Title: Ensembles of random quantum states

Date: Jul 05, 2010 01:30 PM

URL: http://pirsa.org/10070012

Abstract: TBA

Ensembles of Random Quantum States

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Perimeter Institute Workshop, **Waterloo,** July 5, 2010

How to generate an ensemble of random density operators?

Reduction of random pure states

1) Consider an ensemble of **random pure states** $|\psi\rangle$ of a **composite system** distributed according to a given measure μ.

2) Perform partial trace over a chosen subsystem B to get a **random mixed state**

$$\rho := \mathrm{Tr}_B|\psi\rangle\langle\psi|$$

Depending on the **structure** of the composite system, the initial **measure** μ in the space of the pure states and the choice of the **subsystem** B, over which the averaging is performed
one obtaines different **ensembles of random mixed states**.

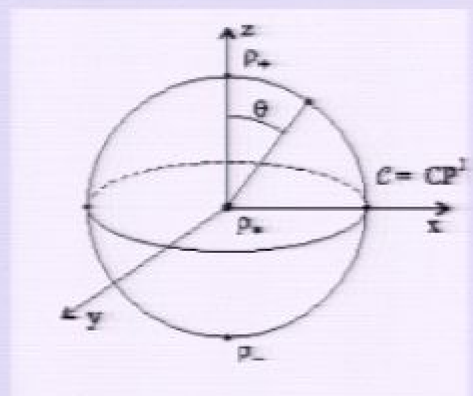
Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Space of normalized complex pure states for an arbitrary N:

Since $\langle\psi|\psi\rangle = 1$ a **normalized** state belongs to the **sphere** S^{2N-1}.

Two states equal up to a phase are identified, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$, so the set of states is equivalent to the **complex projective space** $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

$N=2$: For **qubit** = quantum bit the word **geometry** can be treated literally!



$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

$\mathbb{C}P^1$ = **Bloch sphere of $N=2$ pure states**

Random Pure states in $\mathcal{H}_N$

'Quantum chaotic' dynamics **(pseudo-random evolution)**

described by a **random unitary** matrix U acting on a pure state produces (almost surely) a **'generic pure state'** $|\psi\rangle = U|\phi_0\rangle$.

• Formally one defines an (unique) **Fubini–Study measure** μ on complex projective spaces which is **unitarily invariant**: for any (measurable) set A of states one requires $\mu(A) = \mu(U(A))$.

• This measure covers the entire space $\mathbb{C}P^{N-1}$ **uniformly**, and for $N = 2$ it is just equivalent to the uniform, **Lebesgue measure on the sphere** S^2.

How to obtain numerically a **random pure state** $|\psi\rangle$?

a) Take a column (a row) of a **random unitary** U so that $|\psi\rangle = U|i\rangle$.
b) generate N **independent complex random numbers** z_i according to the **normal** distribution. Write $|\psi\rangle = \sum_{i=1}^{N} c_i|i\rangle$ where the expansion coefficients read $c_i = z_i/\sqrt{\sum_i |z_i|^2}$.

Properties of 'typical' pure states in $\mathcal{H}_N$

Expansion coefficients: $|\psi\rangle = \sum_{i=1}^{N} c_i |i\rangle$

Expand a 'typical' state $|\psi\rangle$ in an (arbitrary) basis $|i\rangle$.
What is the distribution of the components $y_i = |c_i|^2$?

• To characterize the distribution $P(y)$ define the **entropy**
$S(\psi) = -\sum_{i=1}^{N} y_i \ln y_i$

• Compute the **mean entropy** averaged over the set of pure quantum states of size N

$$\langle S\rangle_\psi = \Psi(N+1) - \Psi(2) = \sum_{k=2}^{N} 1/k \ \sim\ \ln N - (1-\gamma),$$

where $\Psi(x)$ represents Digamma function,
while $\gamma = 0.5772...$ is the **Euler constant**.
Study of the distribution $P(y)$ - the **eigenvector statistics**,
Kuś, Mostowski, Haake, 1988

One quantum state fixed, one random...

Fix an arbitrary state $|\psi_1\rangle$. Generate randomly the other state $|\psi_2\rangle$.

- What is the average angle χ between these states ?
- What is the distribution $P(\chi)$ of the angle $\chi := \arccos|\langle\psi_1|\psi_2\rangle|$?

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Measure concentration phenomenon

'Fat hiper-equator' of the sphere S^N in $\mathbb{R}^{N+1}$...
It is a consequence of the Jacobian factor for expressing the volume element of the $N-$ sphere. Let $z = \cos\vartheta_1$, so that

$$J \sim (\sin\vartheta_1)^{N-1} J_2(\vartheta_2, \ldots, \vartheta_N)$$

Hence the typical angle χ is 'close' to $\pi/2$ and two 'typical random states' are orthogonal and the distribution $P(\chi)$ is **'close' to** $\delta(\chi - \pi/2)$.

How close?

Quantitative description of **Measure Concentration**

Levy's Lemma (on higher dimensional spheres)

Let $f : S^N \to \mathbb{R}$ be a **Lipschitz function**,
with the constant η and the mean value $\langle f \rangle = \int_{S^N} f(x) d\mu(x)$.
Pick a point $x \in S^N$ **at random from the sphere**. For large N it is then **unlikely** to get a value of f much different then the average:

$$P\Big(|f(x) - \langle f \rangle| > \alpha\Big) \leq 2\exp\Big(-\frac{(N+1)\alpha^2}{9\pi^3\eta^2}\Big)$$

Simple application: the distance from the 'equator'

.

Take $f(x_1, ...x_{N+1}) = x_1$. Then **Levy's Lemma** says that the probability of finding a random point of S^N outside a band along the **equator of width** 2α converges **exponentially** to zero as $2\exp[-C(N+1)\alpha^2]$.

As $N >> 1$ then **every equator** of S^N is **'FAT'**.

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $d = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

