Title: Isotropic Entanglement

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Abstract: One of the major problems hindering progress in quantum many body systems is the inability to describe the spectrum of the Hamiltonian. The spectrum corresponds to the energy spectrum of the problem and is of out-most importance in accounting for the physical properties of the system. A perceived difficulty is the exponential growth of the Hamiltonian with the number of particles involved. Therefore, even for a modest number of particles, direct computation appears intractable. This work offers a new method, using free probability and random matrix theory, of approximating the spectrum of generic frustrated Hamiltonians of arbitrary size with local interactions. In addition, we show a number of numerical experiments that demonstrate the accuracy of this method.

Isotropic Entanglement

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Acknowledgments

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- Jeffrey Goldstone
- Aram W. Harrow

Outline

Entanglement: Blessing

$$|\Psi_{12}\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle$$

- Can be used as a resource for quantum information science: e.g. Shor's factoring algorithm, teleportation, superdense coding, etc. etc.
- Gives rise to new physics that is, as far as we can tell, absent in the classical limit.
- Allows for existence of quantum information theory and quantum information scientists.

Entanglement: Curse

- Makes quantum systems very hard to study on classical computers.
- Its hard to grasp with our classical intuitions.
- It's fragile: makes it hard to build quantum computers.
- Makes some philosophers very angry.

Preview

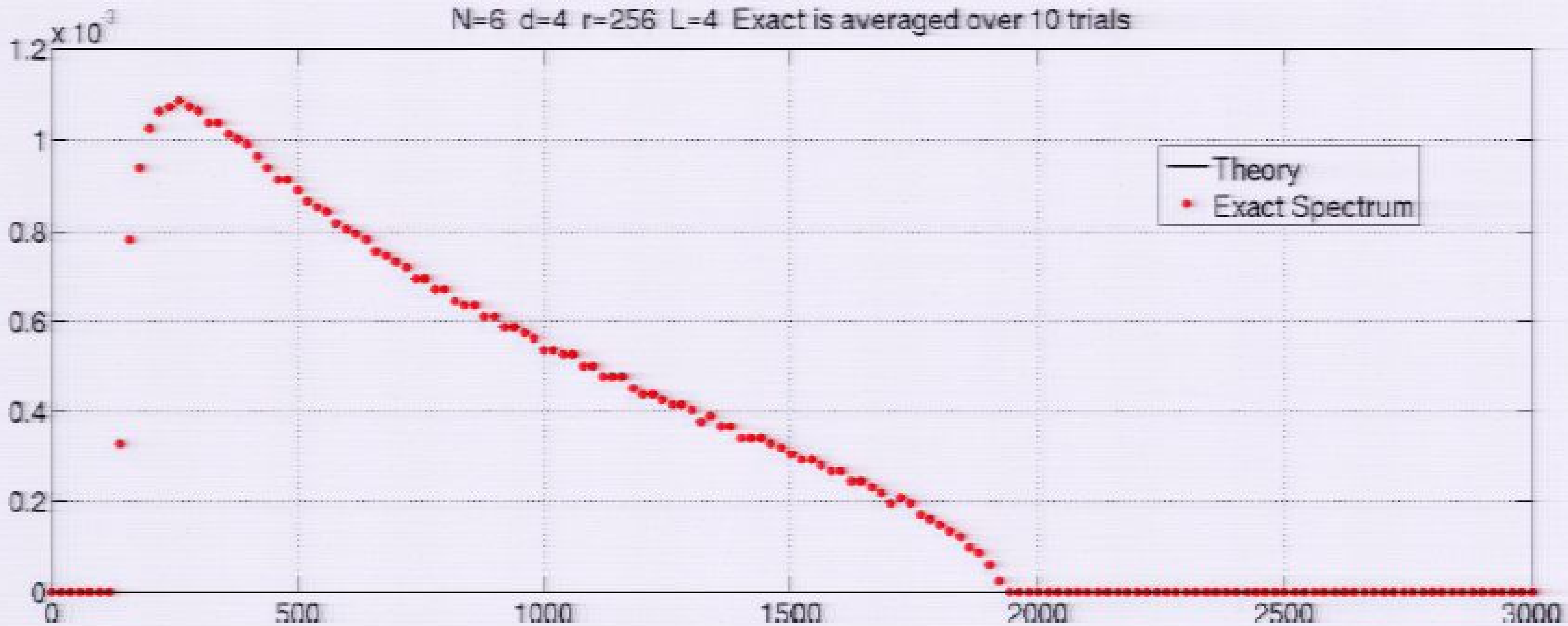

Figure: Exact Spectrum of a quantum chain

What do you think?

