Title: Convergence rates for arbitrary statistical moments of random quantum circuits

Date: Jul 04, 2010 01:30 PM

URL: http://pirsa.org/10070007

Abstract: TBA

Random Matrices from Random Quantum Circuits: convergence rates for arbitrary statistical moments

Lorenza Viola
Dept. Physics & Astronomy
Dartmouth College

☞ Winton G. Brown and L.V., Phys. Rev. Lett. **104**, 250501 (2010)

I. Motivation & Background

Quandom randomness: useful...

- A powerful tool across applied mathematics, statistics, physics (disorder/quantum chaos)...

- A fundamental resource across quantum information theory...

→ A random U is very effective at 'concealing' quantum information:

$$|\psi\rangle|A\rangle \xrightarrow{U} |\tilde{\psi}\rangle$$

Random coding saturates the quantum capacity of the quantum erasure channel.

Bennett et al, PRL 1997.

Applications to superdense coding, remote state preparation, quantum data hiding and approximate encryption protocols, quantum state discrimination...

Arrow et al, PRL 2004; DiVincenzo et al, PRL 2004; Hayden et al, Commun. Math. Phys. 2004; ibid. 2006, Ambainis & Smith, LNCS 2004, Sen, Complexity 2006...

→ A random U can enforce 'typical behavior' (measure concentration) in large Hilbert spaces:

Applications to efficient (scalable) noise estimation in open quantum systems...

'Superoperator twirling' $\Lambda \longrightarrow U^\dagger \circ \Lambda \circ U \quad \Rightarrow \quad Tr[|\psi\rangle\langle\psi| U^\dagger \Lambda(U|\psi\rangle\langle\psi|U^\dagger)U]$

Overlap is an unbiased estimator of average gate fidelity/channel strength.

Emerson et al, J.Opt. B 2005

Quandom randomness: hard!

Randomly distributed unitaries ⟷ Circular Unitary Ensemble (CUE)

Dyson, J. Math. Phys. 1962; Mehta, Random Matrices, 1991.

- Def: A random U acts on a N-dimensional complex Hilbert space $\mathcal{H}$, dim $\mathcal{H} = N = 2^n$, and is drawn uniformly from the (unique) invariant measure on $\mathcal{U}(N)$:

$$\mu_H(U'U) = \mu(U), \quad \forall U, U' \in \mathcal{U}(N)$$ **Haar measure**

Fact: the volume of $\mathcal{U}(N)$ grows exponentially fast with n...

- Any circuit implementing an exact parametrization of CUE requires exponential resources:

→ Explicit procedure provided by the Hurwitz parametrization;

Pozniak et al, J. Phys. A 1998.

→ $O[N^2 (\log N)^3] = O[n^3\, 2^{2n}]$ single- and two- qubit gates from a fixed universal set, plus
2^{2n} independent classical 'input' parameters ⇒ exponential quantum and classical resources.

Genuine quantum randomness is too strong a notion

Quandom *pseudo*-randomness: random circuits

...Weaker requirement: seek efficient approximations of the Haar measure.

Emerson et al, Science 2003; Phys. Rev. A 2005.

- Def: A pseudo-random unitary ensemble is indistinguishable from CUE limited to a restricted set of statistical properties/test functions.

→ Pseudo-random operators can be generated by implementing a random quantum circuit = sequence $U_k \ldots U_1$ of gates drawn independently at random from a biased distribution $\mu(U)$.

- Thm: Provided that μ has support on a universal gate set, the measure over random circuits converges exponentially fast (in circuit depth k) to the Haar measure:

$$\mu_k(U) \equiv \mu^{*k}(U) = \int d\mu(U_k) \ldots d\mu(U_1)\, \delta(U - U_k \ldots U_1) \quad \Rightarrow \quad \lim_{k\to\infty} \mu_k(U) = \mu_H(U)$$

→ Rate of uniform convergence will in general decrease exponentially with n.

Quandom *pseudo*-randomness: unitary t-designs

Want: distributions on unitaries that match a *sub-set* of moments of the Haar measure.