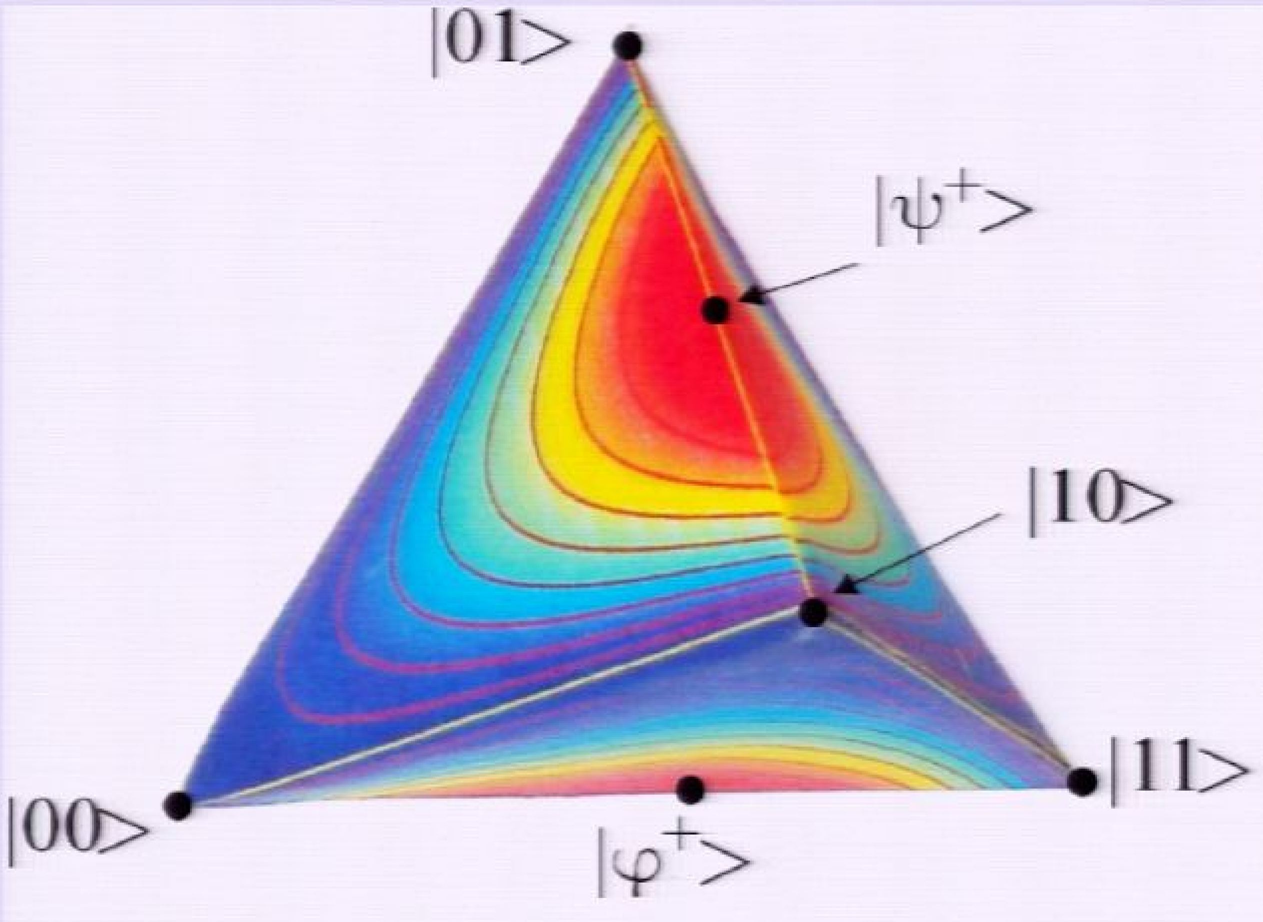
For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := -\mathrm{Tr}\,\sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...

Entanglement of two real qubits

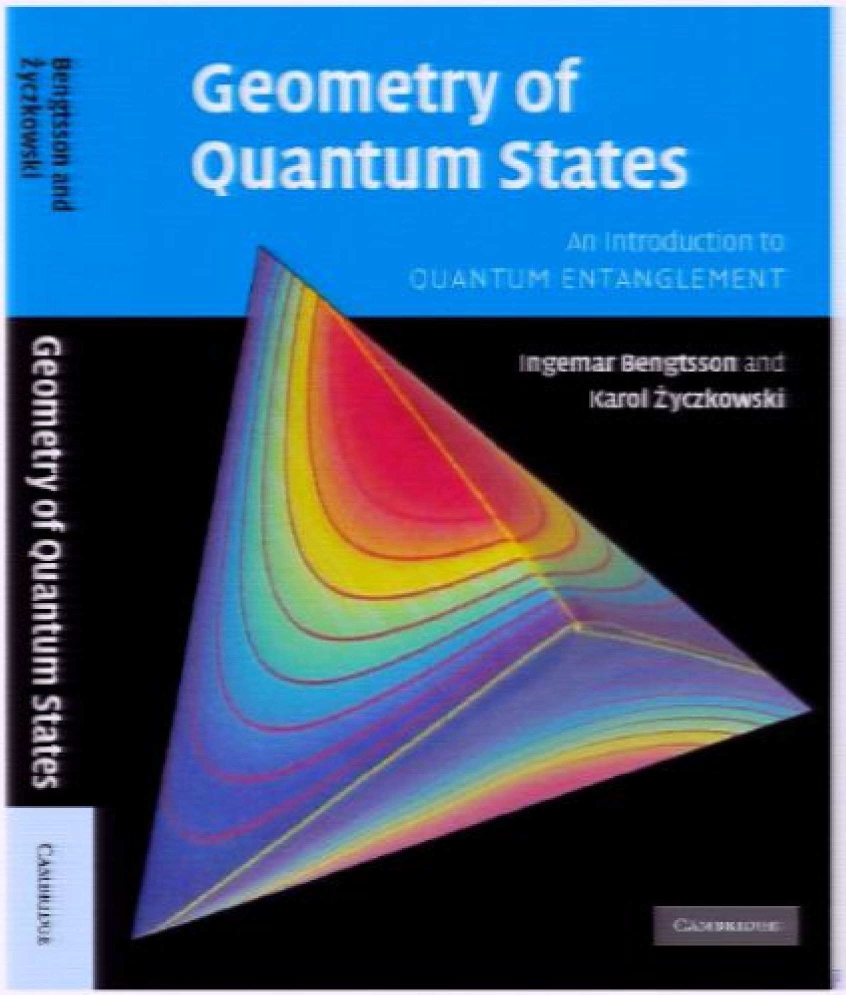
Entanglement entropy at the tetrahedron of $d = 4$ real pure states



More on this is can be found in

I. Bengtsson and K. Życzkowski, Geometry of Quantum States

(Cambridge, 2006, 2008)



Generic pure states of a bi-partite system

'Two quNits' $=$ $N \times N$ quantum system

The space $\mathbb{C}P^{N^2-1}$ of all states in $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N$ has $d_{\rm tot} = N^2 - 2$ dimensions.
The subspace of **separable (product) states** $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$ has only $d_{\rm sep} = 2(N-2)$ dimensions. For large N we observe that $d_{\rm sep} \sim 2N << d_{\rm tot} \sim N^2$ so the **separable states** form a set of measure zero in the space of all states.

Thus a **'typical'** random state is **entangled**!

How much entangled?

Mean entropy of the reduced density matrix ρ

Let us call $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Take any **pure state** $|\psi\rangle \in \mathcal{H}$ and define its partial trace $\rho := {\rm Tr}_B|\psi\rangle\langle\psi| = {\rm Tr}_A|\psi\rangle\langle\psi|$.
The **von Neumann entropy** S of the reduced **mixed state** ρ is a measure of **entanglement** of the initially **pure** bi-partite state $|\psi\rangle$.

