Title: Hastings' additivity counterexample and a sharp version of Dvoretzky theorem

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Abstract: In this talk we will explain how the main step technical steps in the proofs by Hastings and Hayden-Winter of the non-additivity of the minimal output von Neumann and p-Renyi entropy (for any $p>1$) can be reduced to a sharp version of Dvoretzky's theorem on almost spherical sections of convex bodies. This substantially simplifies their analysis, at least on the conceptual level, and provides an alternative point of view on these and related questions.

Joint work with G. Aubrun and E. Werner

Additivity conjectures for quantum channels and a sharp version of Dvoretzky's theorem

Stanislaw Szarek

Paris 6/Case Western Reserve

Perimeter Institute, July 6, 2010

Collaborators:
G. Aubrun, E. Werner

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- the setup : quantum channels as subspaces
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- the Hayden-Winter counterexample
- the Hastings's counterexample
- measure concentration: the union bound vs. chaining

QI objects (geometric functional analysis angle)

- a complex Hilbert space $\mathcal{H}$, usually $\mathcal{H} = \mathbb{C}^d$
- the C^*-algebra $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathbb{C}^d) = \mathcal{M}_d$
- the real space $\mathcal{M}_d^{sa}$ of $d \times d$ Hermitian matrices
- the Schatten p-norm on $\mathcal{M}_d$ or $\mathcal{M}_d^{sa}$ or $\mathcal{M}_{d\times k}$

$$\|\sigma\|_p = \left(\mathrm{tr}(\sigma^\dagger \sigma)^{p/2}\right)^{1/p}$$

- the positive semi-definite cone $\mathcal{PSD} \subset \mathcal{M}_d^{sa}$
- $\mathcal{D} = \mathcal{D}(\mathcal{H})$, the set of states of $\mathcal{B}(\mathcal{H})$, or of density matrices
  - ◇ $\mathcal{D}(\mathcal{H}) = \mathcal{PSD} \cap \{\mathrm{tr}(\cdot) = 1\}$, or the base of $\mathcal{PSD}$
  - ◇ $\mathcal{D}(\mathcal{H}) =$ the positive face of the unit ball in the trace class (Schatten 1-norm)
  - ◇ $\mathcal{D}(\mathcal{H}) = \mathrm{conv}\{|\psi\rangle\langle\psi| : \psi \in \mathcal{H}, |\psi| = 1\}$

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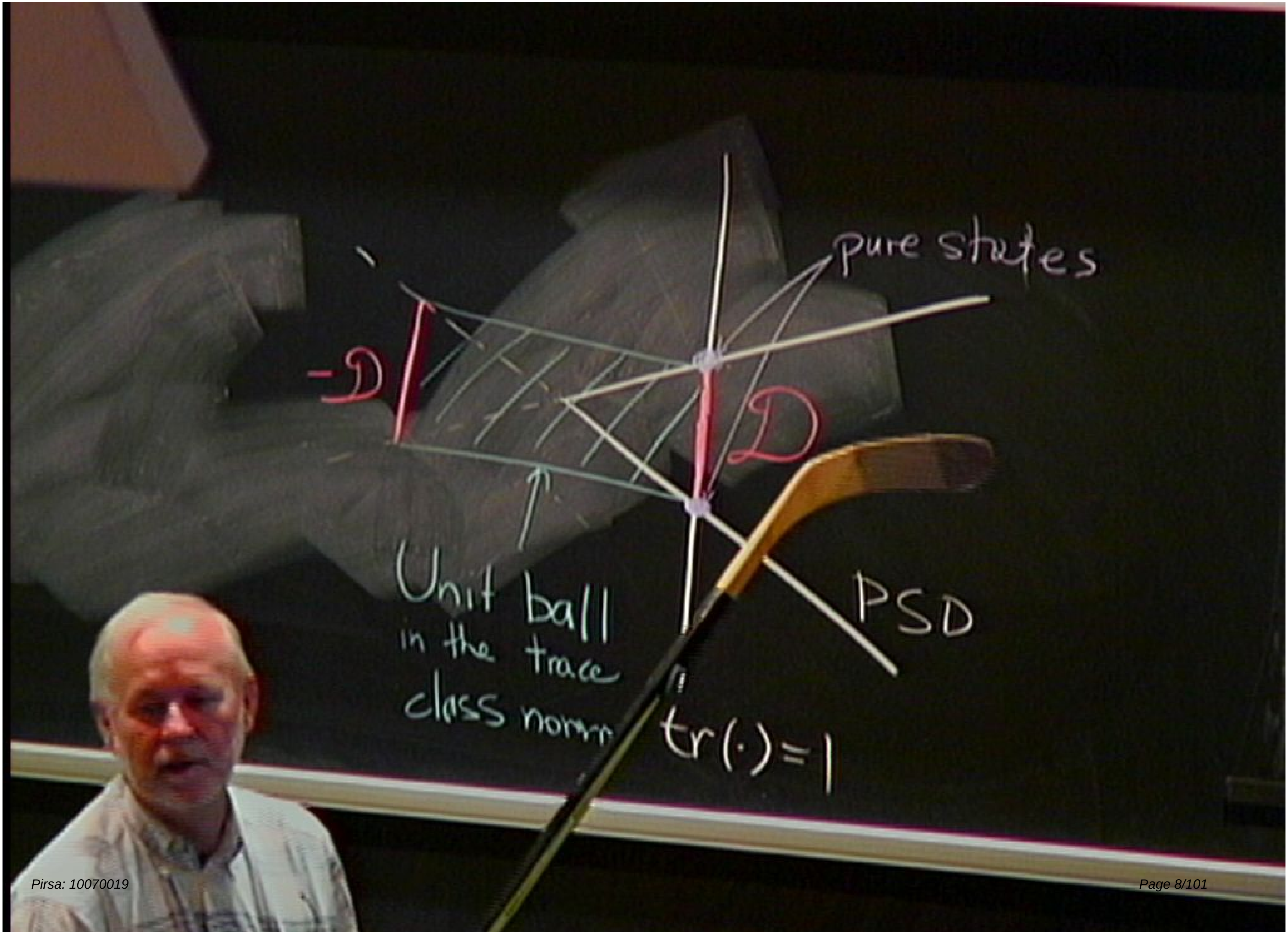
pure states
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Unit ball
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tr(·)=1