Ambainis & Emerson, IEEE Conf. Comp. Complexity 2007; Dankert et al, Phys. Rev. A 2009; Law, PhD Thesis, arXiv:1006.5227.

- Def: (i) t-order moments of a distribution μ on $\mathcal{U}(N)$ = expectations of 'balanced' t-monomials in the matrix elements of U: $$E_\mu\{\underbrace{u_{ij}\dots u_{kl}}_{t}\,\underbrace{u^*_{mn}\dots u^*_{op}}_{t}\} \equiv \int_{U(N)} u_{ij}\dots u_{kl}\,u^*_{mn}\dots u^*_{op}\,d\mu(U)$$

  (ii) μ is an exact unitary t-design on $\mathcal{U}(N)$ if for all N^t x N^t complex matrices ϱ

$$E_\mu\{U^{\otimes t}\varrho(U^\dagger)^{\otimes t}\} = E_{\mu_H}\{U^{\otimes t}\varrho(U^\dagger)^{\otimes t}\} \quad \Leftrightarrow \quad E_\mu\{M(U)\} = E_{\mu_H}\{M(U)\}$$

→ No efficient constructions of exact unitary t-designs are known except $t=2, N=2^n$ [Clifford].

- Def: μ is an ε-approximate unitary t-design on $\mathcal{U}(N)$ if for all balanced monomials of degree t

$$\left|E_\mu\{M(U)\} - E_{\mu_H}\{M(U)\}\right| \le \frac{\varepsilon}{N^t}$$

→ Slightly different definitions depending on specific choice of norm...

A unitary t-design cannot be operationally distinguished from the Haar measure with respect to any test that uses at most t copies of a selected unitary U.

Unitary t-designs: applications

- Approximate unitary t-designs are a practical resource for a variety of quantum tasks:
  - → $t = 1$ suffices for implementing a private quantum channel... Ambainis et al, FOCS 2004.
  - → $t = 2$ suffices for quantum data hiding and protocols relying on Clifford twirls;

    Abeyesinghe at el, Proc. R. Soc. A 2010.

    ...estimation of Haar-averaged fidelity and selective quantum process tomography;

    Dankert at el, Phys. Rev. A 2009; Bendersky et al, Phys. Rev. Lett. 2009.

    ...generation of typical (subsystem) entanglement;

    Oliveira et al, Phys. Rev. Lett. 2007; Znidaric, Phys. Rev. A 2007.

    ...model internal 'thermalization' dynamics of an evaporating black hole;

    Hayden & Preskill, J. High En. Phys. 2007; Sekino & Susskind, ibid. 2007.
  - → $t = 4$ required for estimation of fidelity variance in randomized benchmarking.

    Magesan et al, arXiv:0910.1315.

  ... ...

- Def: An ε-approximate unitary t-design μ on $\mathcal{U}(N)$ is efficient if there exists an algorithm to sample and implement unitaries from μ that uses $O[\text{poly}(\log N, \log 1/\varepsilon)]$ resources.

Can we devise efficient constructions for arbitrary t?

Unitary t-designs: constructions

- Efficient constructions of state t-designs do not directly generalize to unitary t-designs.

Ambainis & Emerson, Comput. Complexity 2007.

An efficient construction of a ε-approximate unitary t-design on n qubits for any $t = O[n/\log n]$ may be obtained from quantum 't-copy tensor product expanders'.

Harrow & Law, arXiv:0811.2597.

Can t-designs emerge from generic physical models of random dynamics?...
In particular, how are random circuits related to unitary t-designs?...

- A random circuit on n qubits yields ε-approximate (1- and) 2-designs.

A random circuit of length poly$[n, t, \log 1/\varepsilon]$ is conjectured to yield an ε-approximate t-design.

Harrow & Law, Commun. Math. Phys. 2009.
[See also Diniz & Jonathan, arXiv:1006.4202]

→ Partial supporting evidence based on numerical analysis...