Average entanglement entropy for a bipartite system

$N \times N$ system

$$\langle S(\psi)\rangle_\psi \approx \ln N - \frac{1}{2} + \mathcal{O}(\frac{\ln N}{N})$$

$N \times K$ system: formula of Don Page (1993/1995)

valid for random states in $\mathcal{H}_N \otimes \mathcal{H}_K$ with $K \geq N$

$$\langle S(\psi)\rangle_\psi = \Psi(NK+1) - \Psi(K+1) - \frac{N-1}{2K} \approx \ln N - \frac{N}{2K}.$$

$N \times K$ system: probability measure

Let $\lambda = \{\lambda_1, \ldots \lambda_N\}$ denote the spectrum of the reduced matrix $\rho := \mathrm{Tr}_B|\psi\rangle\langle\psi|$. If $|\psi\rangle$ is taken **uniformly** on $\mathcal{H}_N \otimes \mathcal{H}_K$ then

$$P_{N,K}(\lambda) = C_{N,K}\, \delta(1 - \sum_i \lambda_i) \prod_i \lambda_i^{K-N} \prod_{i<j}(\lambda_i - \lambda_j)^2$$

normalization constants $C_{N,K}$ derived in **Sommers, Życzkowski (2001)**

Concentration of entropy of the partial trace

Consider an $N \times K$ system with $K \geq N$

The maximal entropy (achieved for $\rho_* = \mathbb{1}_N/N$) is equal to $S_{\max} := \ln N$. Since the **mean entropy**, $\langle S \rangle_\psi \approx S_{\max} - \frac{N}{2K}$, is close to the maximal value a **concentration effect** has to occur...

Levy's lemma and concentration of entanglement

Consider the sphere S^{2NK-1} which represents pure states of a $N \times K$ system with $K \geq N \geq 3$. Use **Levy's lemma** with $f = S(\rho)$. It implies

$$P\Big(S(\mathrm{Tr}_B|\psi\rangle\langle\psi|) < \ln N - N/2K - \alpha\Big) \;\leq\; \exp\Big(-\frac{(NK-1)}{8(\pi \ln N)^2}\alpha^2\Big)$$

Hayden, Leung, Winter (2006)

Thus the **reduced density matrix** ρ is close to the maximally mixed state $\rho_* = \mathbb{1}_N/N$, while the initial **random pure state** is close to a **maximally entangled state** $|\psi^+\rangle$ with entropy $S_{\max} = \ln N$.

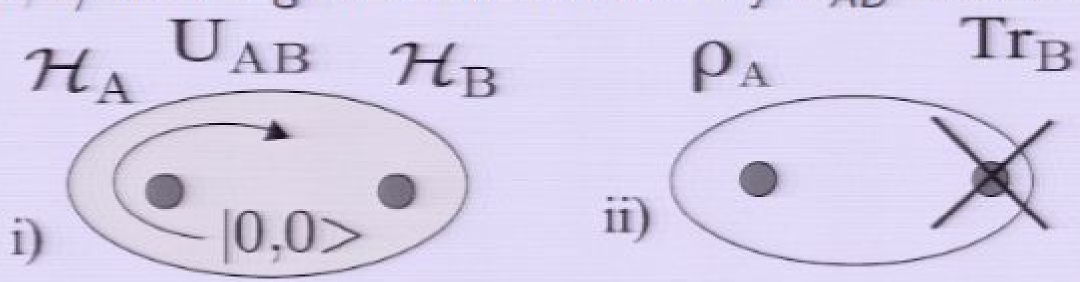
Composed bi–partite systems on $\mathcal{H}_A \otimes \mathcal{H}_B$

Partial trace over one subsystem produces mixed state

Consider an **ensemble of random pure states** $|\psi\rangle$ distributed accoridng to a given measure μ. Define a reduced **mixed state** $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.

Ensembles obtained by partial trace: **a) induced measure**

i) natural measure on the space of pure states obtained by acting on a fixed state $|0,0\rangle$ with a global random unitary U_{AB} of size KN.



ii) partial trace over the K dimensional subsystem B leads to the **induced measure** $P_{N,K}(\lambda)$ in the space of mixed states of size N. Integrating out all eigenvalues but λ_1 one arrives (for large N) at the **Marchenko–Pastur** distribution $P_c(x = N\lambda_1)$ with the parameter $c = K/N$.

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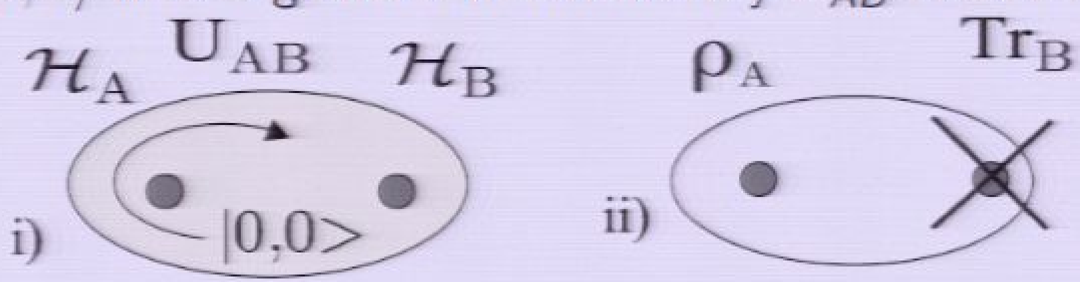

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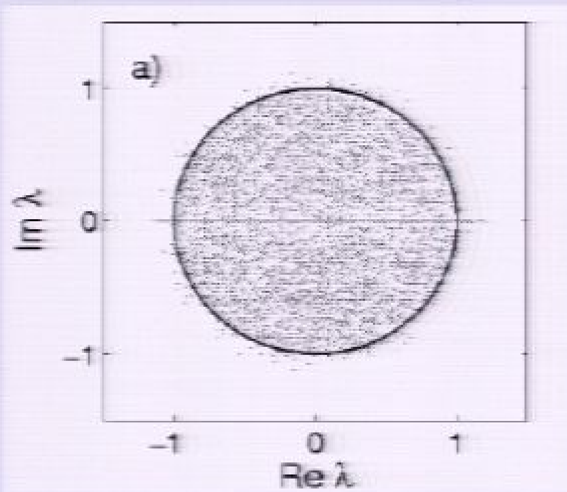
Spectral properties of random matrices

Non-hermitian matrix G of size N of the Ginibre ensemble



Under normalization $\mathrm{Tr} GG^\dagger = N$ the spectrum of G fills **uniformly** (for large N!) the **unit disk**

The so–called **circular law** !

