Title: Random Quantum Repeated Interactions and Random Invariant states

Date: Jul 05, 2010 04:15 PM

URL: http://pirsa.org/10070015

Abstract: Within the framework of quantum repeated interactions we investigate the large time behaviour of random quantum channel. We focus on generic quantum channels generated by unitary operators which are randomly distributed along the Haar measure. After studying the spectrum of these channels, we state a convergence result for the iterations of generic channels. This allows to define a set of random quantum states called "asymptotic induced ensemble".

Random Quantum Repeated Interactions and Random Invariant State

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Waterloo: July 2010
Random Matrix Techniques in Quantum Information Theory

Main Points

- Repeated Quantum Interactions and Quantum Channel.
- Random Invariant State.
- Random Environment and i.i.d Interactions

Introduction

- **What is the Quantum Repeated Interactions model?**
  1. **Framework:** Open System Quantum Dynamics
  2. **Used as a useful approximation of Quantum Langevin Equation**
     - S.Attal, Y.Pautrat: *From repeated to continuous quantum interactions.* Ann. Henri Poincaré, 7(1):59–104, 2006.
  3. **Used for developping a theory of discrete quantum repeated measurement and approximation of Stochastic Master Equations**
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Basic model

- **A small system $\mathcal{H}_0$ in contact with an infinite chain $\bigotimes_{\mathbb{N}} \mathcal{H}_k$**

$$\mathcal{H}_0 \leftrightsquigarrow \mathcal{H}_1 + \mathcal{H}_2 + \ldots + \mathcal{H}_k + \ldots$$

- Each $\mathcal{H}_k = \mathcal{H}$ interacts with $\mathcal{H}_0$, one after the others, during a time τ
- The first copy $\mathcal{H}_1$ interacts with $\mathcal{H}_0$ during a time τ

$$(\mathcal{H}_0 \leftrightsquigarrow \mathcal{H}_1) + (\mathcal{H}_2 + \ldots + \mathcal{H}_k + \ldots)$$

then disappears

- $\mathcal{H}_2$ comes to interact

$$(\mathcal{H}_0 \leftrightsquigarrow \mathcal{H}_2) + (\mathcal{H}_3 + \ldots + \mathcal{H}_k + \ldots)$$

and so on ....

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Basic model

- **Single interaction:** $(\mathcal{H}_0, \rho) + (\mathcal{H}, \beta)$ described by a **total Hamiltonian** on $\mathcal{H}_0 \otimes \mathcal{H}$:
  $H_{tot} = H_0 \otimes I + I \otimes H + H_{int}, \quad U = e^{-i\tau H_{tot}}$.

- **Schrödinger picture:** $\mu = U(\rho \otimes \beta)U^\star$
- **Partial trace over** $\mathcal{H}$, we get a new state $\rho_1 = \Phi^{U,\beta}(\rho) = Tr_{\mathcal{H}}[\mu]$.

- Interaction with the second copy of $\mathcal{H}$:
  $(\mathcal{H}_0, \rho_1) + (\mathcal{H}, \beta) \Rightarrow \rho_2 = \Phi(\rho_1)$ and so on...

$$\rho_k = \left(\Phi^{U,\beta}\right)^{\circ(k)}(\rho).$$

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Link with Quantum Channel

- A linear map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is a quantum channel if and only if one of two properties holds
  1. Stinespring dilation There exists a finite dimensional Hilbert space $\mathbb{C}^{d'}$, a density matrix β on $\mathbb{C}^{d'}$ and an unitary operation $U \in \mathcal{U}(dd')$ such that
  $$\Phi(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right], \quad \forall X \in M_d(\mathbb{C}).$$
  2. Kraus decomposition There exists an integer k and matrices $L_1, \dots, L_k \in M_d(\mathbb{C})$ such that
  $$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*, \quad \forall X \in M_d(\mathbb{C})$$
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  $$\sum_{i=1}^{k} L_i^* L_i = I_d.$$

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Main points

- Initially, S.Attal and Y.Pautrat introduce a **time-renormalization of the total hamiltonian** and show that when τ goes to zero the sequence of state ρ_k converges to the solution of usual **Lindblad equation** which describes the evolution of open system in contact with large reservoir.

