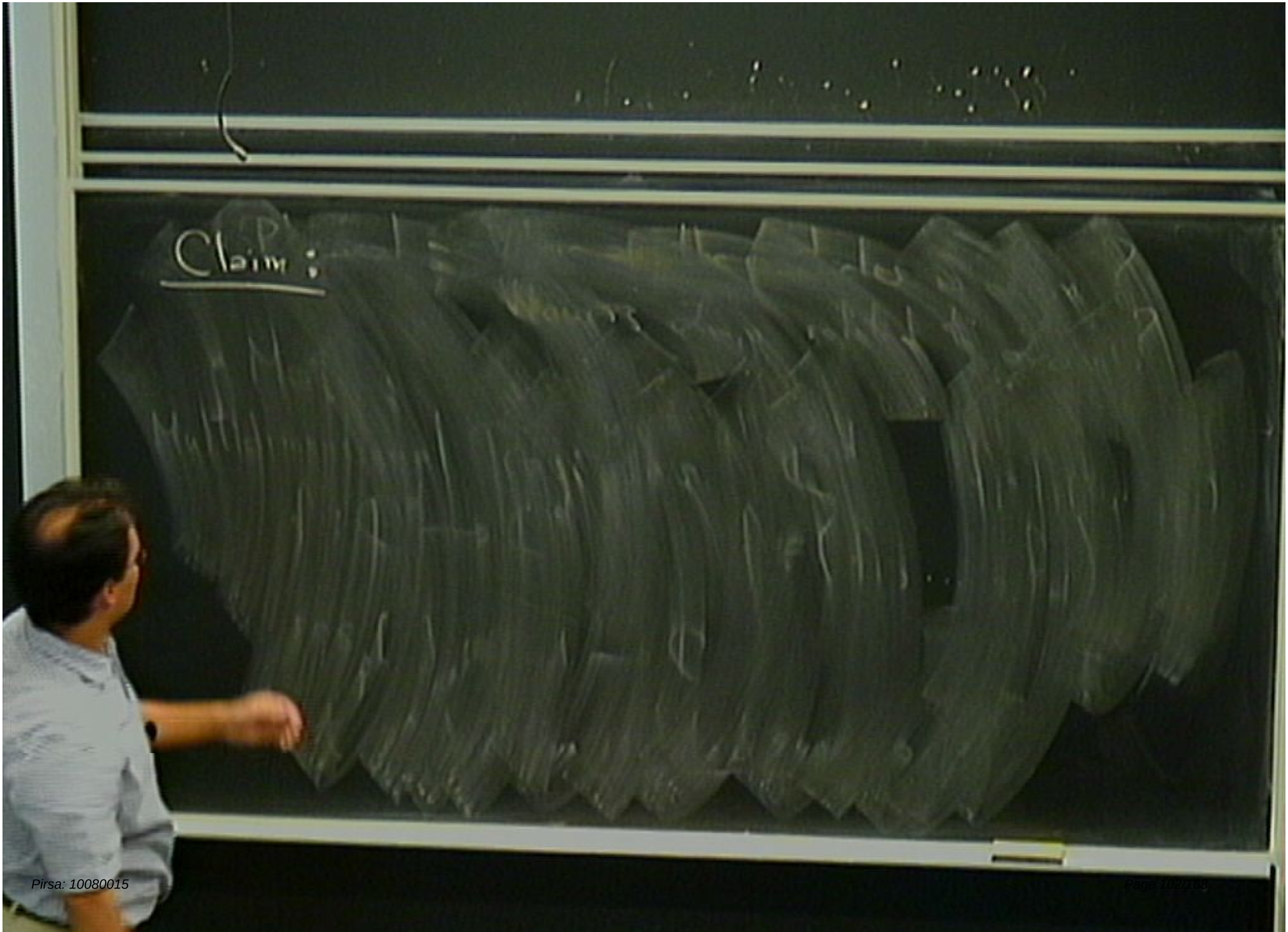
$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(456)(561)(612)} \qquad f^{(6)} = (234)(456)(612)$$

$$G(3,5) \hookrightarrow G(3,6)$$

$$\mathcal{L}_{3,5} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \left[\frac{1}{(561)} \frac{(451)(512)(234)}{f^{(6)}} \right]$$

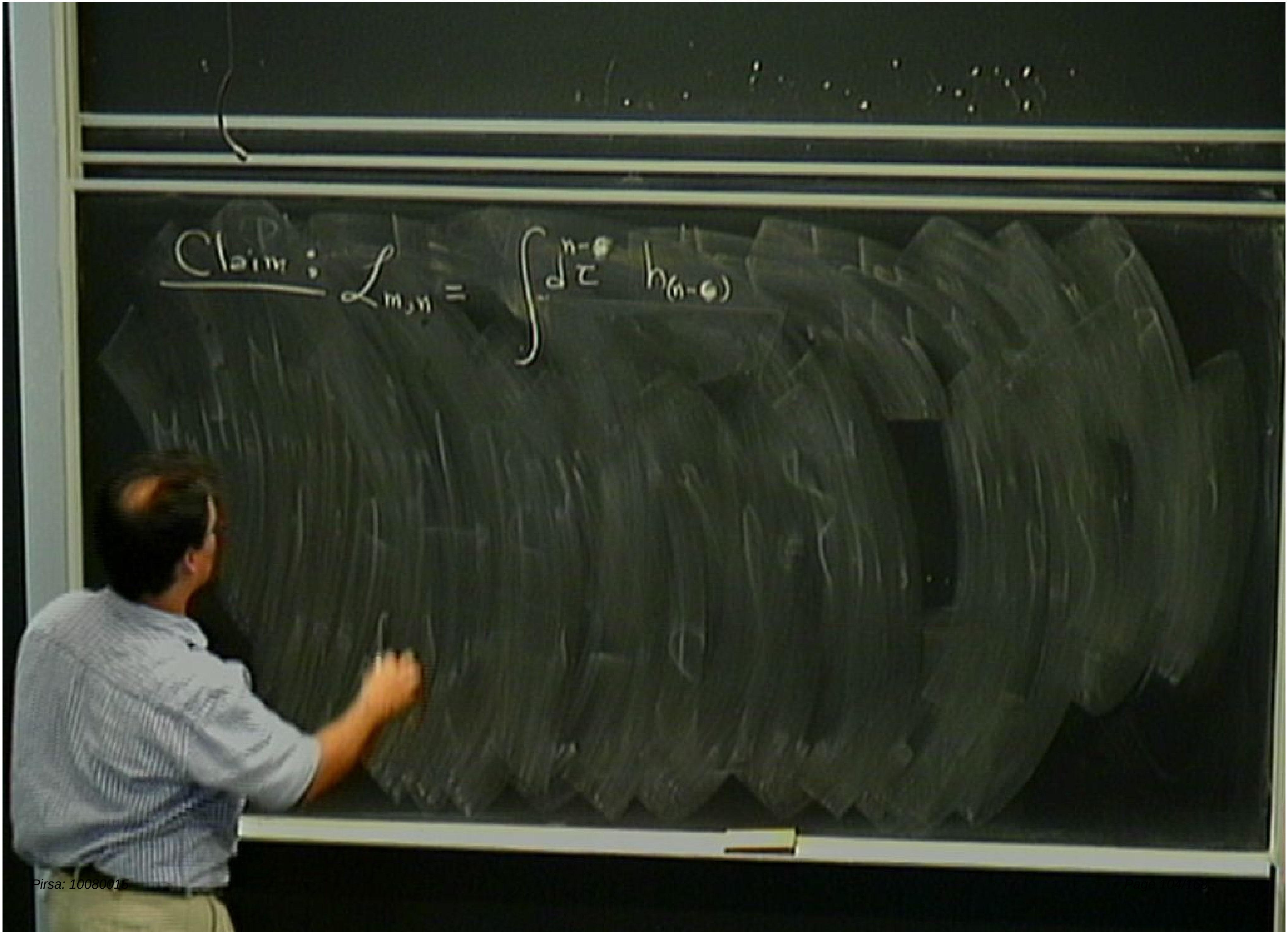
$$f = \mathbb{C} \to \mathbb{C} \qquad f = f^{(6)}$$

$$\underset{5 \to 6.}{\mathcal{A}}$$

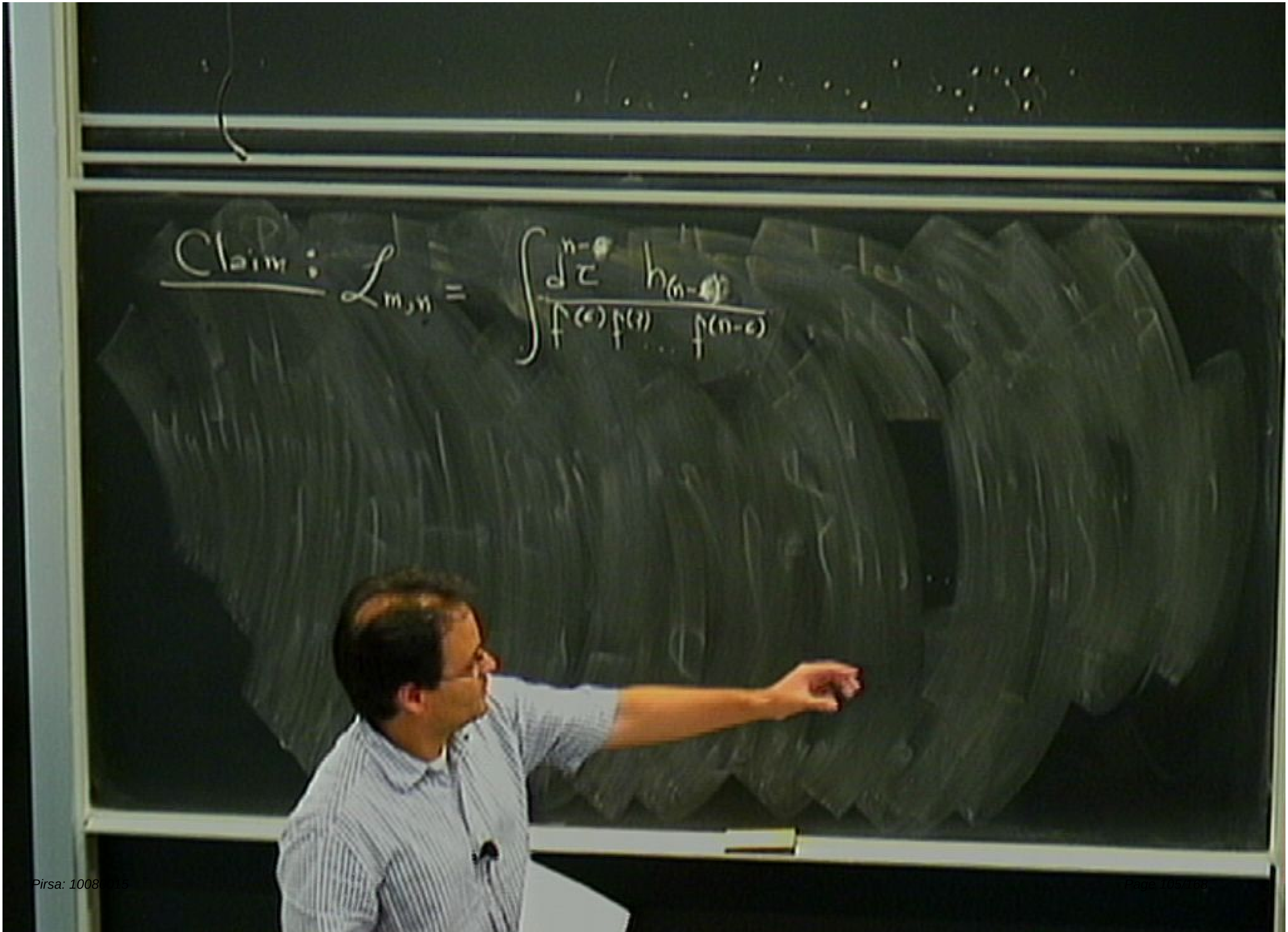


Claim :

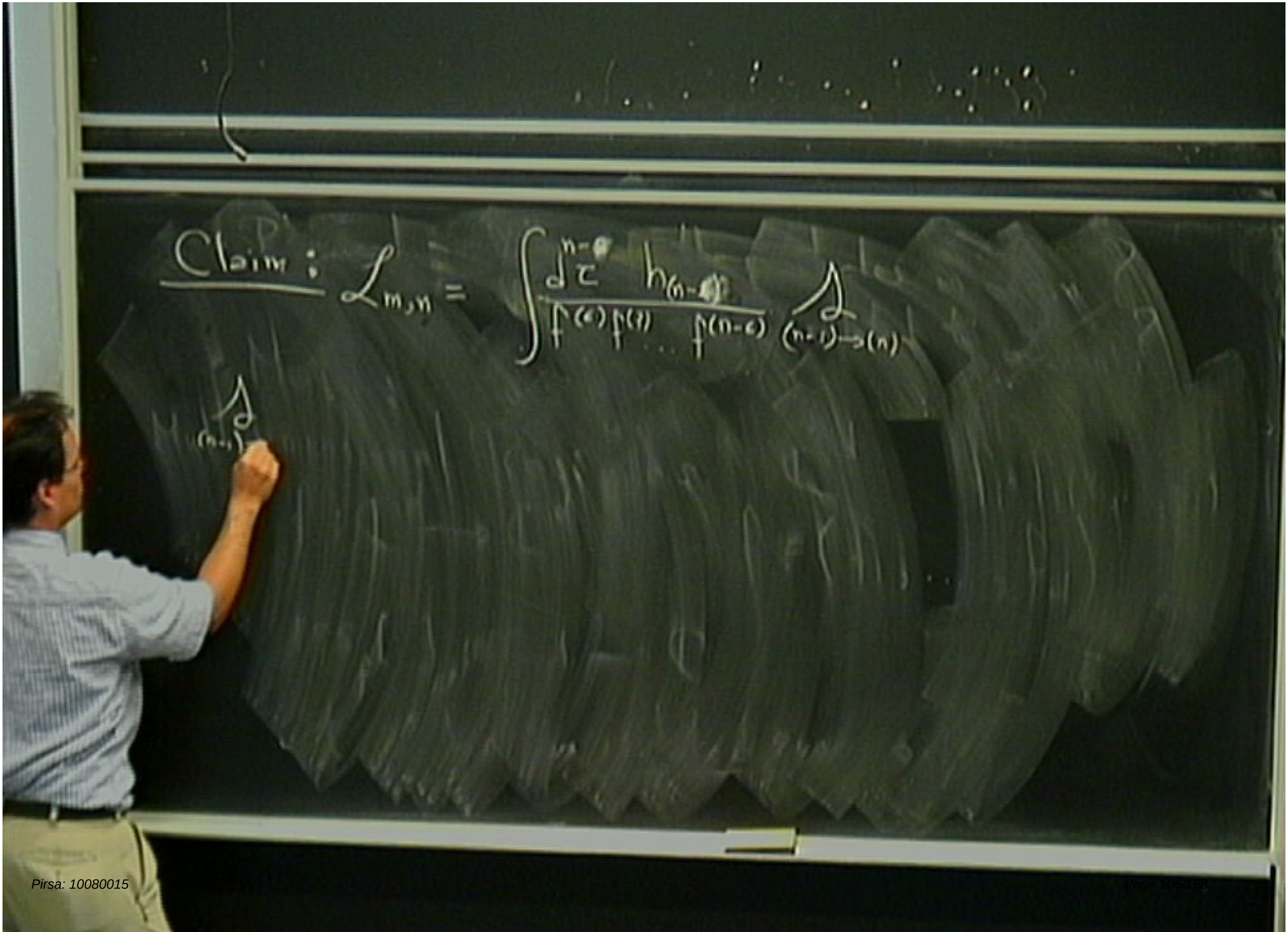
Claim: $\mathcal{L}_{m,n} =$



<u>Claim</u>: $\mathcal{L}_{m,n} = \int d^{n-6}\tau \;\; h_{(n-6)}$

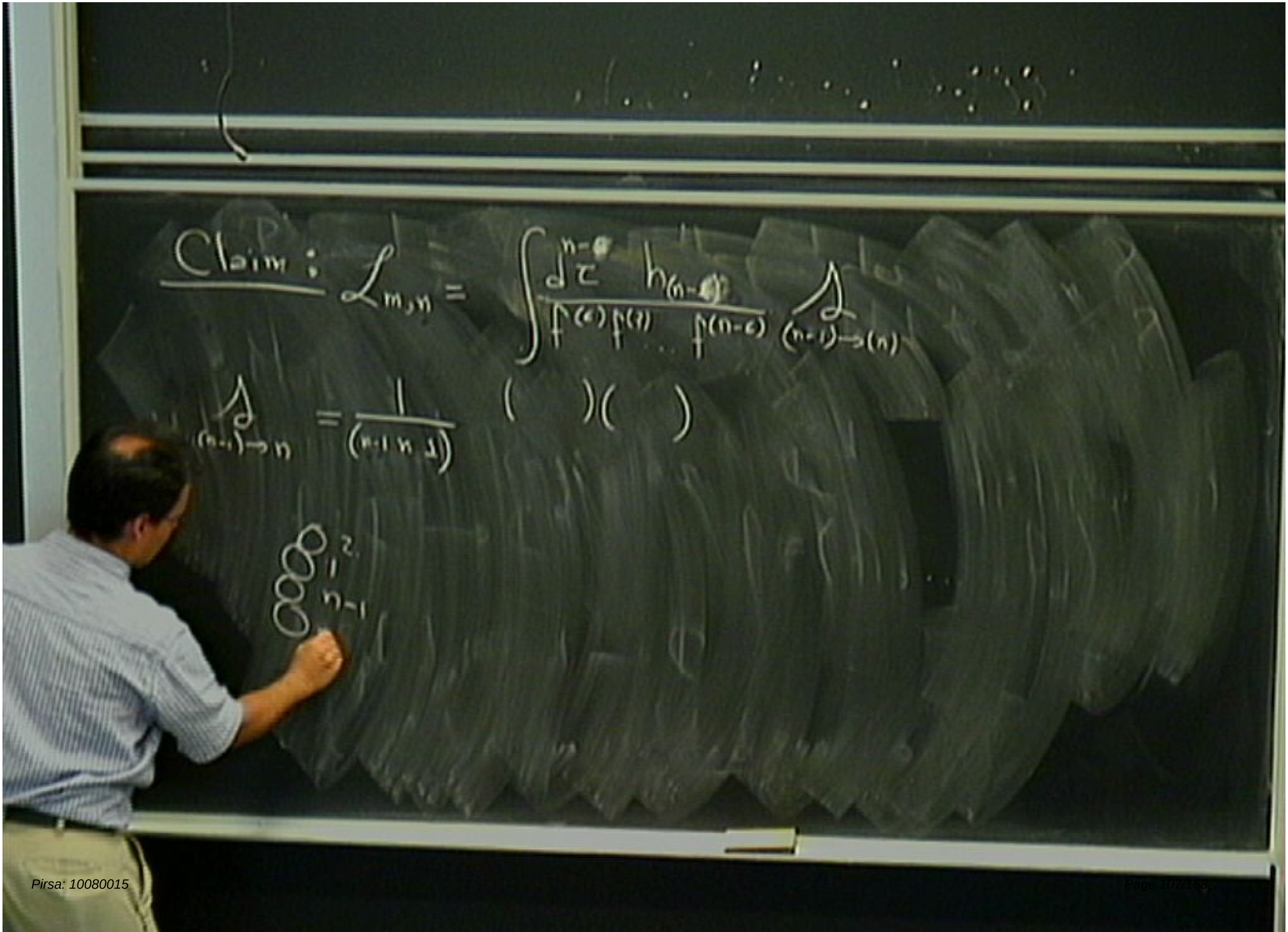


Claim: $\mathcal{L}_{m,n} = \int \frac{d^{n-\text{[illegible]}}\tau \quad h_{(n-\text{[illegible]})}}{f^{(0)} f^{(1)} \cdots f^{(n-1)}}$



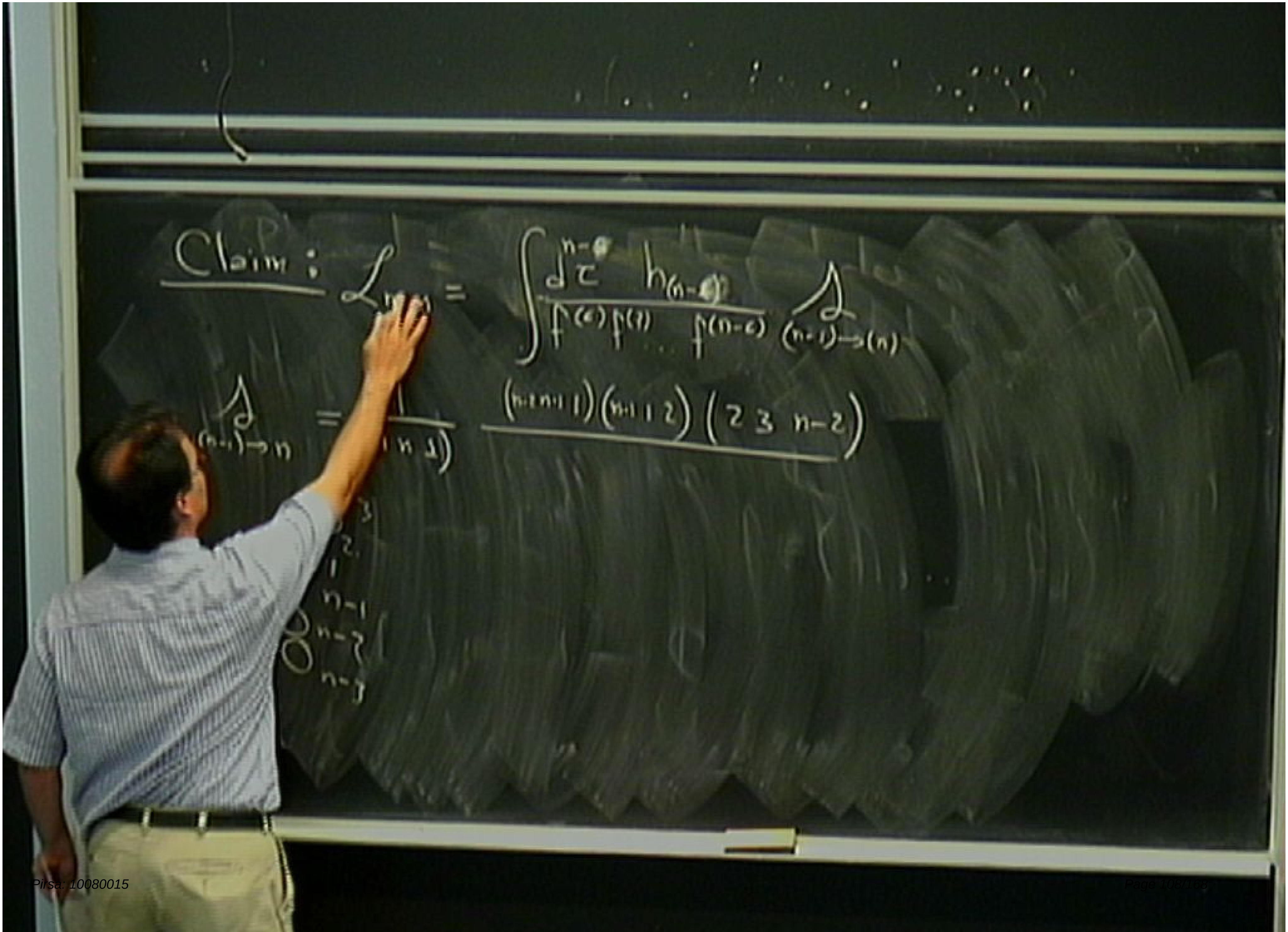
Claim: $\mathcal{L}_{m,n} = \int \frac{d^{n-}\tau \; h_{(n-}}{f^{(1)} f^{(2)} \cdots f^{(n-1)}} \partial_{(n-1) \to (n)}$

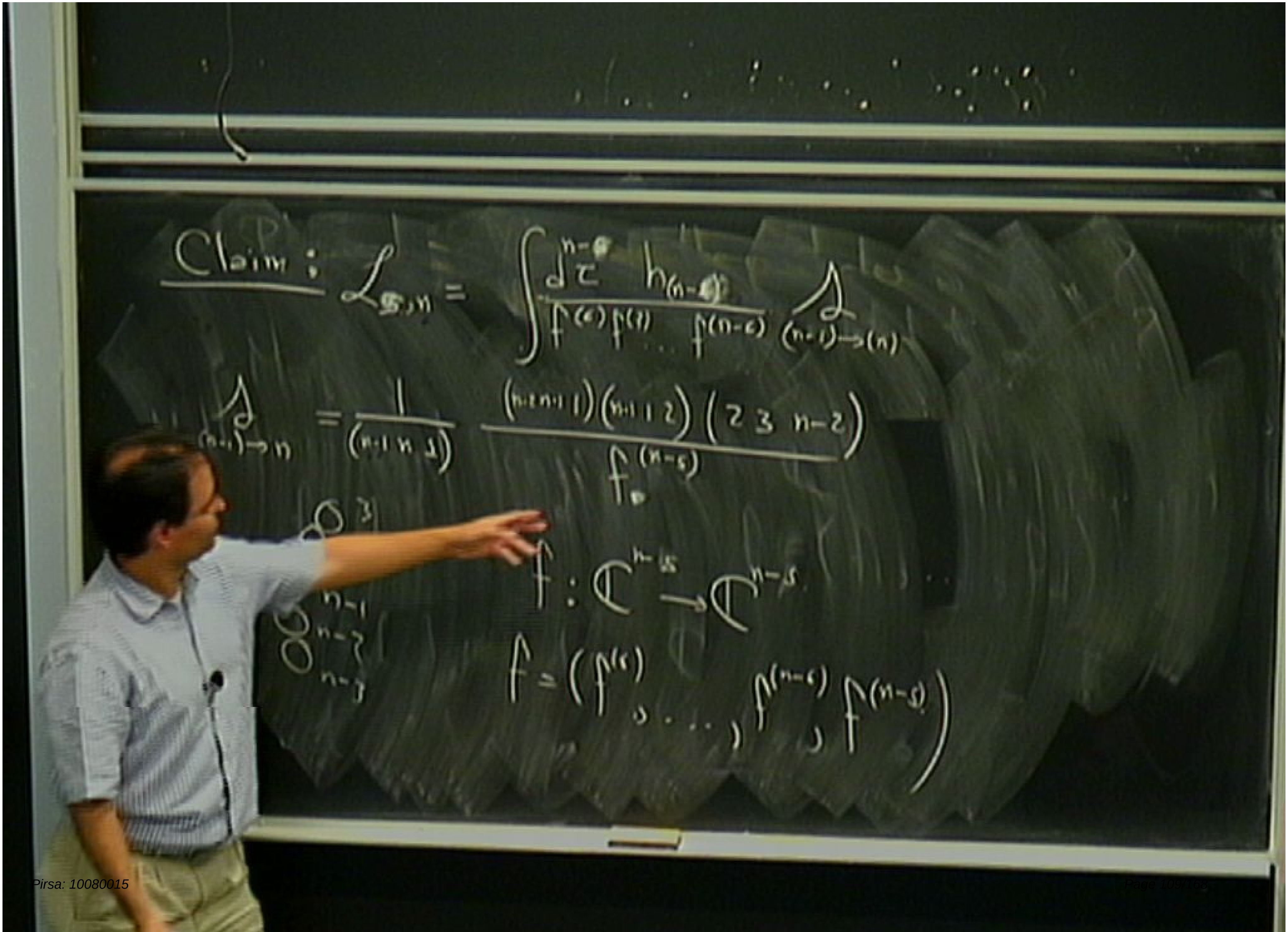
$\partial_{(n-1)}$



<u>Claim :</u> $\mathcal{L}_{m,n} = \int \frac{d^{n-}\tau \;\; h_{(n-}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \, \mathcal{J}_{(n-1)\to(n)}$

$\mathcal{J}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \quad (\quad)(\quad)$





Claim: $\mathcal{L}_{5,n} = \int \frac{d^{n-5}\tau \; h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \mathcal{A}_{(n-1)\to(n)}$

$$\mathcal{A}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \frac{(n-2\; n-1\; 1)(n-1\; 1\; 2)(2\; 3\; n-2)}{f_{\bullet}^{(n-5)}}$$