QI morphisms : quantum operations, or channels

Completely positive (CP) maps $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$, or $\Phi : \mathcal{M}_m \to \mathcal{M}_d$, usually also required to be trace preserving (TP)

Fact 0 : For $\Phi : \mathcal{M}_m \to \mathcal{M}_d$ T.F.A.E. (Stinespring-Kraus-Choi)

- Φ is CPTP
- for some $B_1, \ldots, B_k \in \mathcal{M}_{d\times m}$ with $\sum_i B_i^\dagger B_i = \mathrm{Id}_{\mathbb{C}^d}$

$$\Phi(\rho) = \sum_i B_i \rho B_i^\dagger$$

- for some isometry $V : \mathbb{C}^m \to \mathbb{C}^d \otimes \mathbb{C}^k$

$$\Phi(\rho) = \mathrm{tr}_{\mathbb{C}^k}(V\rho V^\dagger) = \mathrm{tr}_2(V\rho V^\dagger)$$

The relation between V and B_j's: $V = \sum_i B_i \otimes e_i$

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For a pure state $\varphi = V\psi \in \mathcal{W}$, its image by Φ is simply encoded in its "Schmidt decomposition" :

If $\varphi = \sum_j s_j \, u_j \otimes v_j$, then $\Phi(|\psi\rangle\langle\psi|) = \mathrm{tr}_2(|\varphi\rangle\langle\varphi|) = \sum_j s_j^2 \, |u_j\rangle\langle u_j|$

Verification:

$$\mathrm{tr}_2(|\varphi\rangle\langle\varphi|) \;=\; \mathrm{tr}_2\Big(|\sum_i s_i\, u_i \otimes v_i\rangle\langle\sum_j s_j\, u_j \otimes v_j|\Big)$$

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&= \sum_{i,j} s_i s_j\, |u_i\rangle\langle u_j|\, \mathrm{tr}(|v_i\rangle\langle v_j|) \\
&= \sum_j s_j^2\, |u_j\rangle\langle u_j|
\end{aligned}
$$

Recall: $(u_j), (v_j)$ are orthonormal sequences in $\mathbb{C}^d$ and $\mathbb{C}^k$

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The output $\quad \Phi(|\varphi\rangle\langle\varphi|) = \sum_j s_j^2 \, |u_j\rangle\langle u_j| = AA^\dagger$

The bottom line: To understand quantum channels, we need to understand the patterns of singular numbers of A as A varies over an m-dimensional subspace $\mathcal{W}$ of the space of $d \times k$ matrices