Hermitian, positive matrix $\rho = GG^\dagger$ of the Wishart ensemble

Let $x = N\lambda_i$, where $\{\lambda_i\}$ denotes the spectrum of ρ. As $\mathrm{Tr}\rho = 1$ so $\langle x \rangle = 1$. Distribution of the spectrum $P(x)$ is asymptotically given by the **Marchenko–Pastur law**

$$\pi^{(1)}(x) = P_{\mathrm{MP}}(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x} - 1} \quad \text{for} \quad x \in [0, 4]$$

'Biased' ensembles of bi–partite states

Superposition of locally transformed states

Consider a superposition of a given **bi-partite state** $|\phi_{AB}\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$ with the same state transformed by a random **local unitary** U_A

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\phi_{AB}\rangle + (U_A \otimes \mathbb{1}_N)|\phi_{AB}\rangle\Big)$$

Is the outcome superposition state $|\psi\rangle$ (on average) **more entangled** than the initial $|\phi_{AB}\rangle$?

What reduced states are **(on average) more mixed**:

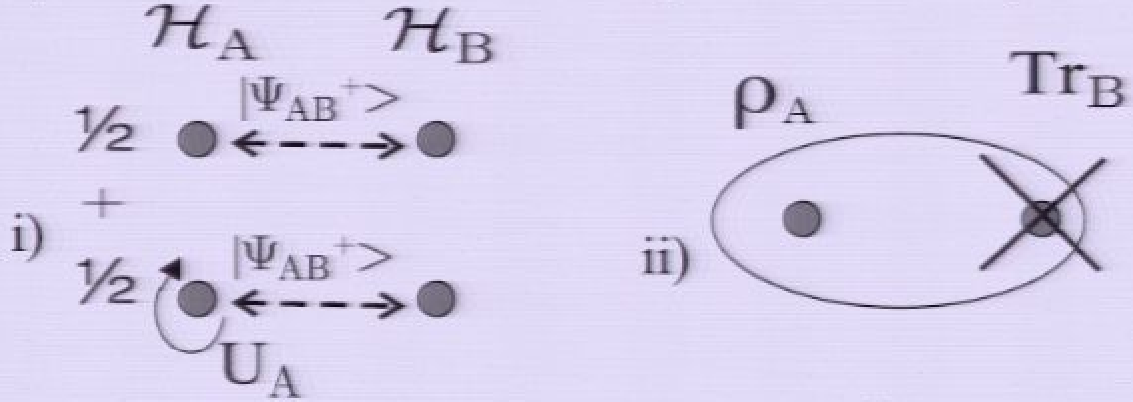
$$\rho = \mathrm{Tr}|\phi_{AB}\rangle\langle\phi_{AB}| \text{ or } \rho' = \mathrm{Tr}|\psi\rangle\langle\psi| \text{ ??}$$

Composed bi–partite systems II

b) Arcsine ensemble

i) Consider a superposition of **two maximally entangled** states on $\mathcal{H}_N \otimes \mathcal{H}_N$

$|\phi\rangle = |\psi^+_{AB}\rangle + (U_A \otimes \mathbb{1}_N)|\psi^+_{AB}\rangle$, where $|\psi^+_{AB}\rangle = (1/\sqrt{N})\sum_{i=1}^{N} |i,i\rangle$,
while $U_A \in U(N)$ is a **Haar random unitary matrix** with phases α_i.



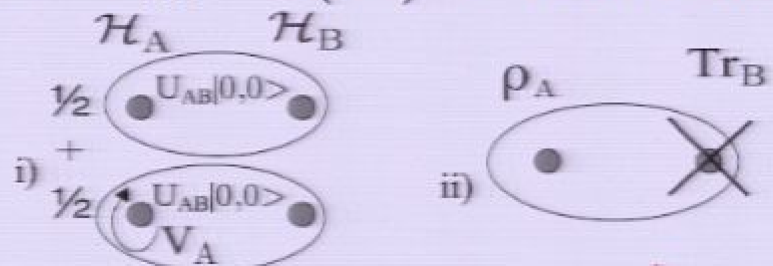
ii) The reduced state $\rho_A = \frac{\mathrm{Tr}_B |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{2\mathbb{1} + U_A + U_A^\dagger}{2N + \mathrm{Tr}(U_A + U_A^\dagger)}$.
has the spectrum $\lambda_i = (1 + \cos\alpha_i)/N$ for $i = 1, \ldots, N$. Thus for large N the spectral density has the form of the **arcsine distribution**,
$P_{\mathrm{arc}}(x) = \frac{1}{\pi\sqrt{x(2-x)}}$ with support $x \in [0, 2]$, where $x = N\lambda$.

c) Bures ensemble

i) Consider a superposition of two pure states: a random state $|\psi_1\rangle$ and the same state transformed by a **local unitary** V_A,

$$|\phi\rangle := (\mathbb{1} \otimes \mathbb{1} + V_A \otimes \mathbb{1})|\psi_1\rangle, \quad \text{where } |\psi_1\rangle = U_{AB}|0,0\rangle$$

while $V_A \in U(N)$ and $U_{AB} \in U(N^2)$ are **Haar random unitary matrices**.



ii) The reduced state $\rho_B = \frac{(\mathbb{1}+V_A)GG^\dagger(\mathbb{1}+V_A^\dagger)}{\mathrm{Tr}[(\mathbb{1}+V_A)GG^\dagger(\mathbb{1}+V_A^\dagger)]}$ is distributed according to the **Bures measure**, $P_B(\lambda_1, \ldots \lambda_N) = C_N^B \prod_i \lambda_i^{-1/2} \prod_{i<j}^{1\ldots N} \frac{(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}$

(Osipov, Sommers, Życzkowski, 2010) characterized by the **Bures distribution**,

$$P_B(x) = \frac{1}{4\pi\sqrt{3}} \left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} \right]$$

where $a = 3\sqrt{3}$. Square matrix G of size N from the **Ginibre ensemble** is obtained from the first column of U_{AB} od size N^2 which acts on $|0,0\rangle$.

Composed mutipartite systems & projections

a) Four-partite system & $\pi^{(2)}$ distribution

Take a four-partite product state,

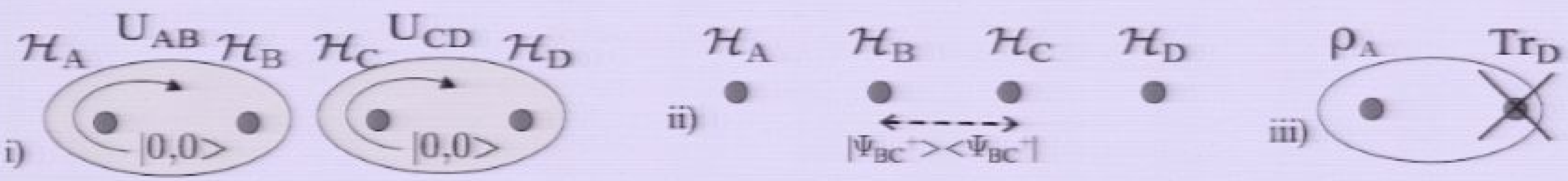
$$|\psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D =: |0,0,0,0\rangle \ \in \mathcal{H}_N{}^{\otimes 4}.$$

i) Apply two random unitary matrices U_{AB} and U_{CD} of size N^2,

$$|\psi\rangle = U_{AB} \otimes U_{CD}|\psi_0\rangle \ = \ \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} G_{ij} E_{kl} \ |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D$$

ii) Consider projector $P := \mathbb{1}_A \otimes |\Psi^+_{BC}\rangle\langle\Psi^+_{BC}| \otimes \mathbb{1}_D$

on the maximally entangled state, $|\Psi^+_{BC}\rangle = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} |\mu\rangle_B \otimes |\mu\rangle_C$