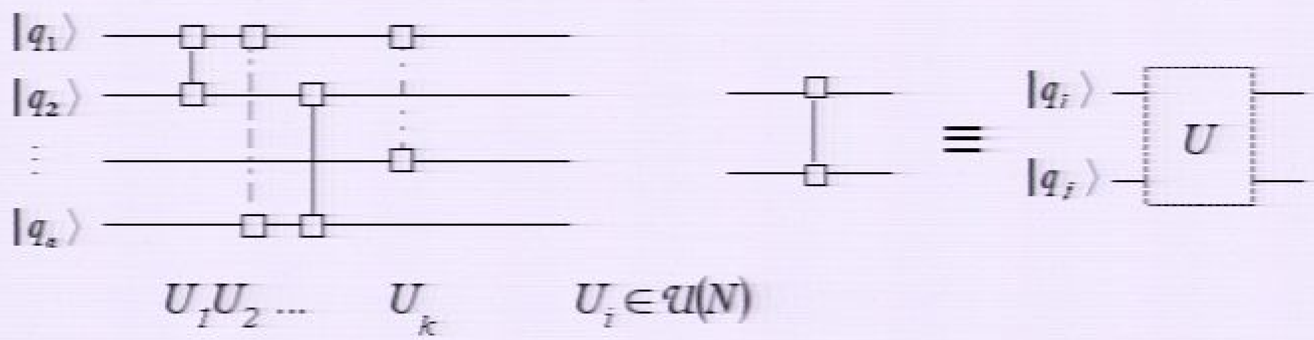
Arnaud & Braun, Phys. Rev. A 2008.

This talk

Can we rigorously characterize the convergence of statistical moments of arbitrary order t?

II. Main Result

Random circuit assumptions



Consider pseudo-random circuits consisting of a sequence $U_k \ldots U_1$ of n-qubit unitaries, constructed as follows:

(i) Pick a pair of distinct qubits (i, j) uniformly and independently at each time step k;

(ii) Apply a gate selected from a universal (single- and two- qubit) gate set on $\mathcal{U}(4)$, according to a probability distribution $\tilde{\mu}(U)$ that obeys

$$\tilde{\mu}(U) = \tilde{\mu}(U^\dagger), \quad \forall\, U \in \mathcal{U}(4)$$ 'Reversibility condition'

(iii) Optional condition: Assume that $\tilde{\mu}(U)$ is invariant under the subgroup $\mathcal{U}(2) \times \mathcal{U}(2) \subset \mathcal{U}(4)$.

- Def: (Reversible) permutationally invariant random circuits = class of circuits obeying (i)-(ii).
  (Reversible) locally invariant random circuits = class of circuits obeying (i)-(iii).

Asymptotic convergence rates

- Claim 1: For any permutationally invariant random circuit on n qubits, and for any fixed $t > 0$, arbitrary t-order moments converge exponentially fast (in circuit depth k) to their Haar value, with an asymptotic rate Δ_t that decreases linearly with n:

$$\Delta_t = \sum_{p=1}^{\infty} \frac{a_p}{n^p} = \frac{a_1}{n} + O\left(\frac{1}{n^2}\right),$$

for expansion coefficients $\{a_p\}$ that may in general depend on t.

- Claim 2: For any locally invariant random circuit on n qubits, and for any fixed $t > 0$, the asymptotic convergence rate Δ_t is independent on t to leading order in n.

Implication: Upper bound on scaling of minimum circuit length k_c for arbitrary t as n grows:

A random circuit of length $k_c \sim [n \log 1/\varepsilon]$ yields an ε-approximate t-design.

III. Technical Approach

Asymptotic convergence rates

- Claim 1: For any permutationally invariant random circuit on n qubits, and for any fixed $t > 0$, arbitrary t-order moments converge exponentially fast (in circuit depth k) to their Haar value, with an asymptotic rate Δ_t that decreases linearly with n:

$$\Delta_t = \sum_{p=1}^{\infty} \frac{a_p}{n^p} = \frac{a_1}{n} + O\left(\frac{1}{n^2}\right),$$

  for expansion coefficients $\{a_p\}$ that may in general depend on t.