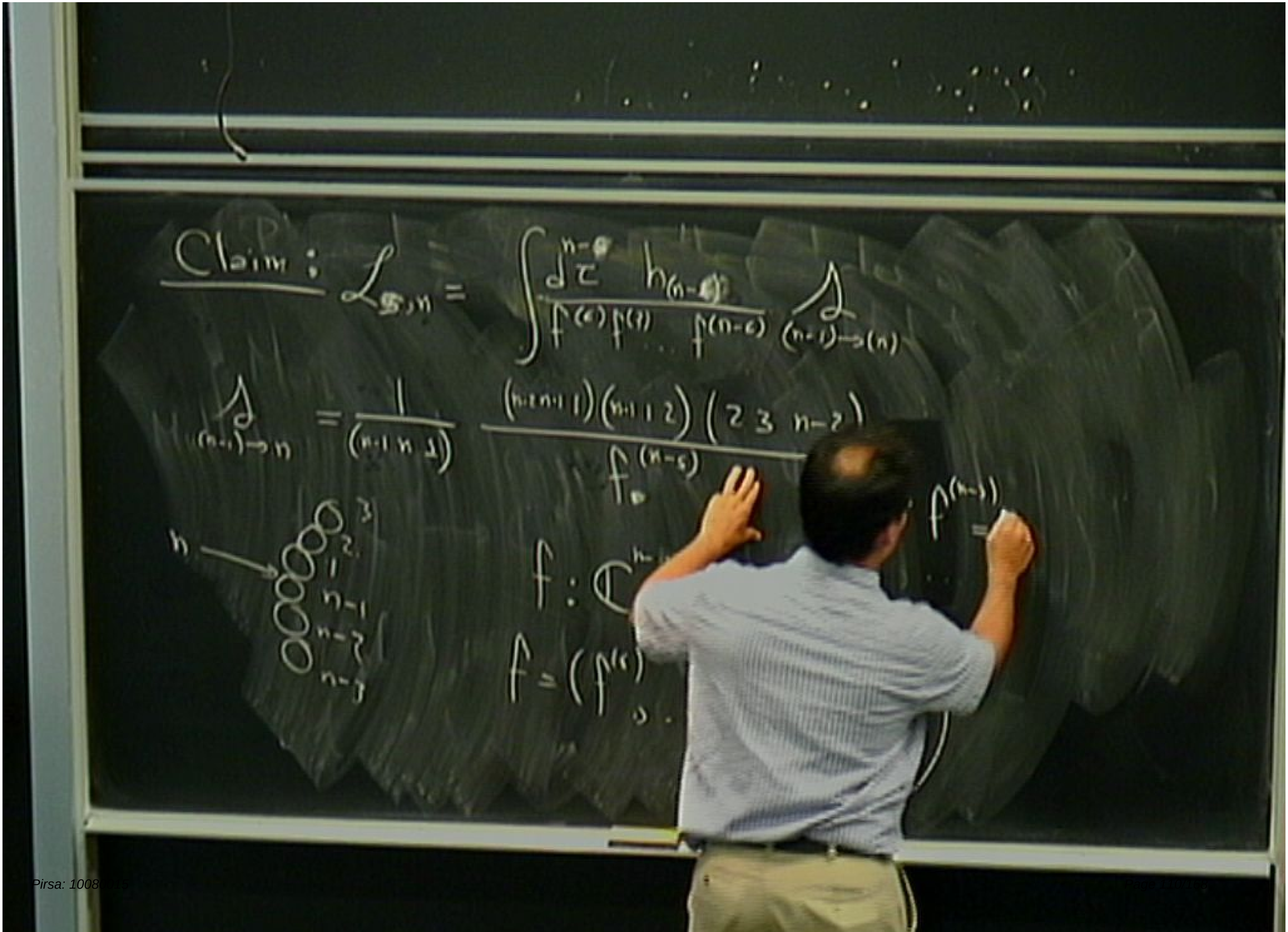
$\mathcal{O}^{3}$

$\mathcal{O}^{n-1}$

$\mathcal{O}^{n-2}_{n-3}$

$$f : \mathbb{C}^{n-5} \to \mathbb{C}^{n-5}$$

$$f = (f^{(6)}, \ldots, f^{(n-6)}, f^{(n-5)})$$



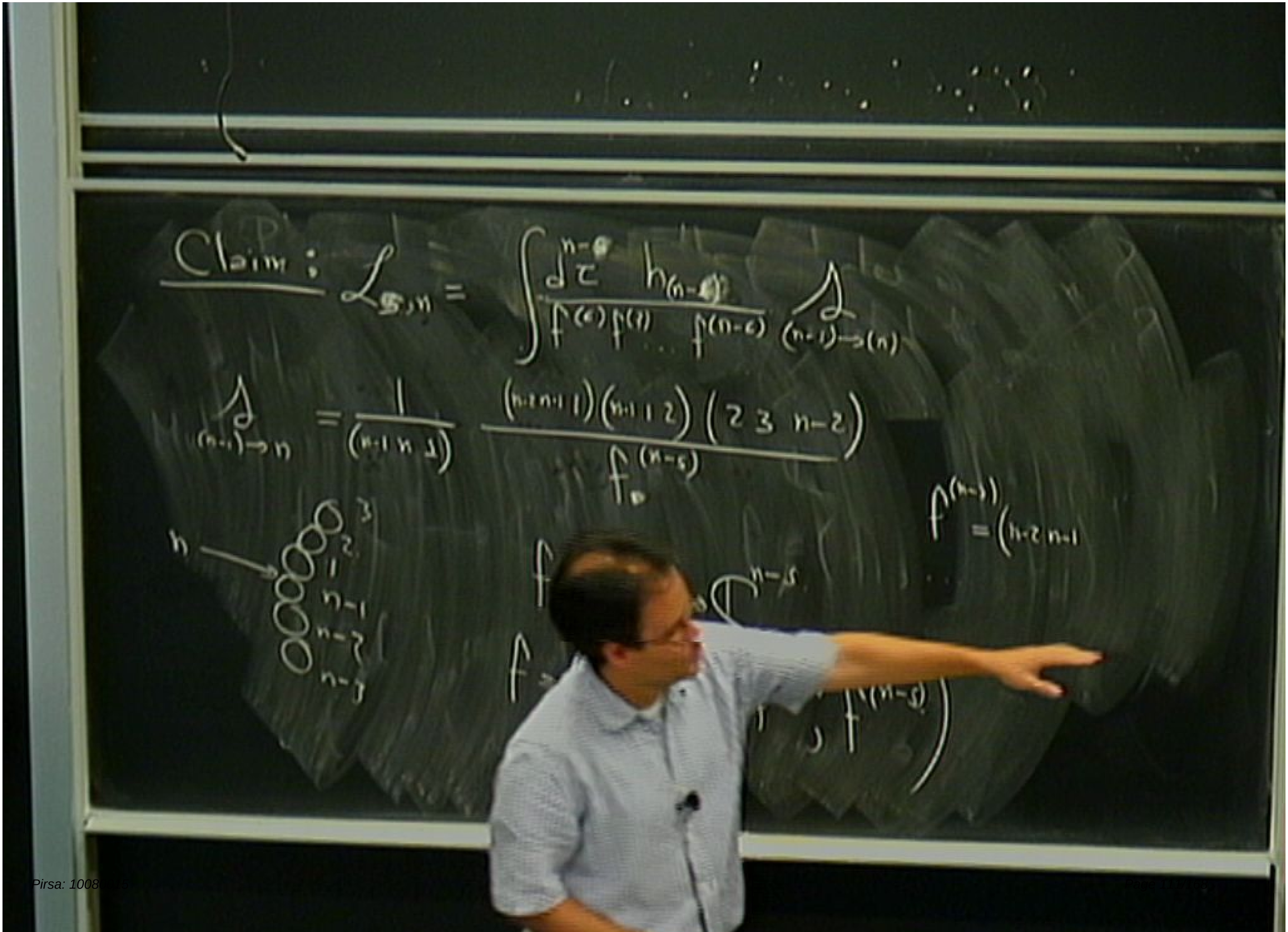
Claim: $\mathcal{L}_{5,n} = \int \frac{d^{n-5}\tau \quad h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \mathcal{A}_{(n-1)\to(n)}$

$\mathcal{A}_{(n-1)\to n} = \frac{1}{(n-1\ n\ 1)} \frac{(n-2\ n-1\ 1)(n-1\ 1\ 2)\ (2\ 3\ n-2)}{f_0^{(n-6)}}$

$f : \mathbb{C}^{n-}$

$f = (f_0^{(6)}$

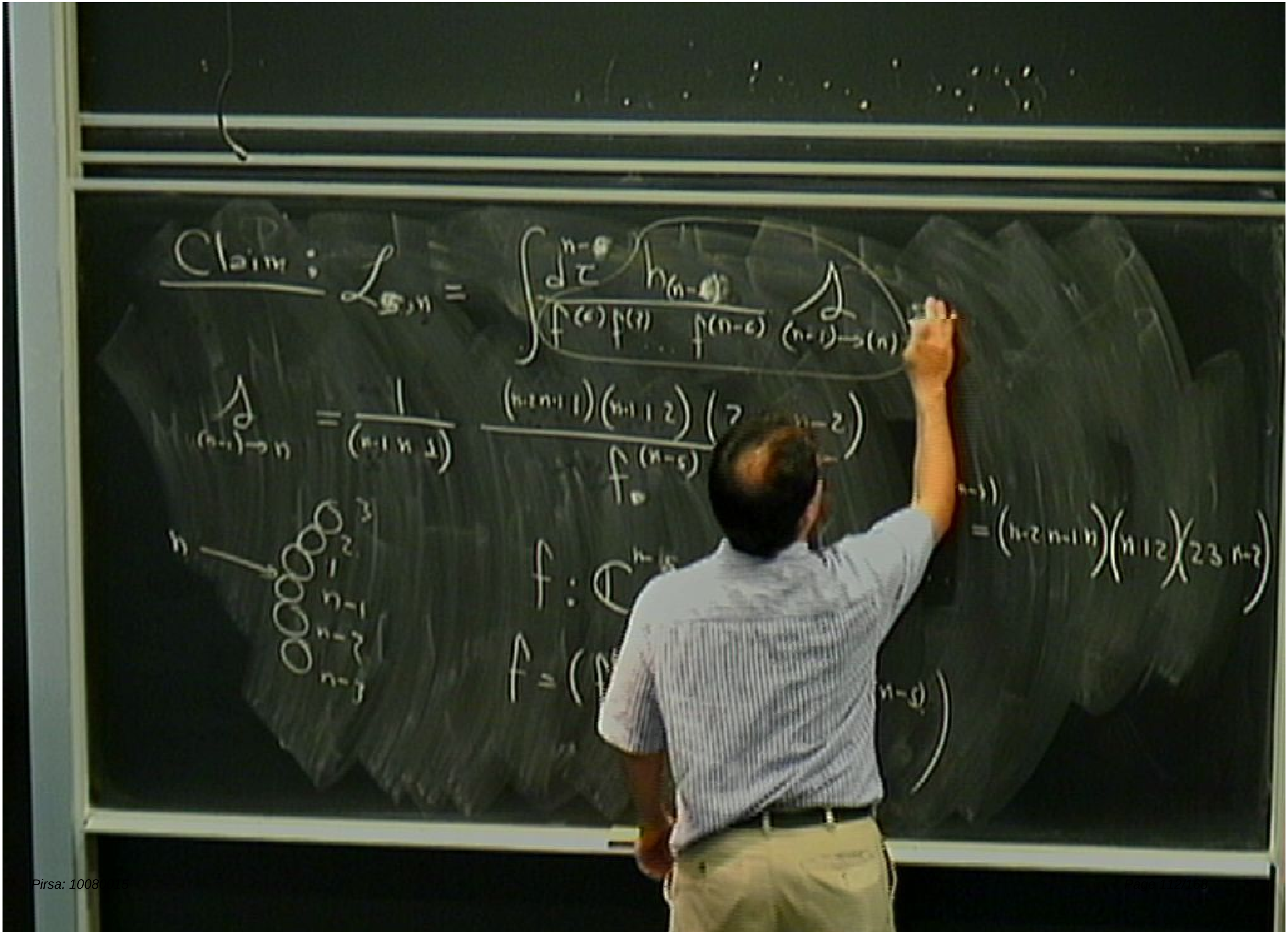
$f^{(n-1)}$



Claim: $\mathcal{L}_{\cdot,n} = \int \frac{d^{n-\cdot}\tau \; h_{(n-\cdot)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \mathcal{A}_{(n-1)\to(n)}$

$$\mathcal{A}_{(n-1)\to n} = \frac{1}{(n{-}1\; n\; 1)} \frac{(n{-}2\; n{-}1\; 1)(n{-}1\; 1\; 2)\,(2\; 3\; n{-}2)}{f^{(n-6)}}$$

$f^{(n-5)} = (n{-}2\; n{-}1$



Claim: $\mathcal{L}_{6,n} = \int \frac{d^{n-6}\tau \; h_{(n-6)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \, \mathcal{J}_{(n-1)\to(n)}$

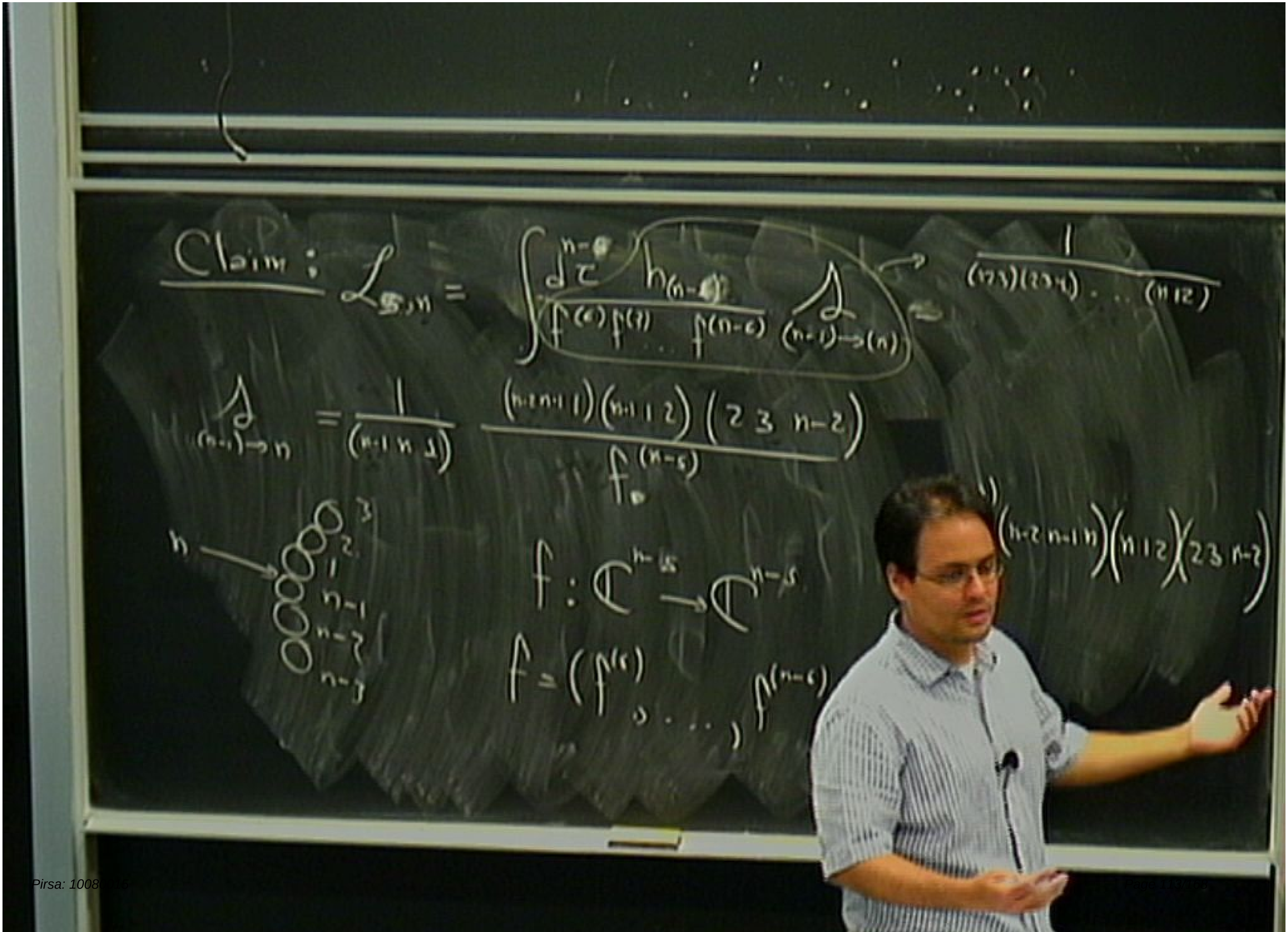
$\mathcal{J}_{(n-1)\to n} = \frac{1}{(n-1\ n\ 1)} \; \frac{(n-2\ n-1\ 1)(n-1\ 1\ 2)(2\ \ldots\ n-2)}{f_0^{(n-6)}}$

$n-1)$

$= (n-2\ n-1\ n)(n\ 1\ 2)(2\ 3\ n-2)$

$f : \mathbb{C}^{n-6}$

$f = (f \ldots n-6))$



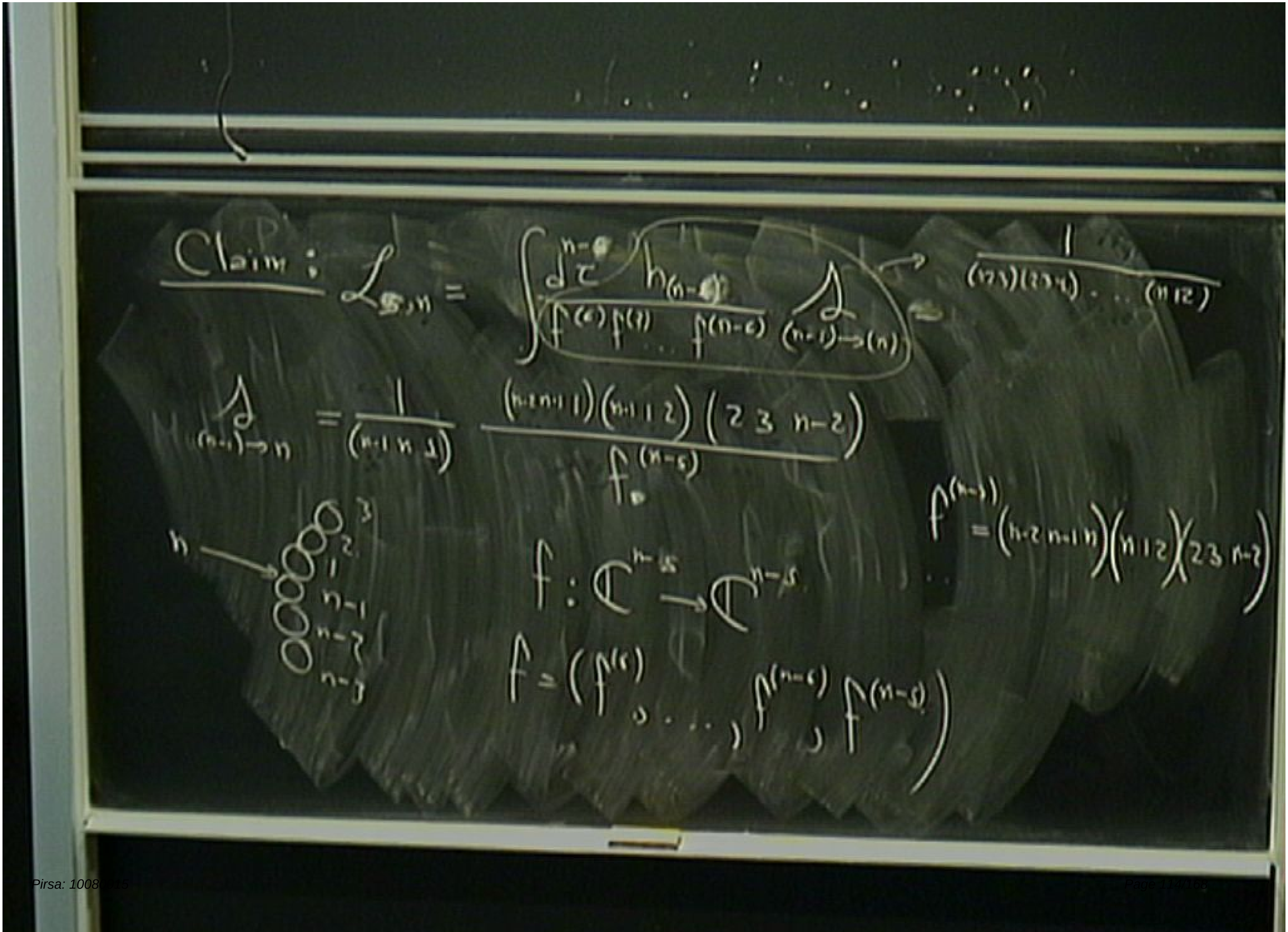
Claim: $\mathcal{L}_{6,n} = \int d^{n-6}\tau \, \frac{h_{(n-6)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \, \mathcal{J}_{(n-1)\to(n)} \longrightarrow \frac{1}{(123)(234)\cdots(n12)}$

$$\mathcal{J}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \, \frac{(n-2\; n-1\; 1)(n-1\; 1\; 2)(2\; 3\; n-2)}{f^{(n-6)}}$$

$$(n-2\; n-1\; n)(n\; 1\; 2)(2\; 3\; n-2)$$

$f : \mathbb{C}^{n-6} \to \mathbb{C}^{n-6}$

$f = (f^{(6)}, \ldots, f^{(n-6)})$



<u>Claim</u>: $\mathcal{L}_{5,n} = \int \frac{d^{n-5}\tau \; h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \, \mathcal{R}_{(n-1)\to(n)} \to \frac{1}{(123)(234)\cdots(n12)}$

$$\mathcal{R}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \frac{(n-2\; n-1\; 1)(n-1\; 1\; 2)(2\; 3\; n-2)}{f^{(n-5)}}$$

$$f : \mathbb{C}^{n-5} \to \mathbb{C}^{n-5}$$

$$f = (f^{(6)}, \ldots, f^{(n-6)}, f^{(n-5)})$$

$$f^{(n-4)} = (n-2\; n-1\; n)(n\; 1\; 2)(2\; 3\; n-2)$$

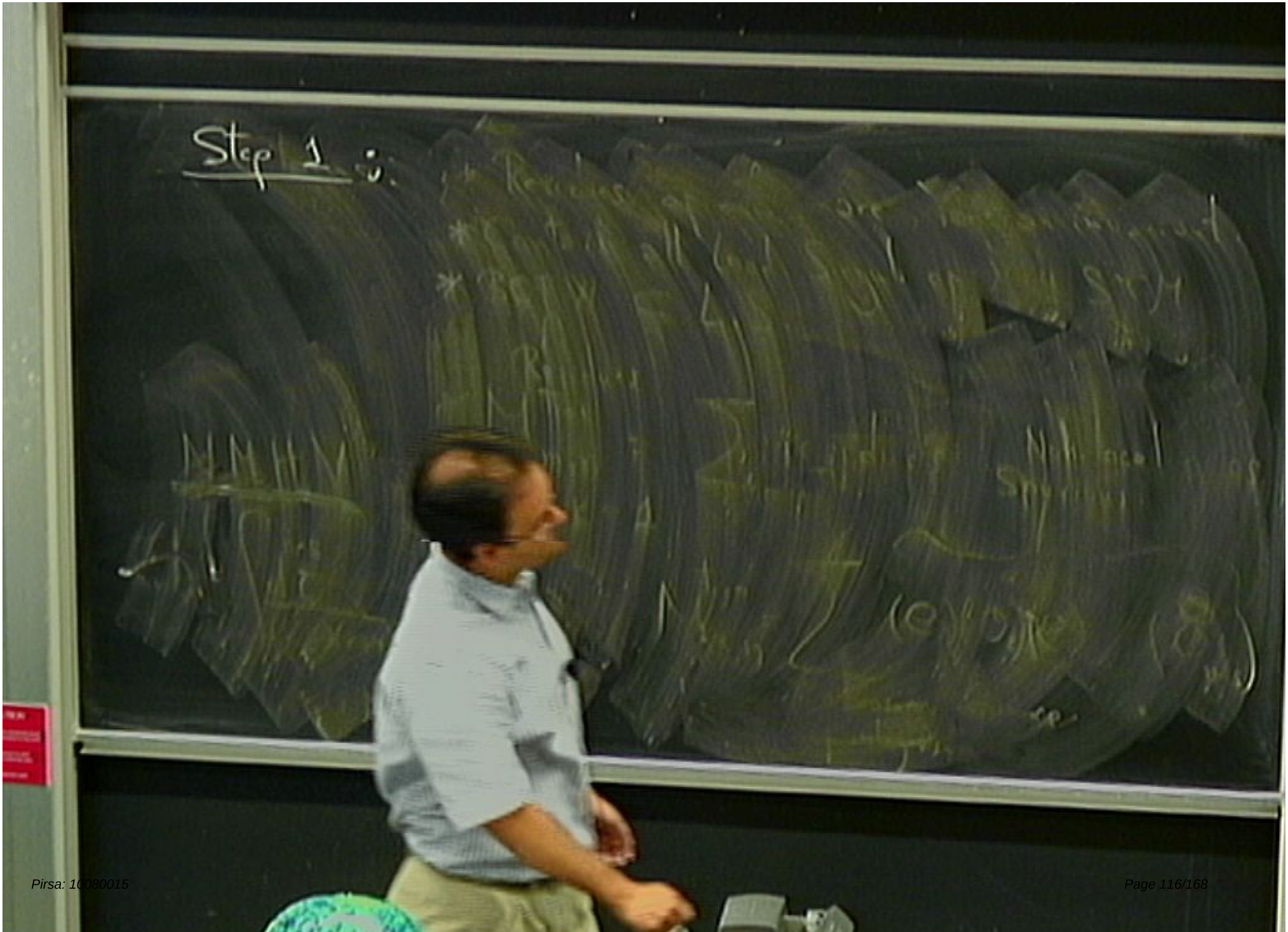
Claim: $\mathcal{L}_{5,n} = \int \boxed{\frac{d\tau^{n-5}\ h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}}\ \mathcal{A}_{(n-1)\to(n)}} \to \frac{1}{(123)(234)\cdots(n12)}$

$$\mathcal{A}_{(n-1)\to n} = \frac{1}{(n-1\,n\,1)}\ \frac{(n-2\,n-1\,1)(n-1\,1\,2)(2\,3\,n-2)}{f^{(n-6)}}$$

