Title: Minimum output entropy of quantum channels is hard to approximate

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Abstract: The headline result of this talk is that, based on plausible complexity-theoretic assumptions, many properties of quantum channels are computationally hard to approximate. These hard-to-compute properties include the minimum output entropy, the 1->p norms of channels, and their "regularized" versions, such as the classical capacity.

The proof of this claim has two main ingredients. First, I show how many channel problems can be fruitfully recast in the language of two-prover quantum Merlin-Arther games (which I'll define during the talk). Second, the main technical contribution is a procedure that takes two copies of a multipartite quantum state and estimates whether or not it is close to a product state.

This is based on arXiv:1001.0017, which is joint work with Ashley Montanaro.

Hardness of approximation, quantum Merlin-Arthur games and product state testing

arXiv:1001.0017

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6 July, 2010

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Outline

1. Things we want to compute
2. A test for product states, a.k.a. a quantum e-meter
3. Implications

Norms, entropies and channels

Schatten norms and Rènyi entropies

For $\alpha \geq 0$ and ρ a density matrix, define

$$\|\rho\|_\alpha := (\operatorname{tr} \rho^\alpha)^{1/\alpha} \qquad S_\alpha(\rho) = \frac{\alpha}{1-\alpha} \log \|\rho\|_\alpha$$

In particular $S(\rho) = S_1(\rho) = -\operatorname{tr} \rho \log \rho$ and $S_\infty(\rho) = -\log \|\rho\|_\infty$.

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superoperator norms and minimum-output Rényi entropy

Let $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ be a CPTP map. $|A| = |B| = d$.

$$\|\mathcal{N}\|_{1\to\alpha} = \max_{\substack{\rho\geq 0 \\ \operatorname{tr}\rho=1}} \|\mathcal{N}(\rho)\|_\alpha$$

Goal 1a: superoperator norms

$$S_\alpha^{\min}(\mathcal{N}) = \min_{\substack{\rho\geq 0 \\ \operatorname{tr}\rho=1}} S_\alpha(\rho)$$

Goal 1b: minimum output entropy

Subspaces

Stinespring dilation

$$\mathcal{N}(\rho) = \mathrm{tr}_E\, U_{\mathcal{N}} \rho U_{\mathcal{N}}^\dagger \qquad U_{\mathcal{N}} : A \mapsto B \otimes E$$

$$V = \mathrm{range}(U_{\mathcal{N}})$$

$$S_\alpha^{\min}(\mathcal{N}) = \min_{\substack{|\psi\rangle \in V \\ \langle\psi|\psi\rangle = 1}} S_\alpha(\mathrm{tr}_E\, \psi) \qquad \text{where } \psi := |\psi\rangle\langle\psi|$$

$$=: E_\alpha(V)$$

Goal 1c: min. entanglement of subspace

[Hayden, Leung, Winter. "Aspects of generic entanglement." quant-ph/0407049] proved that random subspaces have E_α nearly maximal.

Can also interpret this as a special case of Dvoretzky's theorem. See also [Hayden,Winter; 0807.4753], [Hastings; 0809.3972], [Aubrun, Szarak, Werner; 1003.4925], plus talks this morning.

Specialize to $\alpha = \infty$

(All maximizations are over unit vectors.)

$$\begin{aligned}
\exp(-E_\infty(V)) &= \max_{|\psi\rangle \in V} \|\operatorname{tr}_E \psi\|_\infty \\
&= \max_{|\psi\rangle \in V} \max_{|\beta\rangle \in B, |\epsilon\rangle \in E} |\langle\beta| \otimes \langle\epsilon| \cdot |\psi\rangle|^2 \\
&= \max_{|\beta\rangle \in B, |\epsilon\rangle \in E} \operatorname{tr}(\beta \otimes \epsilon)\Pi_V \\
&= \max_{\rho \in \mathrm{Sep}} \operatorname{tr} \rho \Pi_V \\
&=: \|\Pi_V\|_{\mathrm{Sep}}
\end{aligned}$$

where Sep = {separable density matrices}
$= \{\sum_i p_i \beta_i \otimes \epsilon_i : p_i \geq 0, \sum_i p_i = 1, \beta_i = |\beta_i\rangle\langle\beta_i|, \epsilon_i = |\epsilon_i\rangle\langle\epsilon_i|\}$.
Also define $\|M\|_{\mathrm{Sep}} := \max_{\rho \in \mathrm{Sep}} \operatorname{tr} M\rho$. (Equivalently $\|M\|_{S_\infty \otimes_\epsilon S_\infty}$.)
Goal 2a: compute $\|\cdot\|_{\mathrm{Sep}}$.