- Here we are interested in the **large time behaviour** of ρ_k when k goes to infinity (U being fixed without renormalization). In other terms what is the **asymptotic behaviour of quantum channels**.

  1. U initially random and β fixed: $\lim_k (\phi^{U,\beta})^{\circ(k)}(.)$.

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Spectral properties

- Study of "iterations" of random quantum channel

$$\Phi^{U,\beta}(\rho) = Tr_{d'}[U(\rho \otimes \beta)U^{\star}],$$

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- In general, in order to attack such problem we need to study the **properties of the spectrum** of quantum channels.

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Spectral properties of channels

Proposition

Let $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C}$ a quantum channel. Then

1. *Φ has at least one invariant element which is a density matrix;*
2. *Φ has trace operator norm 1;*
3. *Φ has spectral radius 1;*
4. *Φ satisfies the Schwarz inequality*

$$\forall X \in M_d(\mathbb{C}), \quad \Phi(X)^*\Phi(X) \leqslant \|\Phi(I)\|\, \Phi(X^*X).$$

Asymptotic states

Let $\mathcal{C}$ be the set of all quantum channels that have **1 as a simple eigenvalue and all other eigenvalues are contained in the open unit disc.**

Proposition

Consider a quantum channel $\Phi \in \mathcal{C}$. Then, for all density matrices ρ_0,

$$\lim_{n \to \infty} \Phi^n(\rho_0) = \rho_\infty,$$

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Examples

- For $U \in \mathcal{U}(d)$, define the **unitary conjugation channel**

$$\Phi_U(X) = UXU^*.$$

One can check that the spectrum of Φ_U is

$$spec(\Phi_U) = \{\lambda_i \overline{\lambda_j} \mid \lambda_i, \lambda_j \in spec(U)\}.$$

For $U = I$, one gets the identity channel $\Phi_I(X) = X$.

- The **depolarizing channel** $\Phi_{dep} : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is given by

$$\Phi_{dep}(X) = \mathrm{Tr}(X)\frac{I}{d}.$$

It has eigenvalues 1 (with multiplicity 1) and 0 (with multiplicity $d^2 - 1$).

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Generic quantum channels

- Fix two integers $d, d' \geqslant 2$ and a density matrix β. To an unitary matrix $U \in \mathcal{U}(dd')$, associate the channel

$$\Phi^{U,\beta}(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right].$$

- Choosing U random from the **Haar distribution** on the unitary group, we obtain a quantum channel-valued random variable (β is fixed)

$$\begin{aligned}\mathcal{U}(dd') &\to \mathcal{L}(M_d(\mathbb{C}))\\ U &\mapsto \Phi^{U,\beta}.\end{aligned}$$

- Question:
  What are the properties of a generic quantum channel ?

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Almost all quantum channels are in $\mathcal{C}$

Theorem

Let β be a fixed density matrix of size d'. If U is a random unitary matrix distributed along the Haar invariant probability $\mathrm{Haar}_{dd'}$ on $\mathcal{U}(dd')$, then $\Phi^{U,\beta} \in \mathcal{C}$ almost surely.

Corollary

For almost all unitary matrices $U \in \mathcal{U}(dd')$, the channel $\Phi^{U,\beta}$ has an unique invariant state ρ_∞ and for all density matrices ρ_0,

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A key property: irreducibility

Définition

A positive map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is called

- **strictly positive** *(or positivity improving) if $\Phi(X) > 0$ for all $X \geqslant 0$;*
- **irreducible** *if there is no (non-trivial) projector P such that $\Phi(P) \leqslant \lambda P$ for some $\lambda > 0$.*

Proposition

A positive linear map $\Phi : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ is irreducible if and only if the map $(I + \Phi)^{d-1}$ is strictly positive.

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Strictly positive and irreducible channels

Theorem

If Ψ is a **unital, irreducible map** *on $M_d(\mathbb{C})$ which satisfies the* **Schwarz inequality** *(eg. the dual of an irreducible quantum channel Φ), then the set of peripheral (i.e. modulus one) eigenvalues is a (possibly trivial) subgroup of the unit circle $\mathbb{T}$.*

Moreover, every peripheral eigenvalue is simple and the corresponding eigenspaces are spanned by unitary elements of $M_d(\mathbb{C})$.