$f : \mathbb{C}^{n-5} \to \mathbb{C}^{n-5}$

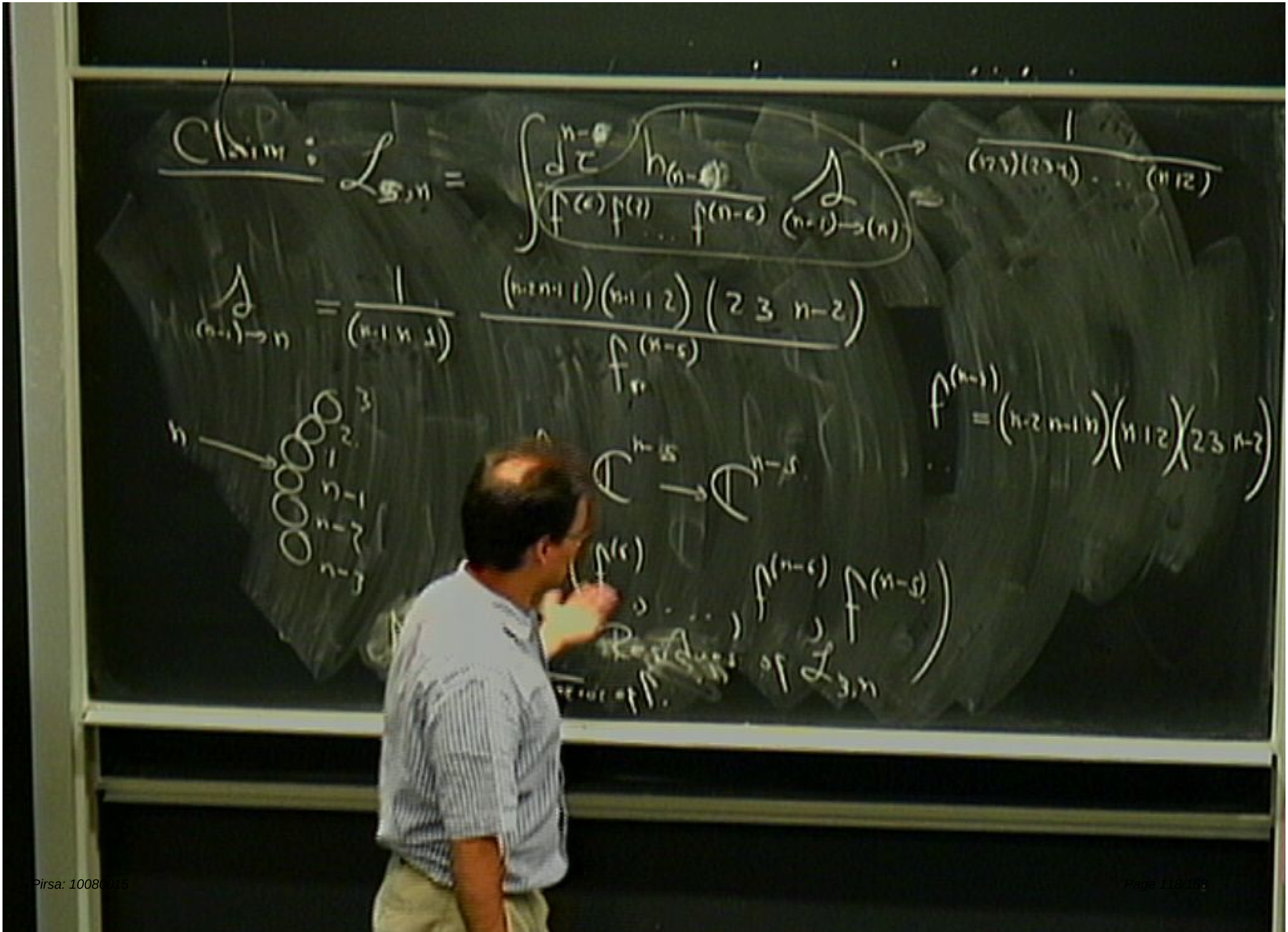
$f = \left(f^{(6)}, \ldots, f^{(n-6)}, f^{(n-5)}\right)$

$f^{(n-5)} = (n-2\,n-1\,n)(n\,1\,2)(2\,3\,n-2)$



Step 1 :

Step 1 : Claim: $\mathcal{L}_{3,n}\,(\;,\;)(\;,\;)\ldots(n-1,1) = 0$



Claim: $\mathcal{L}_{3,n} = \int d^{n-5}\tau \, \frac{h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \, \mathcal{A}_{(n-1)\to(n)} \to \frac{1}{(123)(234)\cdots(n12)}$

$$\mathcal{A}_{(n-1)\to n} = \frac{1}{(n-1\, n\, 1)} \, \frac{(n-2\, n-1\, 1)(n-1\, 1\, 2)(2\, 3\, n-2)}{f_n^{(n-5)}}$$

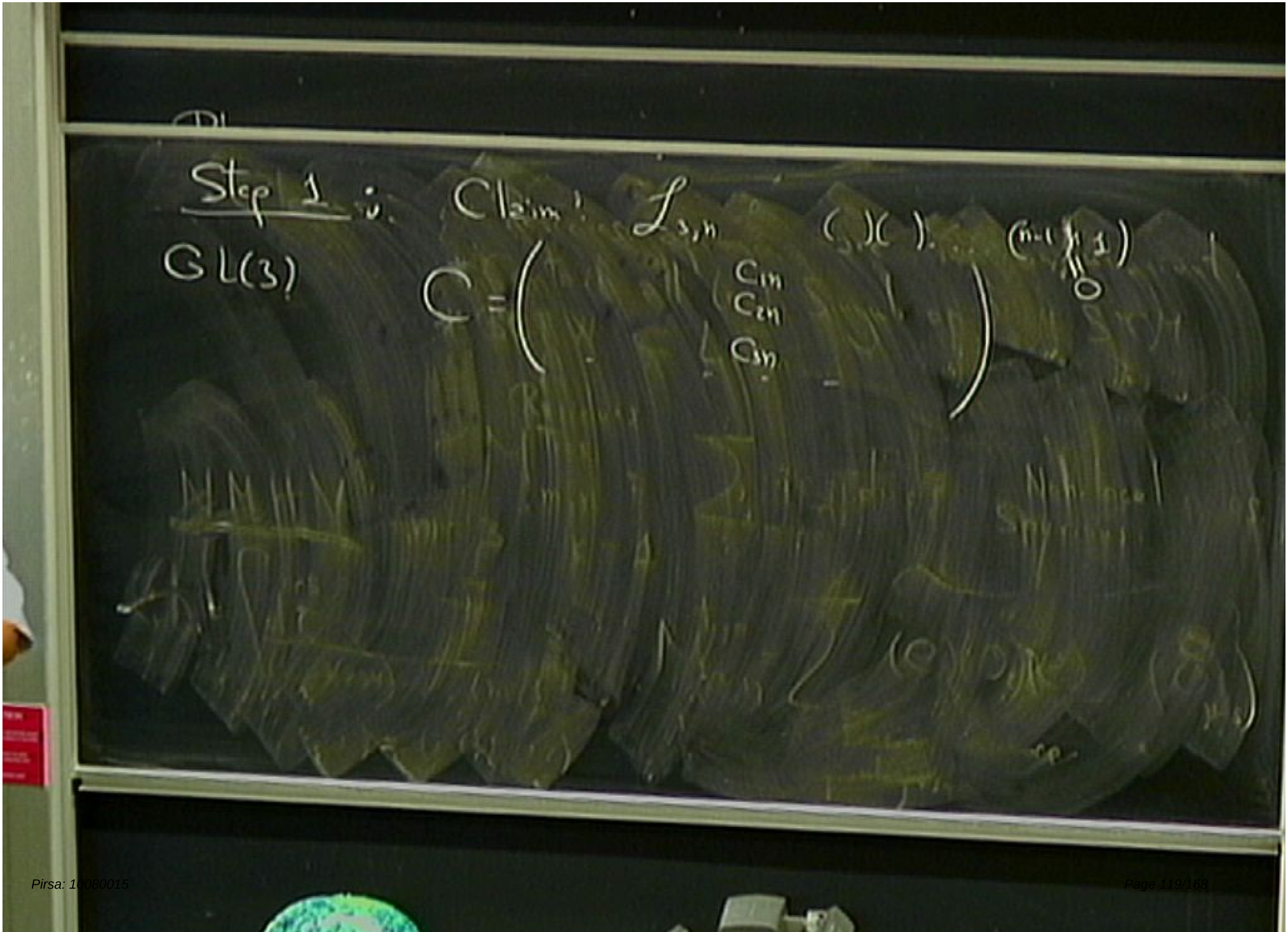
$$f^{(n-1)} = (n-2\, n-1\, n)(n\, 1\, 2)(2\, 3\, n-2)$$

$\mathbb{C}^{n-5} \to \mathbb{C}^{n-5}$

$(f^{(6)}, \ldots, f^{(n-6)}, f^{(n-5)})$

Residues of $\mathcal{L}_{3,n}$

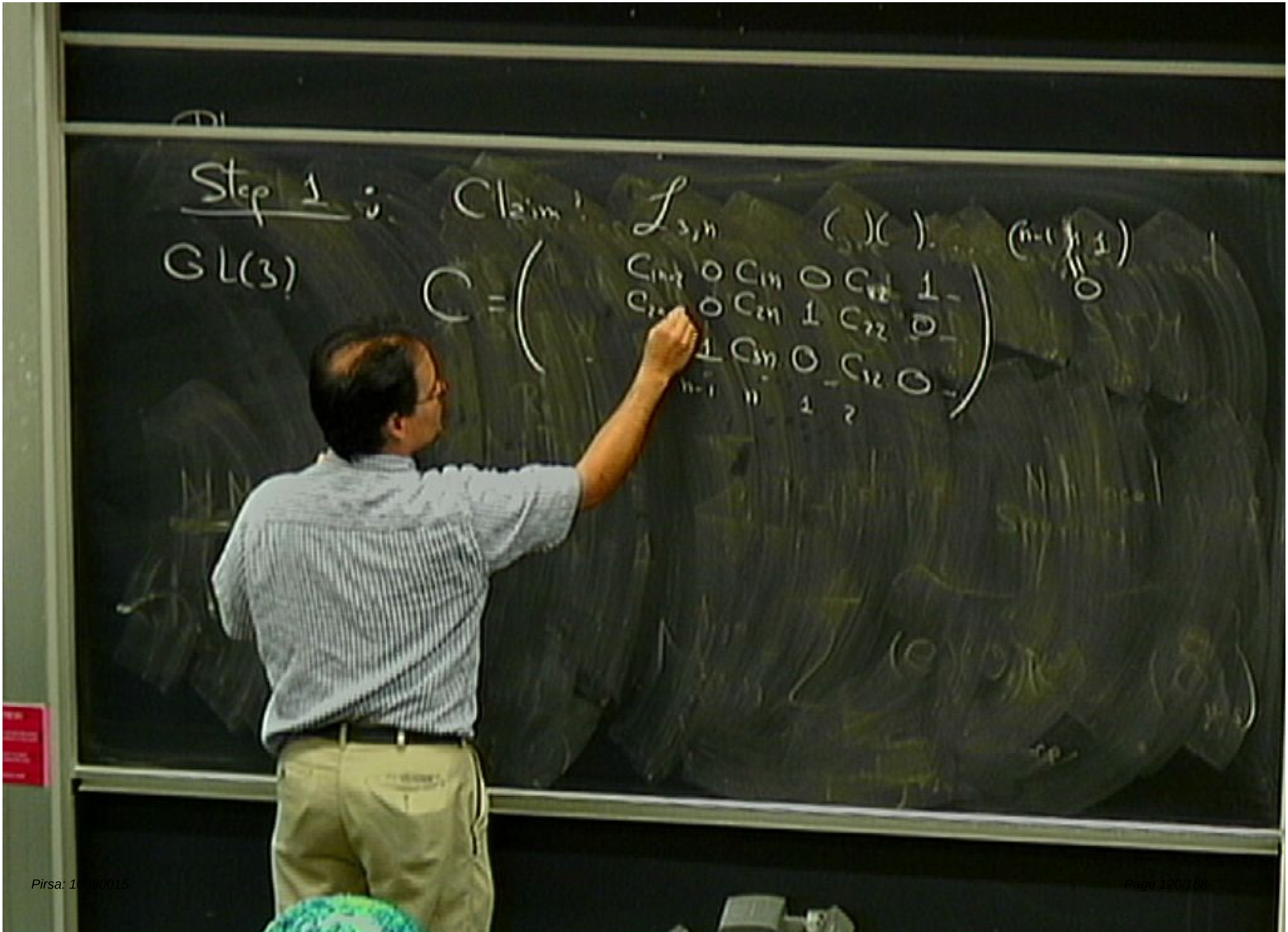
zeroes of f.



Step 1 i. Claim: $\mathcal{L}_{3,n}$ $(,)( ) \ldots (n-1, 1)$ $\neq 0$

GL(3)

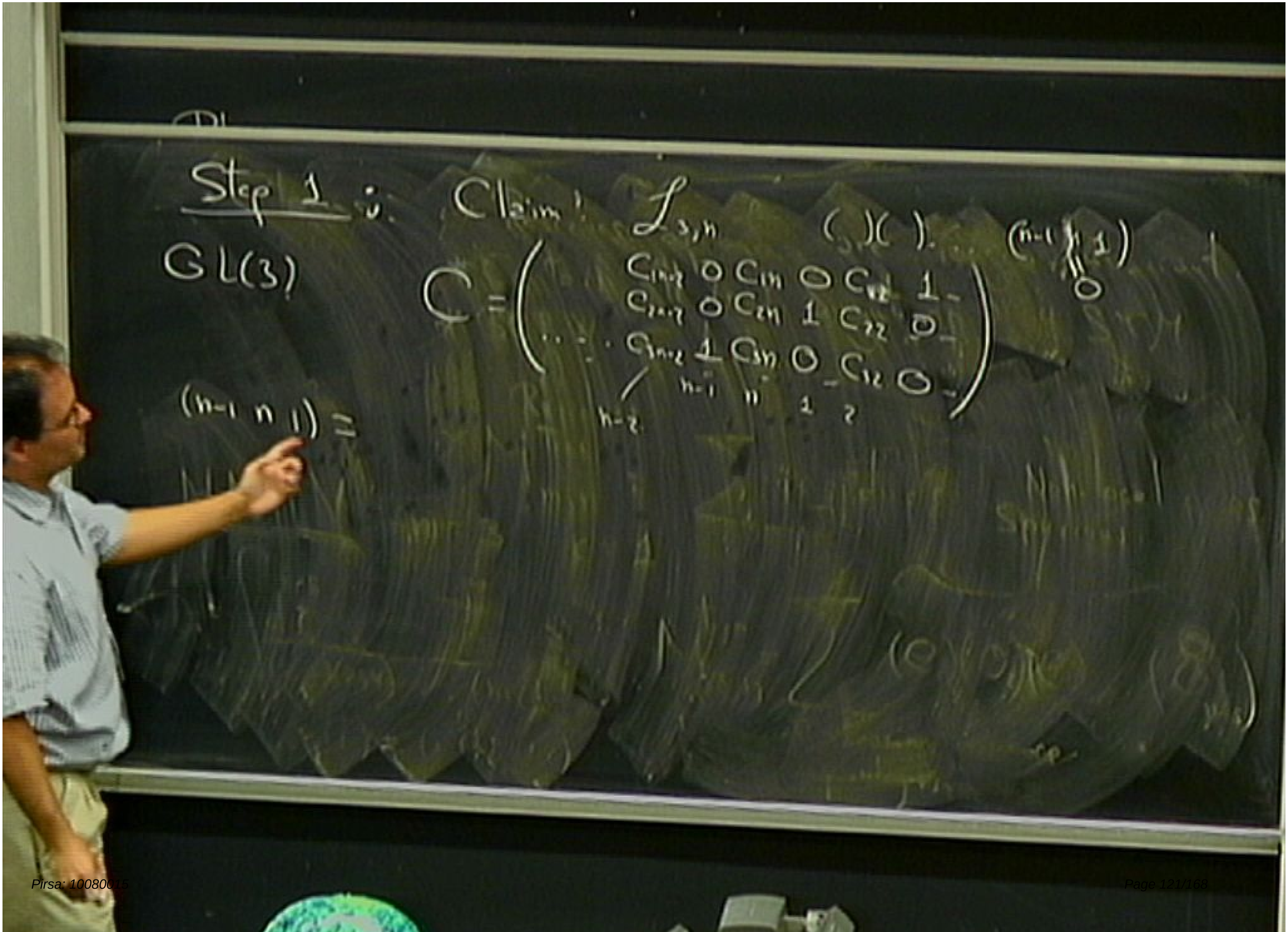
$C = \begin{pmatrix} & c_{1n} & \\ & c_{2n} & \\ & c_{3n} & \end{pmatrix}$



Step 1 : Claim: $\mathcal{L}_{3,n}$ $(\ )(\ ).$ $(n-1)(1)$ $\overset{\shortparallel}{0}$

GL(3)

$$C = \begin{pmatrix} C_{1n-1} & 0 & C_{1n} & 0 & C_{12} & 1 \\ C_{2n-1} & 0 & C_{2n} & 1 & C_{22} & 0 \\ & 1 & C_{3n} & 0 & C_{12} & 0 \\ & n-1 & n & 1 & 2 & \end{pmatrix}$$



Step 1 i. Claim! $\mathcal{L}_{3,n}$ $(\;, \;)(\;).$ $(n-1\; n\; 1)$ $\overset{?}{=} 0$

GL(3)

$$C = \begin{pmatrix} \cdots & c_{1n-2} & 0 & c_{1n} & 0 & c_{12} & 1 & \cdots \\ \cdots & c_{2n-2} & 0 & c_{2n} & 1 & c_{22} & 0 & \cdots \\ \cdots & c_{3n-2} & 1 & c_{3n} & 0 & c_{32} & 0 & \cdots \\ & n-2 & n-1 & n & 1 & 2 & & \end{pmatrix}$$

$(n-1\; n\; 1) =$

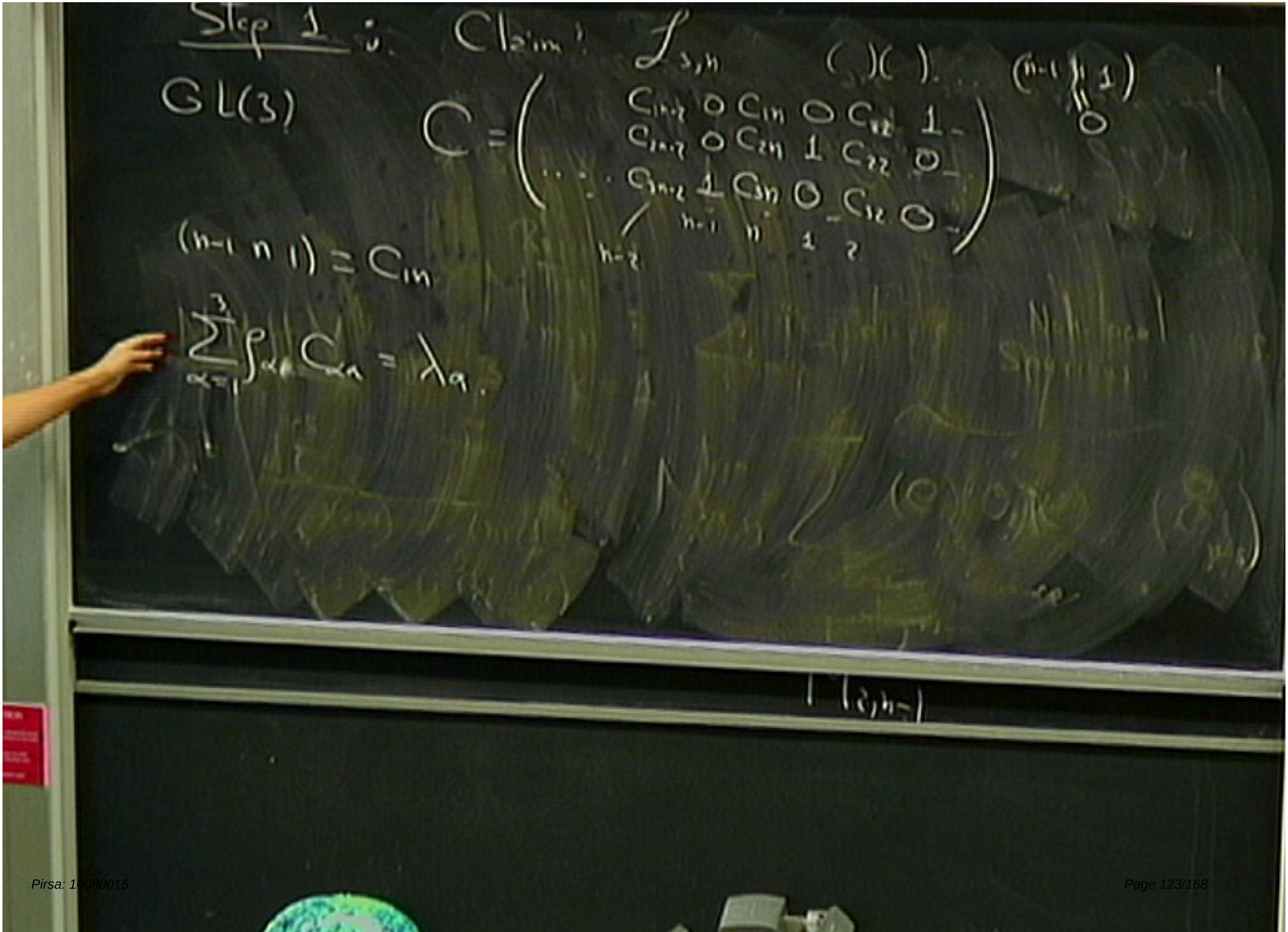
Step 1 : Claim: $\mathcal{L}_{3,n}$ $(\ )(\ )\ldots \underset{\substack{\| \\ 0}}{(n-1\ n\ 1)}$

GL(3)

$$C = \begin{pmatrix} \cdots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 & \cdots \\ \cdots & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 & \cdots \\ \cdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{32} & 0 & \cdots \\ & n-2 & n-1 & n & 1 & 2 & & \end{pmatrix}$$

$(n-1\ n\ 1) = C_{1n}$

$(2, n-)$



Step 1 i. Claim: $\mathcal{L}_{3,n}$ $(\;)(\;)$. $(n-1\, n\, 1) = 0$

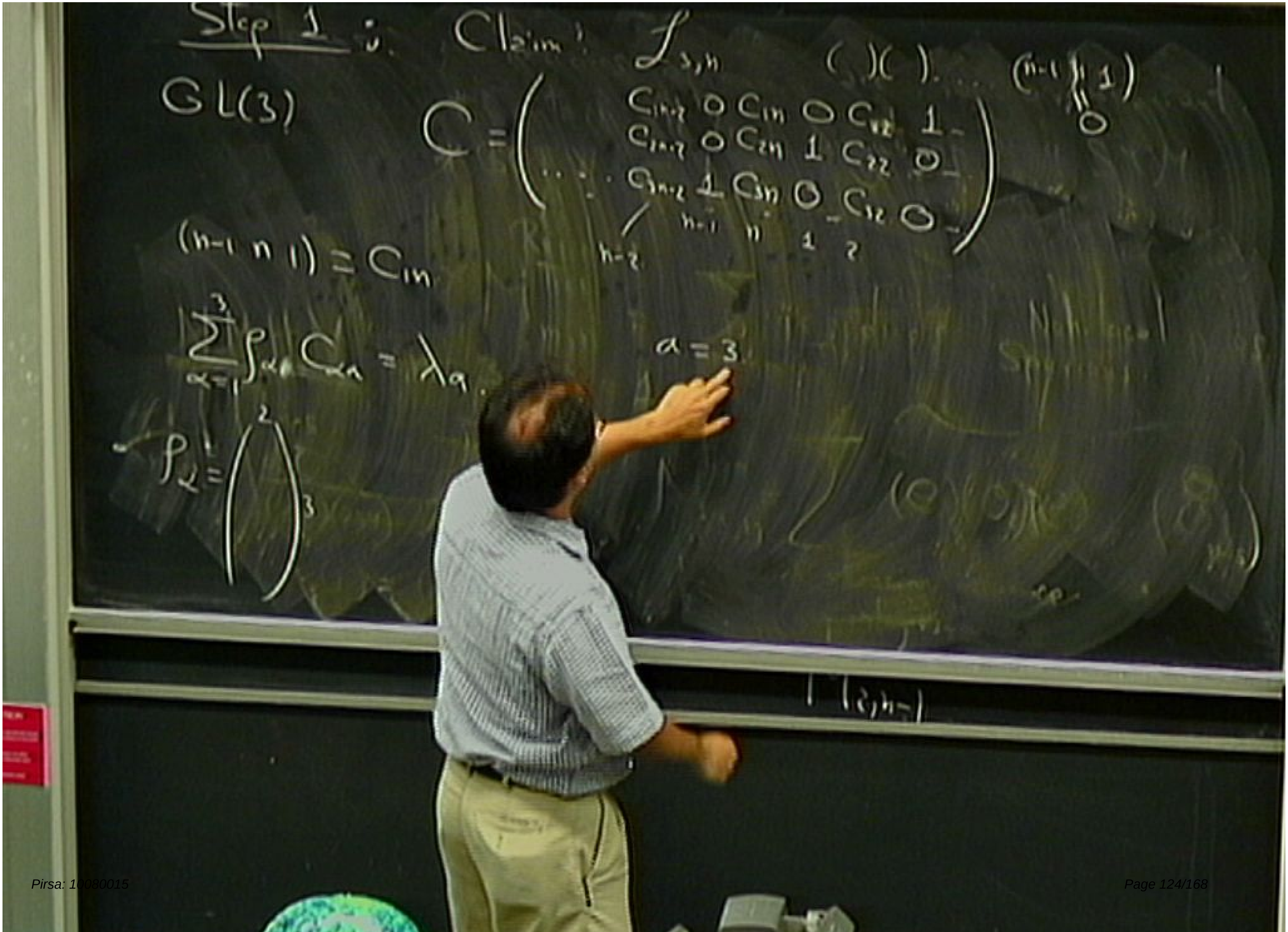
GL(3)

$$C = \begin{pmatrix} \cdots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 & \cdots \\ \cdots & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 & \cdots \\ \cdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{12} & 0 & \cdots \\ & n-2 & n-1 & n & 1 & 2 & & \end{pmatrix}$$

$(n-1\; n\; 1) = C_{1n}$

$$\sum_{\alpha=1}^{3} \rho_{\alpha a} C_{\alpha n} = \lambda_a .$$

$(2, n-)$



Step 1 : Claim: $\mathcal{L}_{3,n}$ $(\ )(\ )\ldots(n-1\ n\ 1) = 0$

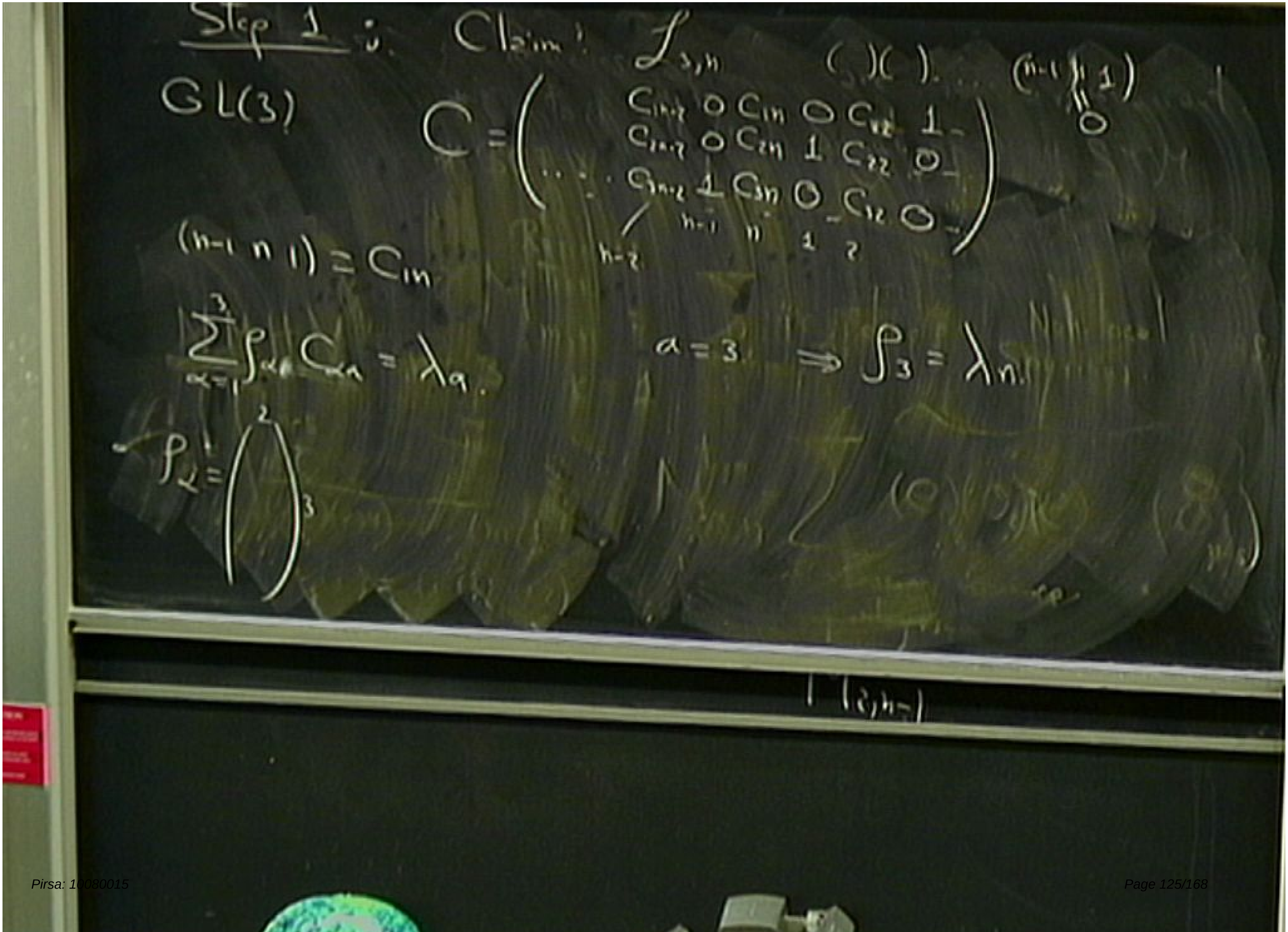
GL(3)

$$C = \left(\begin{array}{cccccccc} \cdots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 & \cdots \\ \cdots & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 & \cdots \\ \cdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{32} & 0 & \cdots \\ & n-2 & n-1 & n & 1 & 2 & & \end{array}\right)$$