The spectrum of the iii) reduced state $\rho_A \ = \ \frac{\mathrm{Tr}_D|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{GEE^\dagger G^\dagger}{\mathrm{Tr}\ GEE^\dagger G^\dagger}$

consists of squared singular values of the product GE

of **two independent Ginibre matrices**, so the spectral density

is described by the **Fuss-Catalan distribution** $\pi^{(2)}(x)$.

b) $2s$-partite system & $\pi^{(s)}$ Fuss-Catalan distribution

Take a $2s$-partite product state,

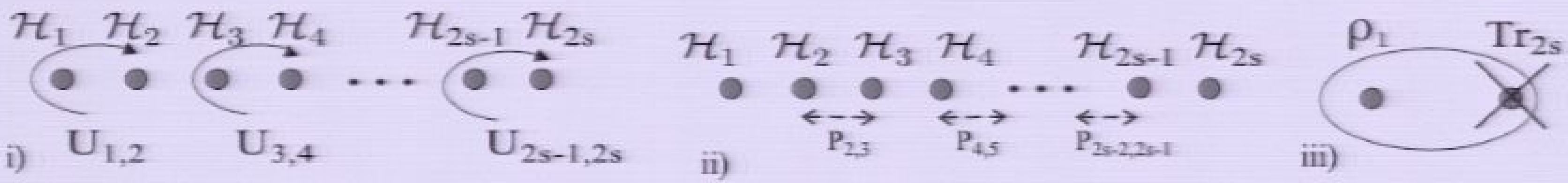
$$|\psi_0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s} \in \mathcal{H}_N{}^{\otimes 2s}.$$

i) Apply s random unitary matrices $U_{1,2}, U_{3,4}, \ldots U_{2s-1,2s}$ of size N^2 each,

$$|\psi\rangle U_{1,2} \otimes \cdots U_{2s-1,2s}|0,\ldots,0\rangle = \sum_{i_1,\ldots i_{2s}} (G_1)_{i_1,i_2} \cdots (G_s)_{i_{2s-1},i_{2s}} |i_1,\ldots,i_{2s}\rangle$$

ii) Project onto the product of $(s-1)$ maximally entangled states,

$$P_s := \mathbb{1}_1 \otimes |\Psi^+_{2,3}\rangle\langle\Psi^+_{2,3}| \otimes \cdots \otimes |\Psi^+_{2s-2,2s-1}\rangle\langle\Psi^+_{2s-2,2s-1}| \otimes \mathbb{1}_{2s}$$



The spectrum of the iii) reduced state

$$\rho_A = \frac{\mathrm{Tr}_{2s}|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger}{\mathrm{Tr}\,[G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger]}$$

consists of squared singular values of the product $G_1 \cdots G_s$ of s **independent Ginibre matrices**, so the spectral density is described by the **Fuss-Catalan distribution** $\pi^{(s)}(x)$.

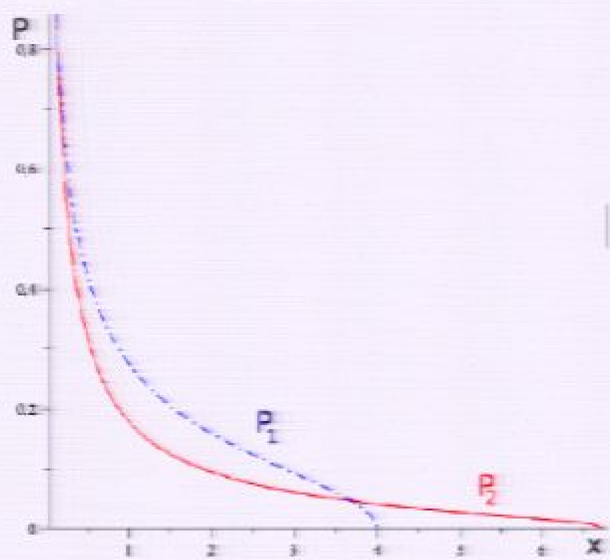
Fuss-Catalan distribution $\pi^{(s)}$

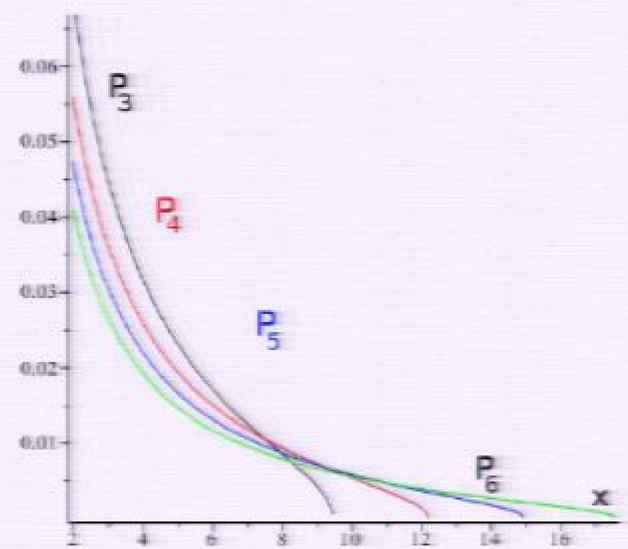
defined for an integer number s is characterized by its **moments**

$$\int x^p \pi^{(s)}(x)dx = \frac{1}{sp+1}\binom{sp+p}{p} \quad =: FC_p^{(s)}$$

equal to the **generalized Fuss-Catalan numbers** .

The density $\pi^{(s)}$ is analitic on the support $[0, (s+1)^{s+1}/s^s]$, while for $x \to 0$ it behaves as $1/(\pi x^{s/(s+1)})$.





The case $s = 1$ is equivalent to the **Marchenko–Pastur distribution**.