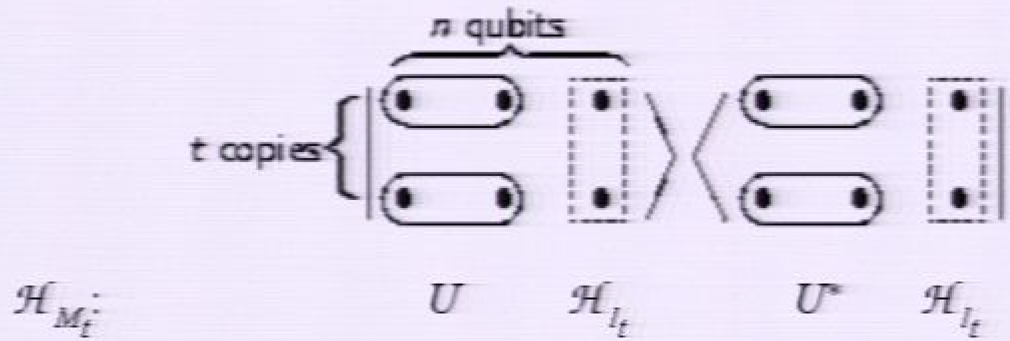
- Claim 2: For any locally invariant random circuit on n qubits, and for any fixed $t > 0$, the asymptotic convergence rate Δ_t is independent on t to leading order in n.

Implication: Upper bound on scaling of minimum circuit length k_c for arbitrary t as n grows:

A random circuit of length $k_c \sim [n \log 1/\varepsilon]$ yields an ε-approximate t-design.

III. Technical Approach

Moment superoperator



Physical n-qubit space: $\mathcal{H} = \bigotimes_i \mathcal{H}_{q_i}$

$$\dim(\mathcal{H}) = N = 2^n$$

Moment space: $\mathcal{H}_{M_t} = \mathcal{H}^{\otimes 2t} \equiv \mathcal{H}_{l_t}^{\otimes n}$

$$\dim(\mathcal{H}_{M_t}) \equiv D = N^{2t}$$

Local moment space: $\mathcal{H}_{l_t} = \mathcal{H}_{q_i}^{\otimes 2t}$

$$\dim(\mathcal{H}_{l_t}) \equiv d = 4^t$$

- Recall that t-order moments are expectations of monomials of degree t,t: $E_\mu\{\underbrace{u_{ij}\ldots u_{kl}}_{t}\ \underbrace{u^*_{mn}\ldots u^*_{op}}_{t}\}$

→ Action of the random circuit on t copies of the n-qubit space induces a superoperator on density operators on nt qubits:

$$\rho \rightarrow M_t[\mu](\rho) = E_\mu\{U^{\otimes t}\rho(U^\dagger)^{\otimes t}\} \equiv \int d\mu(U)\, U^{\otimes t}\rho\,(U^\dagger)^{\otimes t}$$

Moment superoperator

- In Liouville representation, $M_t[\mu]$ acts as a linear operator on D-dim 'operator kets' in $\mathcal{H}_{M_t}$:

$$A \equiv |A\rangle\rangle,\ \ A^\dagger \equiv \langle\langle A|$$
$$U A U^\dagger = U \otimes U^* |A\rangle\rangle,\ \ U \in \mathcal{U}(N^t)$$
$$\Rightarrow \quad M_t[\mu] = \int d\mu(U)\, U^{\otimes t} \otimes U^{*\otimes t}$$

→ The moment superoperator completely characterizes the set of t-order moments.

Convergence properties-I

- Properties of the moment superoperator:
  - Hermicity:
    Reversibility condition, $\tilde{\mu}(U)=\tilde{\mu}(U^{\dagger})$, implies $M_t[\mu]=M_t^{\dagger}[\mu]$ on $\mathcal{H}_{M_t}$.
  - Forward composition property:

$$M_t[\mu_k]=\int\prod_{i=1}^{k}d\mu(U_i)\prod_{i=1}^{k}\left(U_i^{\otimes t}\otimes U_i^{*\otimes t}\right)=\prod_{i=1}^{k}\int d\mu(U_i)U_i^{\otimes t}\otimes U_i^{*\otimes t}=\left(M_t[\mu]\right)^k\equiv M_t^k[\mu]$$

