Title: Random techniques and Bell inequalities

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Abstract: In this talk we will give an overview of how different probabilistic and quantum probabilistic techniques can be used to find Bell inequalities with large violation. This will include previous result on violation for tripartite systems and more recent results with Palazuelos on probabilities for bipartite systems. Quite surprisingly the latest results are the most elementary, but lead to some rather surprsing independence of entropy and large violation.

Random Techniques for Bell inequalities

Marius Junge

University of Illinois

with Palazuelos, Perez-Garica, Villanueovo, Wolff

- Violation for tripartite systems
- Violation for probabilities using quantum probability
- Violation and entanglement

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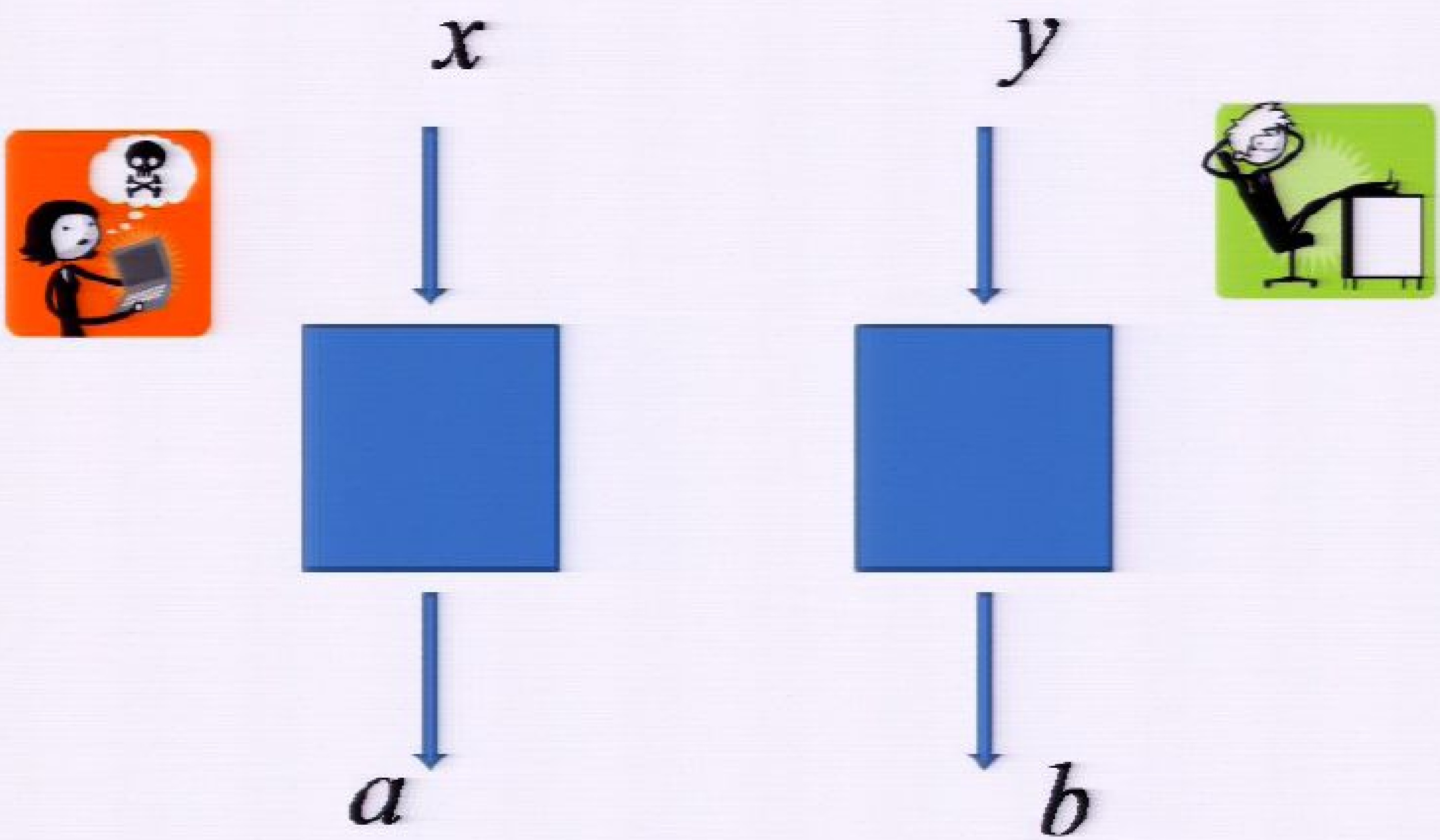

$$p(ab|xy)$$

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The quantum version of this experiment replaces the commuting variables $p_a^x(\lambda)$ and $q_b^y(\lambda)$ by commuting operators

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Theorem: (Bell) There are quantum probabilities which are not local.