Camelot and the complexity zoo

- **P**: Languages that can be decided (yes/no) in time polynomial in the input.
- **NP**: If the answer is yes, then there exists a classical witness (bit string) that the poly-time verifier always accepts. If the answer is no, then no witness will be accepted.
- **MA**: There is still a classical witness but now the verifier is randomized. Yes instances are accepted with probability $\geq c = 2/3$ and No instances are accepted with probability $\leq s = 1/3$.
- **QMA**: The witness is a quantum state $|\varphi\rangle$ and the verifier is quantum.
- **QMA**(k): The witness is a product state $|\varphi_1\rangle \otimes \cdots |\varphi_k\rangle$.

Alternatively $\mathbf{QMA}_m(k)_{s,c}$ means that there are k messages, each with m qubits. Normally $m = \text{poly}(n)$, and n is the length of the input.

QMA(k) as an optimization problem

The verifier performs a measurement $\{M, I - M\}$ on the witness $|\varphi\rangle \in (\mathbb{C}^{2^m})^{\otimes k}$ and accepts with probability $\operatorname{tr} M\varphi$.

$\mathbf{QMA}_m(k)_{s,c}$ = determine whether $\max_\varphi \operatorname{tr} M\varphi$ is $\geq c$ or $\leq s$.

When $k = 1$, this is an eigenvalue problem with an $\exp(m)$-time algorithm

When $k = 2$, this is equivalent to determining whether $\|M\|_{\mathsf{Sep}} \geq c$ or $\|M\|_{\mathsf{Sep}} \leq s$. No $\exp(m)$ time algorithm is known.
Goal 2b: Solve problems in **QMA**(2).

For general k, this is an optimization problem over a degree-k polynomial in the variables $\{\langle i|\varphi\rangle \langle\varphi|j\rangle\}_{i,j}$.

The **QMA**(2) amplification problem

Strong amplification: Suppose we have a coin with Pr(heads) either $\geq c$ or $\leq s$. We can distinguish these cases with probability of error ϵ using $r = O(\frac{\log(1/\epsilon)}{(c-s)^2})$ coin flips.

For **QMA**(1), we can ask for r copies of the proof, a.k.a. parallel repetition. For "no" instances, entanglement doesn't help, because each proof individually succeeds with probability $\leq s$, even conditioned on all the previous measurement outcomes.

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For **QMA**(2) this no longer works because measuring one copy of the proof can introduce entanglement between the remaining $r-1$ copies.

failure of perfect parallel repetition $\cong S_\infty^{\min}$ additivity violations

[Kobayashi et al.; quant-ph/0306051] and [Aaronson et al.; 0804.0802] independently showed how *completeness* could be amplified, while degrading soundness.

Also, strong amplification implies $\mathbf{QMA}(2) = \mathbf{QMA}(k), \forall k > 2$.

Mean-field Hamiltonians

For $M \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ with $0 \leq M \leq I$, define $H \in \mathcal{L}((\mathbb{C}^d)^{\otimes n})$ by

$$H = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} I - M^{(i,j)}.$$

[Fannes-Vanderplas; quant-ph/0605216] showed that the ground state energy is $\approx 1 - \min_{\rho \in \mathsf{Sep}} \operatorname{tr} M\rho$.

Goal 2c: find ground-state energy density

Separability testing

Goal 3: Given a density matrix ρ, determine whether $\rho \in \mathrm{Sep}$.

More realistically, $\mathrm{WMEM}_\epsilon(\mathrm{Sep})$ asks us to determine whether ρ is ϵ-within Sep or ϵ-outside Sep.

