Title: On the comparison of volumes of quantum states

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Abstract: Entangled (i.e., not separable) quantum states play fundamental roles in quantum information theory; therefore, it is important to know the "size" of entanglement (and hence separability) for various measures, such as, Hilbert-Schmidt measure, Bures measure, induced measure, and α-measure. In this talk, I will present new comparison results of α-measure with Bures measure and Hilbert-Schmidt measure. Employing these comparison results to the subsets of separable states and of states with positive partial transpose, we show that the probability of separability is very small, and the well-known Peres-Horodecki PPT Criterion as a tool to detect separability is imprecise for (even moderate) large dimension of Hilbert space. This talk is based on my papers: J. Math. Phys. 50 (2009) 083502, and J. Phys. A: Math. Theor. in press.

On the comparison of volumes of quantum states

Deping Ye

The Fields Institute, Toronto, Ontario, Canada

July 2010

"Random Matrix Techniques in Quantum Information Theory"

workshop at the Perimeter Institute

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Modern quantum algorithms, such as Shor's algorithm for integer factorization, have entangled quantum states as their key ingredients.

To determine the separability and entanglement is a hard problem!

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What is the relative size of the set of separable and entangled quantum states within the set of quantum states in terms of some measure?

Density Matrix

Complex Hilbert space $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2} \cdots \otimes \mathbb{C}^{D_n}$ with complex dimension $N = D_1 \cdots D_n$. Each factor of $\mathcal{H}$ system corresponds to a subsystem of $\mathcal{H}$.

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Density Matrix

The set of quantum states may be identified as the set of density matrices $\mathcal{D}$:

$$\mathcal{D} = \{\rho : N \times N \text{ positive definite matrix with trace 1, i.e., } tr\rho = 1\}$$

Separable and Entangled Quantum states

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A quantum state is called entangled if it is not separable. That is, the set of entangled quantum states is $\mathcal{E} = \mathcal{D} \setminus \mathcal{S}$

Peres-Horodecki Positive Partial Transpose Criterion

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Peres-Horodecki PPT Criterion

Let $\mathcal{PPT} = \{\rho \in \mathcal{D}(\mathcal{H}) : \ s.t. \ T_1(\rho) \geq 0\}$. Then

$$\mathcal{S} \subset \mathcal{PPT} \subset \mathcal{D}.$$

That is, a separable state must satisfy positive partial transpose criterion. Equivalently, a non-PPT state must be entangled.

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If $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$ or $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, then $\mathcal{S} = \mathcal{PPT}$.

How precise is the Peres-Horodecki PPT criterion (as a tool to detect the separability)?

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$dV = d\nu \times d\gamma$,
where $d\nu$ is some measure on Δ_1: the chamber of the simplex (of eigenvalues) $\{(\lambda_1, \cdots, \lambda_N) : \sum_i \lambda_i = 1, \lambda_i \geq 0\}$ with order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$,

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$$d\gamma = \prod_{i<j}^{1\cdots N} 2Re(U^{-1}dU)_{ij} Im(U^{-1}dU)_{ij}$$

where dU is the variation of unitary matrix U such that $U + dU$ is also an unitary matrix.

- The Hilbert-Schmidt measure:

$$dV_{HS} = \sqrt{N} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma;$$

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- Induced measure by partial trace on $\mathcal{H}_N \otimes \mathcal{H}_K$ ($K \geq N$):

$$dV_{N,K} = \prod_{i=1}^{N} \lambda_i^{K-N} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma;$$

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$$dV_\alpha = \prod_{i=1}^{N} \lambda_i^{\alpha-1} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma.$$

Measure induced by Metric on $\mathcal{D}$

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$$d_{HS}(\rho, \sigma) = \|\rho - \sigma\|_2 = \sqrt{tr(\rho - \sigma)^\dagger(\rho - \sigma)}.$$

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where the supremum runs over all the POVM (positive operator valued measurement) $\{\mathbb{E}_i\}$, that is, for some $1 \le k \le N$,

$$\sum_{i=1}^{k} \mathbb{E}_i = Id_N, \text{ and } \mathbb{E}_i = \mathbb{E}_i^\dagger,\ \mathbb{E}_i \ge 0,\ i = 1, \cdots, k.$$

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The partial trace process allows us to view states on $\mathcal{H}_N$ as a (pure) state on (much higher) dimensional space $\mathcal{H}_N \otimes \mathcal{H}_K$. Then the measures induced by partial trace may be considered as a projection of the $(NK-1)$ dimensional simplex of eigenvalues into simplex of $(N-1)$ dimension.

Comparison of α-volume with Hilbert-Schmidt volume

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and the Hilbert-Schmidt volume radii ratio of $\mathcal{K}$ to $\mathcal{D}$ is $\xi = \mathrm{VR}_{HS}(\mathcal{K},\mathcal{D}) = \mathrm{VR}_1(\mathcal{K},\mathcal{D})$.

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D. Ye, (J. Phys. A: Math. Theor. 2010)

Let $\alpha > 0$ be a (fixed) constant, $\alpha_{\max} = \max\{1, \alpha\}$, and $\alpha_{\min} = \min\{1, \alpha\}$. There exist universal constants $c_1, C_1 > 0$, s.t.,

$$c_1 \mathrm{VR}_{HS}(\mathcal{K}, \mathcal{D})^{\alpha_{\max}} \exp\left(\frac{(1 - \alpha_{\max}) \ln \ln(e/\xi)}{N^2 - 1}\right) \leq \mathrm{VR}_\alpha(\mathcal{K}, \mathcal{D})$$

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- $\alpha = 1$: D. Ye, (Journal of Mathematical Physics (JMP), 2009).

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Large number of small subsystems

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D. Ye, (JMP 2009 and J. Phys. A: Math. Theor. 2010)

There exist universal constants $c_3, c_3', C_3, C_3' > 0$, s.t., for $\alpha > 0$,

$$\frac{c_3}{N^{1/2+\alpha_D}}\leq \mathrm{VR}_B(\mathcal{S},\mathcal{D})\leq C_3\sqrt{\frac{(Dn\ln n)^{1/2}}{N^{1/2+\alpha_D}}};$$

$$\left(\frac{c_3'}{N^{1/2+\alpha_D}}\right)^{\max\{1,\alpha\}}\leq \mathrm{VR}_\alpha(\mathcal{S},\mathcal{D})\leq C_3'\left(\frac{(Dn\ln n)^{1/2}}{N^{1/2+\alpha_D}}\right)^{\min\{1,\alpha\}}.$$

G. Aubrun and S. J. Szarek proved:

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Small number of large subsystems

Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$ with (small) n.

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There exist universal constants $c_4, c_4', C_4, C_4' > 0$, s.t., for $\alpha > 0$,

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Numerical examples

Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$. The α-probability of $\mathcal{S}$ in $\mathcal{D}$ is defined as

$$\mathbb{P}_\alpha(\mathcal{S}, n, D) =: \frac{V_\alpha(\mathcal{S})}{V_\alpha(\mathcal{D})}.$$

$\mathbb{P}_2(\mathcal{S}, 8, 2) \leq 2.1 \times 10^{-1595}$, $\mathbb{P}_2(\mathcal{S}, 5, 3) \leq 1.52 \times 10^{-5301}$,
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The conditional α-probability of $\mathcal{S}$ given $\mathcal{PPT}$ is

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$\mathbb{P}_{1.1}(\mathcal{S}|\mathcal{PPT}, 12, 2) \leq 2.5 \times 10^{-2721940}$,
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References

1. D. Ye, *On the Bures volume of separable quantum states.* J. Math. Phys. 50 (2009) 083502;
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