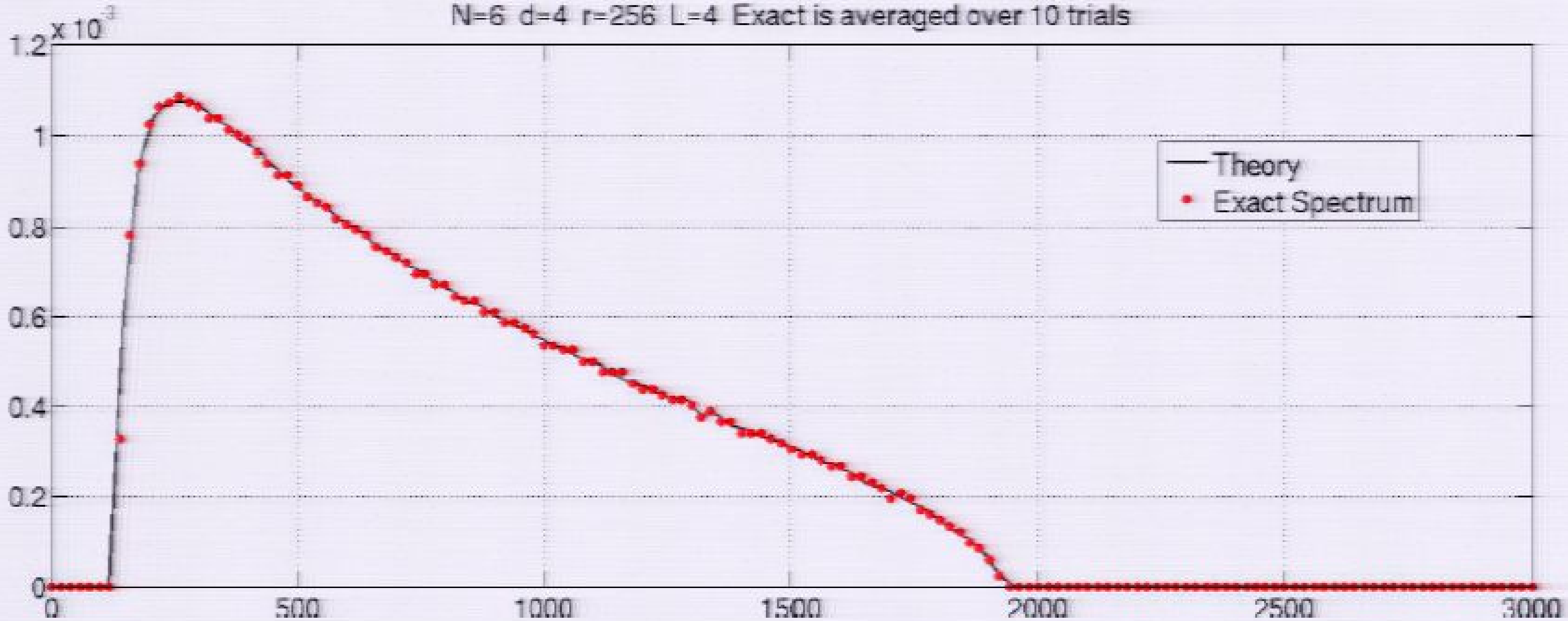

Figure: Exact Spectrum with our theoretical prediction

Lattices

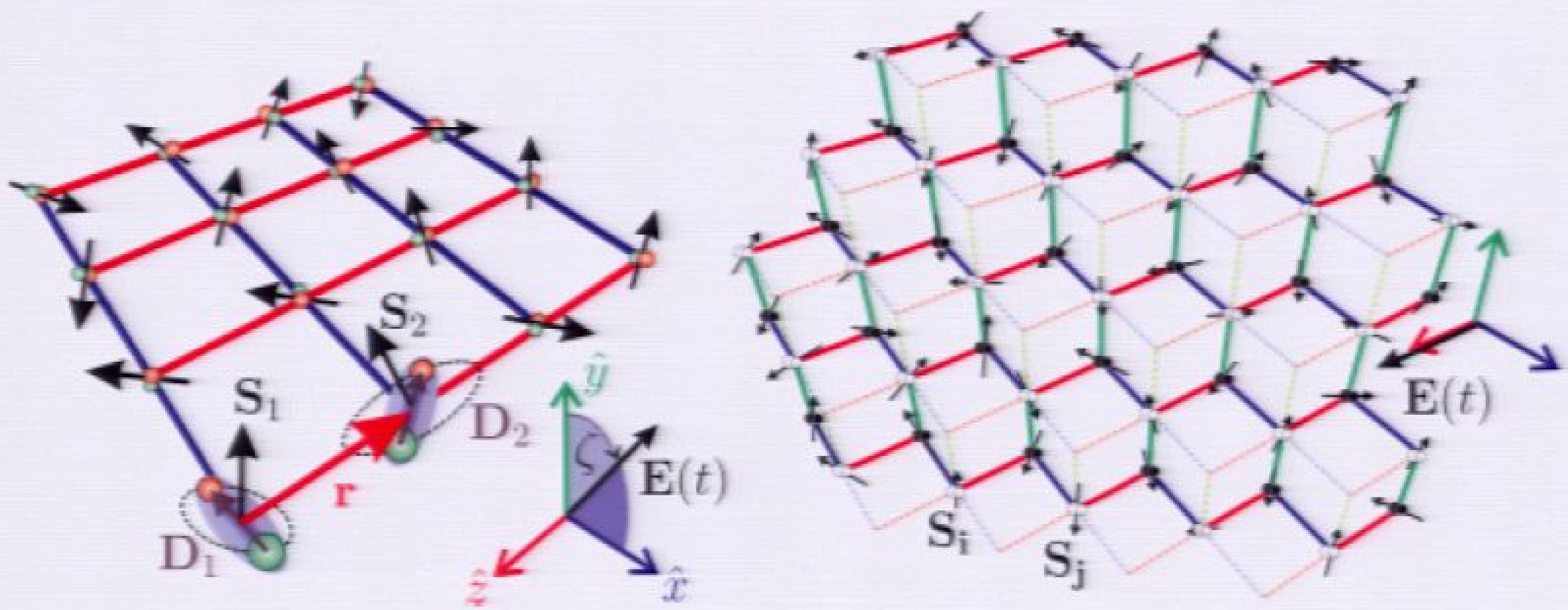

Figure: Picture credit: www.scala-ip.org/public/index.php?s=overview

General state of an N body quantum system with d states per site

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N=1}^{d} c_{i_1,i_2\ldots,i_N} \,|\, i_1\rangle \otimes |\, i_2\rangle \otimes \ldots \otimes |\, i_N\rangle$$

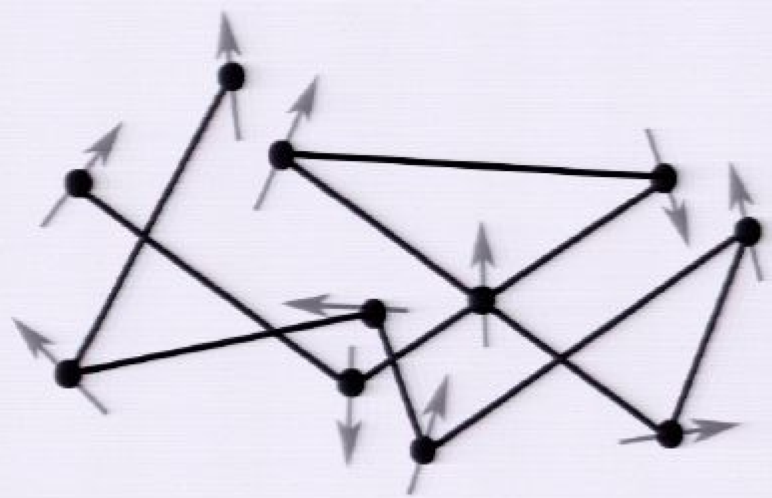

Figure: Each arrow represents a d dimensional spin. Each edge represents an interaction.

Interactions: $H = \sum_{k=1}^{N-1} \mathbb{I} \otimes H_{k,k+1} \otimes \mathbb{I}$

Take the odd summands first