Since optimizing a linear function over a convex set is equivalent to the weak membership problem (up to replacing ϵ by $\epsilon/\operatorname{poly}(d)$), this is equivalent to approximating $\|\cdot\|_{\mathrm{Sep}}$.

Regularized minimum output entropy

Asymptotic minimum output entropy

$$\bar{S}_\alpha^{\min}(\mathcal{N}) := \lim_{n\to\infty} \frac{1}{n} S_\alpha^{\min}(\mathcal{N}).$$

These quantities are relevant for information-theoretic problems.

α	Task	Reference
0	Zero-error capacity	0906.2547
1	Classical capacity	HSW
∞	strong converse $\to$ crypto applications	0906.1030

Goal 4: estimate $\bar{S}_\alpha^{\min}(\mathcal{N})$

Estimating this to poly$(1/d)$ precision is **NP**-hard, but no upper bound on its complexity is known.

Previous results

- [Kobayashi et al., quant-ph/0306051] defined $\mathbf{QMA}(k)$, showed that amplification for $\mathbf{QMA}(2)$ implies that $\mathbf{QMA}(2) = \mathbf{QMA}(k)$, and that multiple provers don't help in the $s = 0$ (perfect soundness) case
- [Beigi-Shor, 0709.2090] showed that it's **NP**-complete to distinguish $S_{\min}(\mathcal{N}) \leq 2$ from $S_{\min}(\mathcal{N}) \geq 2 + 1/\operatorname{poly}(d)$.
- [Blier-Tapp, 0709.0738] showed that 3-SAT$\in \mathbf{QMA}_{\log}(2)_{1-1/\operatorname{poly}(n),1}$
- [Aaronson et al., 0804.0802] showed that 3-SAT$\in \mathbf{QMA}_{\log}(\sqrt{n}\operatorname{poly}\log(n))_{1-\Omega(1),1}$.
- [Gurvitz, quant-ph/0303055] showed that $\mathrm{WMEM}_{1/\exp(d)}(\mathrm{Sep})$ is **NP**-complete.
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Testing productness

The problem

Given a state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, we would like to determine if $|\psi\rangle$ is close to a product state (i.e. something of the form $|\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle$) or far from any product state.

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Our result

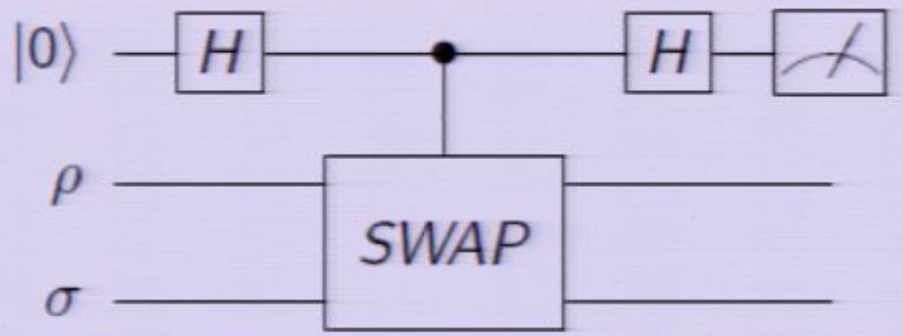
- Given one copy of $|\psi\rangle$, the task is impossible.
- Given $|\psi\rangle \otimes |\psi\rangle$, there is a simple test to determine if $|\psi\rangle$ is close to a product state. If

$$1 - \epsilon = \max_{|\phi_1\rangle,\ldots,|\phi_n\rangle \in \mathbb{C}^d} |\langle \phi_1, \ldots, \phi_n | \psi \rangle|^2$$

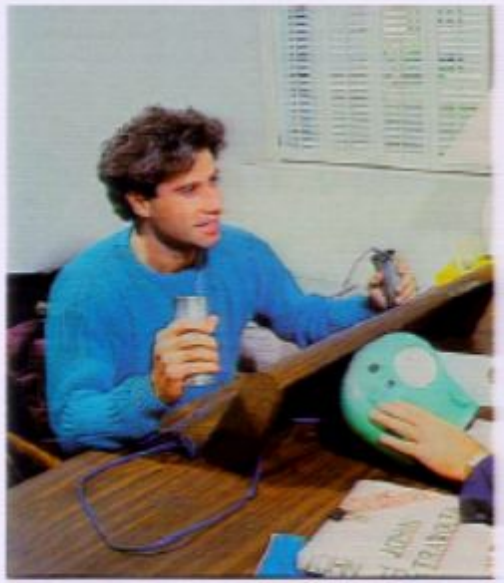
then the test passes with probability $1 - \Theta(\epsilon)$.