Corollary

The peripheral eigenvalues of an irreducible quantum channel are simple and contained in the finite set

$$\{\xi \in \mathbb{T} \mid \exists 1 \leqslant n \leqslant d^2 \ s.t. \ \xi^n = 1\}.$$

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Necessary and sufficient conditions for irreducibility

We denote by $Lat(T)$ the **lattice of invariant subspaces** of an operator $T \in M_d(\mathbb{C})$.

Proposition

Consider a completely positive map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ *defined by*

$$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*,$$

with $L_i \in M_d(\mathbb{C})$, $i = 1, \ldots, k$.
Then Φ *is irreducible if and only if*

$$\bigcap_{i=1}^{k} Lat(L_i)$$

is **trivial**.

Necessary and sufficient conditions for irreducibility

Proposition (the Shemesh criterion)

Two matrices $A, B \in M_d(\mathbb{C})$ *have a* **common eigenvector** *if and only if*

$$\bigcap_{i,j=1}^{d-1} ker\left[A^i, B^j\right] \neq \{0\}.$$

More generally, if A and B have a **common invariant subspace** *of dimension k (for* $1 \leqslant k \leqslant d-1$*), then their k-th wedge powers have a common eigenvector, and hence (we put* $n = \binom{d}{k}$*)*

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Almost all quantum channels are irreducible

- Write the matrix U defining a quantum channel Φ as a $d' \times d'$ matrix of blocks in $M_d(\mathbb{C})$: $U \in M_{d'}(M_d(\mathbb{C}))$. Then, the Kraus matrices L_i are (rescaled copies) of the blocks $U^{s,t} \in M_d(\mathbb{C})$.
- The Shemesh condition on the existence of an common invariant subspace can be written as

$$\det \sum_{i,j=1}^{n-1} [(A^{\wedge k})^i, (B^{\wedge k})^j]^* \cdot [(A^{\wedge k})^i, (B^{\wedge k})^j] = 0.$$

- This is a **polynomial equation** in the real and imaginary parts of the $(dd')^2$ complex coefficients of the matrix U.

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Proposition

Let $P \in \mathbb{R}[X_1, \ldots, X_{2d^2}]$, the set

$$Z = \{U = (u_{ij}) \in \mathcal{U}(d) / P(Re(u_{ij}), Im(u_{ij})) = 0\}$$

is either the whole $\mathcal{U}(d)$ or it has Haar mesure 0.

- For almost all unitary matrices $U \in \mathcal{U}(dd')$, the channel $\Phi^{U,\beta}$ is irreducible,
- For almost all unitary matrices $U \in \mathcal{U}(dd')$, the channel $\Phi^{U,\beta}$ has an unique invariant state ρ_∞ (which depends on U) and for all density matrices ρ_0,

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Asymptotic induced measure

- As a result we have defined almost everywhere an application

$$\begin{array}{rcl} \mathcal{U}(dd') & \to & \mathcal{M}_d^{1,+}(\mathbb{C}) \\ U & \mapsto & \rho_\infty \end{array}$$

- A) Let $b = (b_1, \ldots, b_{d'})$: $b_1 \geqslant \ldots \geqslant b_{d'}$, $\sum_i b_i = 1$ the eigenvalues of the state β of the environment, the image measure of the Haar probability through this application depends only on b and we denote ν_b this measure.
- The measure ν_b is called the

Asymptotic Induced Measure.

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- As a result we have defined almost everywhere an application

$$\begin{array}{ccc} \mathcal{U}(dd') & \to & \mathcal{M}_d^{1,+}(\mathbb{C}) \\ U & \mapsto & \rho_\infty \end{array}$$

- A) Let $b = (b_1, \ldots, b_{d'})$: $b_1 \geqslant \ldots \geqslant b_{d'}$, $\sum_i b_i = 1$ the eigenvalues of the state β of the environment, the image measure of the Haar probability through this application depends only on b and we denote ν_b this measure.
- The measure ν_b is called the

Asymptotic Induced Measure.

Asymptotic induced measure

- B) For all unitary matrices $V \in \mathcal{U}(d)$, ρ_∞ and $V\rho_\infty V^\star$ have the same distribution

- C) There exists a probability measure n_b on the probability simplex Δ_{d-1} such that if D is a diagonal matrix sample from n_b and V is an independent Haar unitary on $\mathcal{U}(d)$, then $VDV^\star$ has distribution ν_b.