$(n-1\ n\ 1) = C_{1n}$

$$\sum_{\alpha=1}^{3} \rho_{\alpha a} C_{\alpha a} = \lambda_a \qquad a = 3$$

$$\rho_\alpha = \left(\ \right)^{2}_{3}$$



Step 1 : Claim : $\mathcal{L}_{3,n}$ ( )( ) … $(n-1\ n\ 1) = 0$

GL(3)

$$C = \left(\begin{array}{cccccc} \cdots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 \\ \cdots & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 \\ \cdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{32} & 0 \\ & n-2 & n-1 & n & 1 & 2 & \end{array}\right)$$

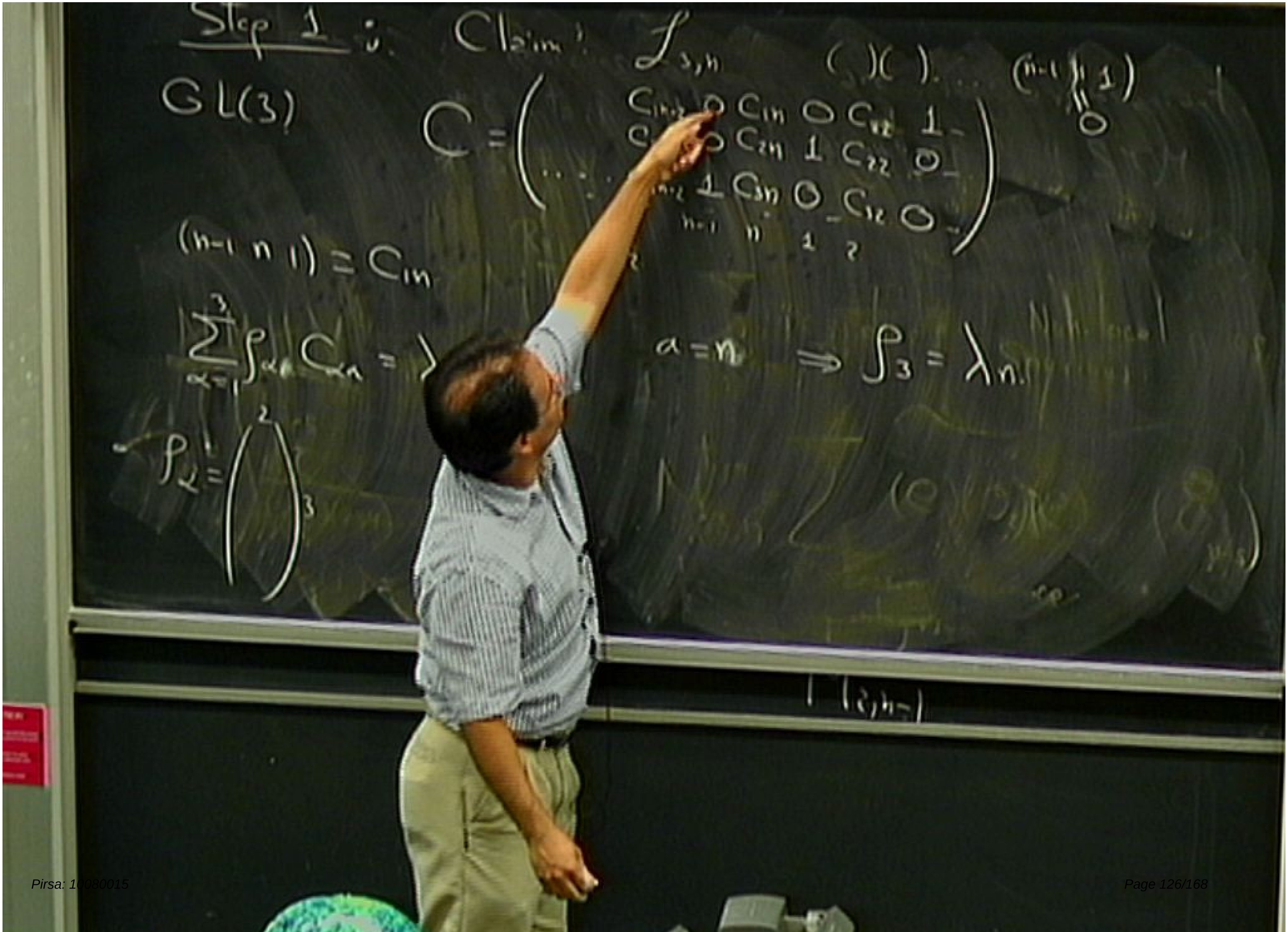
$(n-1\ n\ 1) = C_{1n}$

$$\sum_{\alpha=1}^{3} \rho_\alpha C_{\alpha n} = \lambda_a.$$

$a = 3 \Rightarrow \rho_3 = \lambda_n.$

$\rho_2 = \left(\ \right)^{2}_{3}$

$(2, n-)$



Step 1: Claim: $\mathcal{L}_{3,n}$ $(\;)(\;)\ldots(n-1\; n\; 1) = 0$

GL(3)

$$C = \begin{pmatrix} \ldots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 & \ldots \\ \ldots & C & 0 & C_{2n} & 1 & C_{22} & 0 & \ldots \\ \ldots & C_{n-2} & 1 & C_{3n} & 0 & C_{32} & 0 & \ldots \\ & & n-1 & n & 1 & 2 & & \end{pmatrix}$$

$(n-1\; n\; 1) = C_{1n}$

$\sum_{\alpha=1}^{3} \rho_{\alpha n} C_{\alpha n} = \lambda$ $\quad a = n \Rightarrow \rho_3 = \lambda_n$

$\rho_2 = \left(\right)_3^2$

$| \; (2, n-)$

Step 1 : Claim: $\mathcal{L}_{3,n}$ $\quad (\ )(\ ) \ldots (n-1 \ n \ 1) = 0$

GL(3)

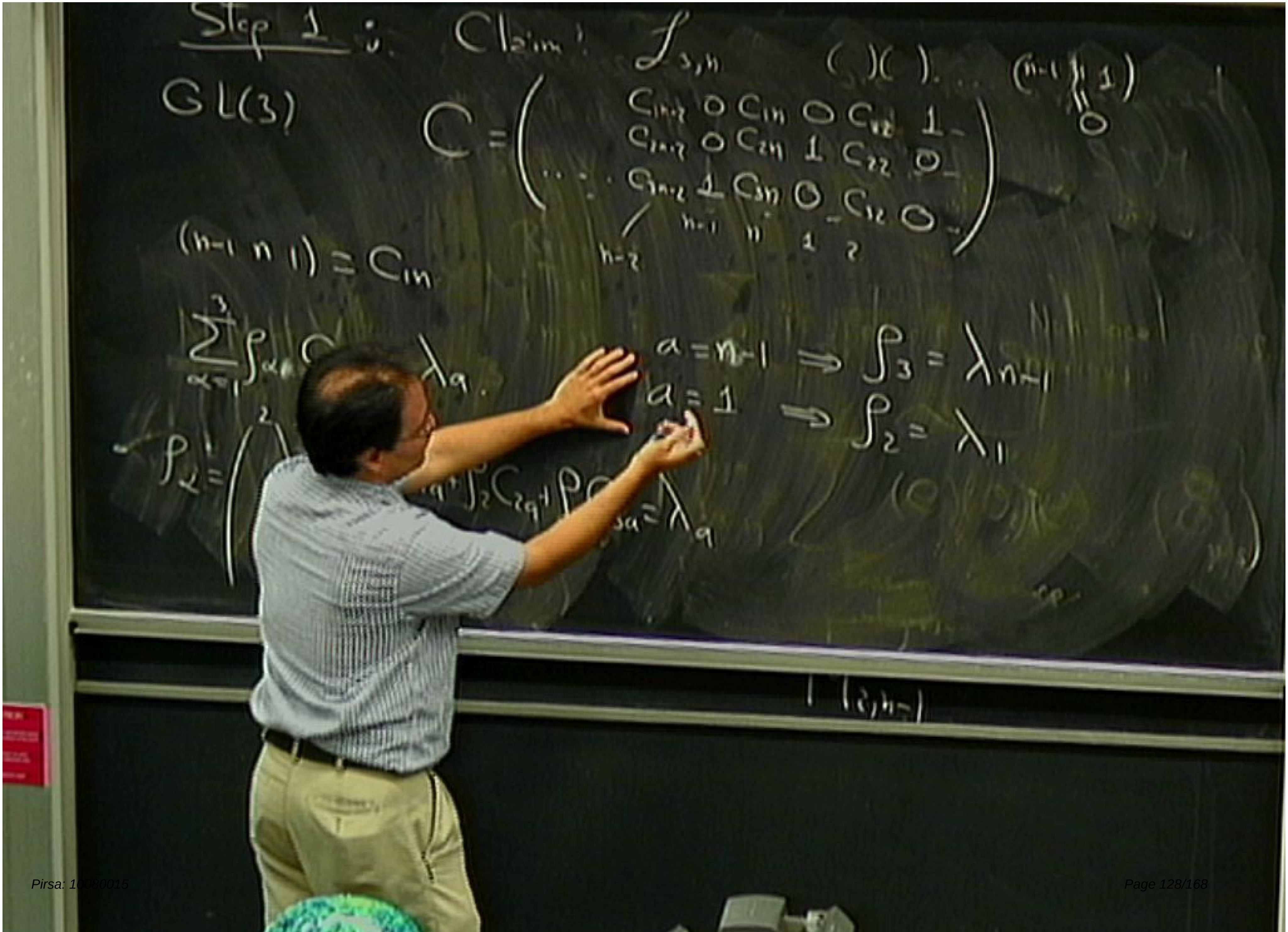
$$C = \left(\begin{array}{cccccc} \cdots & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 \ldots \\ \cdots & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 \ldots \\ \cdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{32} & 0 \ldots \\ & n-2 & n-1 & n & 1 & 2 & \end{array} \right)$$

$(n-1 \ n \ 1) = C_{1n}$

$$\sum_{\alpha=1}^{3} \rho_{\alpha} C_{\alpha a} = \lambda_a . \qquad a = n-1 \Rightarrow \rho_3 = \lambda_{n-1}$$

$$\rho_2 = \left(\quad \right)^{2}_{3} \qquad \rho_1 C_{1a} + \rho_2 C_{2a} + \rho_3 C_{3a} = \lambda_a$$

$(2, n-)$



Step 1 : Claim : $\mathcal{L}_{3,n}$ ( )( ). . . $(n-1\ n\ 1) = 0$

GL(3)

$$C = \begin{pmatrix} \cdots & c_{1n-2} & 0 & c_{1n} & 0 & c_{12} & 1 \\ \cdots & c_{2n-2} & 0 & c_{2n} & 1 & c_{22} & 0 \\ \cdots & c_{3n-2} & 1 & c_{3n} & 0 & c_{32} & 0 \\ & n-2 & n-1 & n & 1 & 2 & \end{pmatrix}$$

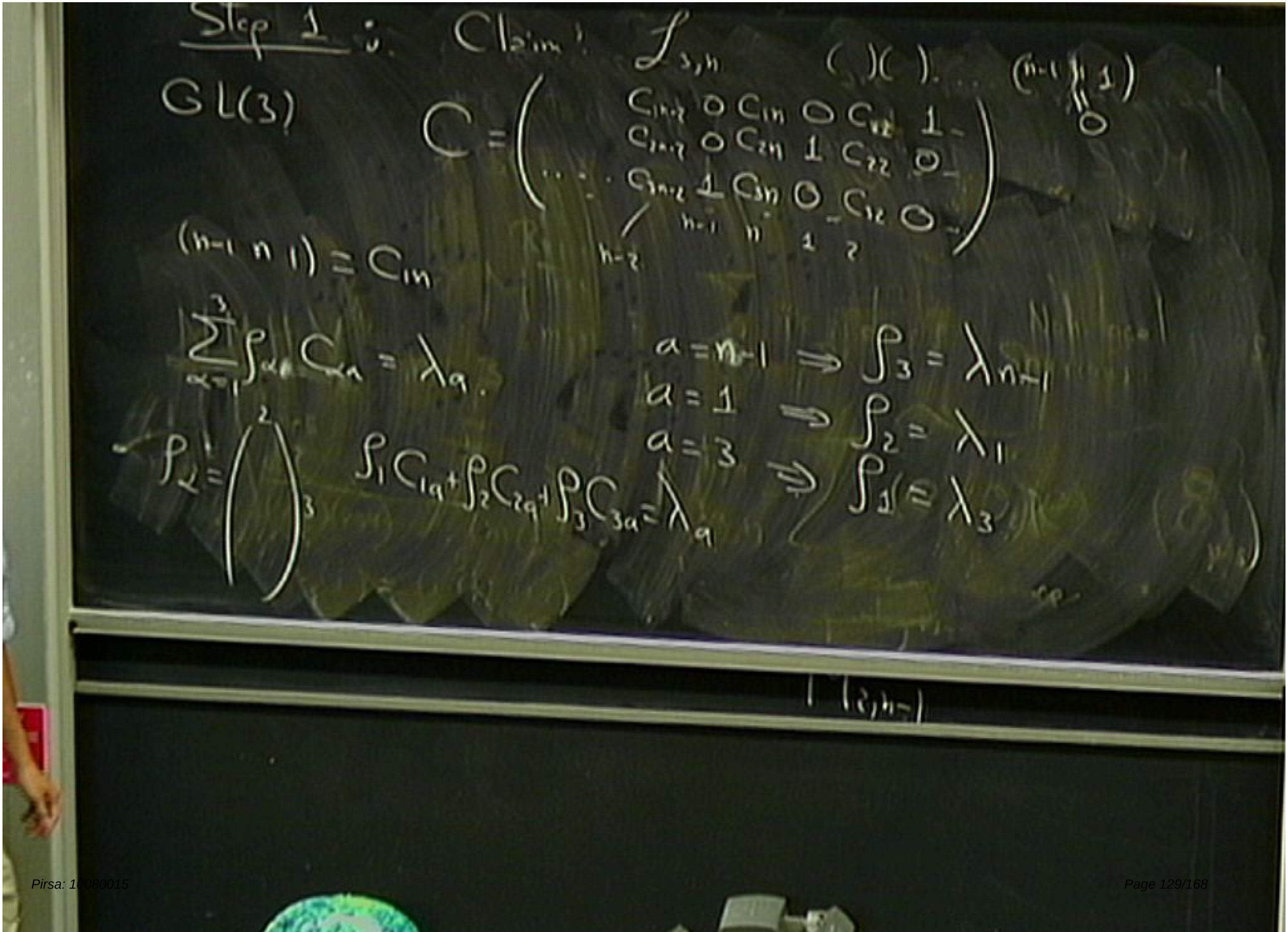
$(n-1\ n\ 1) = c_{1n}$

$\sum_{\alpha=1}^{3} \rho_\alpha c \ldots \lambda_a$.

$a = n-1 \Rightarrow \rho_3 = \lambda_{n-1}$

$a = 1 \Rightarrow \rho_2 = \lambda_1$

$\rho_2 = \Big(\ldots \rho_2 c_{2a+1}\ \rho c \ldots {}_{3a} = \lambda_a$



Step 1: Claim: $\mathcal{L}_{3,n}$ $(\ )(\ )\ldots(n-1\ \underset{\underset{0}{\shortparallel}}{n}\ 1)$

GL(3)

$$C = \begin{array}{c} \left(\begin{array}{cccccccc} & & C_{1n-2} & 0 & C_{1n} & 0 & C_{12} & 1 \\ & & C_{2n-2} & 0 & C_{2n} & 1 & C_{22} & 0 \\ \cdots & \vdots & C_{3n-2} & 1 & C_{3n} & 0 & C_{32} & 0 \end{array}\right) \\ \begin{array}{cccccccc} & & n-2 & n-1 & n & 1 & 2 & \end{array} \end{array}$$

$(n-1\ n\ 1) = C_{1n}$

$$\sum_{\alpha=1}^{3} \rho_{\alpha} C_{\alpha a} = \lambda_a.$$

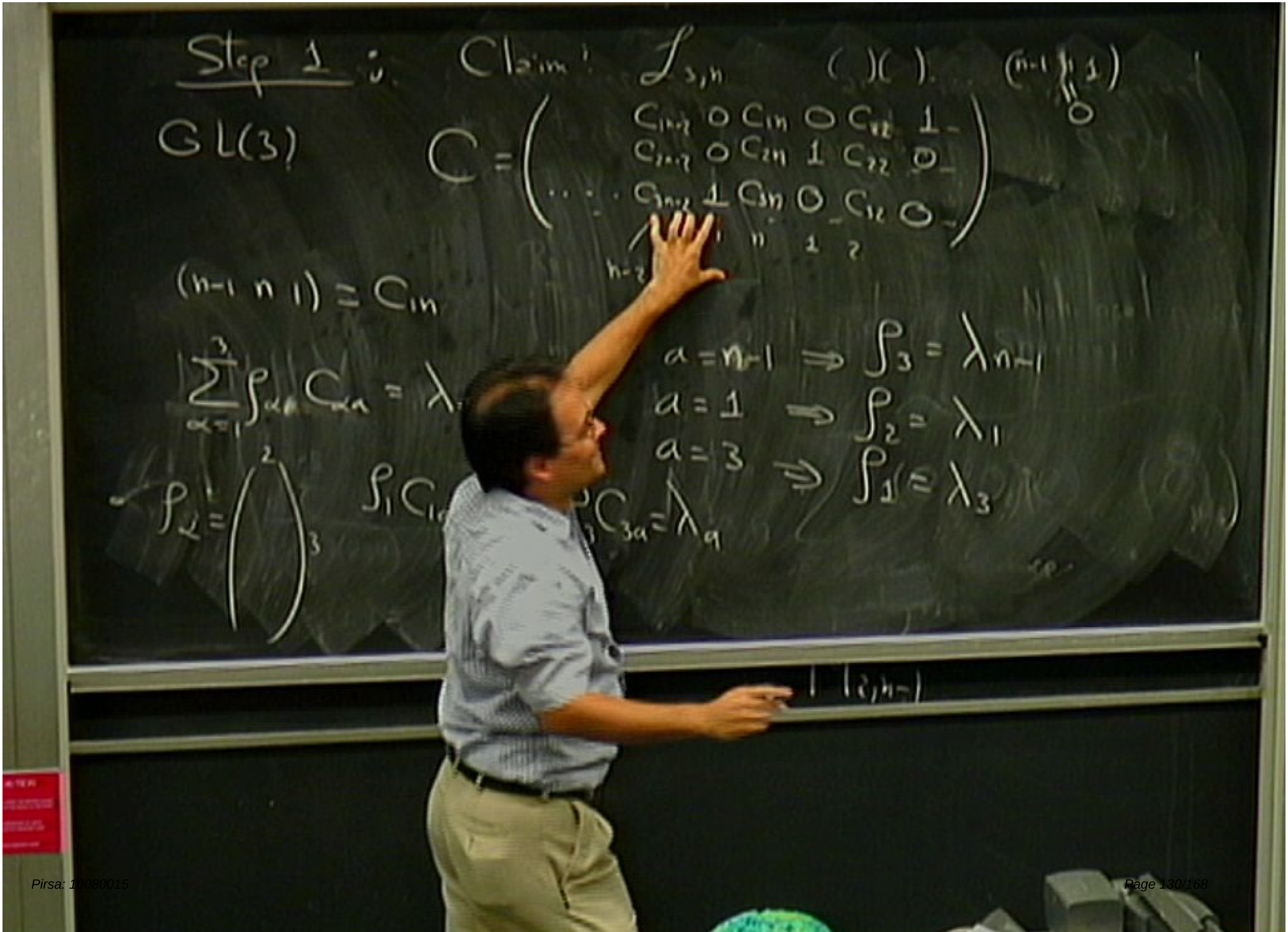
$a = n-1 \Rightarrow \rho_3 = \lambda_{n-1}$

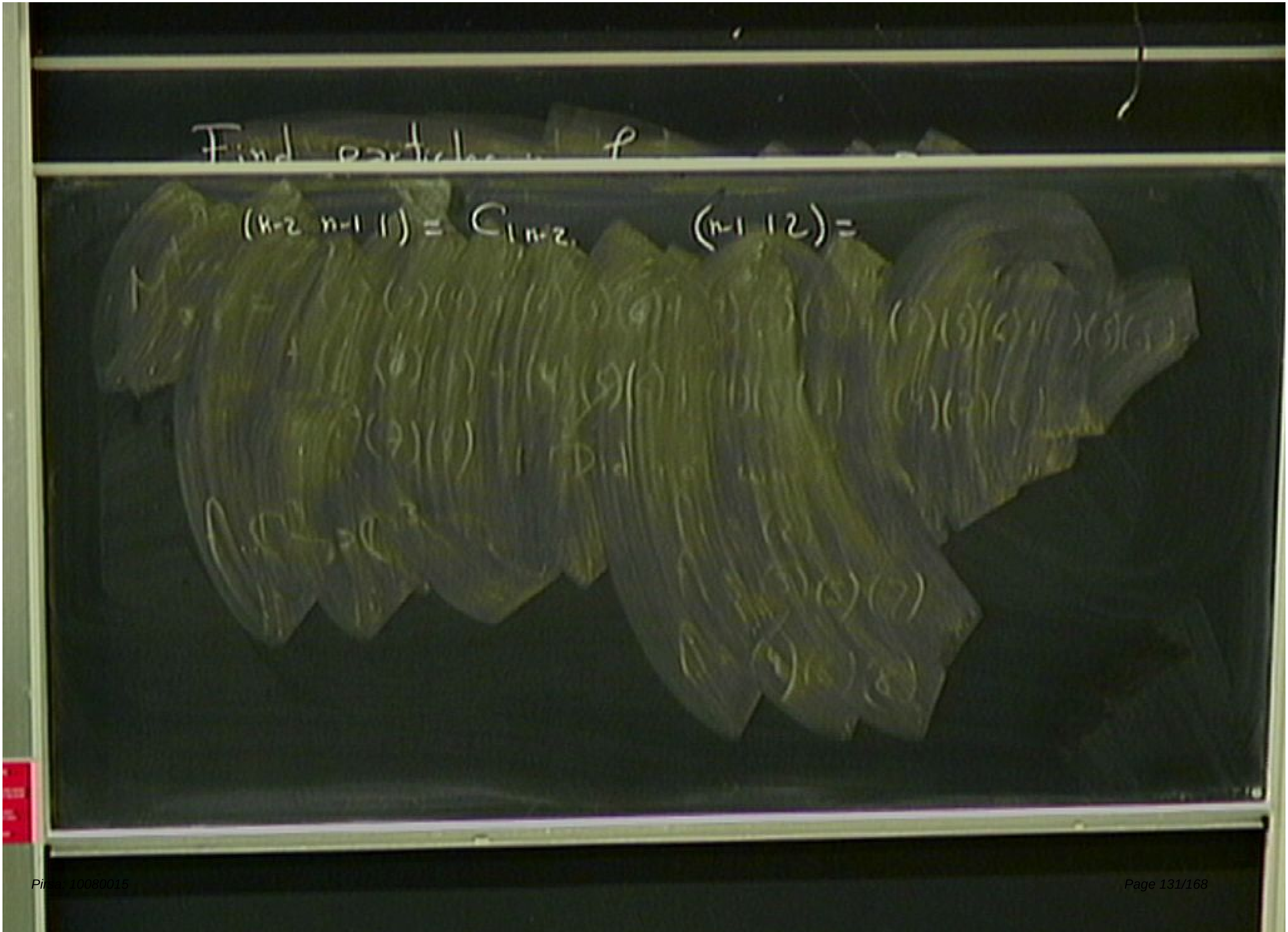
$a = 1 \Rightarrow \rho_2 = \lambda_1$

$a = 3 \Rightarrow \rho_1 = \lambda_3$

$$\rho_{\alpha} = \begin{pmatrix} \\ \\ \end{pmatrix}^{2}_{3} \qquad \rho_1 C_{1a} + \rho_2 C_{2a} + \rho_3 C_{3a} = \lambda_a$$

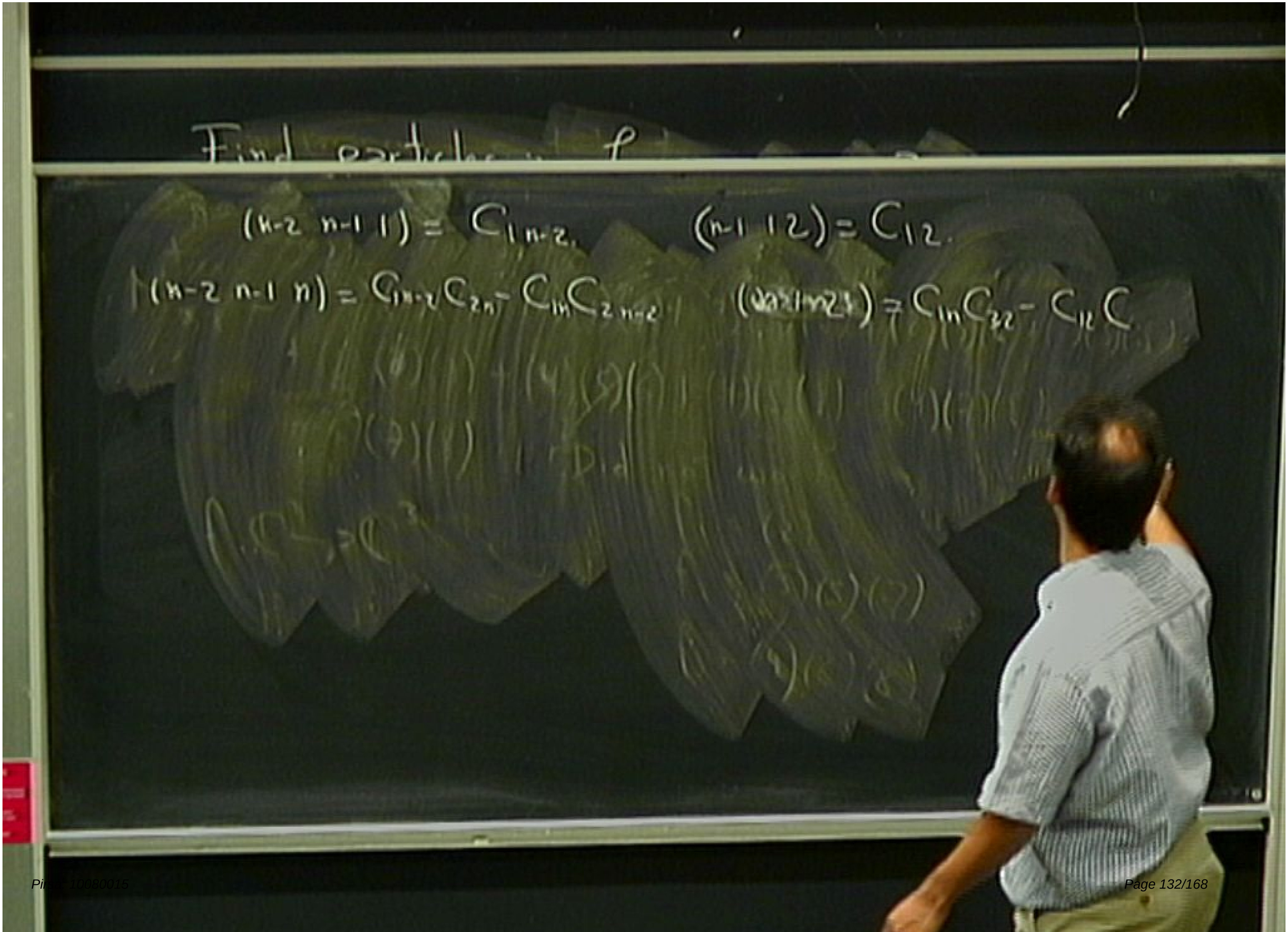
$(2,n-)$


Step 1 : Claim: L_{3,n} ( )( ). (n-1 n 1)
0
GL(3)
C = (C_{1n-2} 0 C_{1n} 0 C_{12} 1
C_{2n-2} 0 C_{2n} 1 C_{22} 0
... C_{3n-2} 1 C_{3n} 0 C_{12} 0)
n-2 n 1 2
(n-1 n 1) = C_{1n}
Σ_{α=1}^{3} ρ_{αa} C_{αa} = λ_a
a = n-1 ⇒ ρ_3 = λ_{n-1}
a = 1 ⇒ ρ_2 = λ_1
a = 3 ⇒ ρ_1 = λ_3
ρ_2 = ( )
2
3
ρ_1 C_{1a} ... ρ_3 C_{3a} = λ_a
(2, n-1)