Key primitive

SWAP test



Accept if the outcome of the measurement is "0", reject if not.



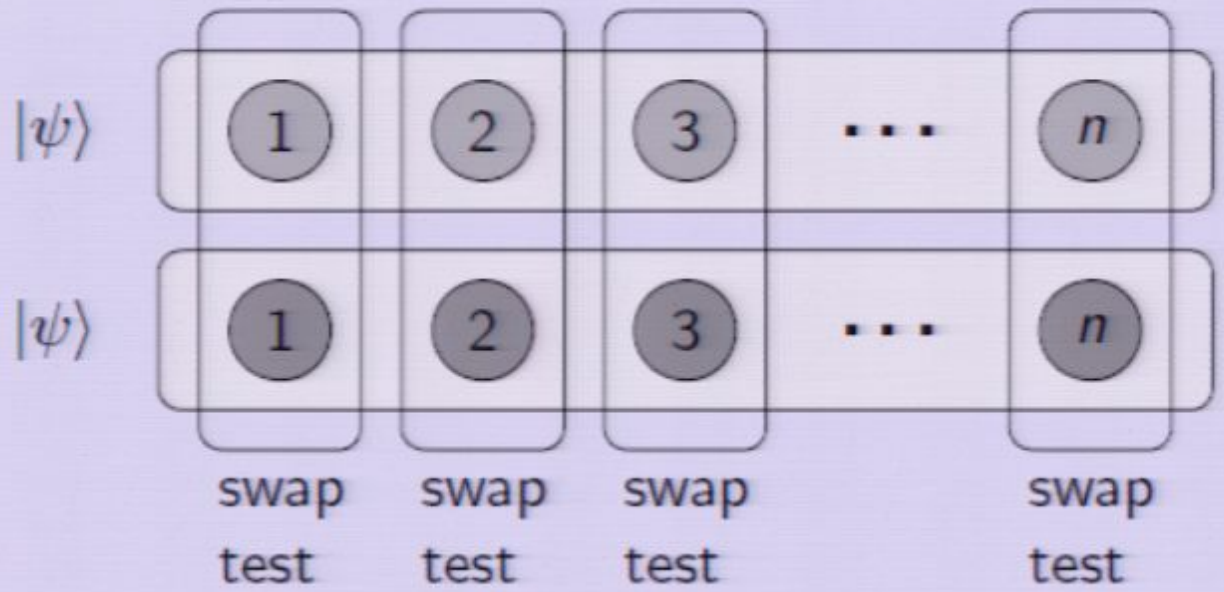
John Travolta - Actor, OT III, currently on his False Purpose Rundown counselling.

The probability of accepting is $\frac{1+\mathrm{tr}\,\rho\sigma}{2}$.

As demonstrated by John Travolta.

Testing productness

Product test algorithm



Accept iff all n swap tests pass.

Why it works. Applied to $\rho \otimes \rho$, the swap test accepts with probability $(1 + \operatorname{tr} \rho^2)/2$. If $|\psi\rangle$ is entangled, some of its subsystems must be mixed and so some swap tests are likely to fail.

$\mathbf{QMA}(2)$ soundness amplification

We obtain $\mathbf{QMA}(2)$ amplification in two different ways.

1. k provers can perform a 2-prover protocol $k/2$ times in parallel. Allows amplification of $c - s \geq 1/\operatorname{poly}(n)$ to $c - s \geq \Omega(1)$.
2. M is now a separable operator. Allows general amplification.