Asymptotic induced measure

- If ψ is a random uniform element on the unit sphere of a product $\mathcal{H} \otimes \mathcal{K}$, the distribution of

$$\rho_1 = \mathrm{Tr}_{\mathcal{K}}[|\psi\rangle\langle\psi|] \tag{1}$$

is called **the induced measure.**

- Let ψ_0 and U Haar distributed the state

$$\rho_1 = \mathrm{Tr}_{\mathcal{K}}[U|\psi_0\rangle\langle\psi_0|U^\star] \tag{2}$$

has the same distribution

- Choosing $\psi_0 = e_0 \otimes f_0$, such that e_0 and f_0 are first vectors of orthonormal basis of $\mathcal{H}$ and $\mathcal{K}$ we see that the induced measure is transported by the iterations of the quantum channel $\Phi^{U,|f_0\rangle\langle f_0|}(|e_0\rangle\langle e_0|) = \mathrm{Tr}_{\mathcal{K}}[U|\psi_0\rangle\langle\psi_0|U^\star]$

Numerical simulations

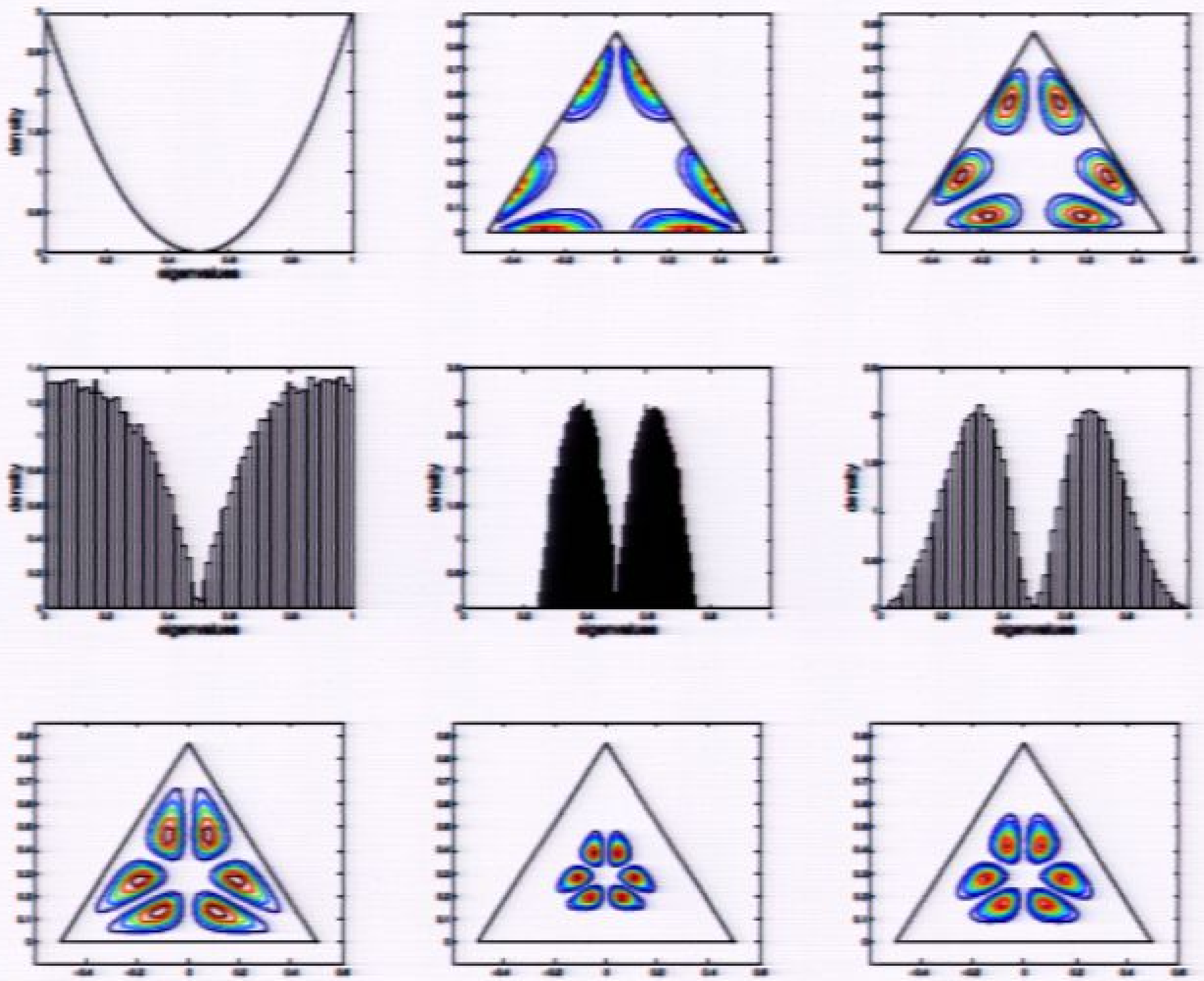

Figure: First row - induced measure d'=2, d'=3, d'=5; Second & third rows - asymptotic measure b=[1, 0], b=[3/4, 1/4],b=[1, 0, 0, 0];b=[1, 0, 0],b=[3/4, 1/8, 1/8] and

Open questions

- For the induced measure, we know asymptotic result when the dimension of the respective system goes to infinity:

$$\frac{dim\,\mathcal{H}}{dim\,\mathcal{K}} \to c,$$

  we recover the **Marcenko-Pastur** distribution.
- Can we obtain a similar result for the asymptotic induced measure?
- Power of disentanglement of repeated interactions.

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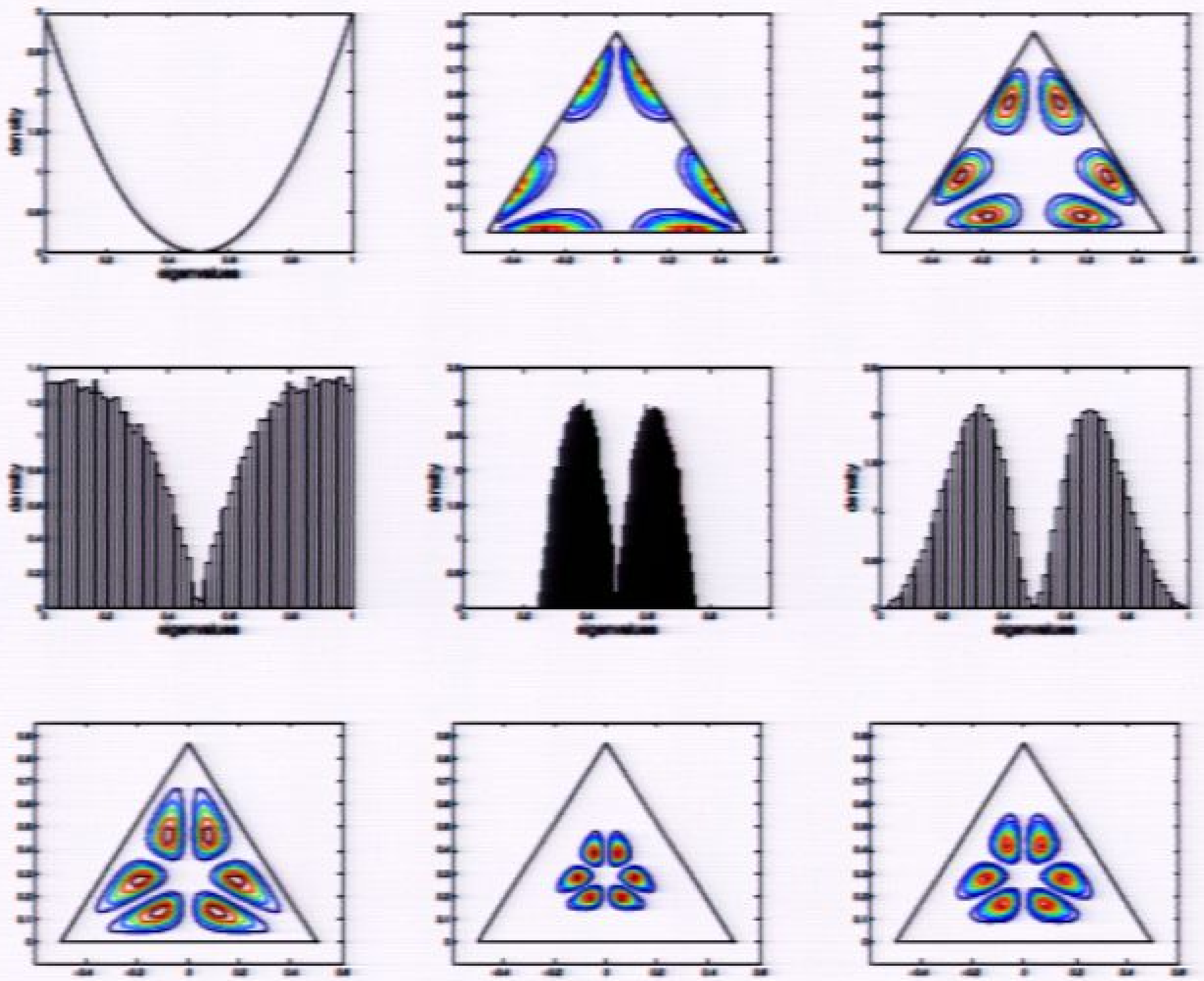