Correlations versus probabilities

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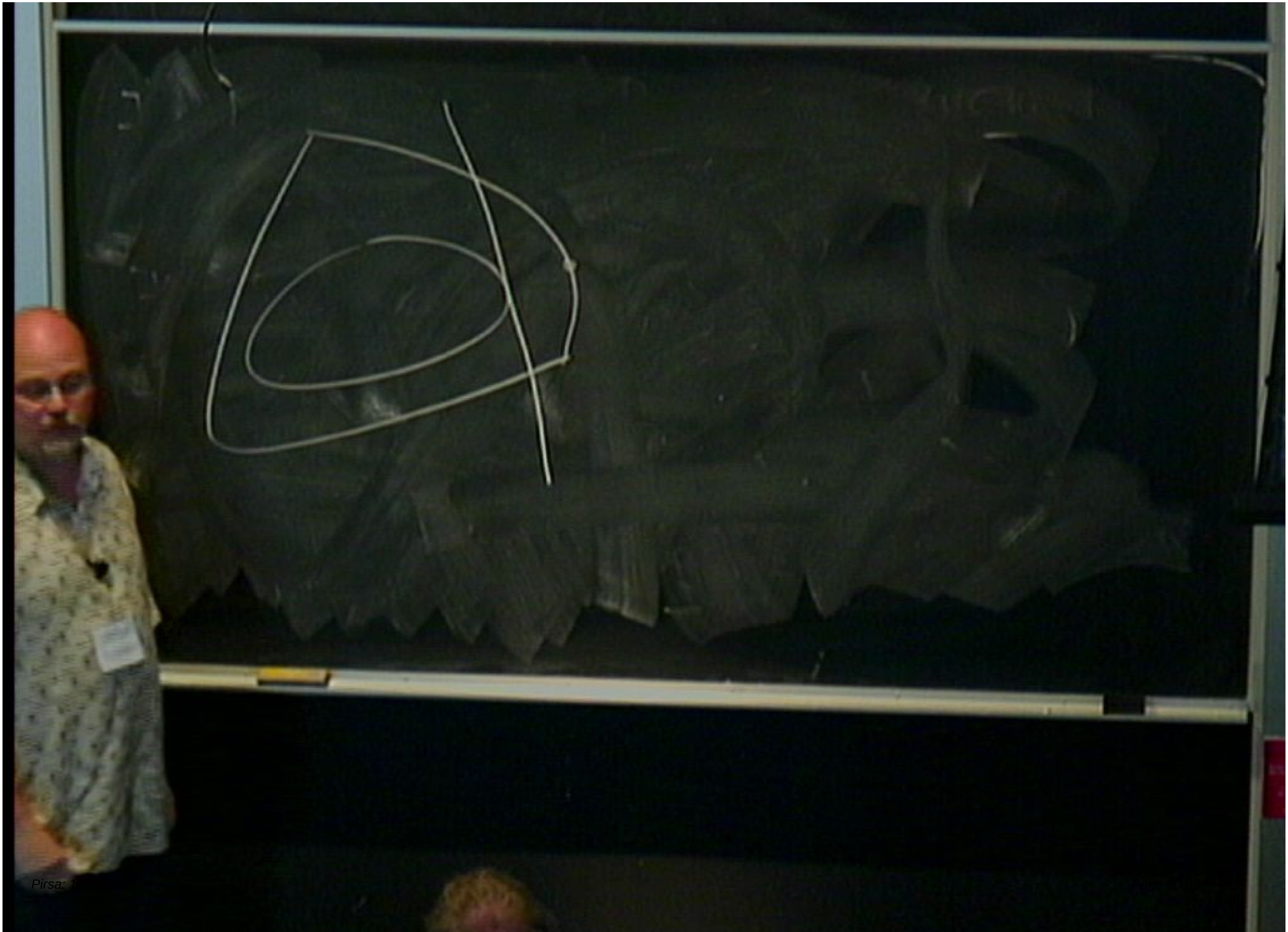