Spectral properties of the ensembles analyzed

Spectral density $P(x)$ of the rescaled eigenvalue $x = N\lambda$

matrix W	$P(x)$	$x \to 0$	support	mean entropy
$\mathbb{1}$	$\pi^{(0)}$	–	$\{1\}$	0
$\mathbb{1} + U$	arcsine	$x^{-1/2}$	$[0, 2]$	$\ln 2 - 1 \approx -0.307$
G	**M.-P.** $\pi^{(1)}$	$x^{-1/2}$	$[0, 4]$	$-1/2 = -0.5$
$(\mathbb{1} + U)G$	**Bures**	$x^{-2/3}$	$[0, 3\sqrt{3}]$	$-\ln 2 \approx -0.693$
$G_1 G_2$	**F–C** $\pi^{(2)}$	$x^{-2/3}$	$[0, 6\frac{3}{4}]$	$-5/6 \approx -0.833$
...	...	...	...	...
$G_1 \cdots G_s$	F–C $\pi^{(s)}$	$x^{-s/(s+1)}$	$[0, b_s]$	$-\sum_{j=2}^{s+1} \frac{1}{j}$

Table: Ensembles of random mixed states obtained as normalized Wishart matrices, $\rho = WW^\dagger / \mathrm{Tr} WW^\dagger$. Here $b_s = (s+1)^{s+1}/s^s$ and the mean entropy $\langle S \rangle = -\int x \ln x P(x) dx$.

Interpolating ensembles of random states

Generalized ensemble of random Wishart matrices

Let

$$W_{a,s} \ := \ \Big(a\mathbb{1} + (1-a)U\Big)\, G_1 \cdots G_s$$

where U is the **Haar random unitary** matrix,
while G_i are independent random **Ginibre matrices**.
Define interpolationg ensemble of normalized random density matrices

$$\rho_{a,s} \ := W_{a,s}W_{a,s}^{\dagger}/\mathrm{Tr}(W_{a,s}W_{a,s}^{\dagger})$$

Special cases:

$s=0,\ a=0 \quad \Rightarrow$ **arcsine ensemble**
$s=1,\ a=1/2 \quad \Rightarrow$ **Bures ensemble**
$s=0,\ a=1 \quad \Rightarrow$ **Hilbert-Schmidt ensemble**
$s,\quad a=1 \quad \Rightarrow$ s **– Fuss Catalan ensemble**

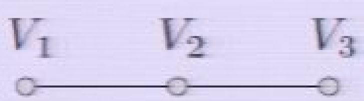
Multi–partite systems: graphs

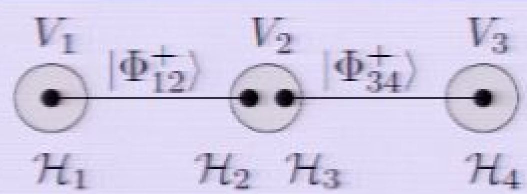
Graph random states

Consider a graph Γ consisting of m edges $B_1, \dots B_m$ and k vertices $V_1, \dots V_k$. It represents a composite **quantum system** consisting of $2m$ sub–systems described in the Hilbert space with $2m$–fold tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m}$ of dimension N^{2m}.

Each **edge** represents the **maximally entangled state** $|\Phi^+\rangle$ in both subspaces, while each **vertex** represents a **random unitary matrix U** (**Haar measure ='generic' Hamiltonian**), coupling connected systems.

A simple example: three vertices & two edges





We define a **random state** $|\psi\rangle = (\mathbf{U_1} \otimes \mathbf{U_{23}} \otimes \mathbf{U_4})\, |\mathbf{\Phi^+_{12}}\rangle \otimes |\mathbf{\Phi^+_{34}}\rangle$ where $|\Phi^+_{kj}\rangle$ denotes the maximally entangled state in subspaces k, j.

Multi–partite graph systems: mixed states

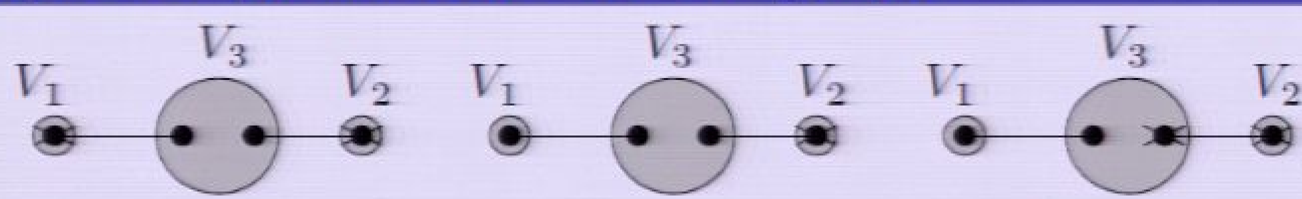
Partial trace over certain subspaces

Consider an **ensemble of random pure states** $|\psi\rangle$ corresponding to a given graph Γ. Select a fixed **subset** T of subspaces and define a (random) **mixed state** $\rho(T) = \mathrm{Tr}_T|\psi\rangle\langle\psi|$.

Tasks

- Determine the **spectral properties** of the ensemble of mixed states $\rho(T)$ associated with the graph Γ.
- Find the mean **entropy** $\langle S(\rho)\rangle_\psi$ of the reduced state ρ averaged over the ensemble of graph random pure states $|\psi\rangle_{\Gamma,T}$.

Examples of partial trace for the graph Γ

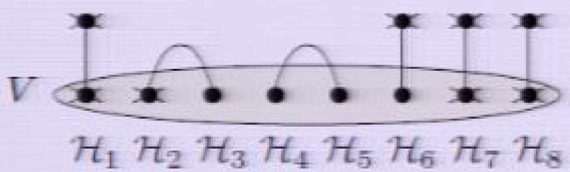


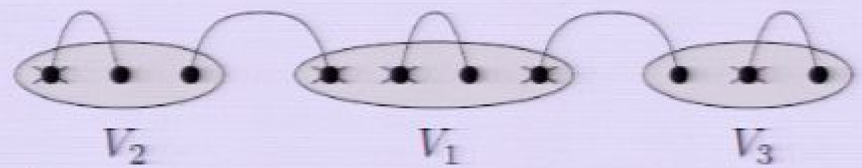
The partial trace is taken over all the subspaces T represented by open symbols.

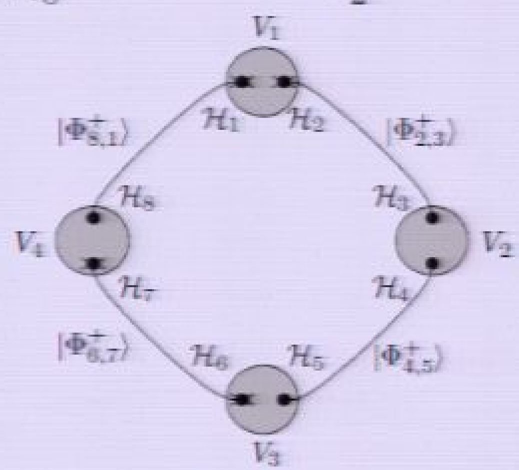
Graphs and random multi–partite systems

Partial trace over certain subspaces

For ensembles of **random states** associated with certain **graphs** Γ and selected subspaces T – cross ($\times$) – over which the partial trace takes place







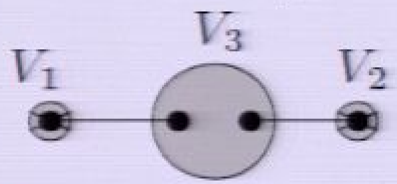
one can compute **moments of the traces** $\mu_q := \langle \mathrm{Tr}\rho^q \rangle_\psi$
and then obtain bounds for the **average entropy** $\langle S \rangle = \langle -\mathrm{Tr}\rho \ln \rho \rangle_\psi$.