Therefore: $\mathbf{QMA}_{\mathrm{poly}}(k)_{s,c}$ is the same for all $k \geq 2$ and $1/\operatorname{poly}(n) \leq c - s \leq 1 - \exp(-n)$.

Implications for **QMA**(k)

Theorem (2 provers can simulate k provers)

$\mathbf{QMA}_m(k)_{s=1-\epsilon,c} \subset \mathbf{QMA}_{mk}(2)_{1-\frac{\epsilon}{50},c}$

Proof.

- If the $\mathbf{QMA}(k)$ protocol had proofs $|\varphi_1\rangle, \ldots, |\varphi_k\rangle$ then simulate in $\mathbf{QMA}(2)$ by asking each prover to submit $|\varphi_1\rangle \otimes \cdots |\varphi_k\rangle$.
- Then use the product test to verify that they indeed submit product states.

Note: the resulting M is now a separable operator: i.e. $M = \sum_i X_i \otimes Y_i$ for $X_i, Y_i \geq 0$.

Corollary: 3-SAT $\in \mathbf{QMA}_{\sqrt{n}\,\mathrm{poly}\log(n)}(2)_{1-\Omega(1),1}$.

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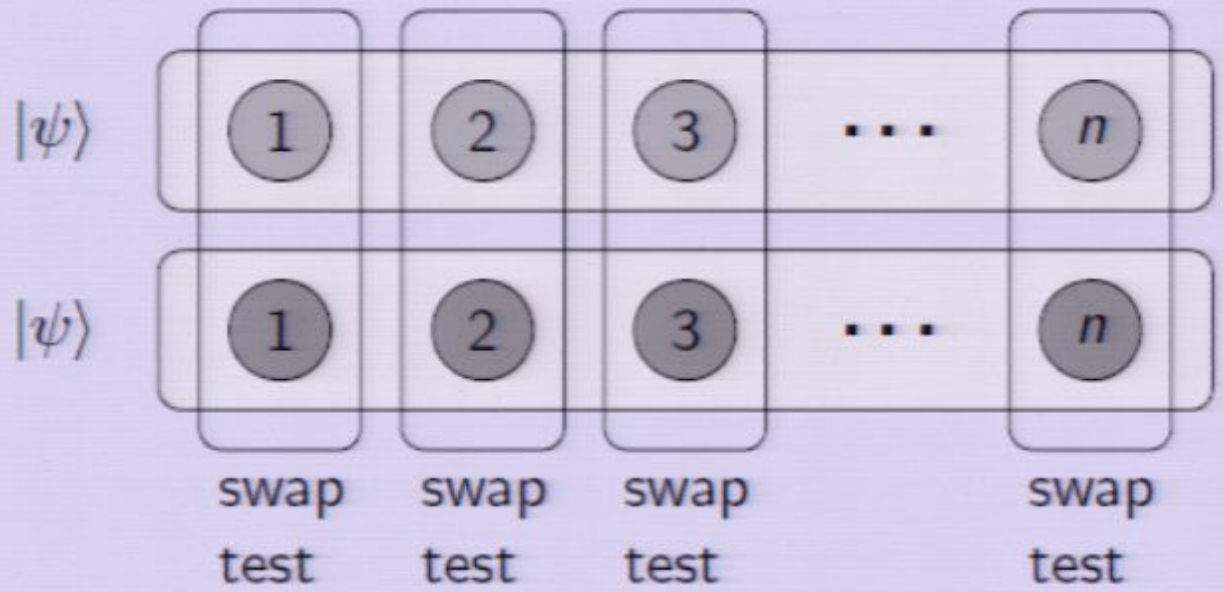