Testing with linear constraints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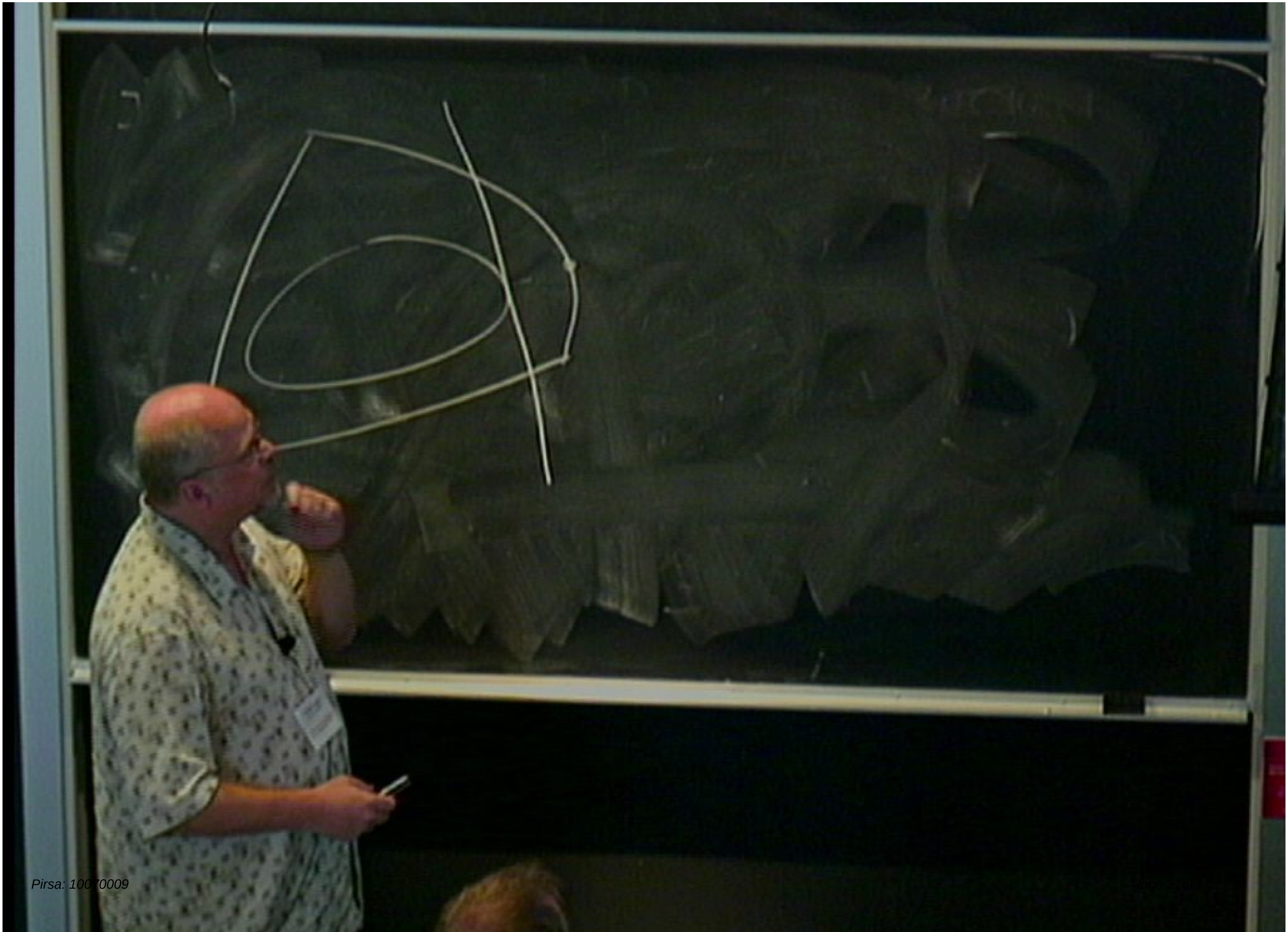