$$H = H_{1,2} \otimes \mathbb{I}_d \otimes \cdots \otimes \mathbb{I}_d + \mathbb{I}_d \otimes \mathbb{I}_d \otimes H_{3,4} \otimes \mathbb{I}_d \otimes \cdots + \mathbb{I}_d \otimes \cdots \otimes \mathbb{I}_d \otimes H_{N-1,N}$$

$$H_{k,k+1} : d^2 \times d^2; \qquad \mathbb{I}_d : d \times d$$

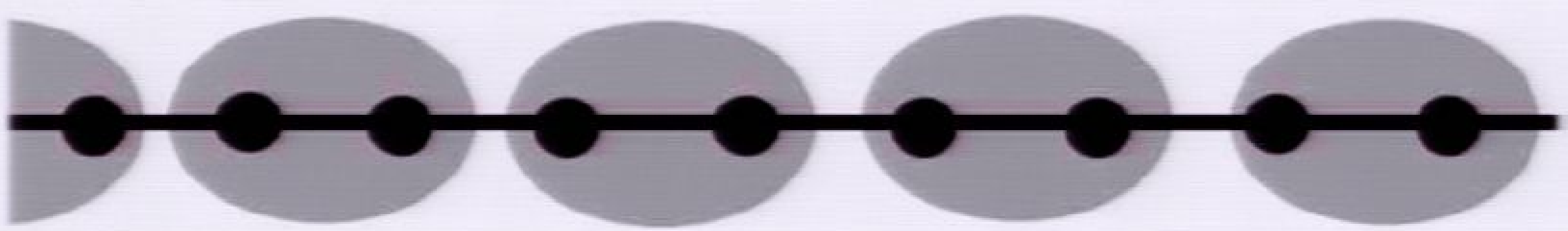

Figure: 2-Local Hamiltonian on a 1D chain

Interactions: $H = \sum_{k=1}^{N-1} \mathbb{I} \otimes H_{k,k+1} \otimes \mathbb{I}$

Add the even pairs as well

$$H = \sum_{k=1}^{N-1} \mathbb{I}_d \otimes \cdots \otimes \mathbb{I}_d \otimes H_{k,k+1} \otimes \mathbb{I}_d \otimes \cdots \otimes \mathbb{I}_d$$

- This overlap is pretty much responsible for the tremendous difficulties met in QMBS.