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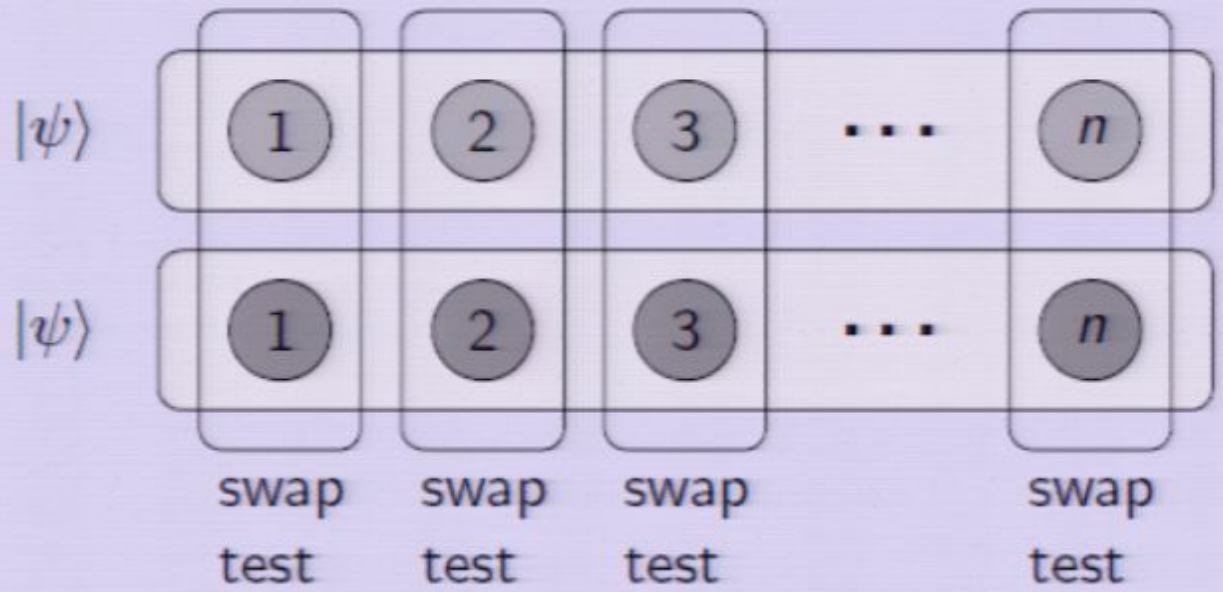

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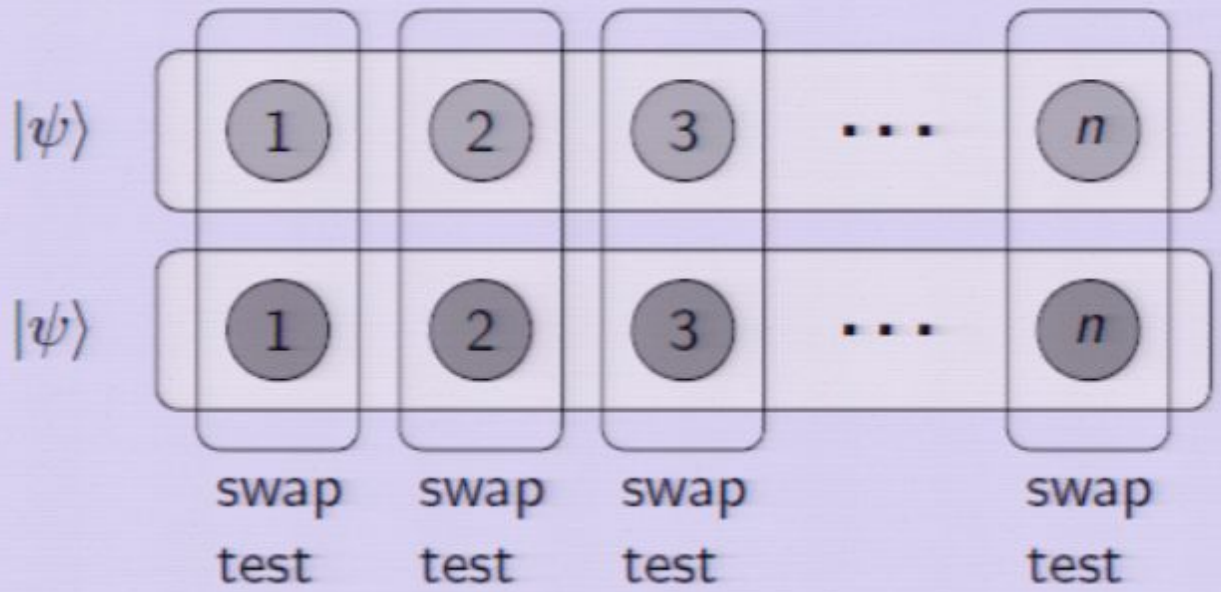

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This also implies that $\bar{S}_{\infty}^{\min}$ is at least as hard to approximate as $S_{\infty}^{\min}$.