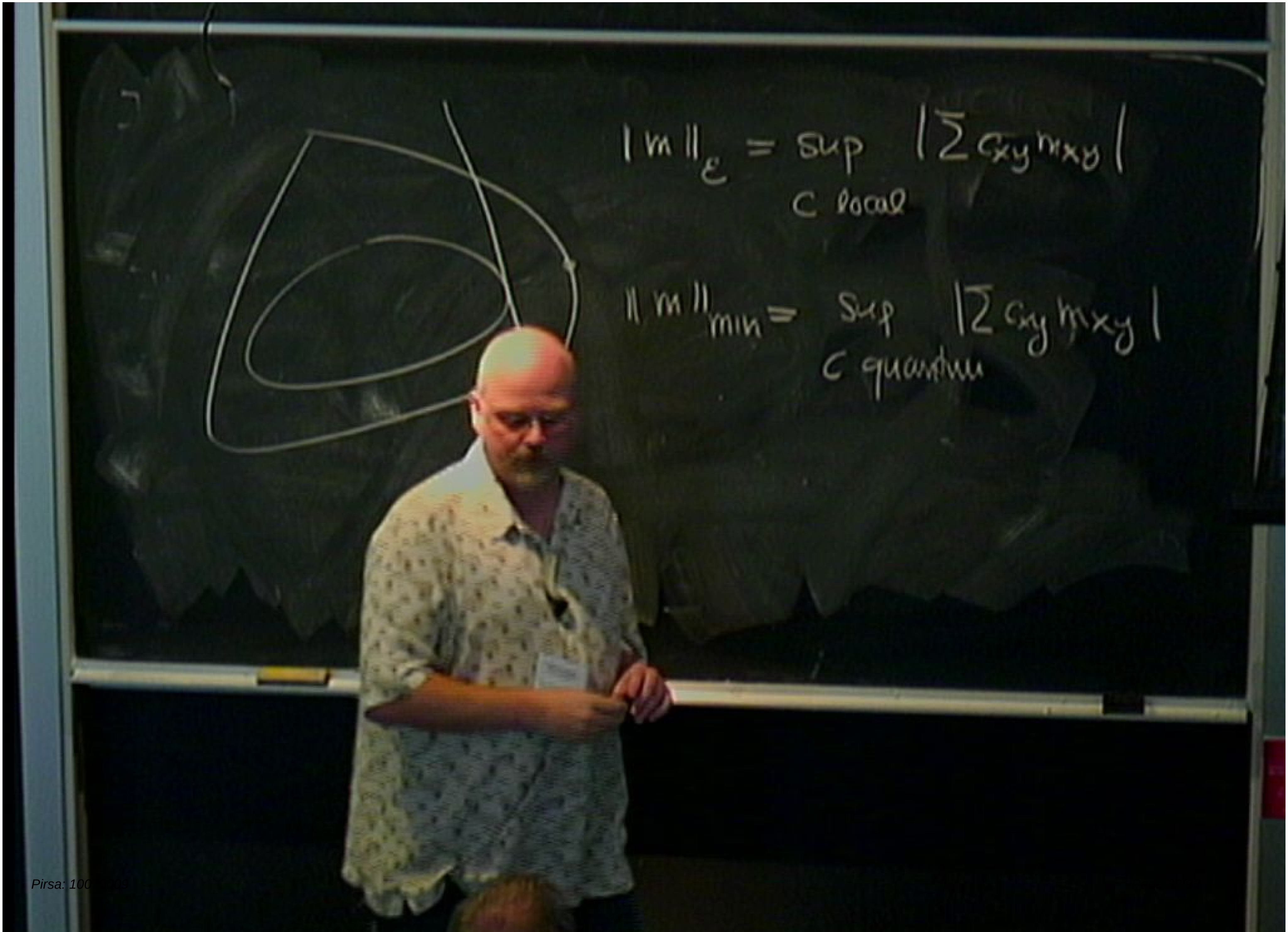

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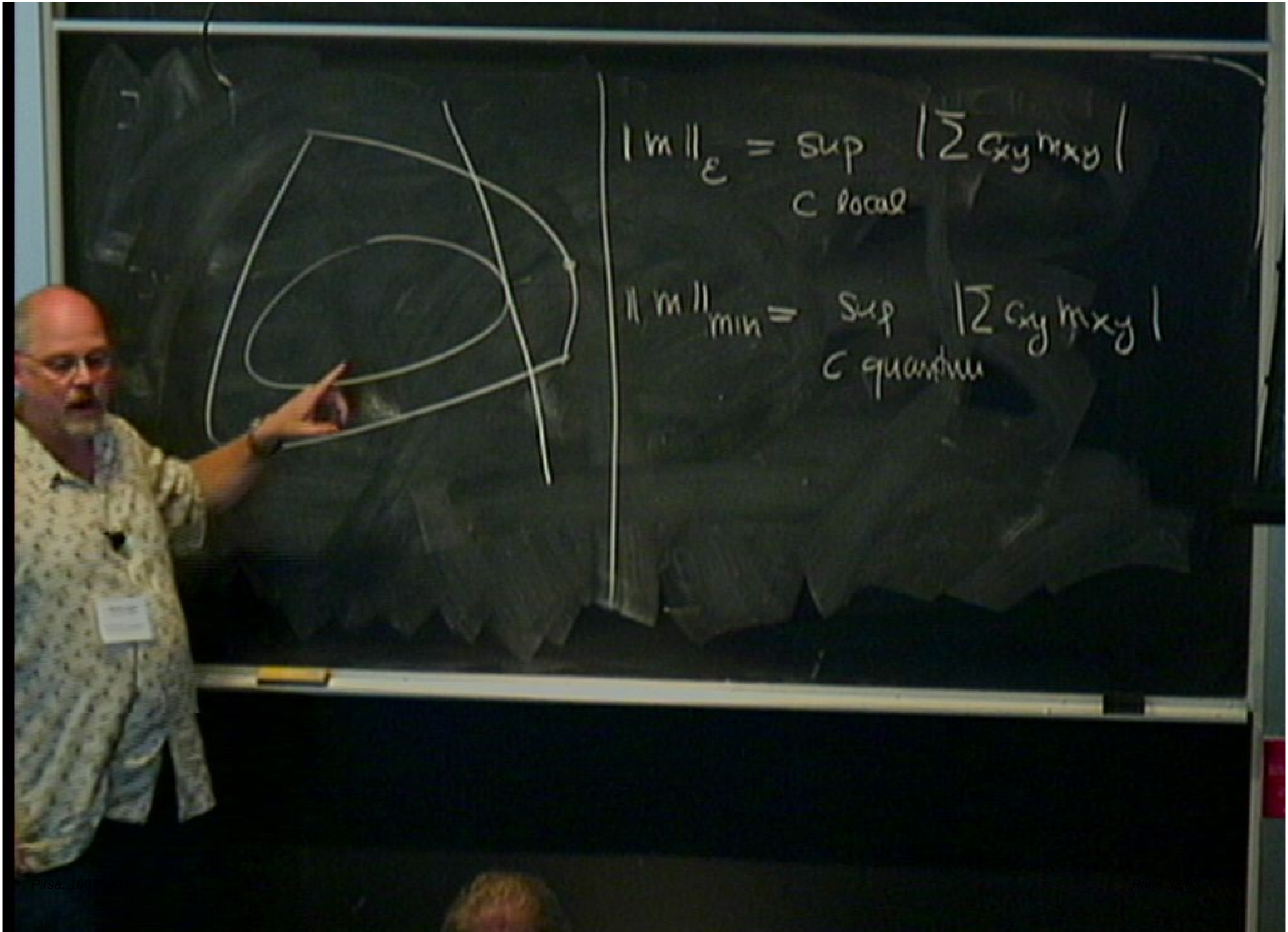
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- Let $m_{x,a,y,b}$ be a matrix. Then we can compare

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$$= \sup_{\sum_a T_a^x = 1 = \sum_b S_b^y, h} |\sum_{x,a,y,b} (h| T_a^x \otimes S_b^y) h) m_{x,y}| \, .$$

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Testing with linear constraints

- Let $m_{x,y}$ be any matrix. By convexity

$$\sup_{C \text{ local}} \left|\sum_{x,y} C_{x,y} m_{x,y}\right| \leq \sup_{\varepsilon_x=\pm1, \delta_y=\pm1} \left|\sum_{x,y} \varepsilon_x \delta_y m_{x,y}\right| .$$

- The quantum analogue is

$$\sup_{C \text{ quantum}} \left|\sum_{x,y} C_{x,y} m_{x,y}\right|$$

$$= \sup_{\|T_x\|, \|S_y\| \leq 1, \|h\| \leq 1} \left|\sum_{x,y} (h|(T_x \otimes S_y)h) m_{x,y}\right| .$$

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Observation: The extreme points of the unit ball in ℓ_∞^n (cube) are exactly ± 1 sequences.