    Convolution product translates into matrix multiplication for moments of fixed order.
  - Eigenvalue structure:
    $|\lambda_i|\leq 1$ for all i, with the extremal eigenvalue $\lambda_0=1$ corresponding to the set of fixed points,

$$V_t=span\left\{|\phi\rangle\rangle\,|\left(U^{\otimes t}\otimes U^{*\otimes t}\right)|\phi\rangle\rangle=|\phi\rangle\rangle,\ \forall U\in\mathcal{U}(N)\right\}$$

- Convergence properties of the measure μ_k over the random circuit translates into convergence of t-order moments:

$$\lim_{k\rightarrow\infty}\left(M_t[\mu]\right)^k=\Pi_{V_t}=M_t[\mu_H],\quad\forall t.$$

[See also Lemma 3.2 in Harrow & Law, Commun. Math. Phys. 2009]

Convergence properties-II

- Approach of $M_t^k[\mu]$ to $M_t[\mu_H]$ depends only on non-extremal eigenvalues/eigenprojectors of $M_t[\mu]$. If k is sufficiently large,

$$\left\| M_t^k[\mu] - M_t[\mu_H] \right\| = \left\| \sum_{\lambda_i \neq 1} \lambda_i^k \Pi_i \right\| \approx |\lambda_1|^k \|\Pi_1\|,$$

$\lambda_1 =$ **Subdominant eigenvalue** $\equiv 1 - \Delta_t$

$\Delta_t =$ **Spectral gap**

→ Convergence is exponential in circuit length, with a rate Γ determined by the spectral gap:

$$\Gamma \equiv -\log \lambda_1 = -\log(1 - \Delta_t) \approx \Delta_t$$

- Knowledge of convergence rate allows to upper bound convergence time = minimum circuit length k_c required for $M_t^k[\mu]$ to become 'operationally indistinguishable' from $M_t[\mu_H]$ within ε.

→ Require that arbitrary nt-qubit density operators be close: $\left\| M_t^{k_c}[\mu](\rho) - M_t[\mu_H](\rho) \right\|_1 \leqslant \varepsilon$

Since $\left\| M_t^{k_c}[\mu](\rho) - M_t[\mu_H](\rho) \right\|_2 < \lambda_1^{k_c}$, we can bound 1-norm using 2-norm:

$$\left\| M_t^{k_c}[\mu](\rho) - M_t[\mu_H](\rho) \right\|_1 \leqslant 2^{nt} \lambda_1^{k_c}$$

→ Letting $2^{nt} \lambda_1^{k_c} \leqslant \varepsilon$ yields upper bound

$$k_c \leqslant \Delta_t^{-1} \left(\log 1/\varepsilon + nt \log 2 \right)$$

Problem reduces to determination of the spectral gap...

Moment superoperator structure

- For arbitrary permutationally invariant circuits, the moment superoperator obeys:

(1) Locality properties – $M_t[\mu]$ is a sum of operators with support only on $d \times d$ dim bi-local moment spaces $\mathcal{H}_{l_t} \otimes \mathcal{H}_{l_t}$ for a fixed qubit pair (i, j).

(2) Symmetry properties – $M_t[\mu]$ is invariant under arbitrary qubit re-labeling.

$$M_t[\mu] = \frac{2}{n(n-1)} \sum_{i<j} m_t^{ij}[\tilde{\mu}], \qquad m_t^{ij}[\tilde{\mu}] \to m_t[\tilde{\mu}] = \int d\tilde{\mu}(U) U^{\otimes t} \otimes U^{*\otimes t}, \quad U \in \mathcal{U}(4)$$

- $M_t[\mu]$ defines a permutationally invariant qudit Hamiltonian, $d=4^t$:

→ Outer product basis for operators on $\mathcal{H}_{l_t}$: $\{ b^i_{\alpha\beta} = |\alpha\rangle\rangle\langle\langle\beta|, \ \alpha, \beta = 1, \ldots, d \}$ ⇒

$$m_t^{ij}[\tilde{\mu}] = \sum_{\alpha\beta\gamma\delta=1}^{d} c_{\alpha\beta\gamma\delta} b^i_{\alpha\beta} b^j_{\gamma\delta}, \quad c_{\alpha\beta\gamma\delta} \in \mathbb{R}$$