Find particle f

$(n-2\ n-1\ 1) = C_{1\,n-2}$ $\qquad$ $(n-1\ 1\ 2) =$



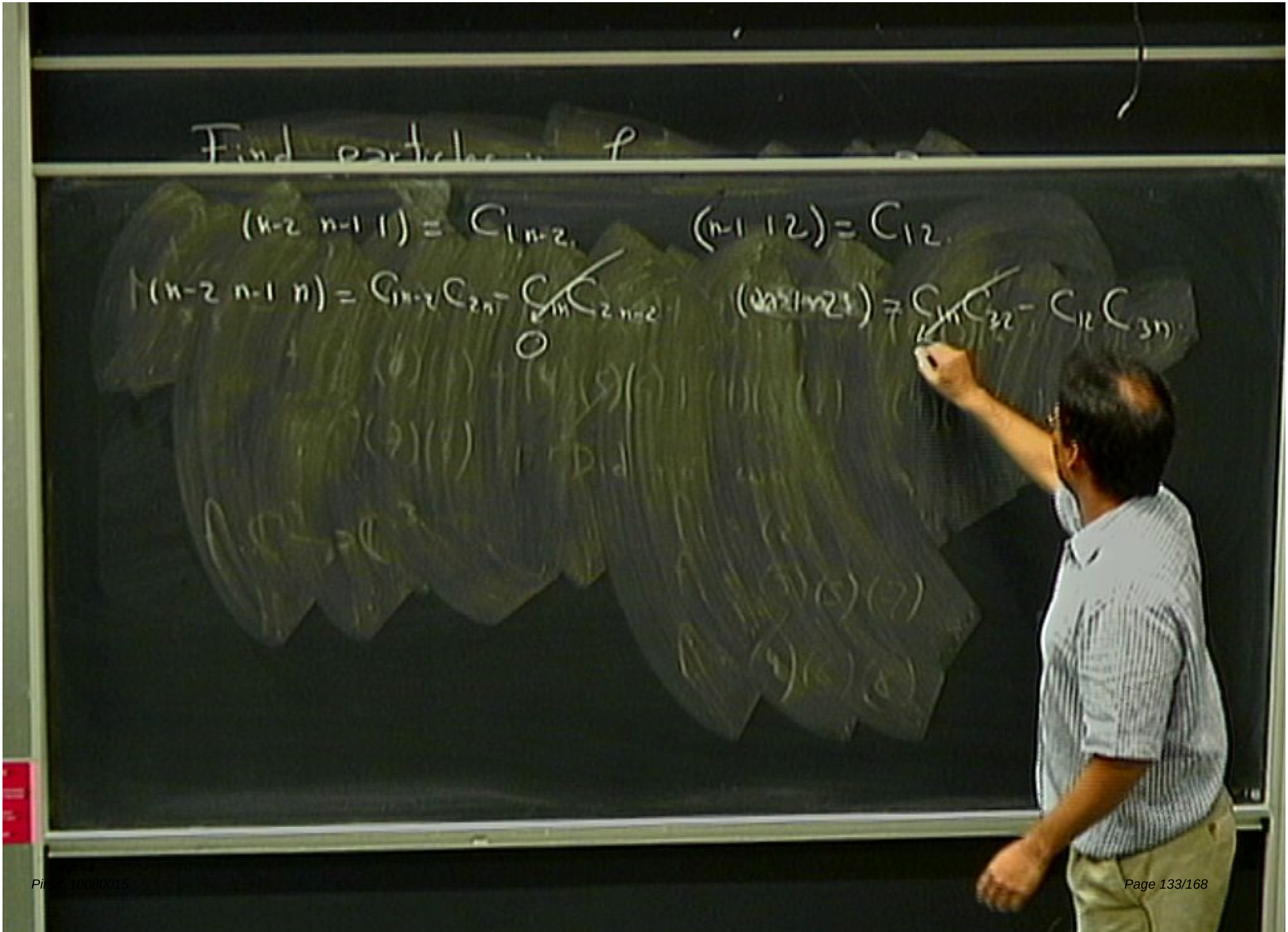
Find partic

$(n-2\ n-1\ 1) = C_{1\,n-2}$

$(n-1\ 1\ 2) = C_{12}$

$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - C_{1n}C_{2\,n-2}$

$($[illegible]$) = C_{1n}C_{32} - C_{12}C$



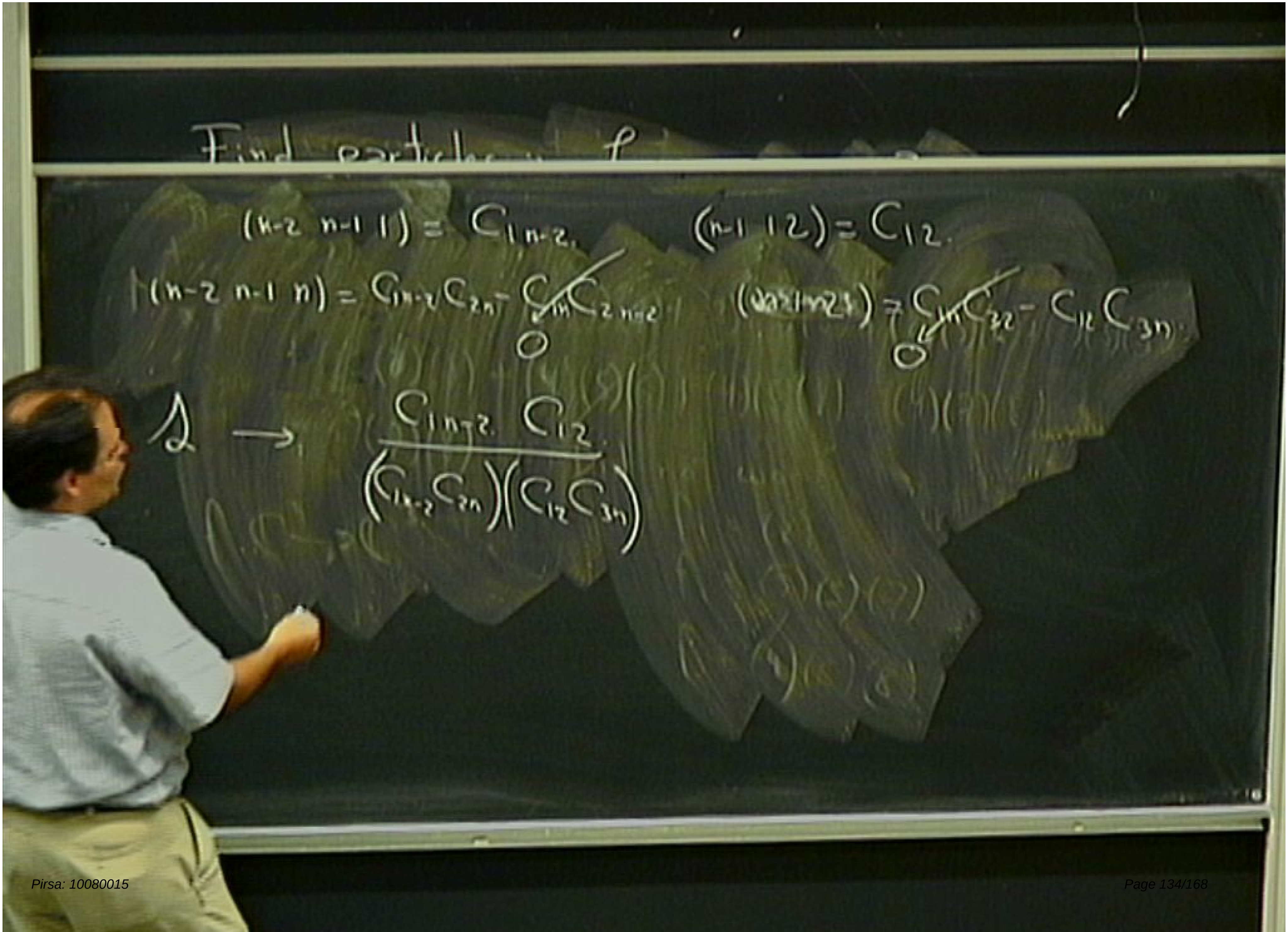
Find particle [illegible]

$$(n-2\ n-1\ 1) = C_{1\,n-2}$$

$$(n-1\ 1\ 2) = C_{12}$$

$$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$$

$$([illegible]) = \cancel{C_{1n}}C_{32} - C_{12}C_{3n}$$

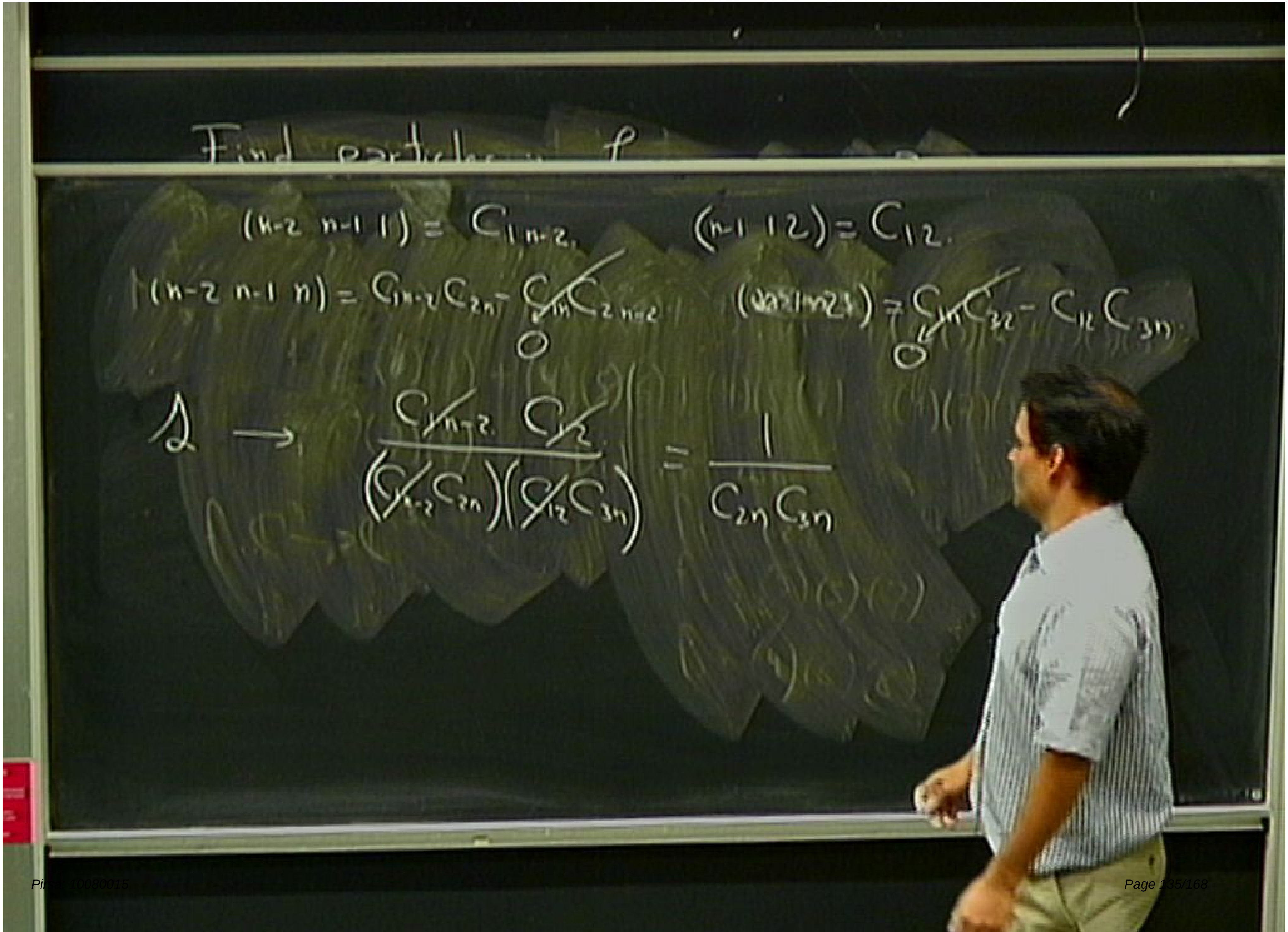


Find partic

$(n-2\ n-1\ 1) = C_{1\,n-2}$ $\qquad (n-1\ 1\ 2) = C_{12}$

$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$ $\qquad$ ([illegible]) $= \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$

$$\Omega \longrightarrow \frac{C_{1\,n-2}\ C_{12}}{(C_{1\,n-2}C_{2n})(C_{12}C_{3n})}$$



Find particle

$(n-2\ n-1\ 1) = C_{1\,n-2}$ $\quad$ $(n-1\ 1\ 2) = C_{12}$

$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$ $\quad$ ([illegible]) $= \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$

$$\mathcal{A} \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

Find particular f

$(n\text{-}2\ n\text{-}1\ 1) = C_{1\,n-2}$ $\qquad$ $(n\text{-}1\ 1\ 2) = C_{12}$

$(n\text{-}2\ n\text{-}1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$ $\qquad$ $([illegible]) = \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$

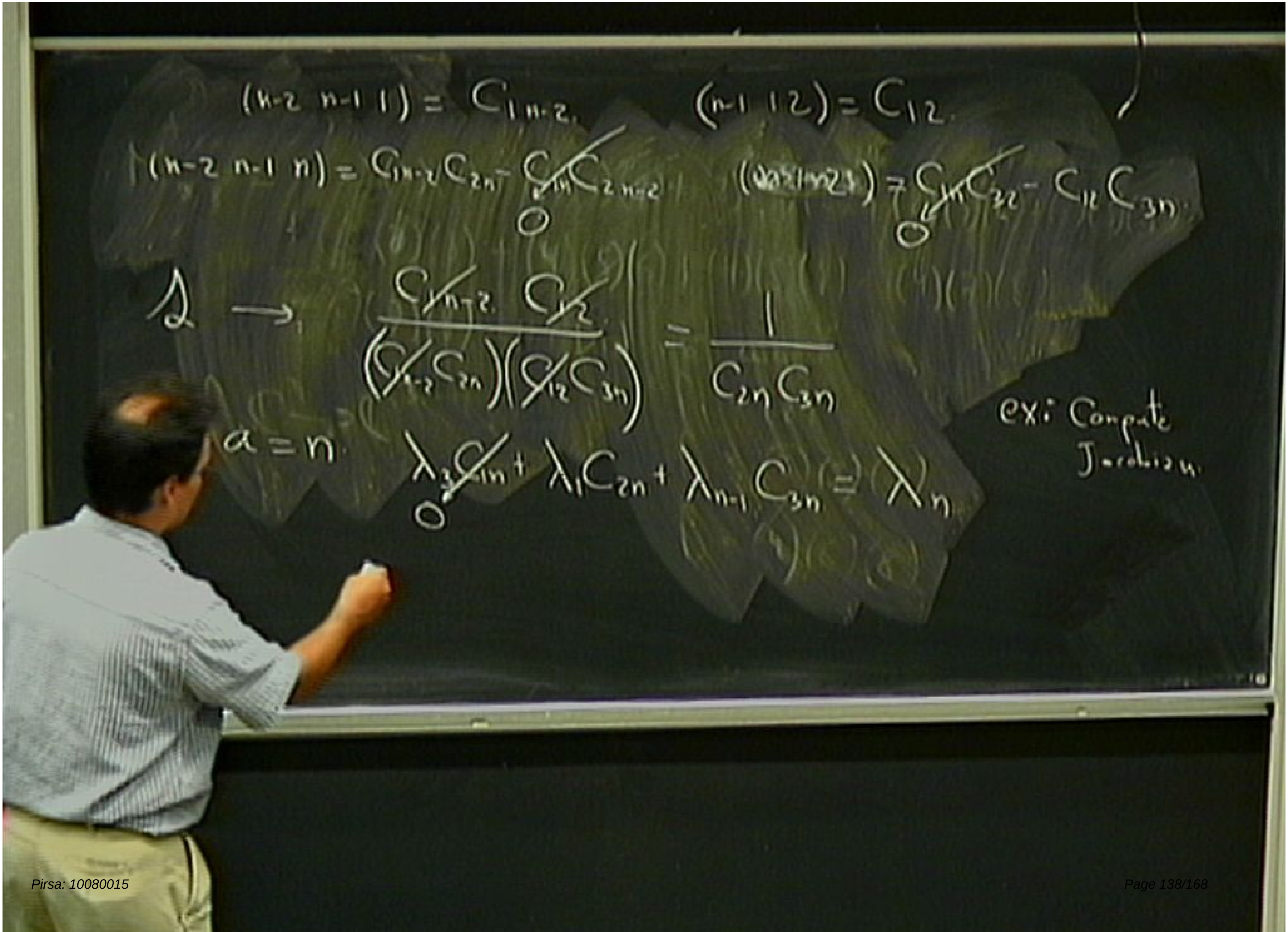
$$\Omega \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$$(n\text{-}2\ \ n\text{-}1\ \ 1) = C_{1\,n-2} \qquad (n\text{-}1\ \ 1\ 2) = C_{12}$$

$$(n\text{-}2\ \ n\text{-}1\ \ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2} \qquad (\text{[illegible]}) = \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$$

$$\Delta \longrightarrow \frac{\cancel{C_{1n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$$a = n \qquad \lambda_3 C_{1n} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$$



$(n\text{-}2\ \ n\text{-}1\ \ 1) = C_{1\,n\text{-}2}$

$(n\text{-}1\ \ 1\ 2) = C_{12}$

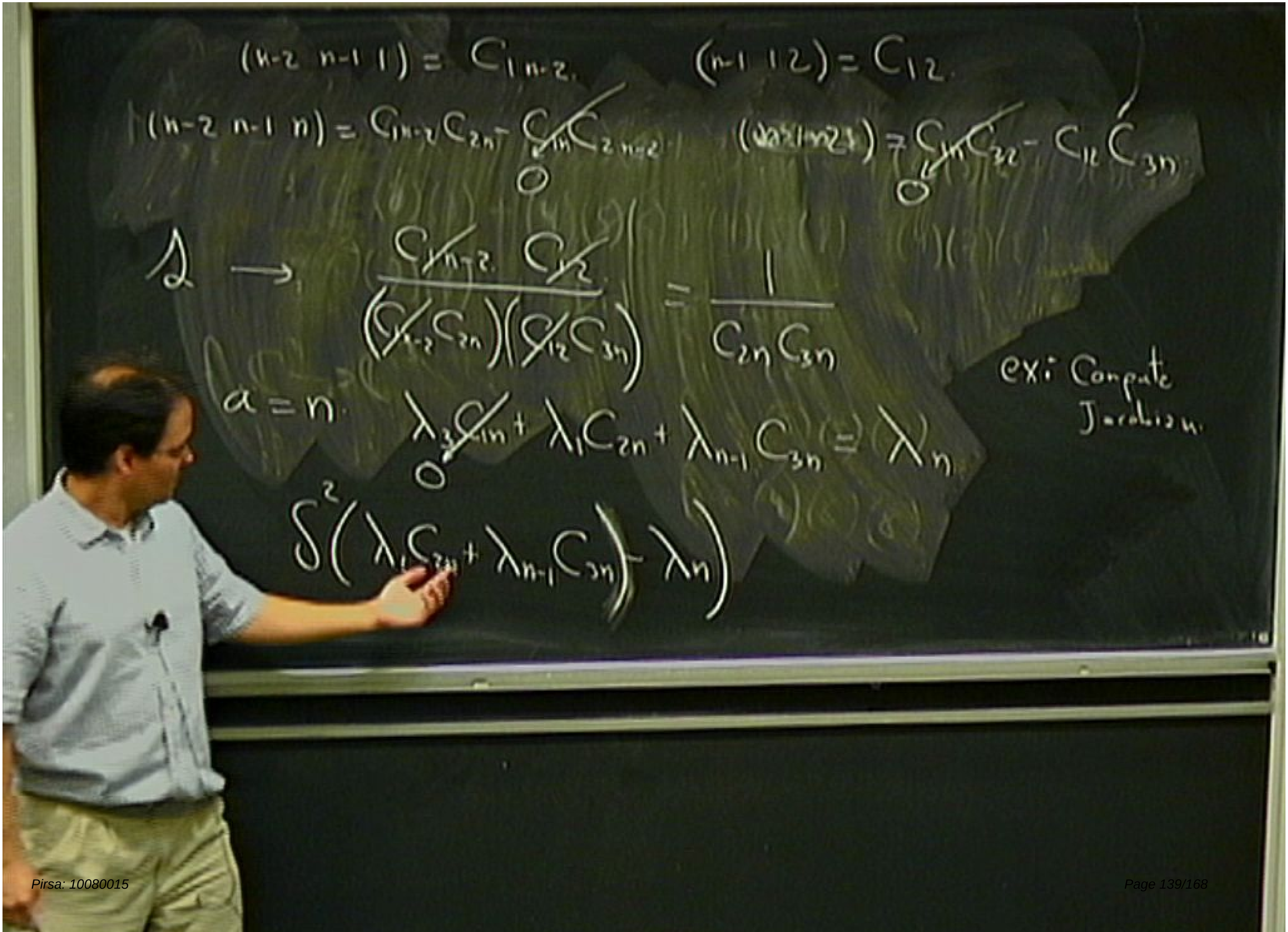
$(n-2\ \ n\text{-}1\ \ n) = C_{1n\text{-}2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n\text{-}2}$

([illegible]) $= \underset{0}{\cancel{C_{1n}}}C_{22} - C_{12}C_{3n}$

$$\mathcal{Q} \longrightarrow \frac{\cancel{C_{1n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$a = n$. $\lambda_3 \underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$

ex: Compute Jacobian.



$(n-2\ n-1\ 1) = C_{1\,n-2}$ $\qquad$ $(n-1\ 1\ 2) = C_{12}$

$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$ $\qquad$ ([illegible]) $= \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$

$$\Delta \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$a = n$. $\quad \lambda_3 \underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1} C_{3n} - \lambda_n)$$

ex: Compute
Jacobian.

$(n-2\ \ n-1\ \ 1) = C_{1\,n-2}$ $\qquad$ $(n-1\ \ 1\ \ 2) = C_{12}$

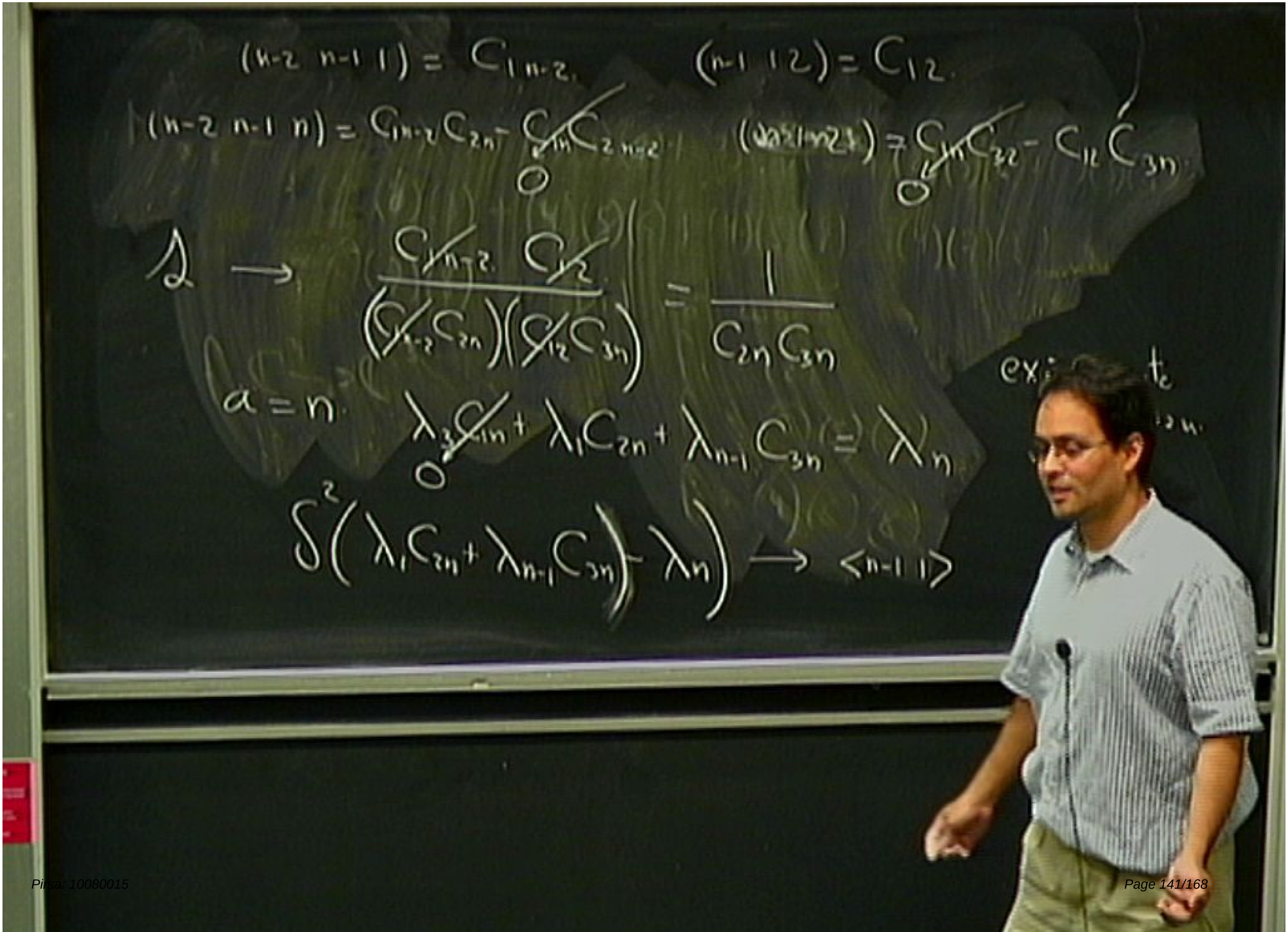
$(n-2\ \ n-1\ \ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$ $\qquad$ ([illegible]) $= \underset{0}{\cancel{C_{1n}}}C_{32} - C_{12}C_{3n}$

$$\Delta \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

ex: Compute Jacobian.

$a = n.$ $\quad \lambda_3 \underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$

$$\delta^2\left(\lambda_1 C_{2n} + \lambda_{n-1} C_{3n}) - \lambda_n\right)$$



$(n-2\ n-1\ 1) = C_{1\,n-2}$. $(n-1\ 1\ 2) = C_{12}$.