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Now $M_n = S_\infty^n$ satisfies $M_n \otimes_{\min} M_k = M_{nk}$. Also there is a operator space version of π which produces the all states, not only separable states

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A Bell inequality with violation for a two-partite correlation is a matrix $m_{x,y}$ such that

$$\|\sum_{x,y} m_{x,y} e_x \otimes e_y\|_{\ell_1 \otimes_\varepsilon \ell_1} < \|\sum_{x,y} m_{x,y} e_x \otimes e_y\|_{\ell_1 \otimes_{\min} \ell_1}$$

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(J-P-10) There exists a rank n matrix in $\ell_1^n(\ell_\infty^n) \otimes \ell_1^n(\ell_\infty^n)$ *such that*

$$c\frac{\sqrt{n}}{\log n} \leq \frac{\|m\|_{\min}}{\|m\|_{\varepsilon}} \leq C\sqrt{n}\,.$$

Comments: Previous estimates of polynomial order $n^{-10^{-5}}$,

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- We lift this via an ultraproduct argument to find large perturbations of large matrices.

Random techniques-bipartite probabilities

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- Important observation: Maps of the from T^*T.

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- Step 1: Take a random n-dimensional subspace H_n of $\ell_1^n(\ell_\infty^n)$. Then H_n is $\sqrt{\log n}$ complemented and the norm is $(1+\varepsilon)$ hilbertian.
- By Grothendieck's inequality every bounded map $V : \ell_1(\ell_\infty) \to R_n = \mathrm{span}\{e_{k1} : 1 \leq k \leq n\}$ is completely bounded. Hence we obtain violation of the order

$$\| id : \ell_2^n \otimes_\varepsilon \ell_2^n \to R_n \otimes_{\min} R_n = R_{n^2} \| \sim \sqrt{n}\,.$$

Bell inequalities

Step 2: The matrix

$$m_{x,a,y,b} = \frac{1}{K}\sum_{k=1}^{n} \varepsilon_{x,a}^{k}\varepsilon_{y,b}^{k} \,.$$

Then $\|m\|_{\varepsilon} \ \leq \ C \log n.$

POVM's

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$$T_{x,a} = \begin{cases} \frac{1}{nK}\begin{pmatrix} 1 & \epsilon^1_{x,a} & \cdots & \epsilon^n_{x,a} \\ \epsilon^1_{x,a} & 1 & \cdots & \epsilon^1_{x,a}\epsilon^n_{x,a} \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon^n_{x,a} & \epsilon^n_{x,a}\epsilon^1_{x,a} & \cdots & 1 \end{pmatrix} & \text{for } a = 1, \cdots, n, \\ id - \sum_{a=1}^n T^a_x & \text{for } x = 1, \cdots, n. \end{cases}$$

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For a state $\psi = \sum_i \alpha_i |ii\rangle$ we define the indicator

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Theorem

Let $\mathrm{viol}(\psi) \;\geq\; 2$*. Then there exists Bell inequalities* $m_{x,a,y,b}$ *and POVM's such that*

$$\|m\|_\varepsilon \;\leq\; C \log n$$

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In short: Violation for almost all states (neither flat nor rank one).

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Given $\delta > 0$, we can find states with $\mathrm{Ent}(\psi) < \frac{\delta}{\log n}$ or $\mathrm{Ent}(\varphi) \ \geq \ \log n - \delta$ such that

$$\left|\sum_{x,y} m_{xy} \langle\psi| T_x^a \otimes S_y^b |\psi\rangle\right| \geq \frac{c\sqrt{n}}{\log^2 n}$$

$\|$ 1_ε

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