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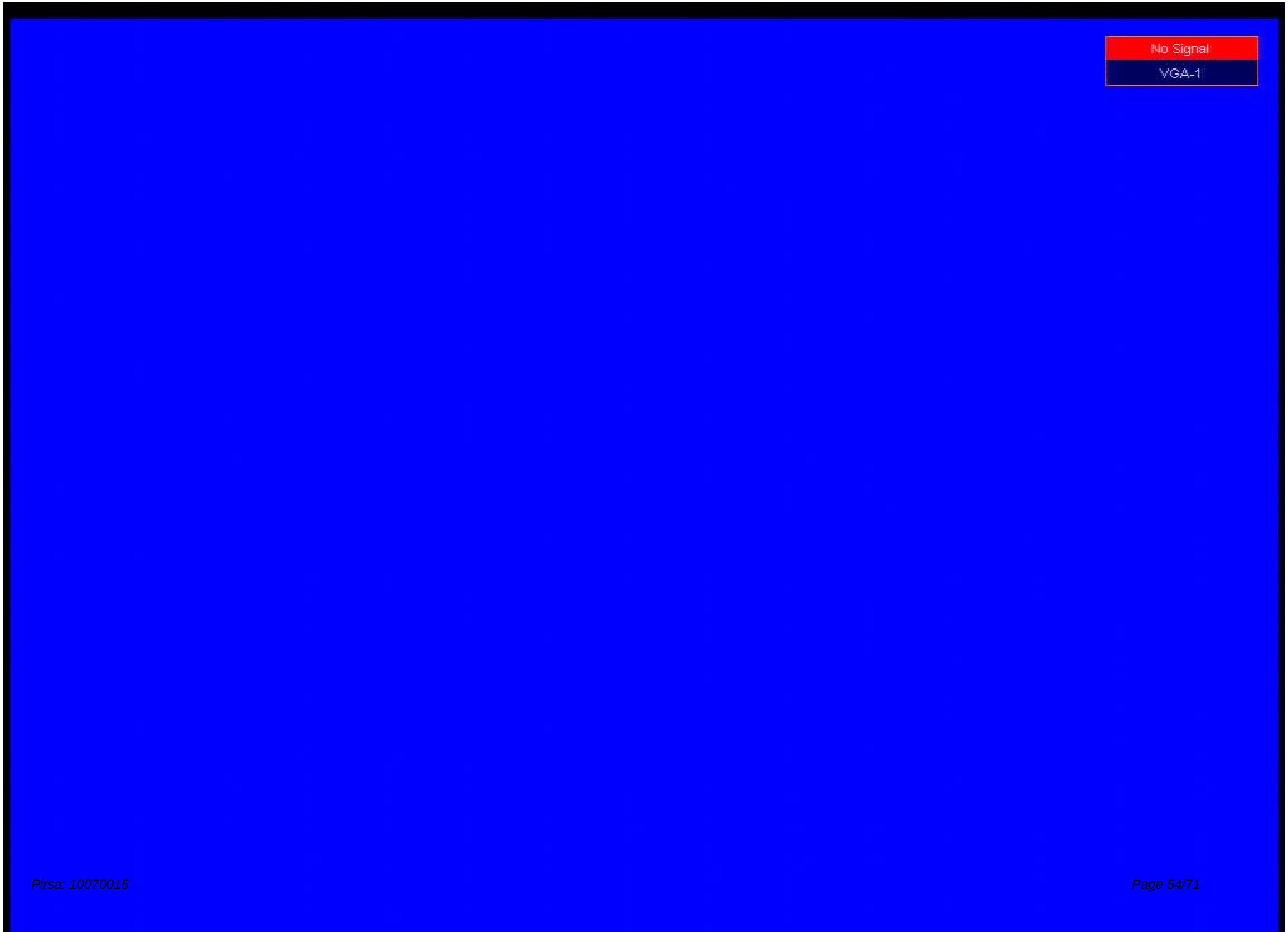
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Asymptotic results: random environment