The promised hardness results

Recall 3-SAT $\in \mathbf{QMA}_{\sqrt{n}\,\mathrm{poly}\log(n)}(2)_{1-\Omega(1),1}$.

Therefore there exists a universal $\epsilon > 0$, such that distinguishing $S_\infty^{\min} = 0$ from $S_\infty^{\min} > \epsilon$ is $\mathbf{NP}_{\log^2}$-hard.

Equivalently, we cannot approximate $S_\infty^{\min}$ (or any of the equivalent quantities) to constant accuracy in poly-time unless 3-SAT can be solved in $2^{\sqrt{n}\,\mathrm{poly}\log(n)}$ time.

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Alternate way to summarize results:

Detecting pure-state entanglement is easy, therefore detecting mixed-state entanglement is hard.

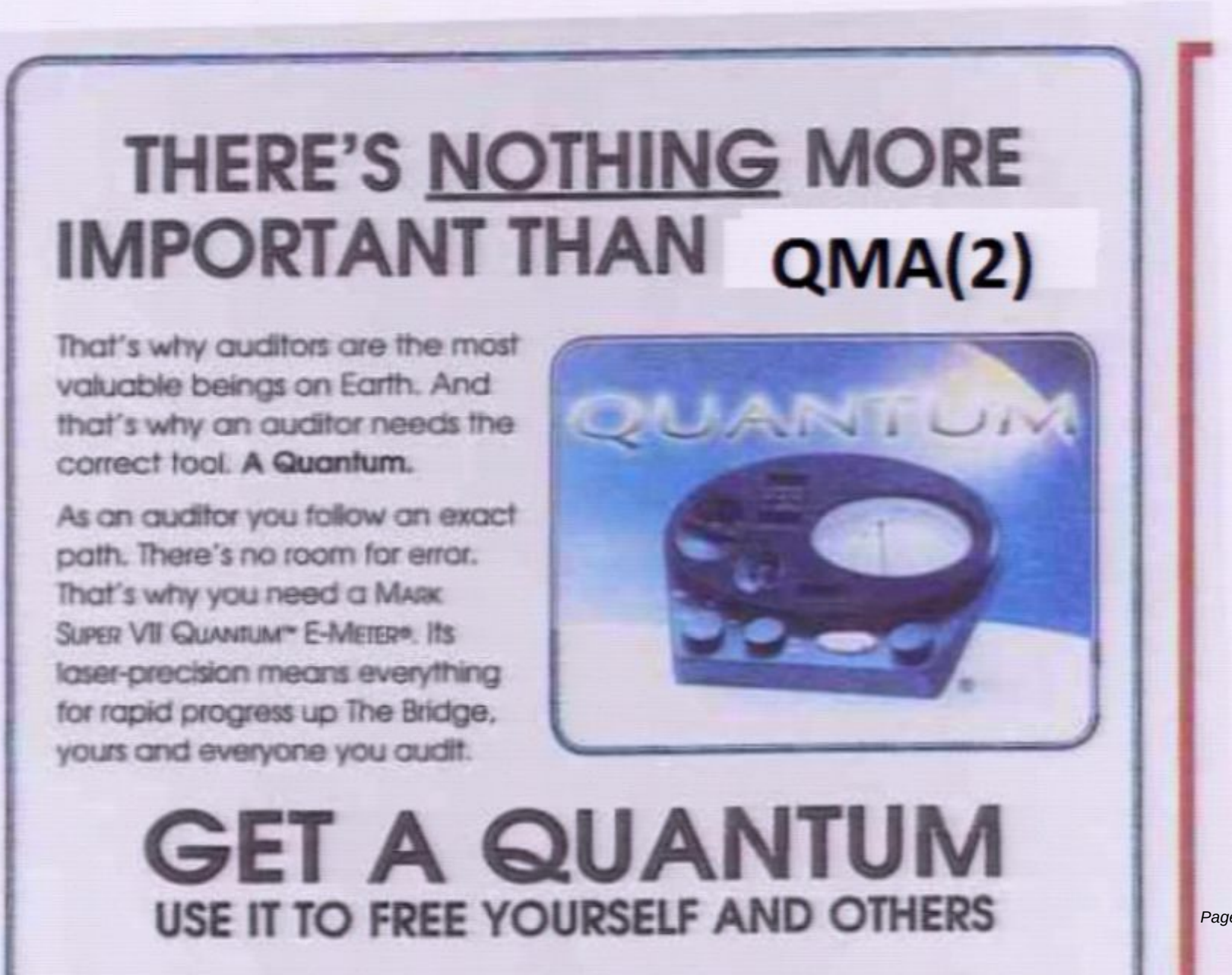
Open questions