Collins, Nechita, Życzkowski, 2010

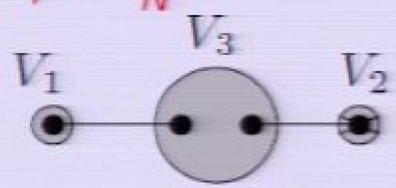
Spectral properties of random mixed states I

Example 1: 2 bonds, 4 subsystems and one bi-partite interaction U_0

a) $\pi^{(0)}$ – maximaly mixed state $\rho = \frac{1}{N}\mathbb{1}$ with **entropy** $S(\rho) = \ln N$



or



b) $\pi^{(1)}$ random mixed state generated according to the induced measure

with **entropy** $S(\rho) \approx \ln N - 1/2$

Let $|\psi\rangle = \sum_i \sum_j G_{ij}|i\rangle \otimes |j\rangle$ be a **random pure state.**

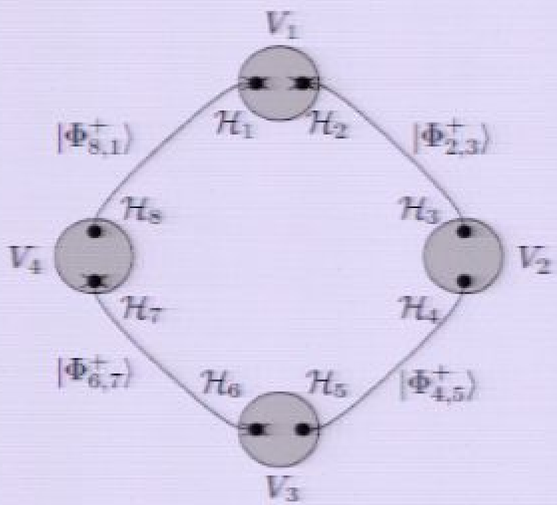
Then G is a random matrix of **Ginibre ensemble** consisting of independent complex Gaussian entries normalized as $|G|^2 = \mathrm{Tr}\, GG^\dagger = 1$.

The distribution of eigenvalues of a **non–hermitian matrix** G is given by the **Girko circular law**, while positive **Wishart** matrices $\rho = \mathrm{Tr}_B|\psi\rangle\langle\psi| = GG^\dagger$ are described by **Marchenko-Pastur** law $\pi^{(1)}$.

Spectral properties of random mixed states II

Example 2: 4 bonds, 8 subsystems and four bi-partite interactions V_i

c) $\pi^{(2)}$ random mixed state generated by the 4–**cycle graph**

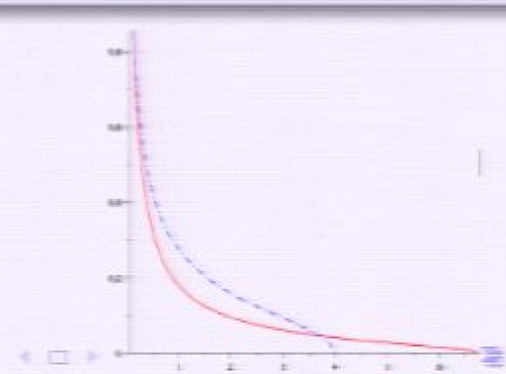


After partial trace over **crossed** subsystems the random mixed state has the structure

$$\rho = \alpha G_2 G_1 G_1^\dagger G_2^\dagger,$$

where G_1 and G_2 are independent **Ginibre** matrices and $\alpha = 1/\mathrm{Tr} G_2 G_1 G_1^\dagger G_2^\dagger$.

Mixed states with spectrum given by the **Fuss-Catalan distribution** $\pi^{(2)}(x)$ characterized by mean **entropy** $S(\rho) \approx \ln N - 5/6$



$P_{\mathbf{MP}}(x) = \pi^{(1)}(x)$ and $\pi^{(2)}(x)$.

Spectral properties of random mixed states II

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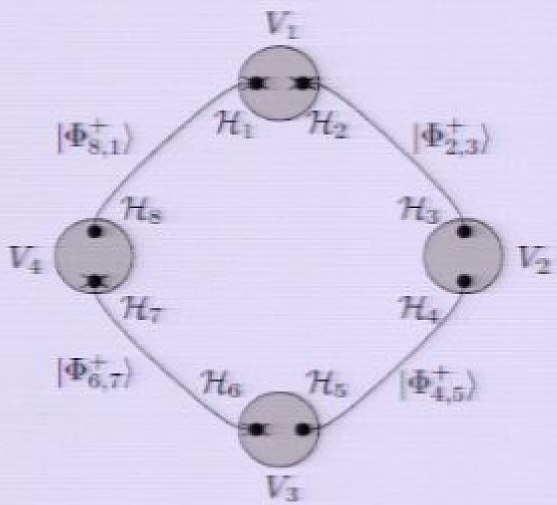

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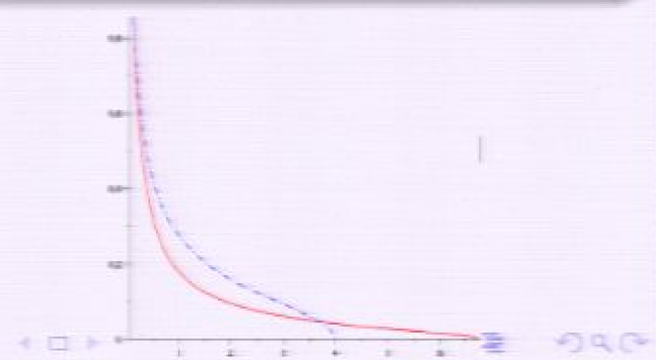


$P_{\mathbf{MP}}(x) = \pi^{(1)}(x)$ and $\pi^{(2)}(x)$.