- Discrete evolution equation

$$\rho_n = \Phi^{\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U(\rho_{n-1} \otimes \beta_n)U^*\right].$$

  In this model, the interaction unitary U is fixed beforehand and the environment states $(\beta_n)_n$ are i.i.d. random density matrices.

- As usual, we are interested in the asymptotic behavior of the states

$$\rho_n = \Phi^{\beta_n} \circ \cdots \circ \Phi^{\beta_1}(\rho_0).$$

- We use results by L. Bruneau, A. Joye and M. Merkli on products of random matrices, applied to the (i.i.d.) channels

$$\Phi^{\beta_n} \in \mathcal{L}(M_d(\mathbb{C})).$$

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Asymptotic results: random environment

Theorem (BJM)

Let $(M_n)_n$ be a sequence of i.i.d. random contractions of $M_D(\mathbb{C})$ with the following properties:

1. *There exists a constant vector $\psi \in \mathbb{C}^D$ such that $M\psi = \psi$ almost surely;*
2. $\mathbf{P}(1 \text{ is a simple eigenvalue of } M) > 0$.

Then the (deterministic) matrix $\mathbb{E}[M]$ has eigenvalue 1 with multiplicity one and there exists a constant vector $\theta \in \mathbb{C}^D$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} M_1(\omega) M_2(\omega) \cdots M_n(\omega) = |\psi\rangle\langle\theta| = P_{1,\mathbb{E}[M]},$$

where $P_{1,\mathbb{E}[M]}$ is the rank-one spectral projector of $\mathbb{E}[M]$ corresponding to the eigenvalue 1.

Asymptotic results: random environment

Using the duality between the Schrödinger and the Heisenberg pictures of Quantum Mechanics, we obtain

Theorem

Let $(\Phi_n)_n$ be a sequence of i.i.d. random quantum channels acting on $M_d(\mathbb{C})$ such that

$$\mathbf{P}(\Phi \text{ has an unique invariant state}) > 0.$$

Then $\mathbb{E}[\Phi]$ is a quantum channel with an unique invariant state θ and, $\mathbf{P}$-almost surely,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}[\Phi_n \circ \cdots \circ \Phi_1](\rho_0) = \theta, \quad \forall \rho_0.$$

Asymptotic results: random environment

Proposition

Let $\{\beta_n\}_n$ be a sequence of i.i.d. random density matrices such that, with positive probability, the random quantum channel Φ^β has an unique invariant state. Then, almost surely, for all initial states ρ_0, one has

$$\lim_{N\to\infty} \frac{\rho_1 + \ldots + \rho_N}{N} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}[\Phi^{\beta_n} \circ \cdots \circ \Phi^{\beta_1}](\rho_0) = \theta,$$

where θ is the unique invariant state of the deterministic channel $\Phi^{\mathbb{E}[\beta]}$. In particular, if $\mathbb{E}[\beta] = I_{d'}/d'$, then θ is the "chaotic" state I_d/d.

Asymptotic results: i.i.d. unitaries

- Discrete evolution equation

$$\rho_n = \Phi^{U_n,\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U_n(\rho_{n-1} \otimes \beta_n)U_n^*\right].$$

- In this model, the interaction unitaries U_n are Haar distributed independent random matrices.
  The environment states $(\beta_n)_n$ are independent of the family $(U_n)_n$ and can have an arbitrary joint distribution.