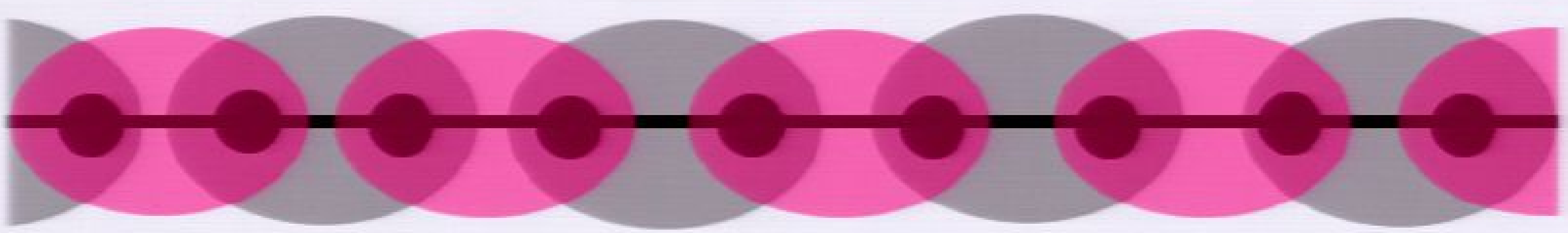

Figure: 2-Local Hamiltonian on a 1D chain

Example of Frustration: Antiferromagnetism

$$H = J \sum_{<i,j>} \sigma_i \sigma_j, \qquad \sigma_i = \pm 1,\ \sigma_j = \pm 1,\ J > 0$$

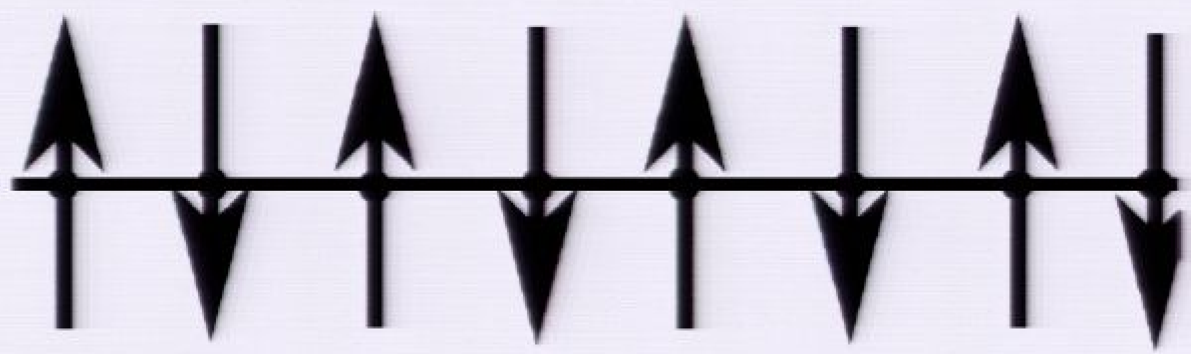

Figure: Spins locally like to be anti-aligned to minimize the total energy

Frustration

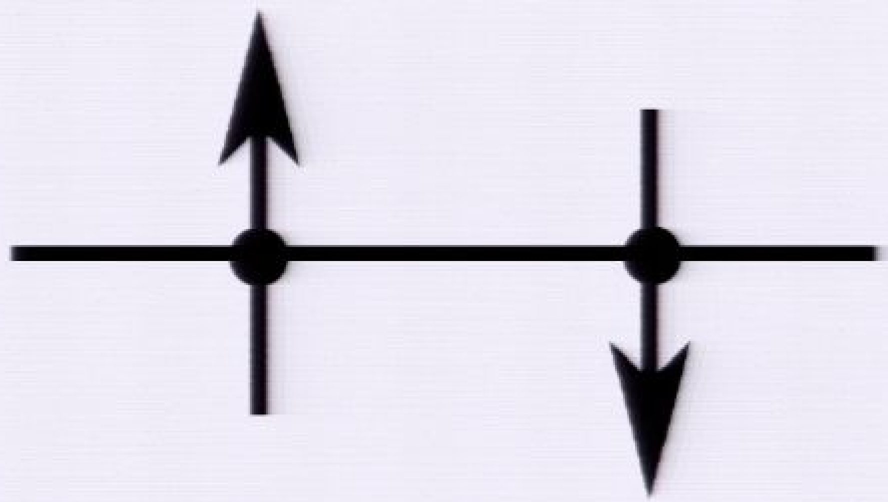

Figure: Story is the same locally: Spin configuration looks the same locally

Definition: We say a quantum system is *unfrustrated* when its ground state is also a common ground state of all of the local terms $H_{k,\dots,k+l-1}$.

Frustration

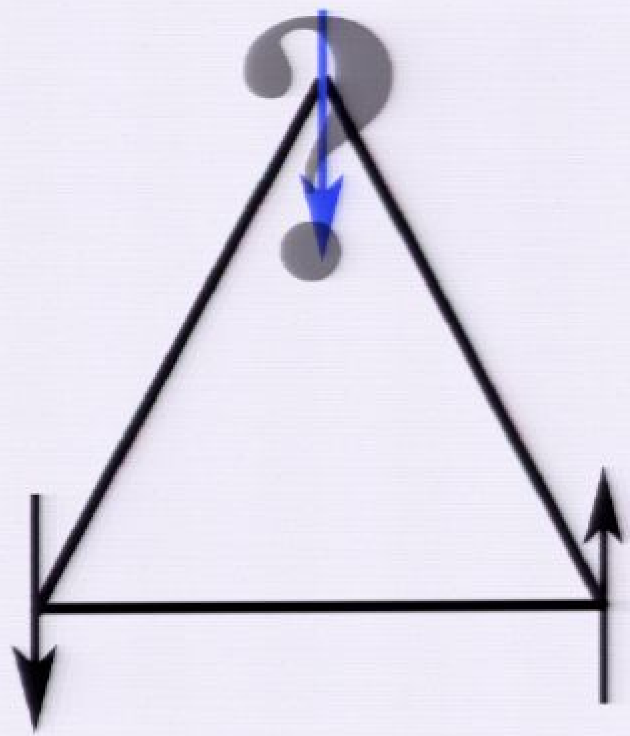

Figure: Magnetic Frustration

$L-$ Local interactions

Suppose we have a chain of d-dimensional quantum particles (spins) with local interactions. The Hamiltonian of the system