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Additivity conjecture: For CPTP maps Φ, Ψ, do we have

$$S_{\min}(\Phi \otimes \Psi) \stackrel{?}{=} S_{\min}(\Phi) + S_{\min}(\Psi)$$

"No" (Hastings 2009)

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Additivity of the minimum output entropy would follow from additivity of the minimum output p-Rényi entropy

$$S_p^{\min}(\Phi) := \min_{\rho \in \mathcal{D}(\mathbb{C}^m)} S_p(\Phi(\rho))$$

for $p > 1$, where $S_p(\sigma) := \frac{1}{1-p} \log(\mathrm{tr}\sigma^p)$

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Modulo normalizing factors and logarithmic change of variables, $S_p^{\min}(\Phi)$ is equivalent to $\max_{\rho \in \mathcal{D}(\mathbb{C}^m)} \|\Phi(\rho)\|_p$, or $\|\Phi\|_{1 \to p}$

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(again, attained on a pure state).

Additivity of $S_p^{\min}(\Phi)$ is equivalent to multiplicativity of $\|\Phi\|_{1\to p}$
"No" (Hayden-Winter 2008)

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$$\|\Phi\|_{1\to p} = \max_{A\in\mathcal{W},\ \|A\|_2=1} \|AA^\dagger\|_p = \|A\|_{2p}^2$$

In other words

$$\|\Phi\|_{1\to p}^{1/2} = \max_{A\in\mathcal{W}\setminus\{0\}} \frac{\|A\|_{2p}}{\|A\|_2}$$

Conference "Perspectives in High Dimensions"

Cleveland, August 2 until August 6, 2010

http://www.case.edu/artsci/math/perspectivesInHighDimensions/

Dvoretzky's theorem

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Given $m \in \mathbb{N}$ *and* $\varepsilon > 0$ *there is* $N = N(m, \varepsilon)$ *such that, for any norm on* $\mathbb{R}^N$ *(or* $\mathbb{C}^N$*) there is an m-dimensional subspace on which the ratio between that norm and the Euclidean norm is (approximately) constant, up to a multiplicative factor* $1 + \varepsilon$.

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Fact 1 : Consider the N-dimensional Euclidean space (over $\mathbb{R}$ or $\mathbb{C}$) endowed with the Euclidean norm $|\cdot|$ and some other norm $\|\cdot\|$ such that, for some $b > 0$, $\|\cdot\| \leq b|\cdot|$. Denote $M = \mathbb{E}\|X\|$, where X is a random variable uniformly distributed on the unit Euclidean sphere. Let $\varepsilon > 0$ and let $m \leq c\varepsilon^2(M/b)^2N$, where $c > 0$ is an appropriate (computable) universal constant. Then, for most m-dimensional subspaces E we have

$$\forall x \in E, \quad (1-\varepsilon)M|x| \leq \|x\| \leq (1+\varepsilon)M|x|.$$

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A similar statement holds for Lipschitz functions in place of norms.

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then for a generic m-dimensional subspace $\mathcal{W}$ of $\mathcal{M}_d$

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which is $\ll 1$ for large d and nearly as small as it can be: $\|\Phi\|_{1\to p} \geq d^{1/p-1}$ always.

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The counterexample follows by showing that for the composite channel $\Phi \otimes \overline{\Phi}$ we have a nontrivial lower bound $\frac{m}{dk} = \frac{m}{d^2} \sim d^{1/p-1}$.

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Obviously $\mathbb{E}\|X\|_2 = 1$

For $q \in (2, \infty)$ we interpolate (Hölder inequality)

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On the other hand, the "large subspace/large eigenvalue" argument gives for the composite channel

$$S^{\min}(\Phi\otimes\bar{\Phi}) \leq \log(d^2) - \Omega\left(\frac{\log d}{d}\right)$$

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For $A \in \mathcal{W}, \|A\|_2 = 1$, we have

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For $A \in \mathcal{W}$, $\|A\|_2 = 1$, we have

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Consequently,

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Note : Applying *directly* Dvoretzky's theorem for the parameters in question ($k \sim d^2, m \sim d^2$) gives only $1 + O\left(\frac{1}{\sqrt[4]{d}}\right)$

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The trick : $g(A) := \left\| AA^\dagger - \frac{\mathrm{Id}}{d} \right\|_2$ is only 2-Lipschitz on the Frobenius sphere S_F, but much more regular on a large subset

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$$\Omega = \{A \in S_F \ : \ \|A\|_\infty \leq 3/\sqrt{d}\}$$