Lemma

Let $(V_n)_n$ be a sequence of i.i.d. Haar unitaries independent of the family $\{U_n, \beta_n\}_n$ and consider the sequence of successive states $(\rho_n)_n$ defined earlier. Then the sequences $(\rho_n)_n$ and $(V_n\rho_n V_n^*)_n$ have the same distribution.

Asymptotic results: i.i.d. unitaries

- Discrete evolution equation

$$\rho_n = \Phi^{U_n,\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U_n(\rho_{n-1} \otimes \beta_n)U_n^*\right].$$

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Asymptotic results: i.i.d. unitaries

Proposition

Let $(\rho_n)_n$ be the successive states of a repeated quantum interaction scheme with i.i.d. random unitary interactions. Then, almost surely,

$$\lim_{n\to\infty} \frac{\rho_1 + \ldots + \rho_n}{n} = \frac{I_d}{d}.$$

Thanks

Open questions

- For the induced measure, we know asymptotic result when the dimension of the respective system goes to infinity:

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 we recover the **Marcenko-Pastur** distribution.
- Can we obtain a similar result for the asymptotic induced measure?
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Necessary and sufficient conditions for irreducibility

Proposition (the Shemesh criterion)

Two matrices $A, B \in M_d(\mathbb{C})$ have a **common eigenvector** *if and only if*

Consider a completely positive map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ defined by

$$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*,$$

with $L_i \in M_d(\mathbb{C})$, $i = 1, \ldots, k$.
Then Φ is irreducible if and only if

$$\bigcap_{i=1}^{k} Lat(L_i)$$

is **trivial**.

Generic quantum channels

- Fix two integers $d, d' \geqslant 2$ and a density matrix β. To an unitary matrix $U \in \mathcal{U}(dd')$, associate the channel

$$\Phi^{U,\beta}(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right].$$

- Choosing U random from the **Haar distribution** on the unitary group, we obtain a quantum channel-valued random variable (β is fixed)

$$\mathcal{U}(dd') \to \mathcal{L}(M_d(\mathbb{C}))$$
$$U \mapsto \Phi^{U,\beta}.$$

- Question:
  What are the properties of a generic quantum channel ?

Main points

- Initially, S.Attal and Y.Pautrat introduce a **time-renormalization of the total hamiltonian** and show that when τ goes to zero the sequence of state ρ_k converges to the solution of usual **Lindblad equation** which describes the evolution of open system in contact with large reservoir.

- Here we are interested in the **large time behaviour** of ρ_k when k goes to infinity (U being fixed without renormalization). In other terms what is the **asymptotic behaviour of quantum channels**.

  1. U initially random and β fixed: $\lim_k (\phi^{U,\beta})^{\circ(k)}(.)$.
  2. U and β random at each step $\phi^{U_k,\beta_k} \circ \ldots \circ \phi^{U_1,\beta_1}$.

Link with Quantum Channel

- A linear map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is a quantum channel if and only if one of two properties holds
  1. Stinespring dilation There exists a finite dimensional Hilbert space $\mathbb{C}^{d'}$, a density matrix β on $\mathbb{C}^{d'}$ and an unitary operation $U \in \mathcal{U}(dd')$ such that
  $$\Phi(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right], \quad \forall X \in M_d(\mathbb{C}).$$
  2. Kraus decomposition There exists an integer k and matrices $L_1, \ldots, L_k \in M_d(\mathbb{C})$ such that
  $$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*, \quad \forall X \in M_d(\mathbb{C})$$
  and
  $$\sum_{i=1}^{k} L_i^* L_i = I_d.$$

Introduction

- **What is the Quantum Repeated Interactions model?**
  1. **Framework:** Open System Quantum Dynamics
  2. **Used as a useful approximation of Quantum Langevin Equation**

     • S.Attal, Y.Pautrat: *From repeated to continuous quantum interactions.* Ann. Henri Poincaré, 7(1):59–104, 2006.
  3. **Used for developping a theory of discrete quantum repeated measurement and approximation of Stochastic Master Equations**

     • C.Pellegrini: *Markov Chains Approximations of jump-Diffusion Stochastic Master Equations.* Ann Instit Henri Poincaré: Probability and Statistic (in press).

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