$$H = \sum_{k=1}^{N-L} \mathbb{I} \otimes H_{k,\cdots,k+L-1} \otimes \mathbb{I} \tag{1}$$

is L-local (each $H_{k,\cdots,k+L-1}$ acts non-trivially only on L neighboring spins).

The Hamiltonian as Sums of "Random" Projectors

We take for the L-local Hamiltonians

$$H_{k,\cdots,k+L-1} = \sum_{k=1}^{r} |\, v^k \rangle_{k,\cdots,k+L-1} \langle v^k \,|, \tag{2}$$

where $|\, v^k \rangle$'s are random vectors of size d^L.

- Also known as Wishart matrices.
- In physics systems with such Hamiltonians are "quantum spin glasses".

Analytical Results: Cases of Interest

- Unfrustrated case: $r \leq \dfrac{d^2}{4}$
  - Easy
- Frustrated case: $r > \dfrac{d^2}{4}$
  - Hard

R. M., E. Farhi, J. Goldstone, D. Nagaj, T. Osborne, and P. Shor ***"Unfrustrated Qudit Chains and Their Ground States"*** to appear in PRA, 2010.

"The Problem" and what we can do about it...

The goal is to find the spectrum of

$$H = \sum_k \mathbb{I}_{1\cdots k-1} \otimes H_{k,\cdots,k+L-1} \otimes \mathbb{I}_{k+L,\cdots,N}$$

That is a hard problem to solve exactly. BUT...

What if the eigenvectors point in random direction? Then we can solve it for arbitrary large N !

We find that the very generation of entanglement "stirs the pot" enough and that we obtain incredible predictive power of the spectrum for highly generic cases.

- Yet another blessing of entanglement!

This we can solve by applying Free Probability

Any problem that promises to act roughly like:

Figure: The matrix of eigenvectors cover the unit sphere living in $O(d^N)$ with Haar measure

Free Probability: The Big Picture

- Free probability theory is well suited for non-commuting random variables e.g. random matrices:
  - It promises to answer: what is the eigenvalue distribution of $A+B$ when A and B are known random matrices?
- Recall that in the commuting case the distributions don't add but the **cumulants** and **log-characteristics** do!
- Analogously, in free probability theory the distributions don't add but the **free cumulants and R-transforms** do!
  - R-transforms can be obtained from the **Cauchy transform** of the distribution: $w \equiv g(z) = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{f(x)}{z-x}$.
  - R-transform: $r(w) = g^{-1}(w) - 1/w$
  - Luckily: $r_C(w) = r_A(w) + r_B(w)$, where $C = A + B$.