$(n-2\ n-1\ n) = C_{1n-2} C_{2n} - \overset{0}{\cancel{C_{1n}}} C_{2\,n-2}$ $([illegible]) = \overset{0}{\cancel{C_{1n}}} C_{22} - C_{12} C_{3n}$

$$s \longrightarrow \frac{\cancel{C_{1n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1n-2}} C_{2n})(\cancel{C_{12}} C_{3n})} = \frac{1}{C_{2n} C_{3n}}$$

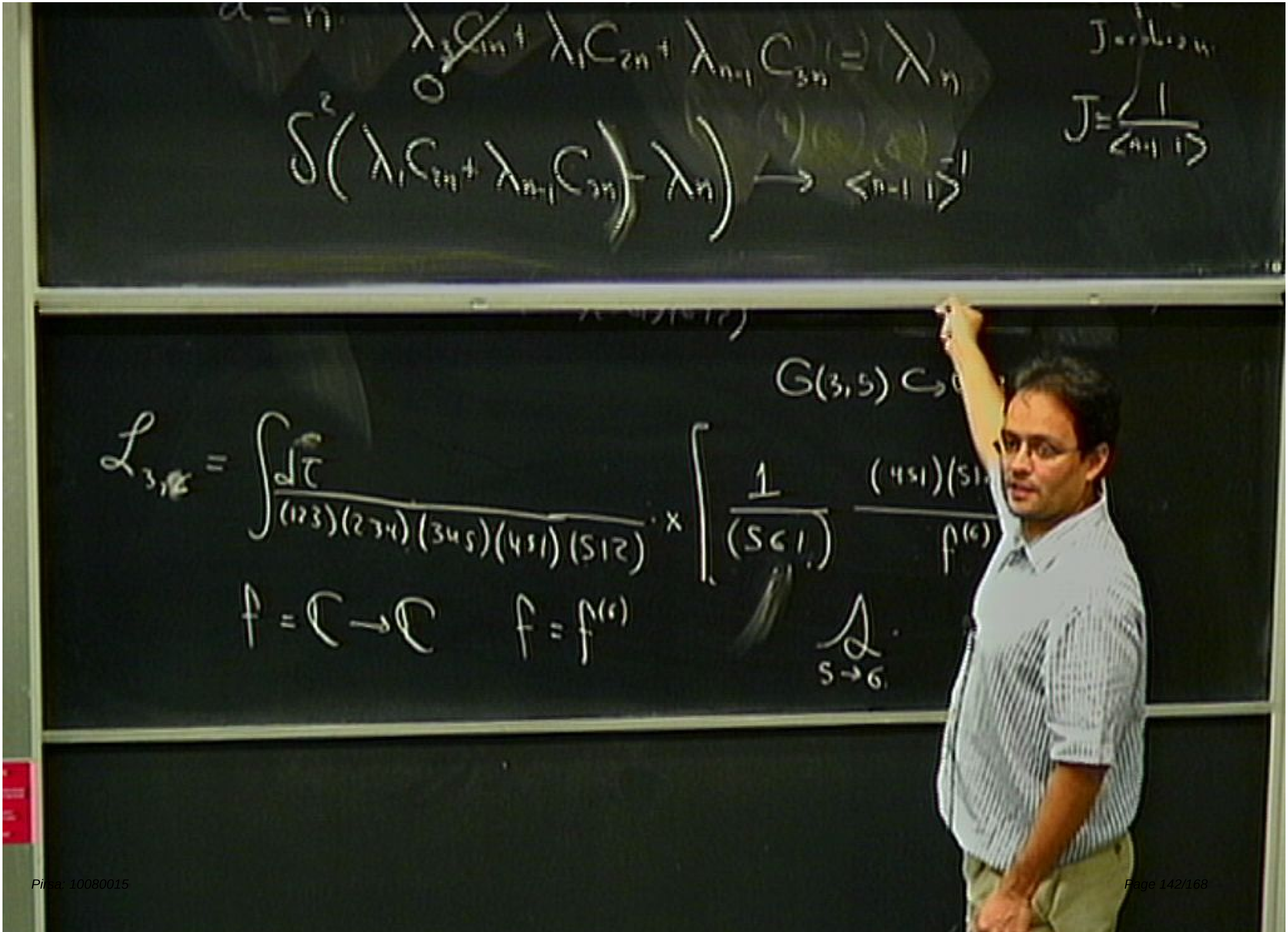
exi [illegible] te

[illegible] 2 n.

$a = n$.

$$\lambda_3 \overset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1} C_{3n} - \lambda_n) \longrightarrow \langle n-1\ 1|$$



$$\cancel{\lambda_3 C_{1n}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$$

$$J = \frac{1}{\langle n{-}1\ 1\rangle}$$

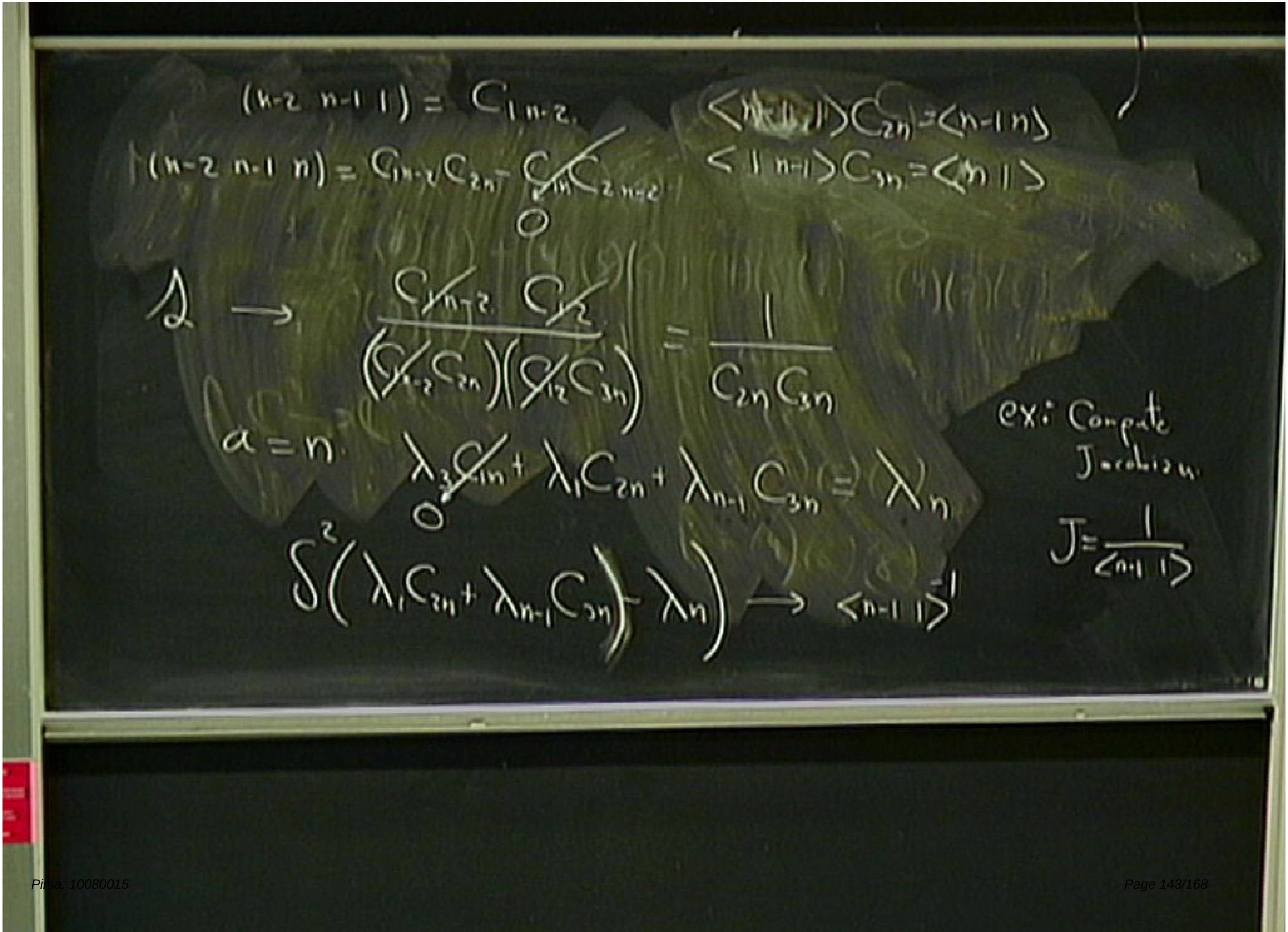
$$\delta^2(\lambda_1 C_{1n} + \lambda_{n-1} C_{3n} - \lambda_n) \to \langle n{-}1\ 1\rangle^{-1}$$

$$G(3,5) \hookrightarrow G$$

$$\mathcal{L}_{3,6} = \int \frac{d\tau}{(123)(234)(345)(451)(512)} \times \int \left(\frac{1}{(561)} \quad \frac{(451)(51}{f^{(6)}} \right.$$

$$f = \mathbb{C} \to \mathbb{C} \qquad f = f^{(6)}$$

$$\underset{5\to 6}{\mathcal{L}}$$



$$(n\text{-}2\ n\text{-}1\ 1) = C_{1\,n\text{-}2}$$

$$(n\text{-}2\ n\text{-}1\ n) = C_{1n\text{-}2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n\text{-}2}$$

$$\langle n\text{-}1\ 1\rangle C_{2n} = \langle n\text{-}1\ n\rangle$$

$$\langle 1\ n\text{-}1\rangle C_{3n} = \langle n\ 1\rangle$$

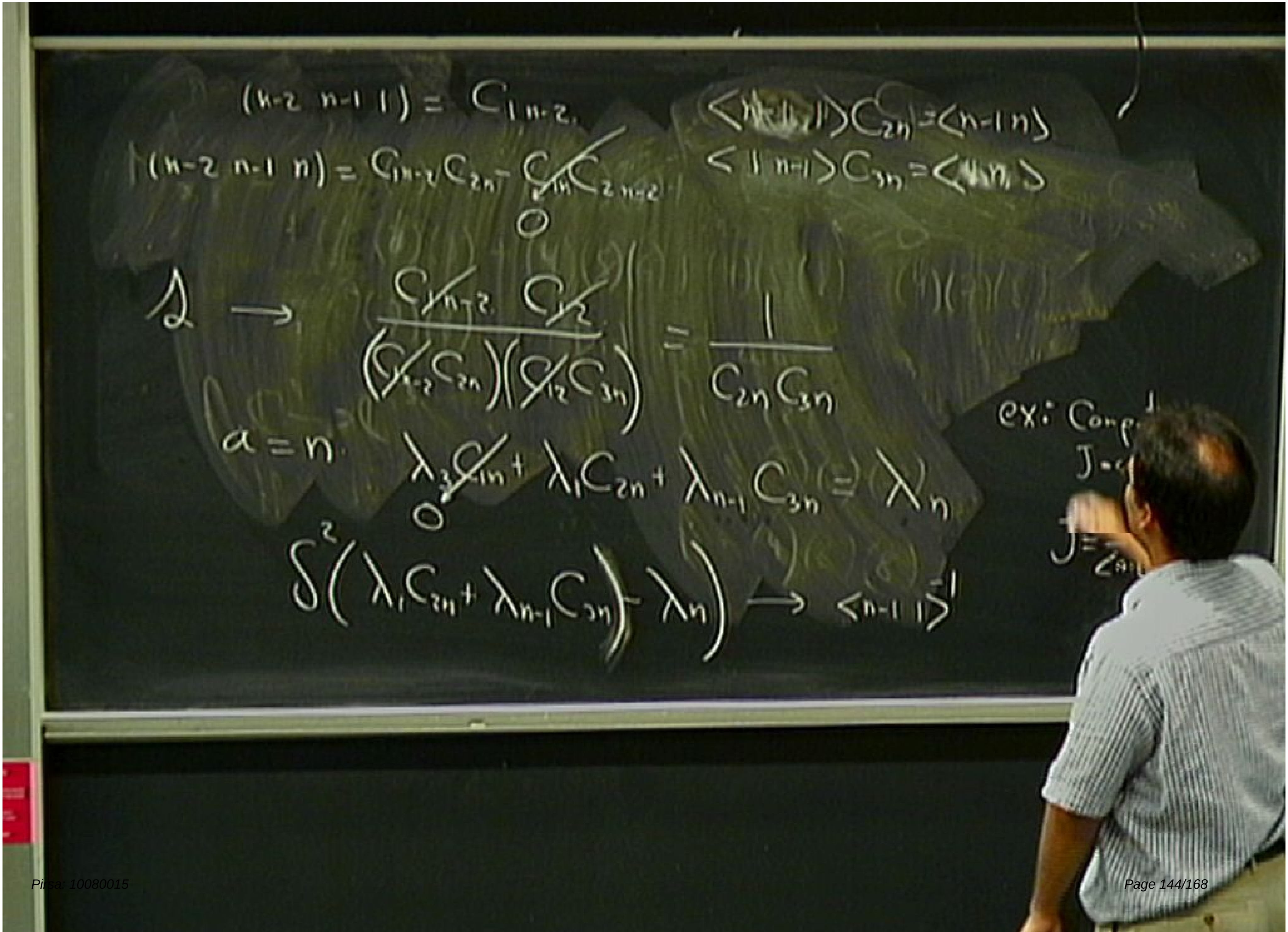
$$\Delta \longrightarrow \frac{\cancel{C_{1n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$$a = n.\quad \lambda_3\underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1}C_{3n} = \lambda_n$$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1}C_{3n} - \lambda_n) \longrightarrow \langle n\text{-}1\ 1\rangle^{-1}$$

ex: Compute Jacobian

$$J = \frac{1}{\langle n\text{-}1\ 1\rangle}$$



$$(n\text{-}2\ n\text{-}1\ 1) = C_{1\,n\text{-}2}$$

$$(n-2\ n\text{-}1\ n) = C_{1\,n\text{-}2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$$

$$\langle n\text{-}1\ 1\rangle C_{2n} = \langle n\text{-}1\ n\rangle$$

$$\langle 1\ n\text{-}1\rangle C_{3n} = \langle 1\ n\rangle$$

$$\Delta \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}}$$

$$a = n.\quad \lambda_3\underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1}C_{3n} = \lambda_n$$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1}C_{3n} - \lambda_n) \longrightarrow \langle n\text{-}1\ 1\rangle^{-1}$$

ex: Comp

J =

J = 2n

$(n-2\ n-1\ 1) = C_{1\,n-2}$

$(n-2\ n-1\ n) = C_{1\,n-2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n-2}$

$\langle n-1\ 1\rangle C_{2n} = \langle n-1\ n\rangle$

$\langle 1\ n-1\rangle C_{3n} = \langle 1\ n\rangle$

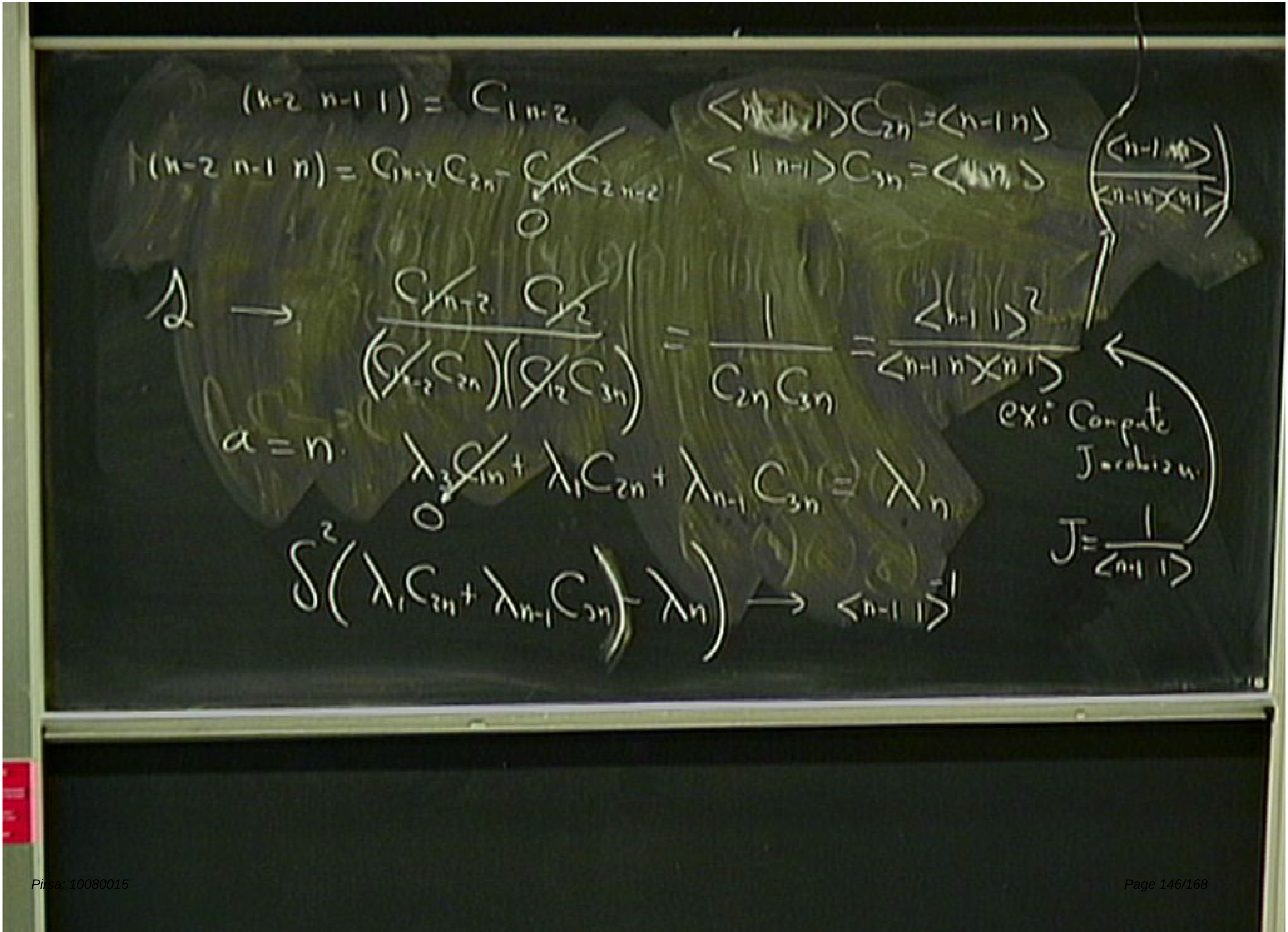
$$\Omega \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}} = \frac{\langle n-1\ 1\rangle^2}{\langle n-1\ n\rangle\langle n\ 1\rangle}$$

$a = n$: $\lambda_3\underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1}C_{3n} = \lambda_n$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1}C_{3n} - \lambda_n) \longrightarrow \langle n-1\ 1\rangle^{-1}$$

ex: Compute Jacobian

$J = \frac{1}{\langle n-1\ 1\rangle}$



$$(n\text{-}2\ n\text{-}1\ 1) = C_{1\,n\text{-}2}$$

$$(n\text{-}2\ n\text{-}1\ n) = C_{1\,n\text{-}2}C_{2n} - \underset{0}{\cancel{C_{1n}}}C_{2\,n\text{-}2}$$

$$\langle n\text{-}1\ 1\rangle C_{2n} = \langle n\text{-}1\ n\rangle$$

$$\langle 1\ n\text{-}1\rangle C_{3n} = \langle 1\ n\rangle$$

$$\left(\frac{\langle n\text{-}1\ n\rangle}{\langle n\text{-}1\ n\rangle\langle n\ 1\rangle}\right)$$

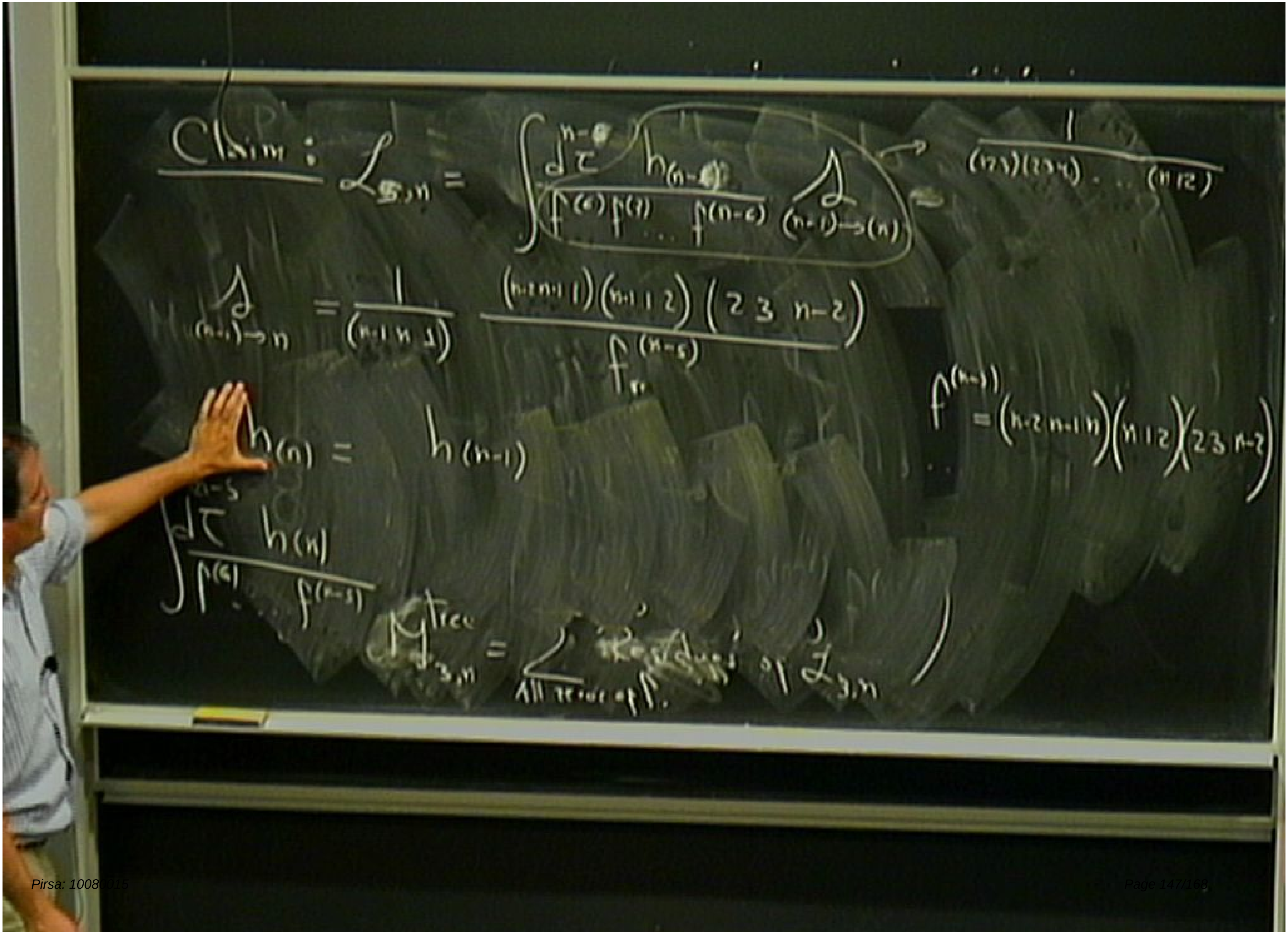
$$2 \longrightarrow \frac{\cancel{C_{1\,n-2}}\ \cancel{C_{12}}}{(\cancel{C_{1\,n-2}}C_{2n})(\cancel{C_{12}}C_{3n})} = \frac{1}{C_{2n}C_{3n}} = \frac{\langle n\text{-}1\ 1\rangle^2}{\langle n\text{-}1\ n\rangle\langle n\ 1\rangle}$$

$$a = n. \quad \lambda_3 \underset{0}{\cancel{C_{1n}}} + \lambda_1 C_{2n} + \lambda_{n-1} C_{3n} = \lambda_n$$

$$\delta^2(\lambda_1 C_{2n} + \lambda_{n-1} C_{3n} - \lambda_n) \longrightarrow \langle n\text{-}1\ 1\rangle^{-1}$$

ex: Compute Jacobian.

$$J = \frac{1}{\langle n\text{-}1\ 1\rangle}$$



Claim: $\mathcal{L}_{5,n} = \int \frac{d^{n-6}\tau \; h_{(n-6)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \mathcal{S}_{(n-1)\to(n)} \to \frac{1}{(123)(234)\cdots(n12)}$

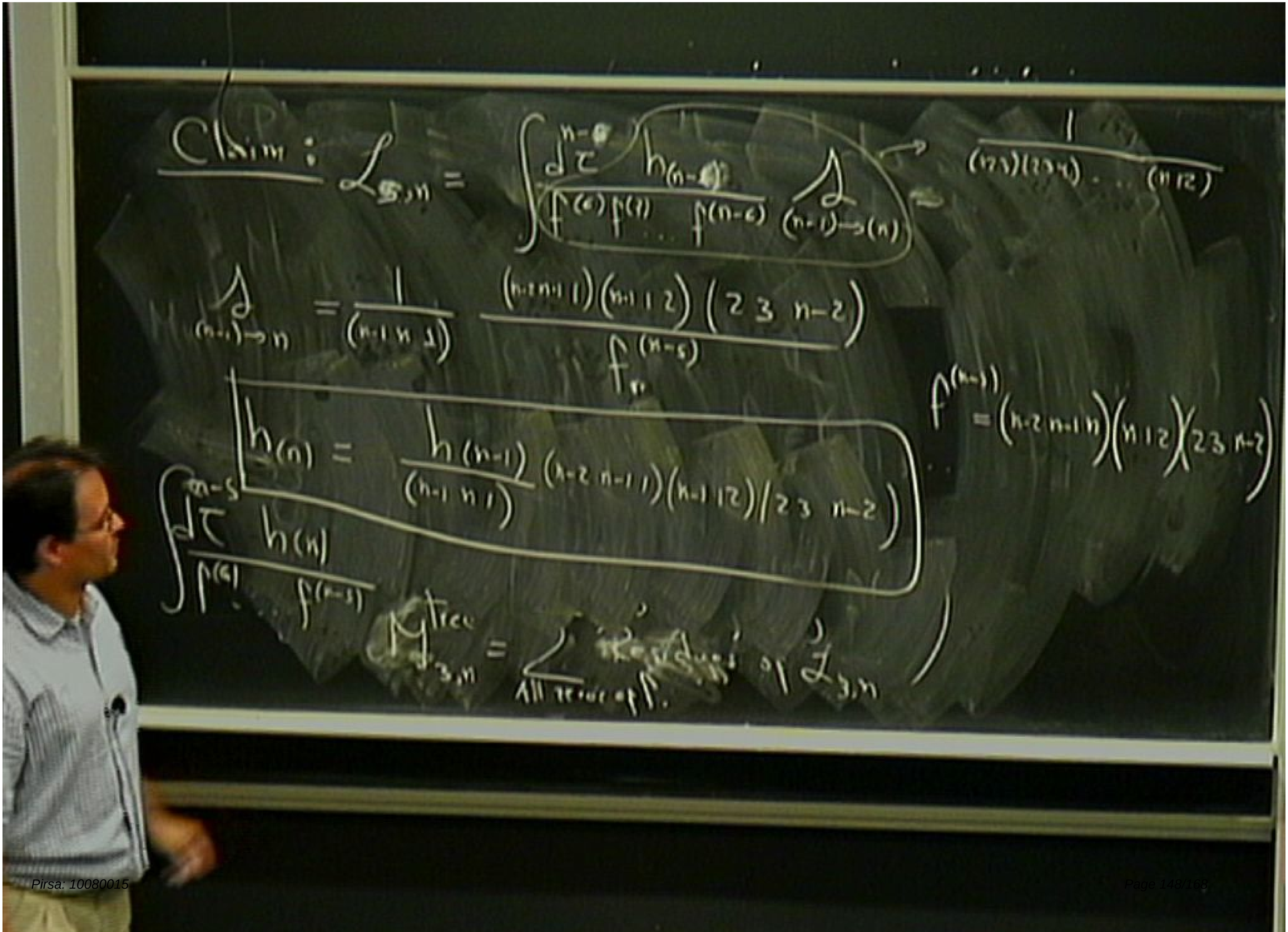
$$\mathcal{S}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \frac{(n-2\; n-1\; 1)(n-1\; 1\; 2)(2\; 3\; n-2)}{f_n^{(n-5)}}$$

$$f^{(n-5)} = (n-2\; n-1\; n)(n\; 1\; 2)(2\; 3\; n-2)$$

$$h_{(n)} = h_{(n-1)}$$

$$\int \frac{d^{n-5}\tau \; h_{(n)}}{f^{(6)} \cdots f^{(n-5)}}$$

$$M^{\text{tree}}_{3,n} = \sum_{\text{All residues of } f} \text{"Residues" of } \mathcal{L}_{3,n}$$