- We know $\mathbf{NP}_{\log^2} \subseteq \mathbf{QMA}_{\log}(2)_{1-\Omega(1),1} \subseteq \mathbf{NP}$. Can we narrow these bounds?
- Similarly the $\mathbf{QMA}_{\mathrm{poly}}(2) \subset \mathbf{NEXP}$ bound seems pretty loose.
- What about parallel repetition without the product test? Can we bound

$$\bar{S}_{\infty}^{\min}(\mathcal{N}) \geq f(S_{\infty}^{\min}, d)$$

  for some f? The best known so far is $f(S, d) \sim S/d \log(d)$.
- The analysis of the product test is not very good for states that are far from product.
- Perhaps also it could be improved when ψ is symmetric across the n systems.

Closing message



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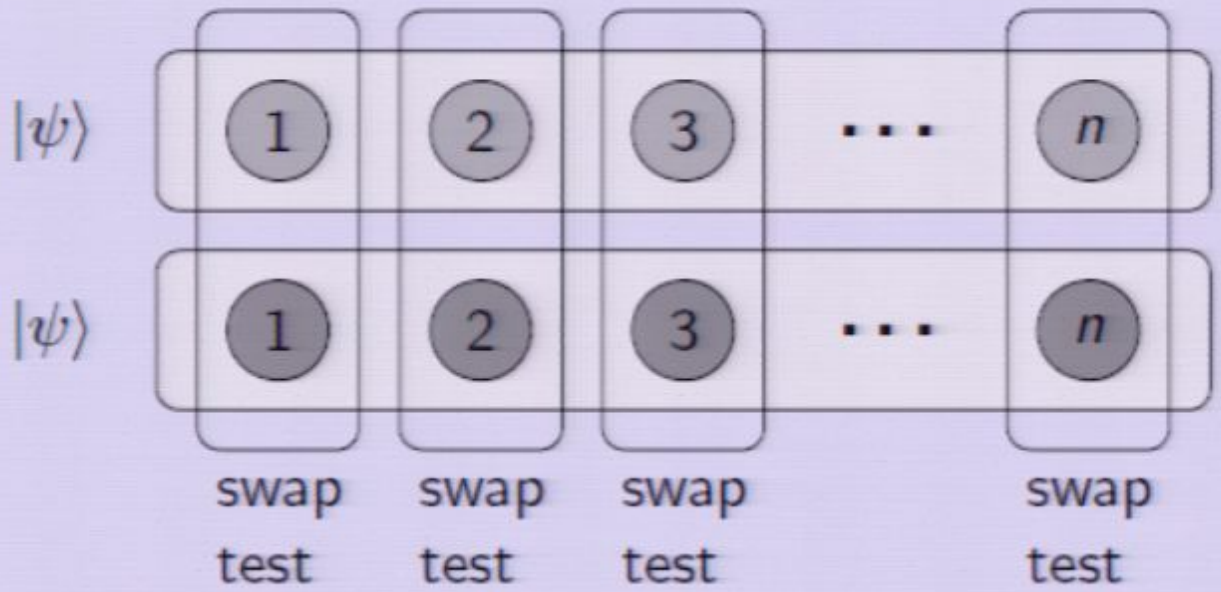

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