A. Nica, R. Speicher: "**Lectures on the Combinatorics of Free Probability**" Cambridge University Press, 2006

Take $d = 4$ and $N = 6$

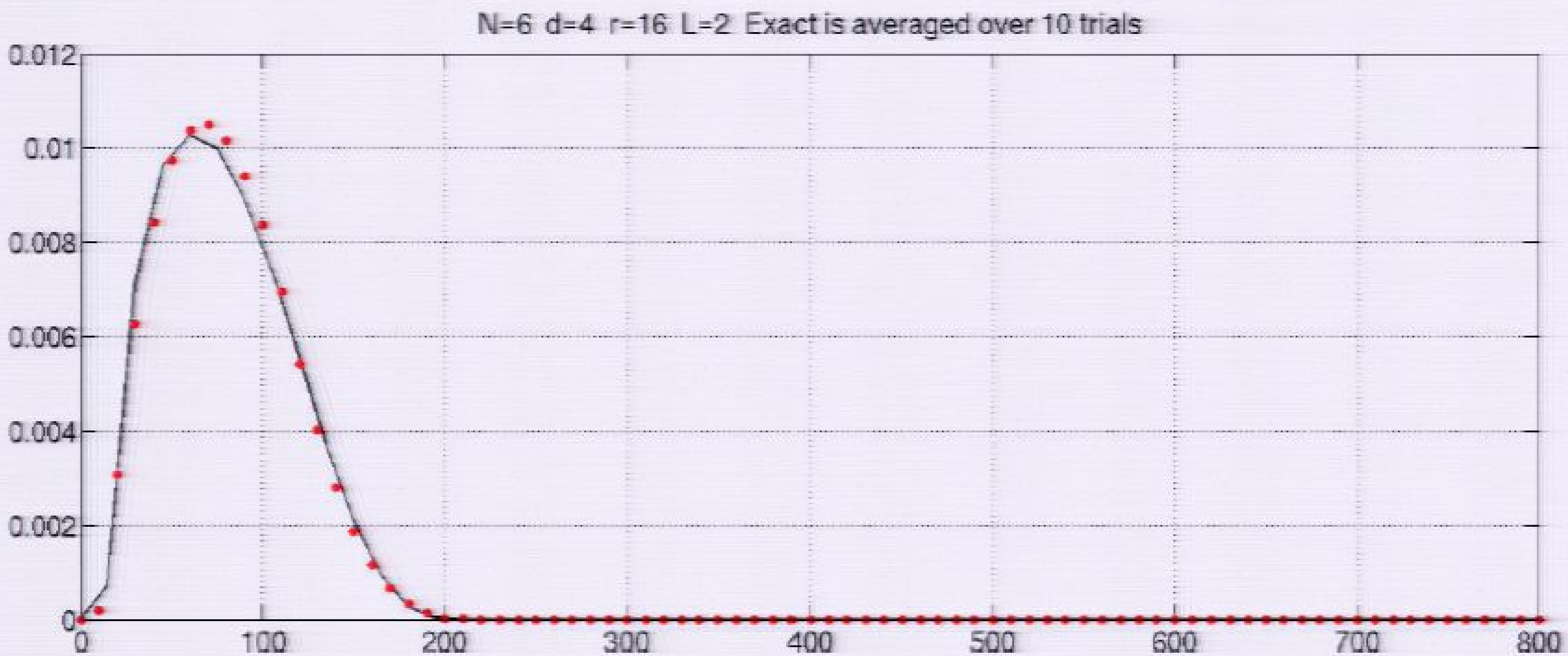

Figure: $L = 2$, $r = 16$

Take $d = 4$ and $N = 6$

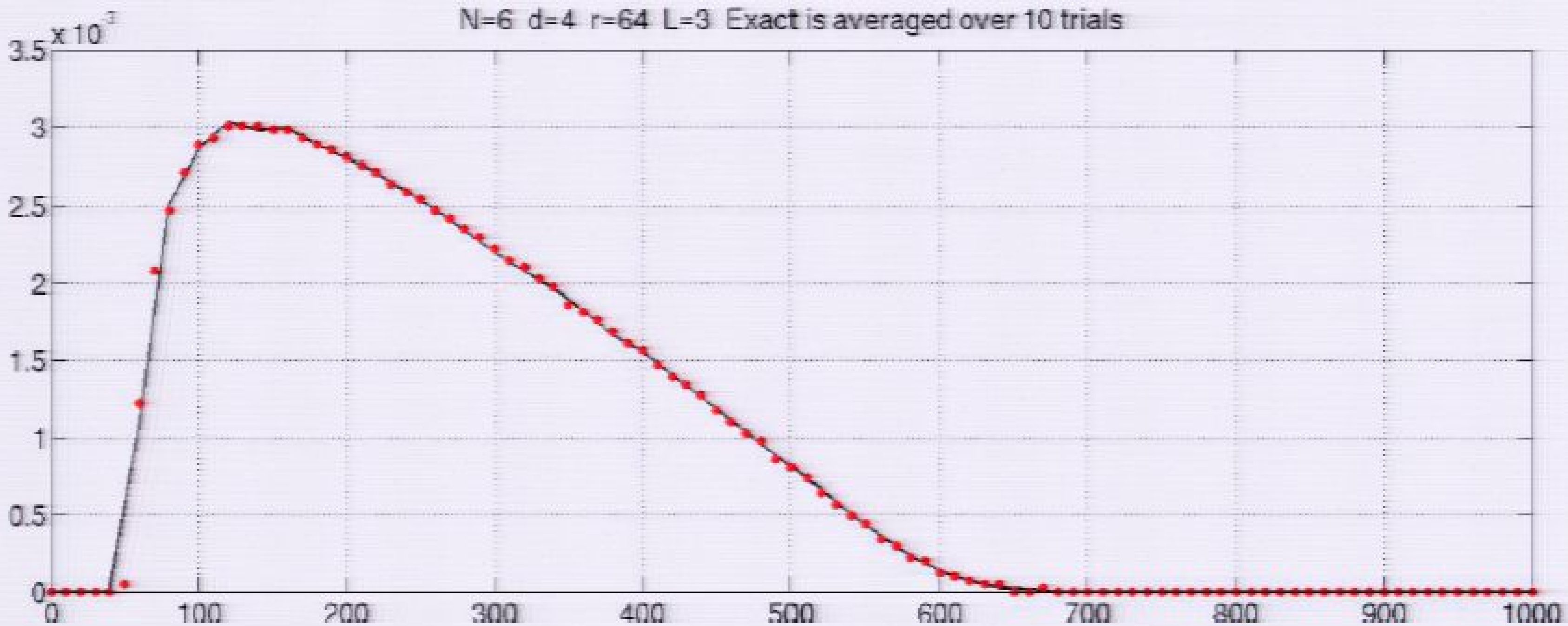

Figure: $L = 3$, $r = 64$

Take $d = 4$ and $N = 6$

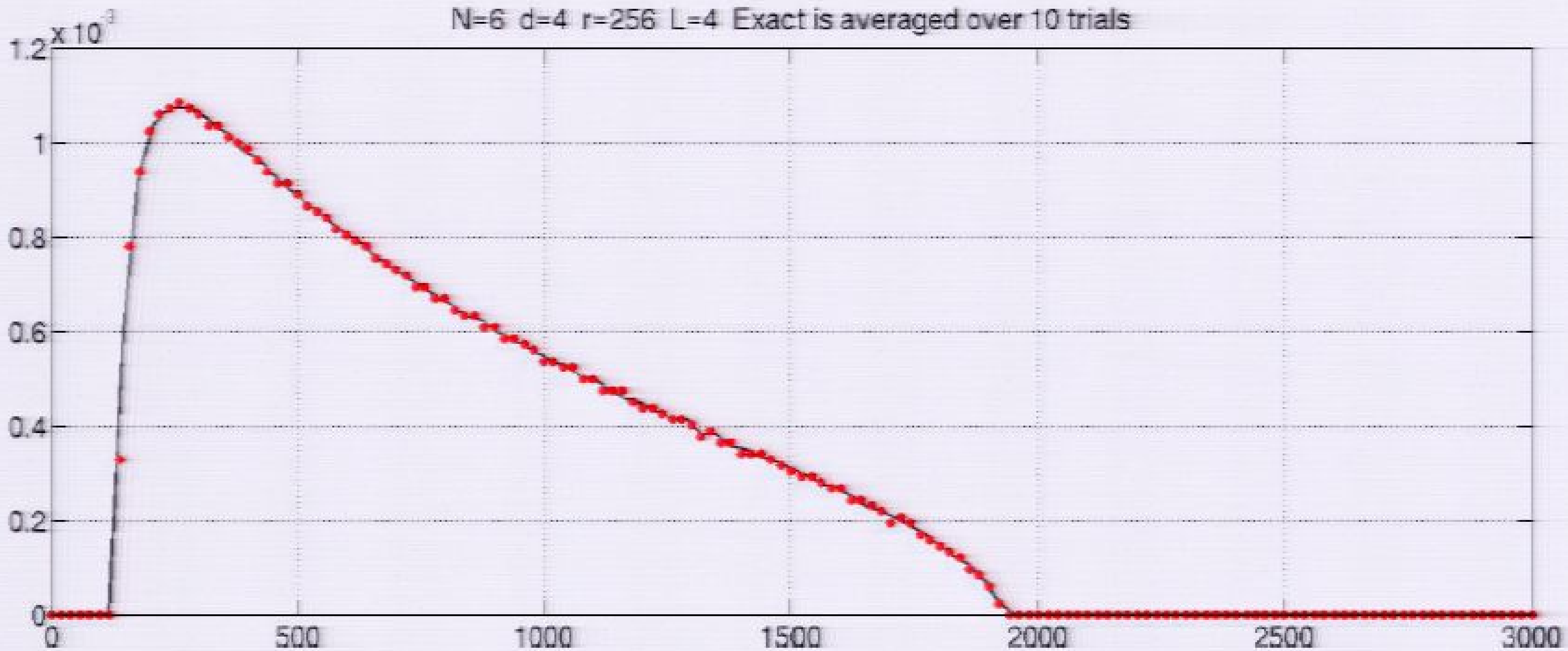