Claim: $\mathcal{L}_{5,n} = \int d\tau^{n-6} \frac{h_{(n-6)}}{f^{(6)} f^{(7)} \cdots f^{(n-6)}} \mathcal{S}_{(n-1)\to(n)} \rightarrow \frac{1}{(123)(234)\cdots(n12)}$

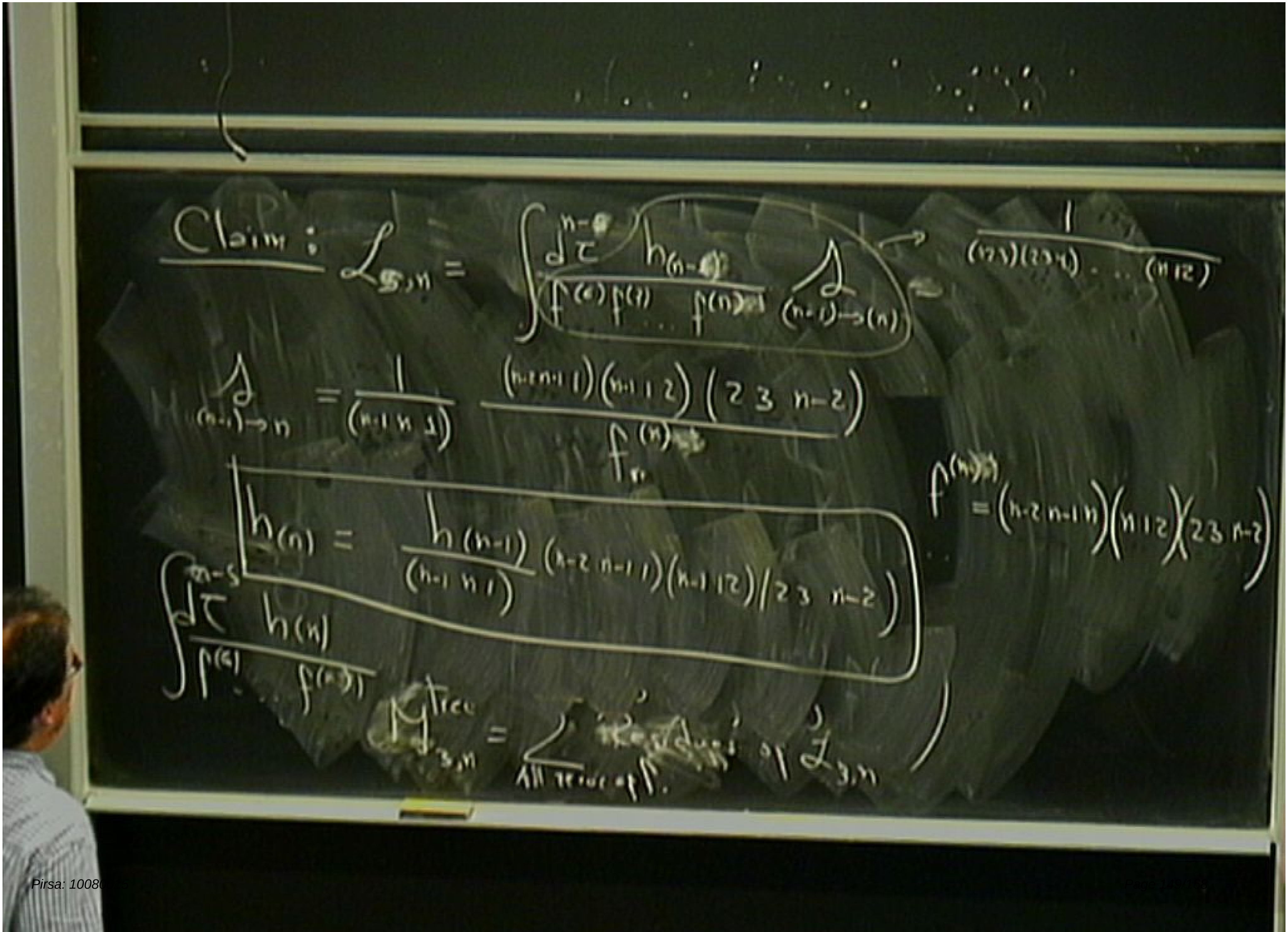
$$\mathcal{S}_{(n-1)\to n} = \frac{1}{(n-1\;n\;1)} \frac{(n-2\;n-1\;1)(n-1\;1\;2)(2\;3\;n-2)}{f_n^{(n-5)}}$$

$$f^{(n-5)} = (n-2\;n-1\;n)(n\;1\;2)(2\;3\;n-2)$$

$$\boxed{h_{(n)} = \frac{h_{(n-1)}}{(n-1\;n\;1)} (n-2\;n-1\;1)(n-1\;1\;2)/(2\;3\;n-2)}$$

$$\int d\tau^{n-5} \frac{h_{(n)}}{f^{(6)} \cdots f^{(n-5)}}$$

$$M^{tree}_{3,n} = \sum_{\text{All zeroes of } f} \text{"Residues" of } \mathcal{L}_{3,n}$$



Claim: $\mathcal{L}_{5,n} = \int \frac{d\tau^{n-5}\, h_{(n-5)}}{f^{(4)} f^{(5)} \cdots f^{(n)}}\, \mathcal{S}_{(n-1)\to(n)} \rightarrow \frac{1}{(123)(234)\cdots(n12)}$

$$\mathcal{S}_{(n-1)\to n} = \frac{1}{(n\text{-}1\; n\; 1)} \frac{(n\text{-}2\; n\text{-}1\; 1)(n\text{-}1\; 1\; 2)(2\; 3\; n\text{-}2)}{f^{(n)}_{n}}$$

$$\boxed{h_{(n)} = \frac{h_{(n-1)}}{(n\text{-}1\; n\; 1)} (n\text{-}2\; n\text{-}1\; 1)(n\text{-}1\; 1\; 2)/(2\; 3\; n\text{-}2)}$$

$$f^{(n)} = (n\text{-}2\; n\text{-}1\; n)(n\; 1\; 2)(2\; 3\; n\text{-}2)$$

$$\int \frac{d\tau^{n-5}\, h_{(n)}}{f^{(4)} \cdots f^{(n-1)}}$$

$$M^{\text{tree}}_{3,n} = \sum_{\text{All residues of}} \text{Residues of } \mathcal{L}_{3,n}$$

Claim: $\mathcal{L}_{5,n} = \int d^{n-5}\tau \, \frac{h_{(n-5)}}{f^{(6)} f^{(7)} \cdots f^{(n-1)}} \, \mathcal{S}_{(n-1)\to(n)} \to \frac{1}{(123)(234)\cdots(n12)}$

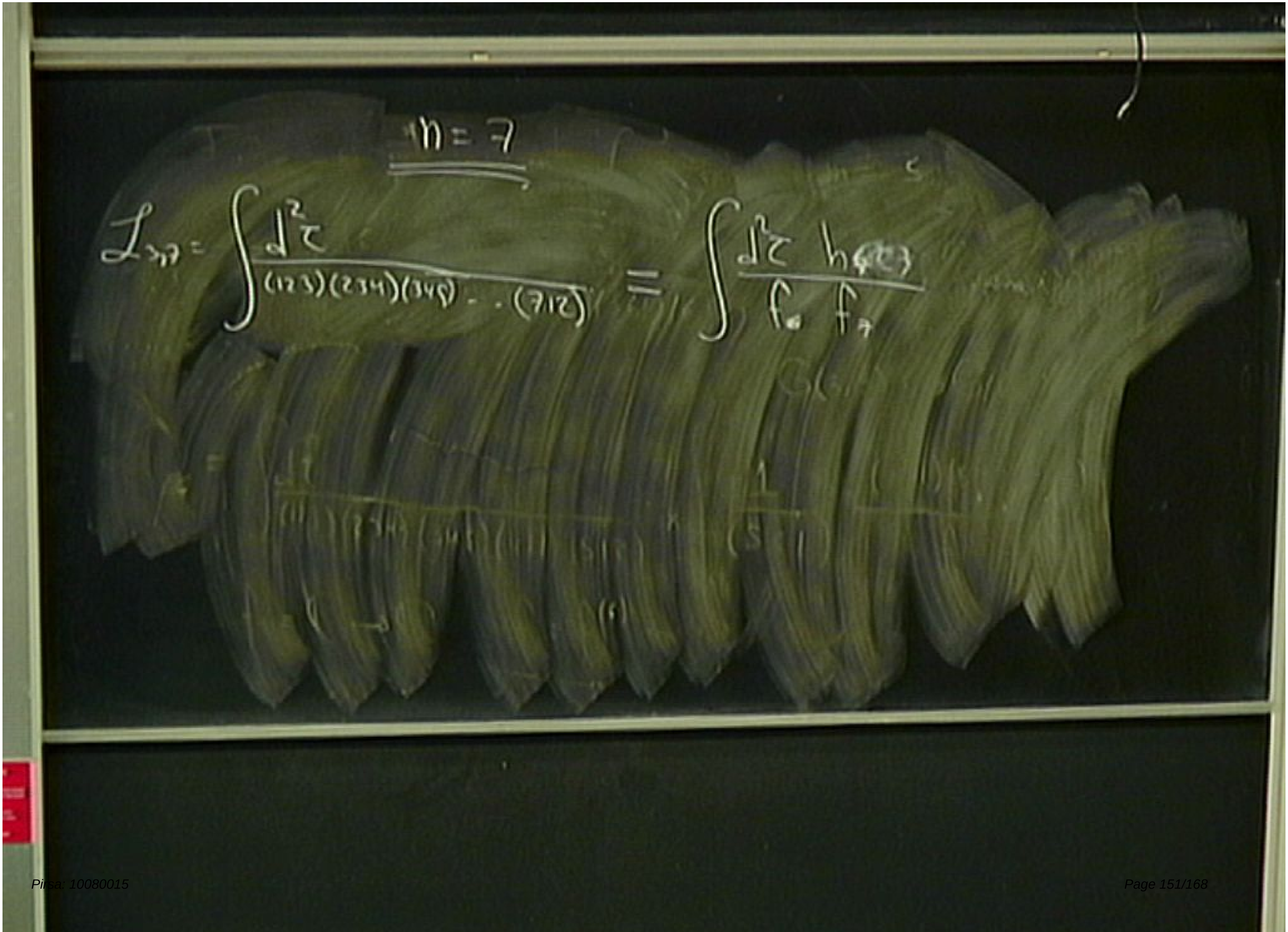
$$\mathcal{S}_{(n-1)\to n} = \frac{1}{(n-1\; n\; 1)} \, \frac{(n-2\; n-1\; 1)(n-1\; 1\; 2)(2\; 3\; n-2)}{f^{(n)}_{n}}$$

$$\boxed{h_{(n)} = \frac{h_{(n-1)}}{(n-1\; n\; 1)} (n-2\; n-1\; 1)(n-1\; 1\; 2)/(2\; 3\; n-2)}$$

$$f^{(n-1)} = (n-2\; n-1\; n)(n\; 1\; 2)(2\; 3\; n-2)$$

$$\int d^{n-5}\tau \, \frac{h(n)}{f^{(6)} \cdots f^{(n-1)}}$$

$$M^{tree}_{3,n} = \sum_{\text{All residues of } \mathcal{L}} \text{Residues of } \mathcal{L}_{3,n}$$



$n = 7$

$$\mathcal{L}_{n,7} = \int \frac{d^2\tau}{(123)(234)(345)\,\cdots\,(712)} = \int \frac{d^2\tau\; h(\tau)}{f_6\, f_7}$$

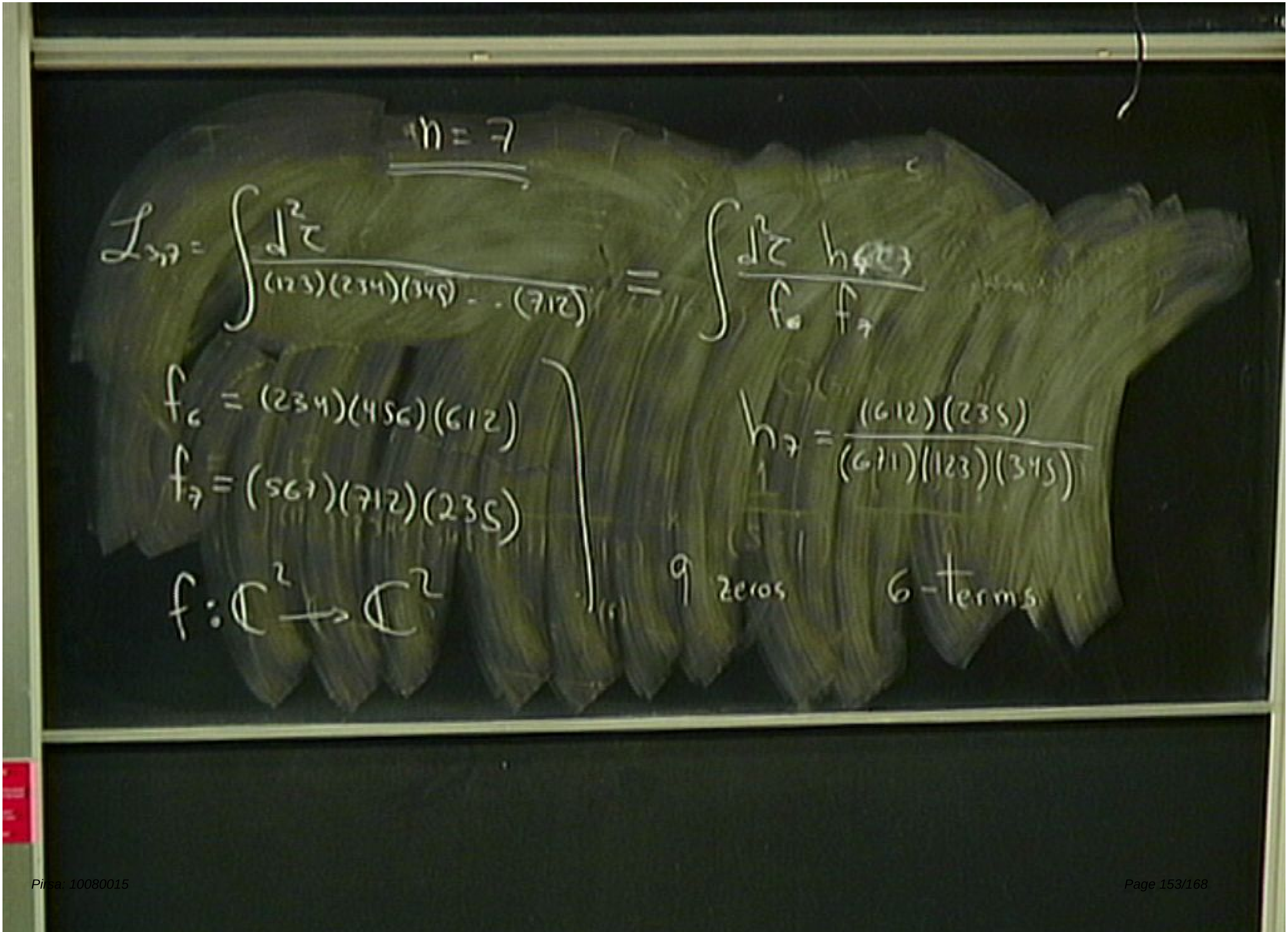
$n = 7$

$$\mathcal{L}_{3,7} = \int \frac{d^2\tau}{(123)(234)(345)\cdots(712)} = \int \frac{d^2\tau \, h_7}{f_6 \, f_7}$$

$$f_6 = (234)(456)(612)$$

$$f_7 = (567)(712)(235)$$

$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$



$n = 7$

$$\mathcal{L}_{3,7} = \int \frac{d^2 \tau}{(123)(234)(345)\cdots(712)} = \int \frac{d^2 \tau \; h_{[illegible]}}{f_6 \; f_7}$$

$f_6 = (234)(456)(612)$

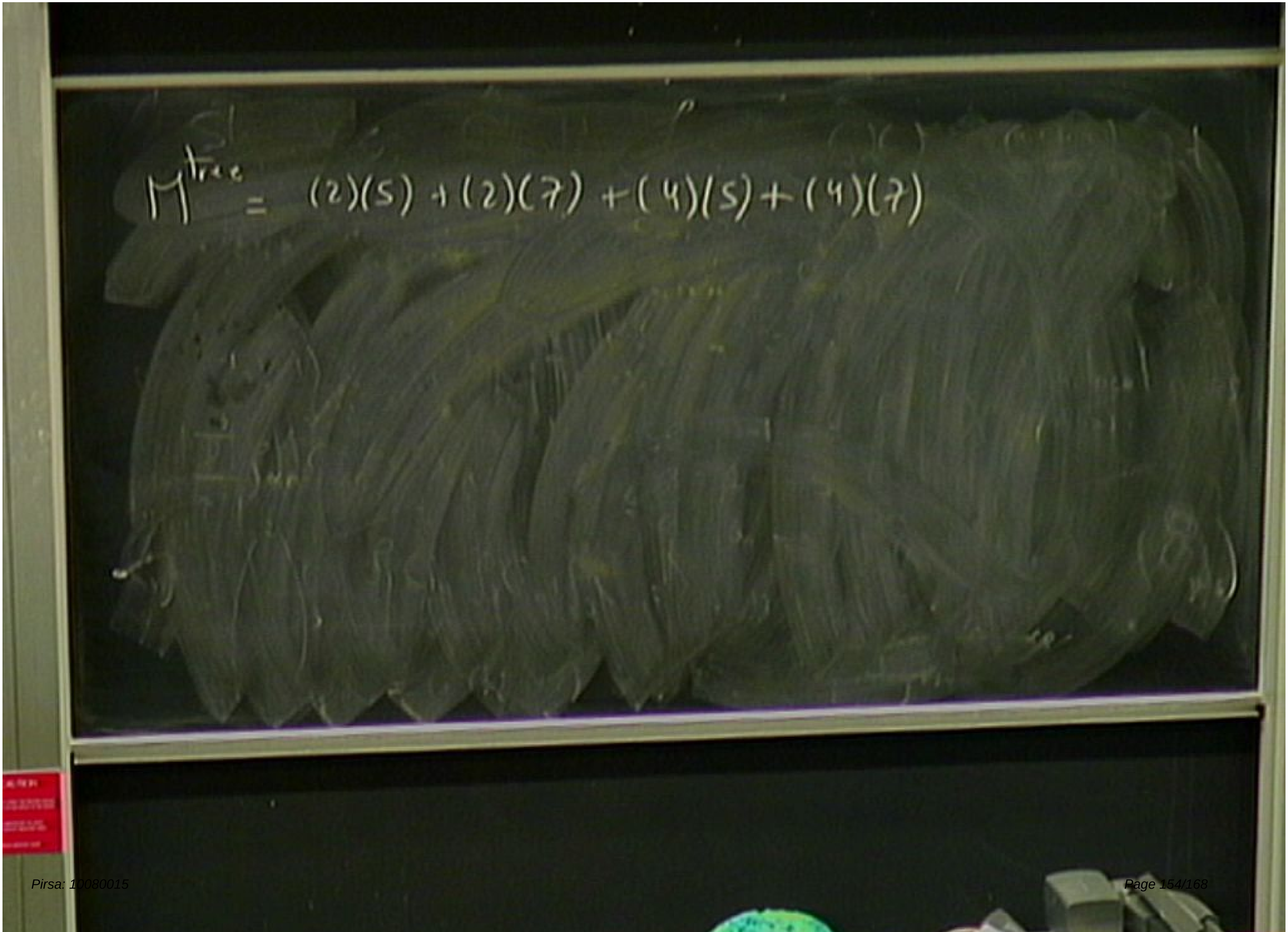
$f_7 = (567)(712)(235)$

$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$

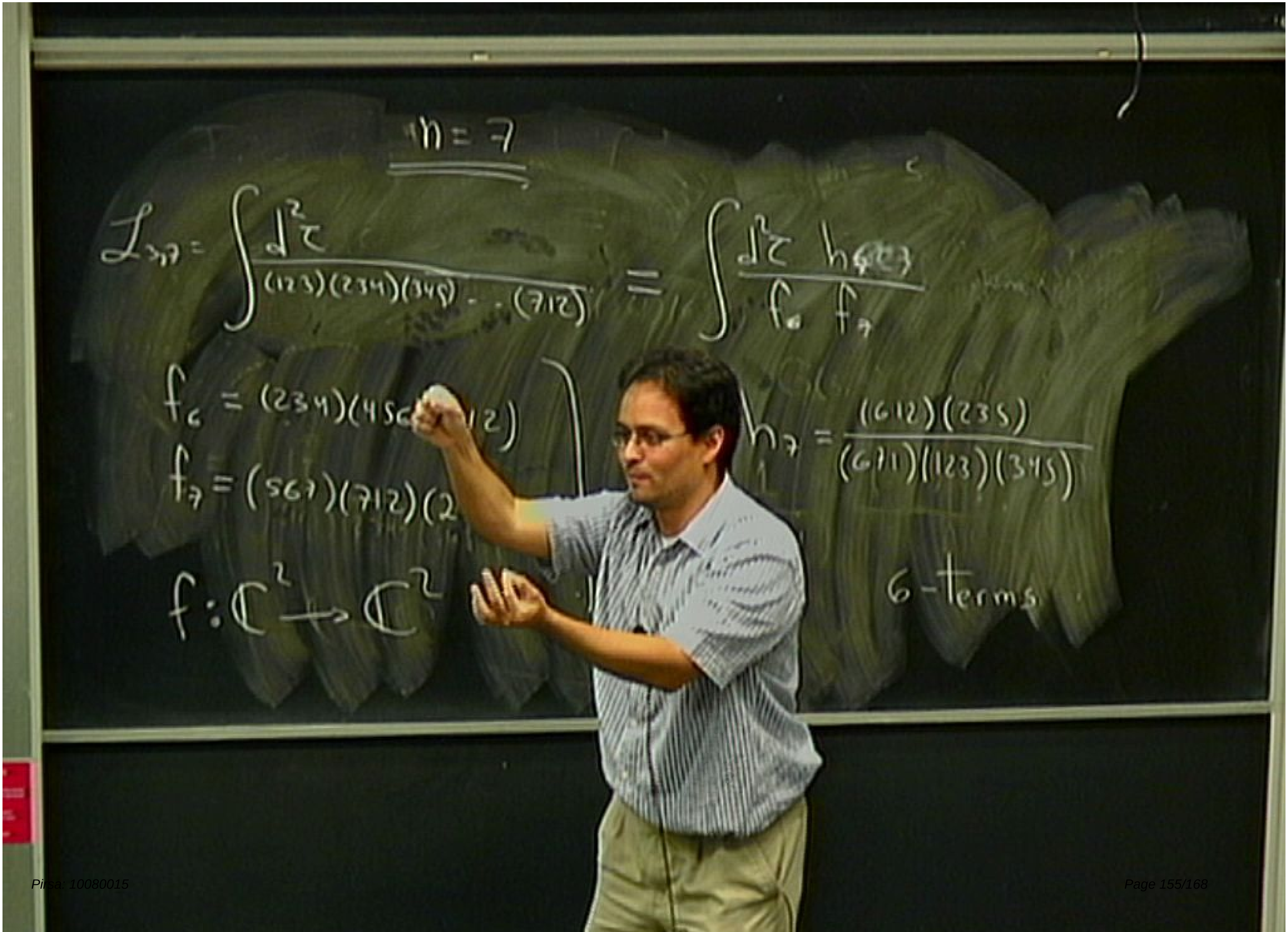
$f : \mathbb{C}^2 \to \mathbb{C}^2$

9 zeros

6-terms



$$M^{tree} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$



$n = 7$

$$\mathcal{L}_{3,7} = \int \frac{d^2\tau}{(123)(234)(345)\cdots(712)} = \int \frac{d^2\tau\, h\text{[illegible]}}{f_6\, f_7}$$

$f_6 = (234)(456$[illegible]$12)$

$f_7 = (567)(712)(2$

$f : \mathbb{C}^2 \to \mathbb{C}^2$

$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$

6-terms

$n = 7$

5

$$\mathcal{L}_{n,7} = \int \frac{d^2\tau}{(123)(234)(345)\cdots(712)} = \int \frac{d^2\tau \; h_{[illegible]}}{f_6 \; f_7}$$

$f_6 = (234)(456)(612)$

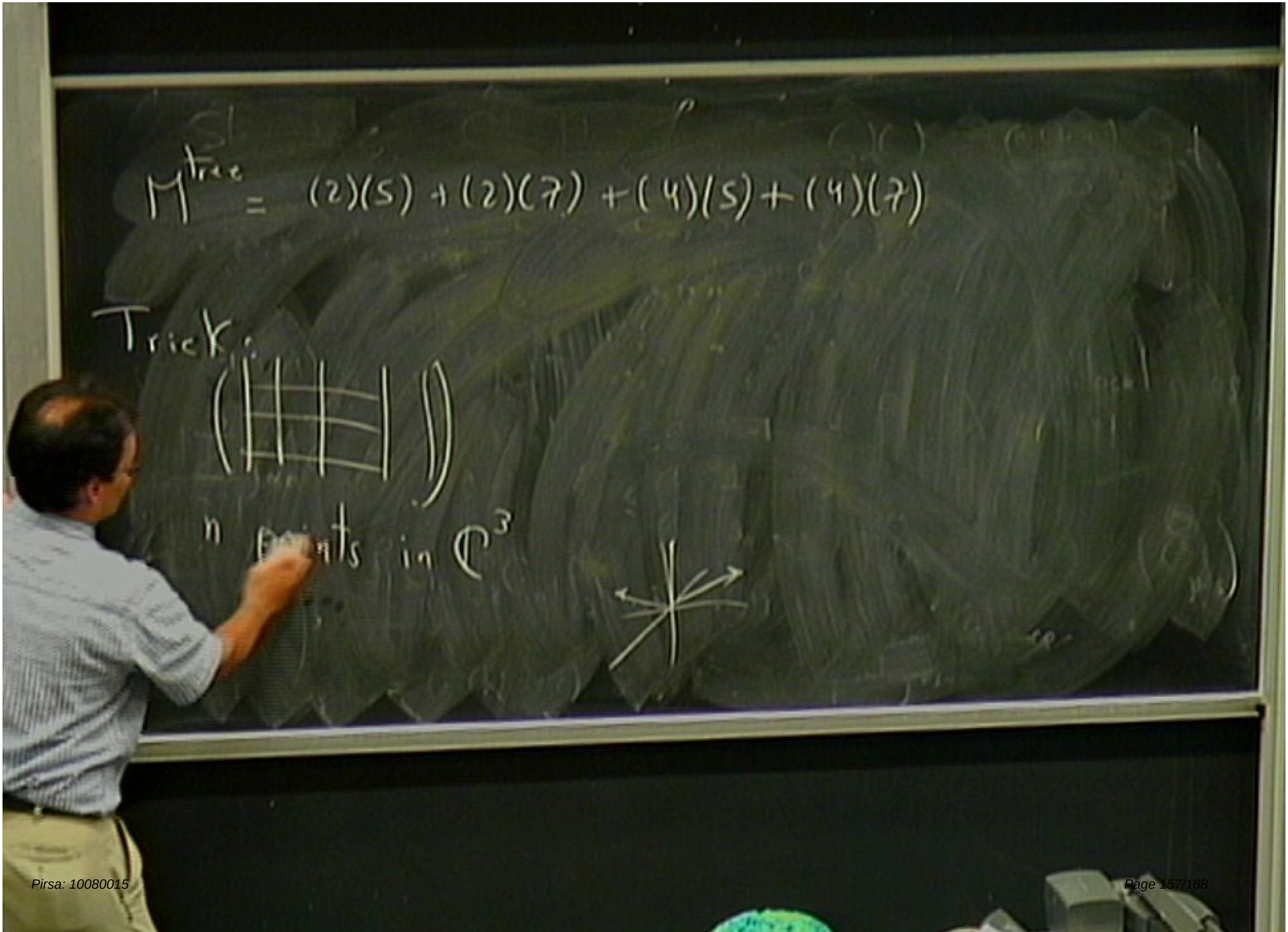
$f_7 = (567)(712)(235)$

$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$

$f : \mathbb{C}^2 \to \mathbb{C}^2$

9 zeros

6-terms



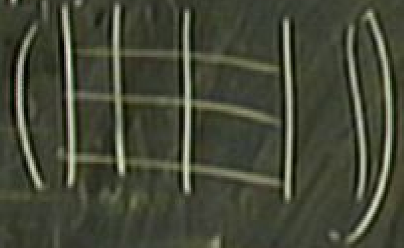
$$M^{\text{tree}} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