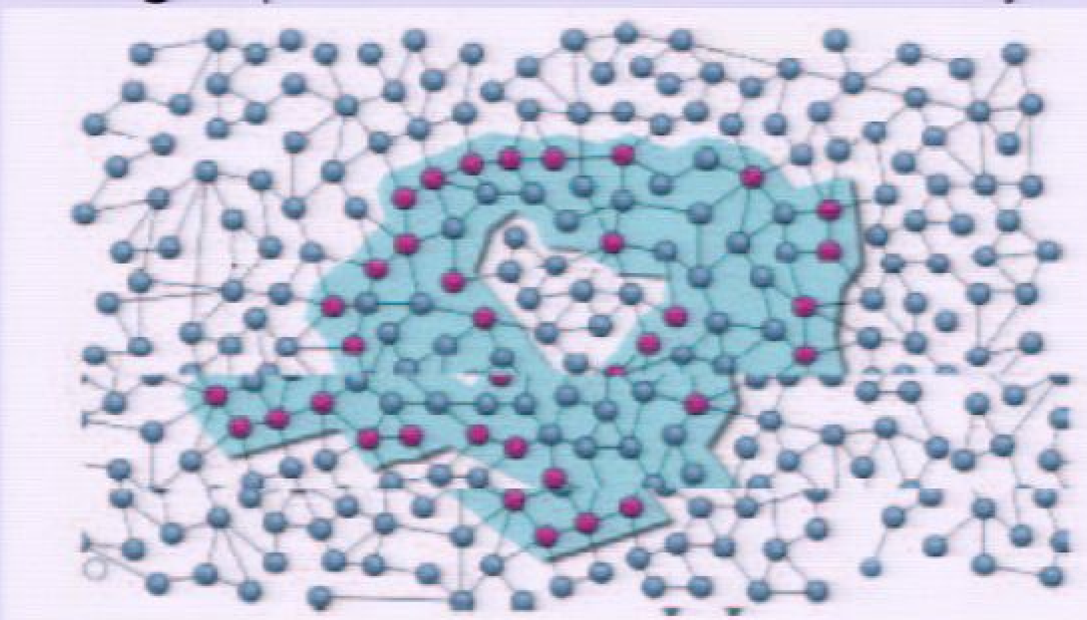
Multi–partite systems: a lattice L

Partition of the lattice into two disjoint sets, $L = A \cup \bar{A}$

Consider lattice (graph), in which each **vertex** denotes a spin

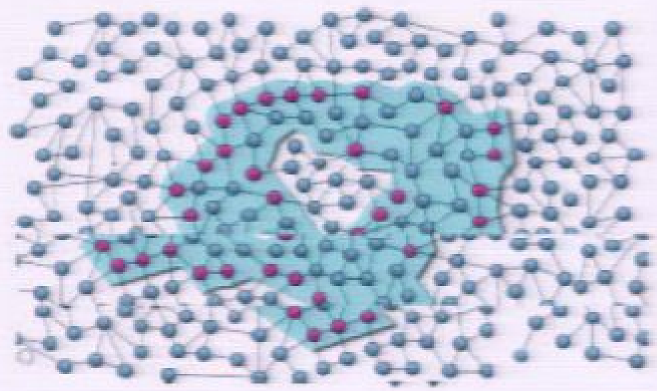
(*different meaning than before!*)

and each **edge** represents an interaction defined by a local Hamiltonian H.



Let A denotes a distinguished set of vertices while ∂A represents **spins** belonging to its **area**, i.e. these spins for which some edges are cut away.

Area law for a partition of the lattice $L = A \cup \bar{A}$



Consider an eigenstate $|\psi\rangle$ of the Hamiltonian H, define **set of spins** A and take the partial trace of the pure state over all spins belonging to the **complementary set** $\bar{A}$.

• **Von Neumann entropy** of the resulting mixed state $\rho := \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$ is proportional to the **area** ∂A of the distinguished subset A.
Hence **entanglement** of the state $|\psi\rangle$ with **respect to** the partition $A \cup \bar{A}$ behaves as the **area** ∂A.

Eisert, Cramer, Plenio 2008, Rev. Mod. Phys. 2008

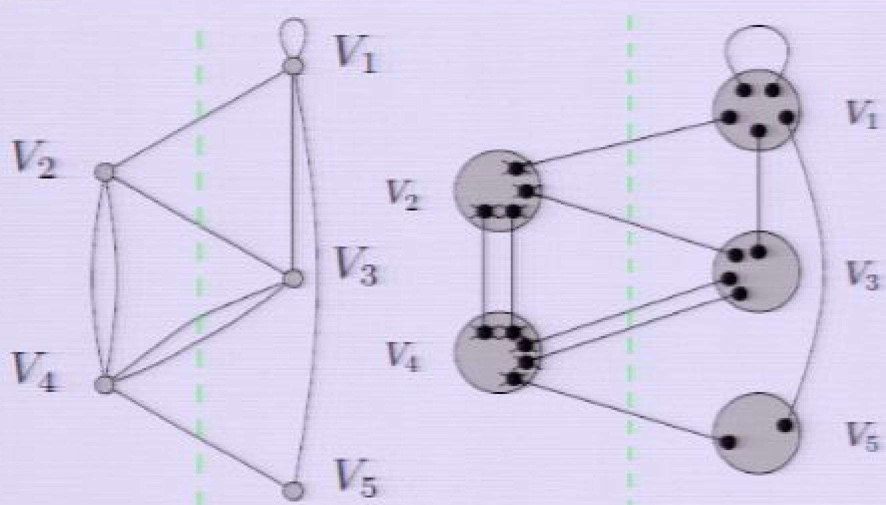
Universal Entanglement Area Law

Area law for random graph states

Theorem. Consider a graph Γ and its partition into two sets A and $\bar{A}$. Let $|\psi\rangle$ be a random graph pure state and $\rho := \mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$.
Then the mean **entropy of** ρ (entanglement entropy of $|\psi\rangle$) is proportional to the **number** M of **bonds cut** ('area' of A) ,

$$\langle \mathbf{S}(\rho)\rangle_{\psi} = \mathbf{M}\ \ln \mathbf{N} \ .$$

Example: graph with 10 bonds, $M = 5$ of them cut



The **area law** $S(\rho) = 5 \ln N$ is **universal** as it does not depend on the choice of Hamiltonians describing the interaction in the vertices.

Only the **topology** of the interaction matters!

Concluding remarks

- There exists a natural, **unitarily invariant** measure in the space $\mathbb{C}P^{N-1}$ of **pure states** of a finite size N. A **quantized chaotic evolution** sends an initial state $|i\rangle$ into a **'typical'** state $|\psi\rangle$.
- A generic pure state of a **bi-partite quantum system** is strongly **entangled**, so its partial trace is **strongly mixed!**
- **'Biased'** ensembles of random pure states + **partial trace** allow one to generate **random states** according to various measures, including **(Arcsine, Hilbert-Schmidt, Bures, s–Fuss-Catalan)** ensembles.
- With any graph one can associate an **ensemble of random pure states**. Selecting a set A of subsystems we define an ensemble of mixed states ρ by performing the **partial trace** over them. Statistics of the spectra of ρ is described by delta distribution $h_0(x) = \delta(x-1)$, **Marchenko–Pastur** distribution $h_1(x)$ or **Fuss–Catalan** distributions $h_s(x)$, with $s \geq 2$, for which mean entropies are known.
- **Universal Entanglement Area law**: For any **graph** Γ and its partition A and $\bar{A}$ the **mean entanglement entropy** of the **random pure state** $|\psi\rangle$ depends on **the area ∂A** (the number of **bonds cut**).

Cracow with the **Wawel Castle** and the **Tatra mountains** in the background.

You are welcome!


No Signal
VGA-1