Figure: $L = 4$, $r = 256$

Take $d = 4$ and $N = 6$

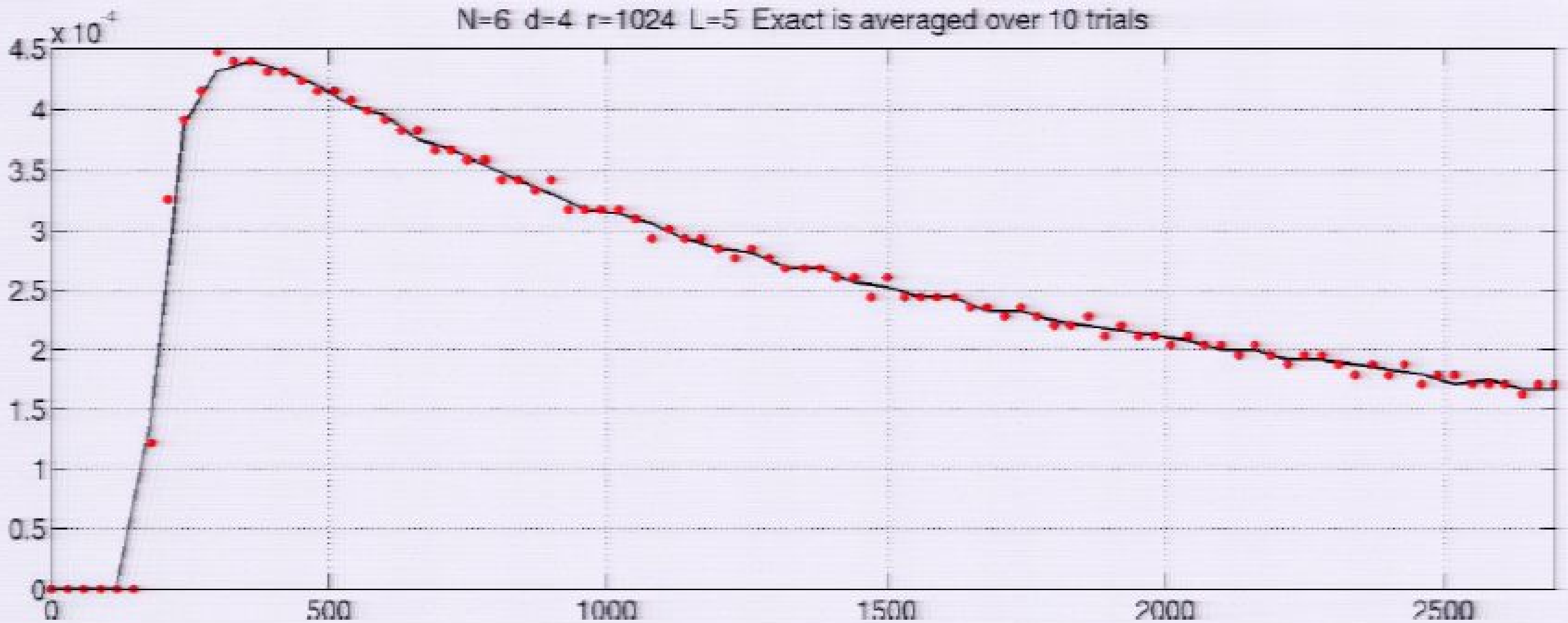

Figure: $L = 5$, $r = 1024$

Take $d = 4$ and $N = 6$

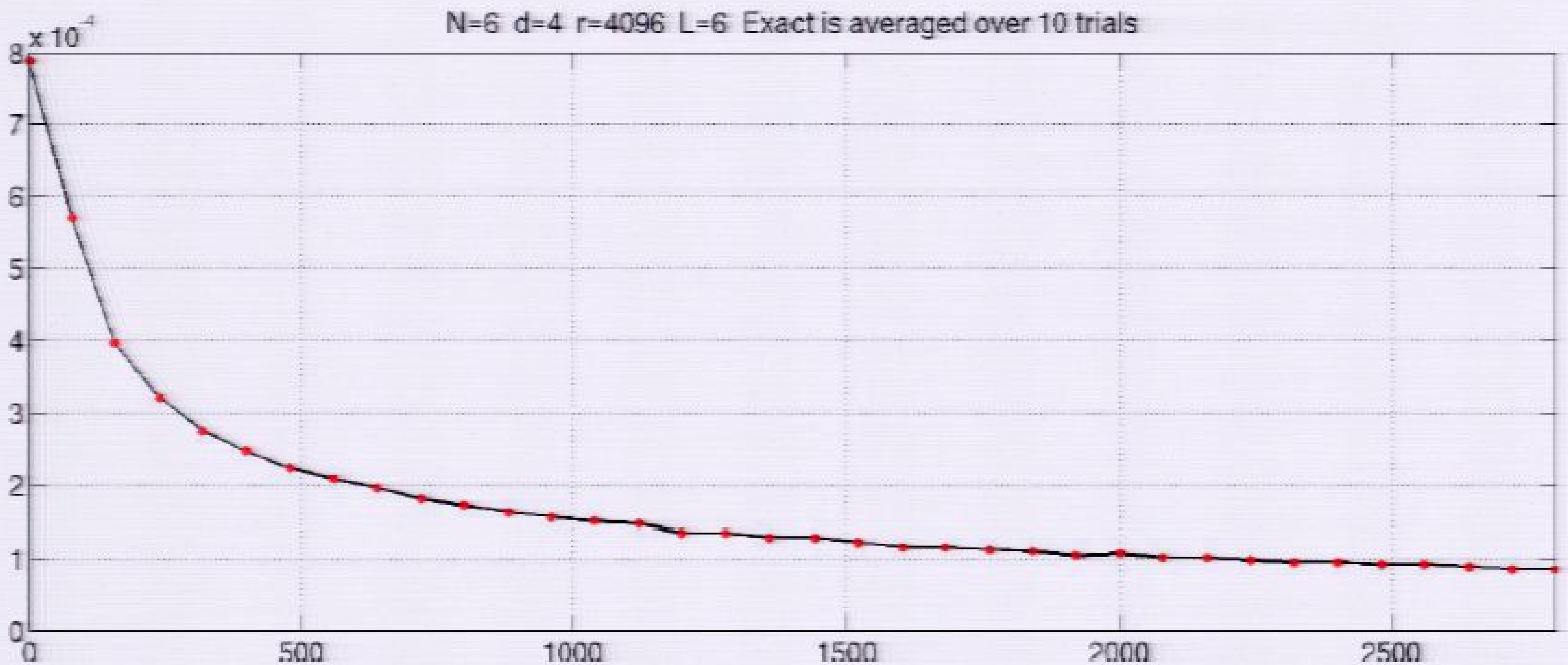

Figure: $L = 6,\ r = 4096$

Conclusions

- We can describe the whole spectrum of generic quantum many body Hamiltonians.
- Our results are universal.
- Having the spectrum, one can make inferences on a variety of issues: e.g. eigenvectors, characterization of different systems, study of the tails, etc. etc. So it is a great start !
- The interface among Quantum Information Science, Random Matrix Theory and Free Probability Theory promises to be very fruitful for future investigations.

THANK YOU FOR YOUR PRESENCE AND ATTENTION