Trick.

n points in $\mathbb{C}^3$

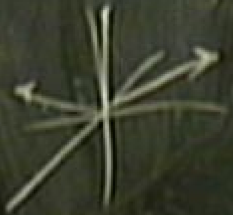
$$M^{\text{tree}} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

Trick:



n vectors in $\mathbb{C}^3$

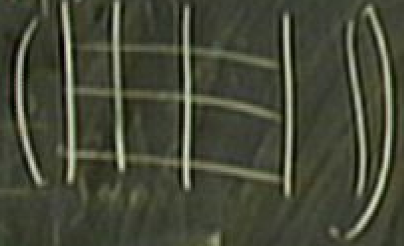
n points in $\mathbb{CP}^2$



$$M^{\text{tree}} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

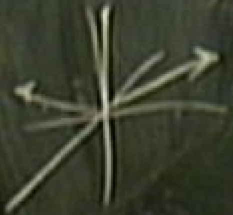
$(i\,j\,k) = 0$

Trick:



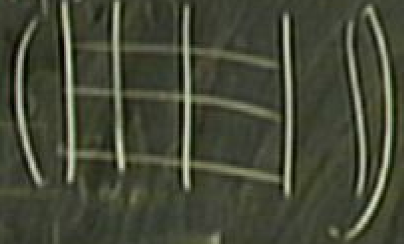
n vector in $\mathbb{C}^3$

n points in $\mathbb{CP}^2$



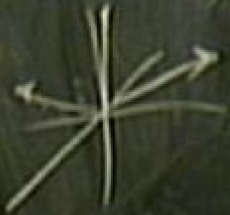
$$M^{\text{tree}} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

Trick:

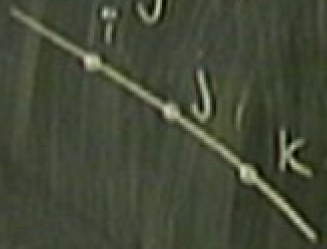


n vector in $\mathbb{C}^3$

n points in $\mathbb{CP}^2$



$(i\ j\ k) = 0$



$n = 7$

$$\mathcal{L}_{3,7} = \int \frac{d^2 z}{(123)(234)(345)\ldots(712)} = \int \frac{d^2 z \; h_{[illegible]}}{f_6 \; f_7}$$

$f_6 = (234)(456)(612)$

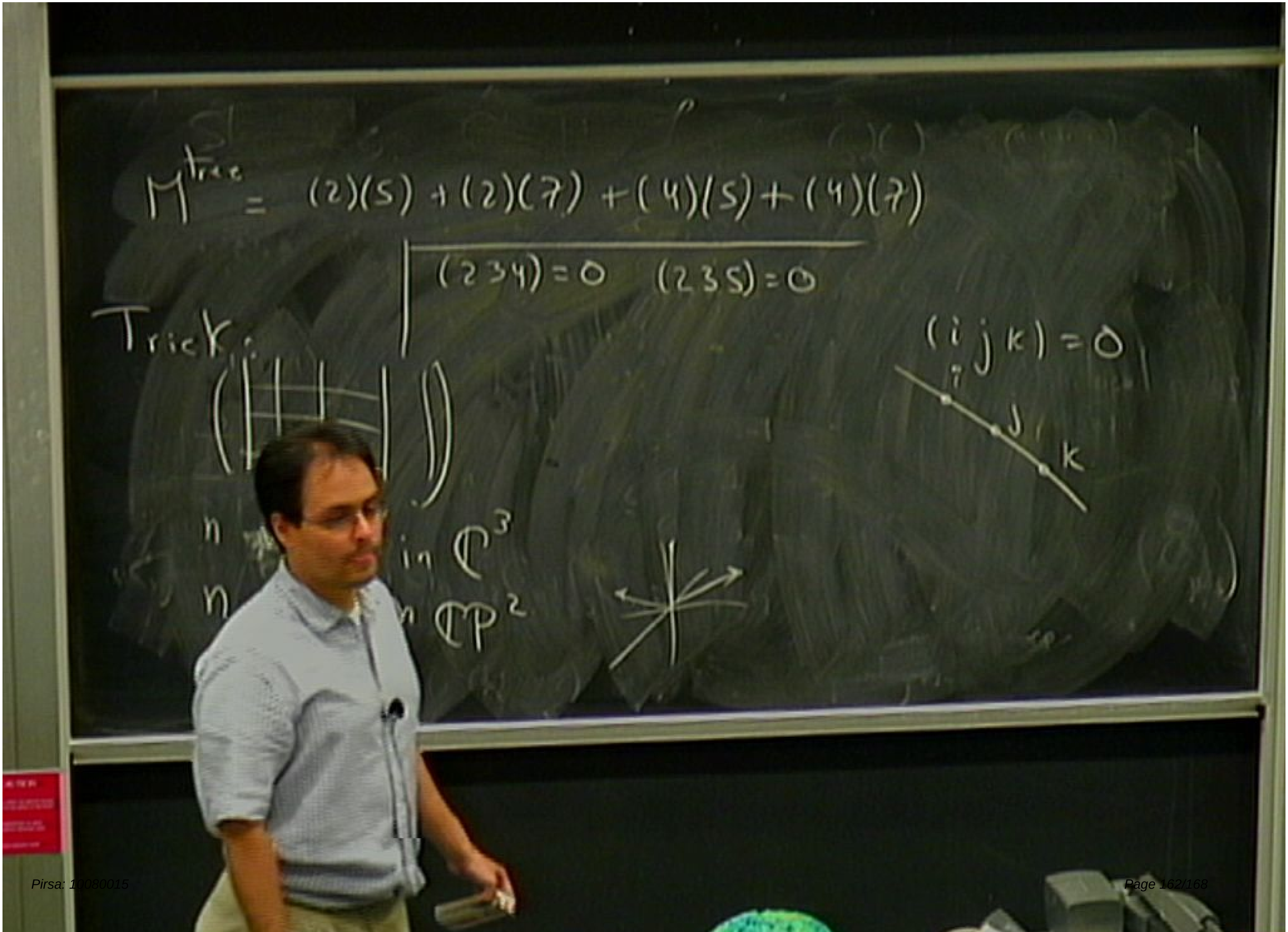
$f_7 = (567)(712)(235)$

$h_7 = \frac{(612)(235)}{(671)(123)(345)}$

$f : \mathbb{C}^2 \to \mathbb{C}^2$

9 zeros

6-terms



$$M^{tree} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

$(234) = 0 \quad (235) = 0$

Trick:

n in $\mathbb{C}^3$

n in $\mathbb{CP}^2$

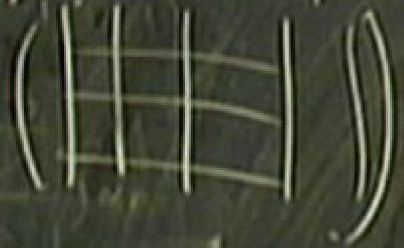
$(i\,j\,k) = 0$

$$M^{tree} = (2)(5) + (2)(7) + (4)(5) + (4)(7)$$

$\boxed{m=4}$

$(234)=0 \quad (235)=0$

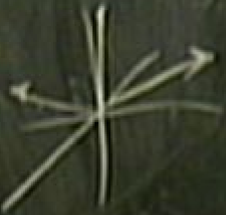
Trick:

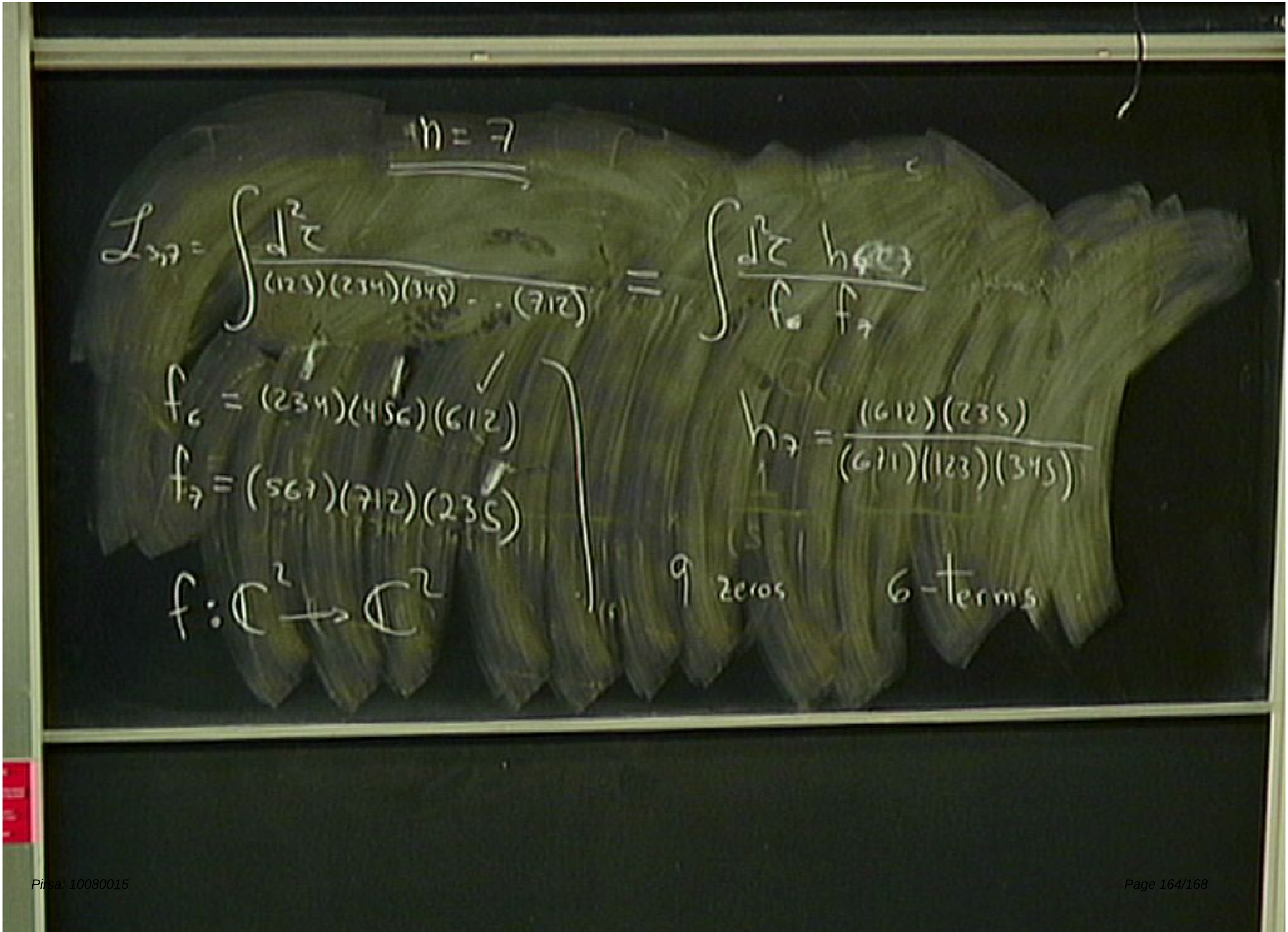


$\Rightarrow (234)=0 \quad (345)=0$

n vectors in $\mathbb{C}^3$

n points in $\mathbb{CP}^2$





$$n = 7$$

$$\mathcal{L}_{3,7} = \int \frac{d^2 z}{(123)(234)(345)\ldots(712)} = \int \frac{d^2 z \; h(z)}{f_6 \; f_7}$$

$$f_6 = (234)(456)(612)$$

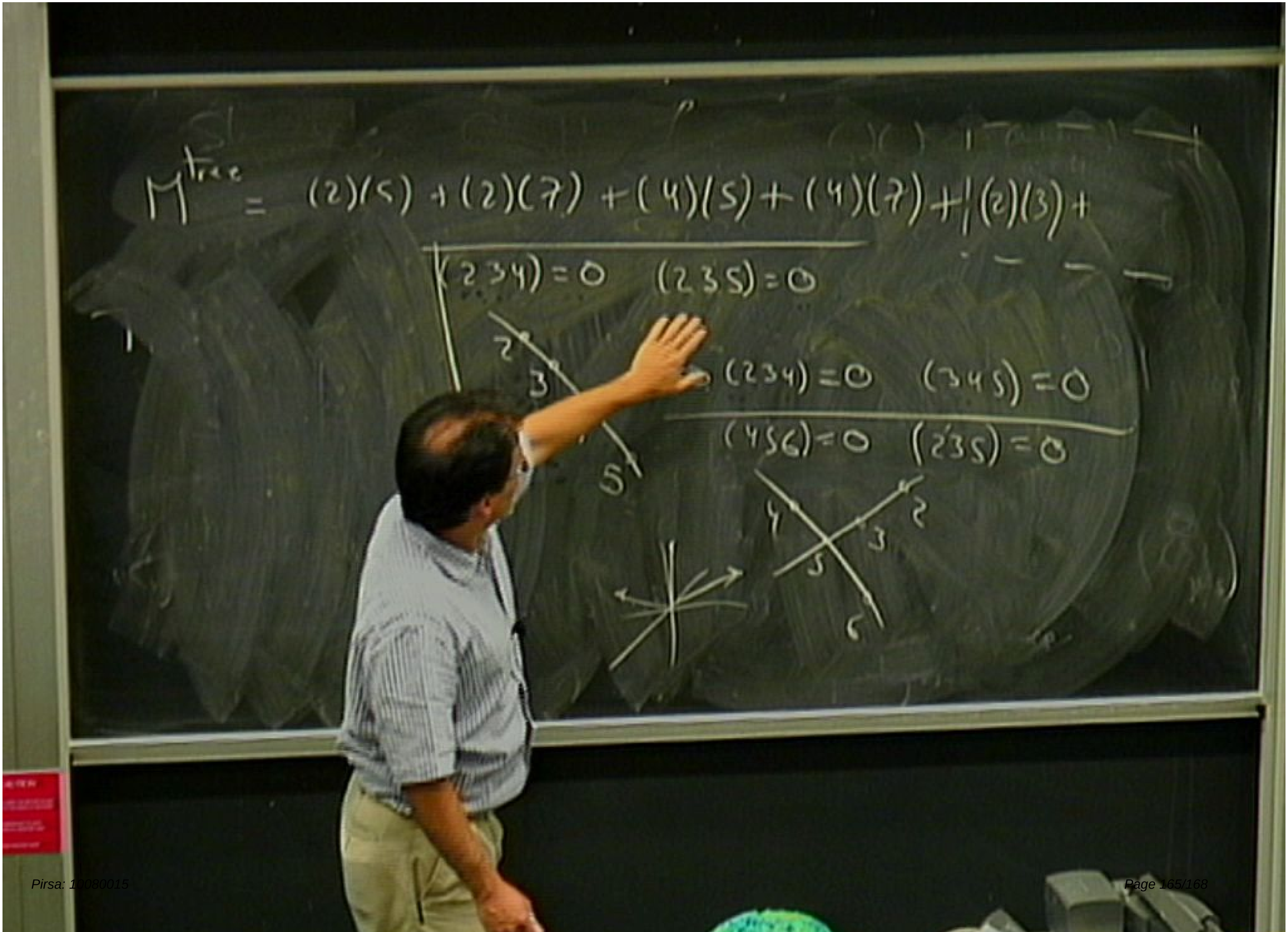
$$f_7 = (567)(712)(235)$$

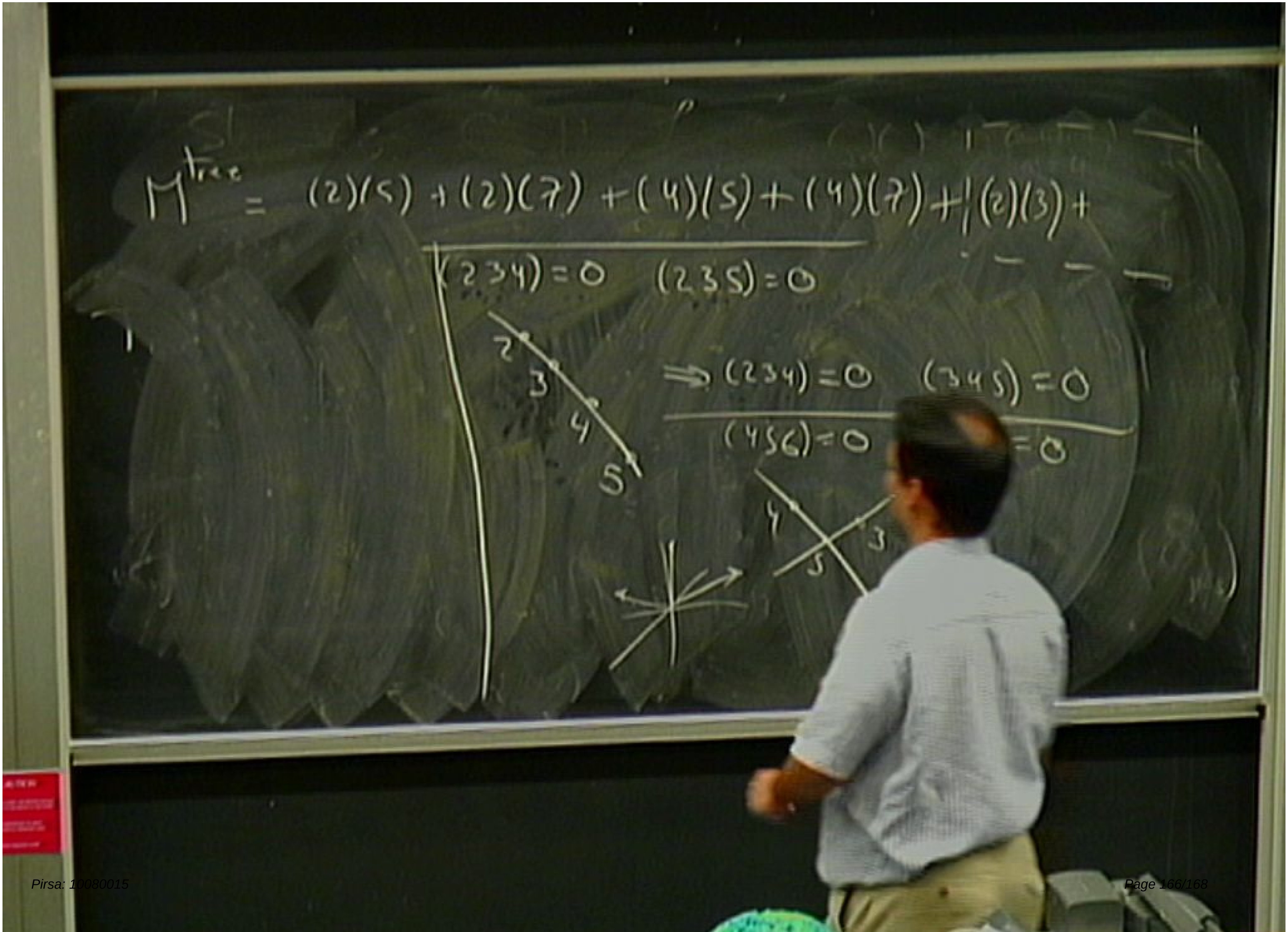
$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$

$$f : \mathbb{C}^2 \to \mathbb{C}^2$$

9 zeros

6-terms


M^tree = (2)(5) + (2)(7) + (4)(5) + (4)(7) + (2)(3) +
(234) = 0   (235) = 0
(234) = 0   (345) = 0
(456) = 0   (235) = 0
2
3
5
4
2
3
5
6

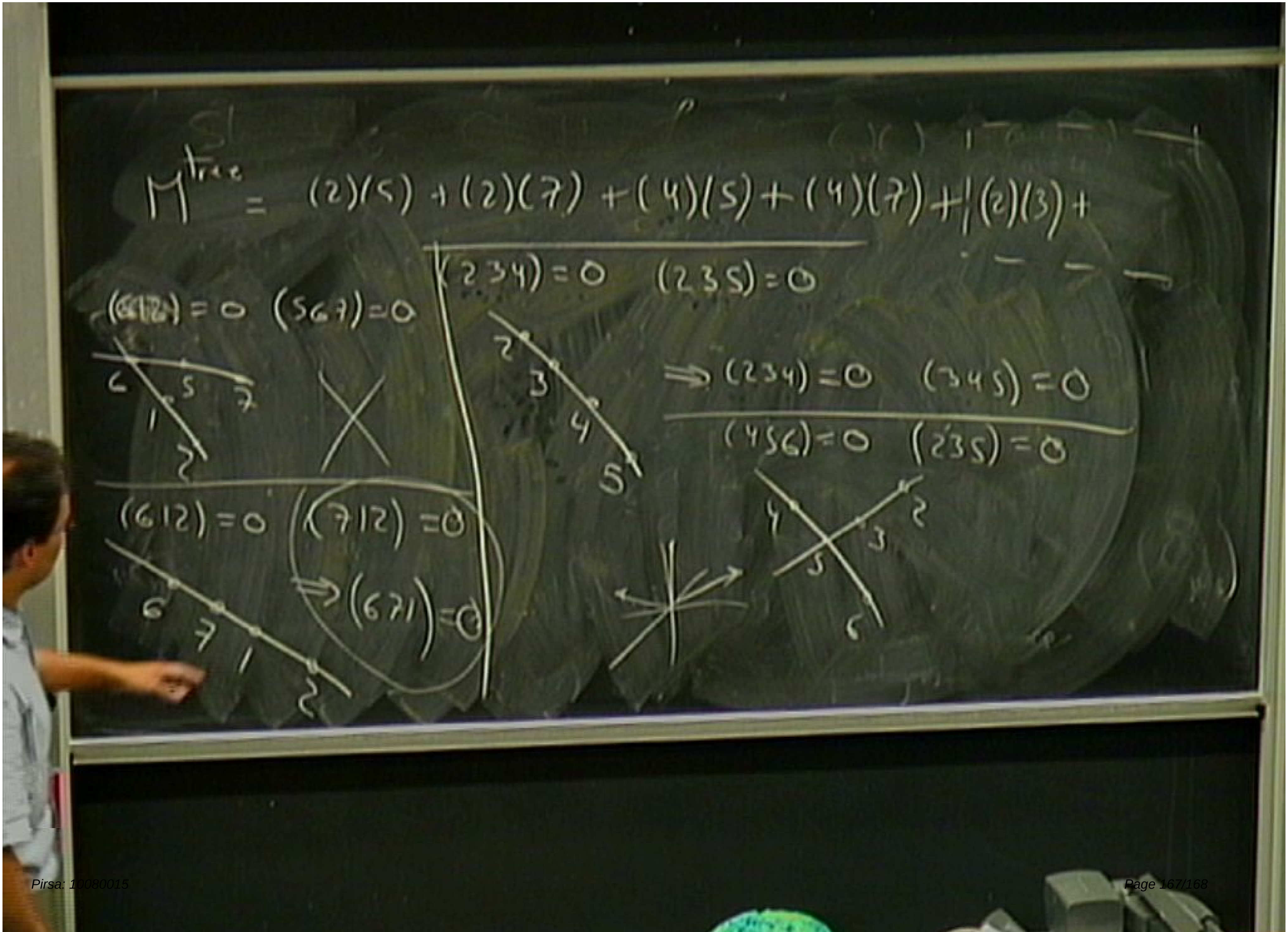


$M^{tree} = (2)(5) + (2)(7) + (4)(5) + (4)(7) + (2)(3) +$

$(234) = 0 \quad (235) = 0$

$\Rightarrow (234) = 0 \quad (345) = 0$

$(456) = 0 \quad = 0$



$M^{tree} = (2)(5) + (2)(7) + (4)(5) + (4)(7) + (2)(3) +$

$(234) = 0 \quad (235) = 0$

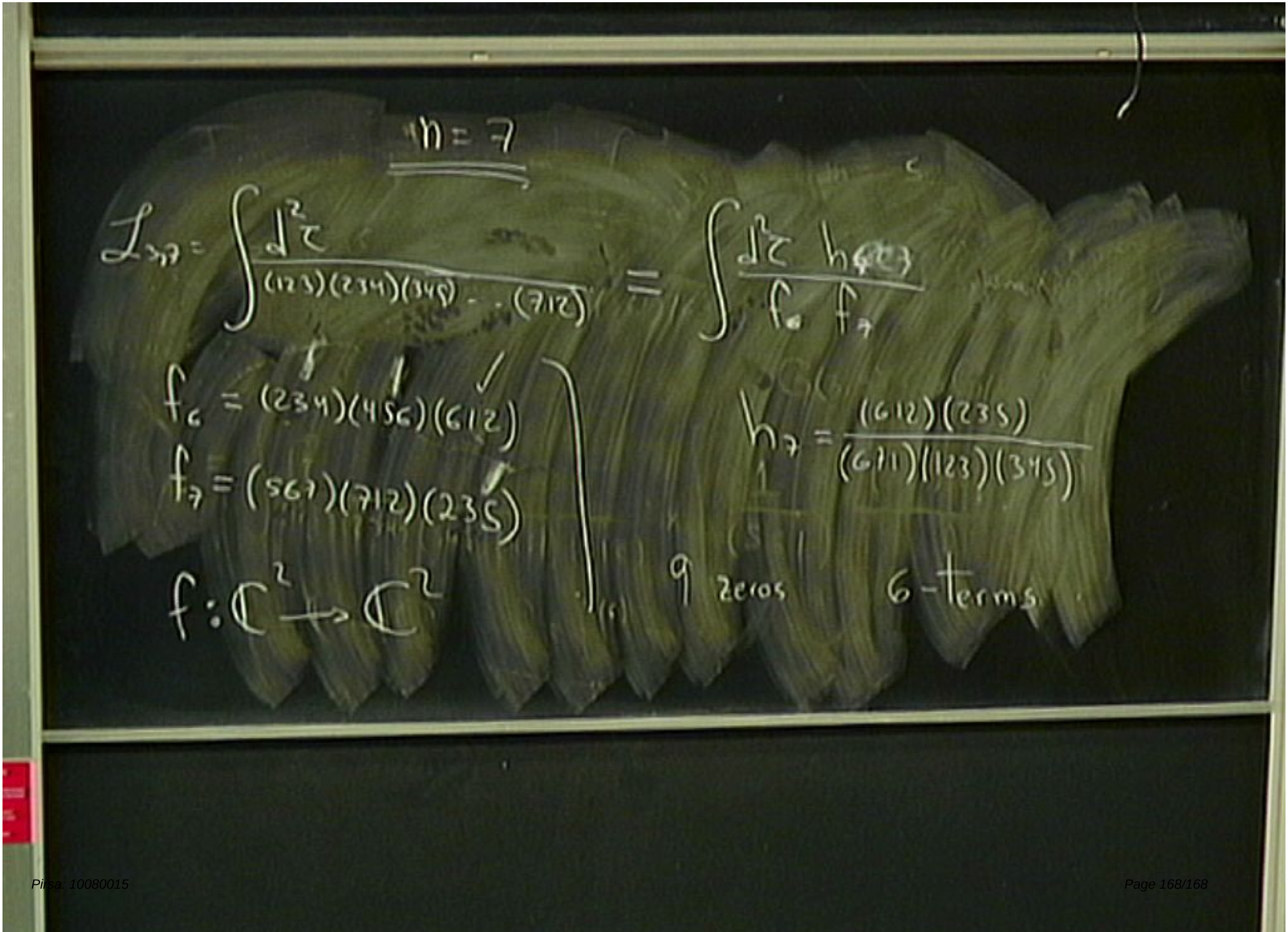
$(6712) = 0 \quad (567) = 0$

$\Rightarrow (234) = 0 \quad (345) = 0$

$(456) = 0 \quad (235) = 0$

$(612) = 0 \quad (712) = 0$

$\Rightarrow (671) = 0$



n = 7

$$\mathcal{L}_{3,7} = \int \frac{d^2\tau}{(123)(234)(345)\cdots(712)} = \int \frac{d^2\tau\, h}{f_6\, f_7}$$

$$f_6 = (234)(456)(612)$$

$$f_7 = (567)(712)(235)$$

$$h_7 = \frac{(612)(235)}{(671)(123)(345)}$$

$$f : \mathbb{C}^2 \to \mathbb{C}^2$$

9 zeros

6-terms