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(3) follows from Dvoretzky's theorem with $\|\cdot\| = \|\cdot\|_\infty$ and $\varepsilon = 1/2$ once we establish that the average M of $\|\cdot\|_\infty$ verifies $M \sim 1/\sqrt{d} + 1/\sqrt{k} \leq 2/\sqrt{d}$

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Levy's lemma

Fact 5 : If $f : S^{N-1} \to \mathbb{R}$ is a 1-Lipschitz function, then for every $\varepsilon > 0$,

$$\mathbb{P}(|f(x) - \mu| > \varepsilon) \leq C_1 \exp(-c_1 N \varepsilon^2),$$

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Here : $\mu = M \leq 2/\sqrt{d}$, $\varepsilon = 1/\sqrt{d}$, $N = 2kd$

$\Rightarrow \mu + \varepsilon \leq 3/\sqrt{d}$ and $c_1 N \varepsilon^2 = 4 c_1 k$

The final step

Restrict $g(A) = \|AA^\dagger - \frac{\mathrm{Id}}{d}\|_2$ to Ω, and then extend the restriction $g_{|\Omega}$ to $\tilde{g} : S_F \to \mathbb{R}$ without increasing the Lipschitz constant, which is $\leq 6/\sqrt{d}$

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- the median of g on S_F is $O(\frac{1}{d})$
- allowing $O(\frac{1}{d})$ deviation from a central value of $\tilde{g}$ leads to a correct value of m, i.e., $m \sim k \sim d^2$

The "non-linear" Dvoretzky's theorem

Fact 1' : Let $f : S_{\mathbb{C}^N} \to \mathbb{R}$ is a 1-Lipschitz circled function and let $\varepsilon > 0$. Let $E \subset \mathbb{C}^n$ be a random subspace (Haar-distributed) of dimension $m = c_0 N \varepsilon^2$. Then, with large probability,

$$|f - \mu| \leq \varepsilon \;\; \text{on} \;\; S_{\mathbb{C}^N} \cap E,$$

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where μ is any central value of f (with respect to the normalized Lebesgue measure on $S_{\mathbb{C}^N}$) and c_0 is an absolute constant. If the function is L-Lipschitz, the dimension changes to $m = c_0 N (\varepsilon / L)^2$.

Final verifications

- $L = 6/\sqrt{d}$, $\varepsilon = O(1/d)$ $\Rightarrow m \sim N\left(\frac{\varepsilon}{L}\right)^2 \sim kd\left(\frac{1/d}{6/\sqrt{d}}\right)^2 \sim k$

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- the median of $\tilde{g} \stackrel{?}{=} O(1/d)$

Marchenko-Pastur: with large probability, all singular values of A are in the interval $\left[\frac{1}{\sqrt{d}} - \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{d}} + \frac{1}{\sqrt{k}}\right]$
$\Rightarrow$ all eigenvalues of $AA^\dagger$ are within $O\left(\frac{1}{\sqrt{kd}}\right)$ of $\frac{1}{d}$

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Remedy : a chaining argument, or a more economical usage of nets

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- given $s \in S$ and $k = 0, 1, 2, \ldots,$, let $z_k(s) \in \mathcal{N}_k$ verify
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Under appropriate assumptions on the continuity of the process $(X_s)_{s \in S}$, this argument leads to surprisingly sharp bounds

Dudley's inequality

Fact 6 : S compact metric space, $(X_s)_{s\in S}$ a subgaussian process, i.e., there are $\alpha, \beta > 0$ such that, for all $t, t' \in S$ and for all $\lambda \geq 0$,

$$\mathbb{P}(|X_t - X_{t'}| \geq \lambda) \leq \beta \exp\left(-\alpha \frac{\lambda^2}{\mathrm{dist}(t,t')^2}\right)$$

Then

$$\mathbb{E} \sup_{t,t'\in S} |X_t - X_{t'}| \leq C\beta\alpha^{-1/2} \int_0^R \sqrt{\log N(S,\eta)}\, d\eta,$$

where $N(S,\eta)$ is the minimal cardinality of a η-net of S and R is the radius of S.

In our context : $f : S_{\mathbb{C}^N} \to \mathbb{R}$, 1-Lipschitz

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- randomness : $U \in U(N)$, the Haar measure
- $X_s := f(Us)$
- subgaussian property, with $\alpha = cN$ and $\beta = O(1)$, is given by Levy's lemma + some standard tricks
- elementary bound $N(S, \eta) \leq \left(\frac{3}{\eta}\right)^m$
- singularity at 0 integrates out; in other words, no $\log(2/\varepsilon)$ effect

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