→ $M_t[\mu]$ may be (exactly) rewritten in terms of $\mathcal{SU}(d)$ collective operators:

$$\begin{cases} B_{\alpha\beta} = \sum_{i=1}^{n} b^i_{\alpha\beta} \\ [B_{\alpha\beta}, B_{\gamma\delta}] = B_{\alpha\delta}\delta_{\beta\gamma} - B_{\beta\gamma}\delta_{\alpha\beta} \end{cases} \Rightarrow \quad M_t[\mu] = \frac{1}{n(n-1)} \sum_{\alpha\beta\gamma\delta} \left(B_{\alpha\beta} B_{\gamma\delta} - \delta_{\beta\gamma} B_{\alpha\delta} \right)$$

Lipkin-Meshkov-Glick model Hamiltonians

Collective Hamiltonians have a long/rich history in many-body physics...

- Paradigmatic case: Quadratic functions of $\mathcal{SU}(2)$ collective spin operators $\Rightarrow$ Spin ½ LMG model

Lipkin, Meshkov, Glick, Nucl. Phys. 1965.

$$H=\frac{1}{n}\left(\gamma_x S_x^2+\gamma_y S_y^2+\gamma_x S_x^2\right)+h S_z, \quad S_a=\frac{1}{2}\sum_{i=1}^{n}\sigma_a, \quad a=x,y,z;\ \gamma_a,h\in\mathbb{R}$$

→ Infinitely-coordinated, exactly solvable via (algebraic) Bethe Ansatz;

→ Low-energy spectrum well understood in the thermodynamic limit $n\to\infty$;

Ribeiro et al, Phys. Rev. Lett. 2007.

→ Mapping to LMG model exploited to analyze convergence of 2nd -order moments:

$$M_2[\mu]=\frac{1}{n(n-1)}c_{xy}\left(S_x^2-S_y^2\right)+\frac{1}{n}c_z S_z$$

Znidaric, Phys. Rev. A 2008.

- For generic t, $M_t[\mu]$ corresponds to a d-level extension of the standard LMG model.

Gilmore, J. Math. Phys. 1979.

$$[M_t[\mu],S_n]=0, \quad \forall t$$

↑ Symmetric group

→ State space $\mathcal{H}_{M_t}$ carries (reducible) n-fold tensor product representation of $\mathcal{SU}(d)$;

→ Each eigenvector of $M_t[\mu]$ belongs to a fixed irrep.

Ground-state manifold

- For ***any*** LMG Hamiltonian, the exact ground state is given in the thermodynamic limit by a variational mean-field Ansatz over $\mathcal{SU}(d)$ coherent states of the totally symmetric S_n-irrep:

$$\lim_{n\to\infty} min_{SU(d)} \langle\langle CS^{(n)}|H|CS^{(n)}\rangle\rangle = \langle\langle GS|H|GS\rangle\rangle, \quad |CS^{(n)}\rangle\rangle = |\phi\rangle\rangle^{\otimes n}$$

Gilmore, J. Math. Phys. 1979.

Stronger result: Mean-field extremal eigenspace of $M_t[\mu]$ is exact for any finite n...

→ Extremal eigenspace V_t consists of fixed points under t-fold tensor products of $\mathcal{U}(2^n)$:

$$|\phi\rangle\rangle \in V_t \Leftrightarrow U^{\otimes t}\phi U^{\dagger\otimes t} = \phi, \ \forall U \in \mathcal{U}(2^n)$$

→ Each such operator must be a linear combination of permutations of the t copies of $\mathcal{H}$ in $\mathcal{H}_{M_t}$:

$$|\sigma^{(n)}\rangle\rangle = \sum_{i_1\ldots i_t=1}^{N} |i_1\ldots i_t\rangle\langle i_{\sigma(1)}\ldots i_{\sigma(t)}|, \quad \sigma\in S_t$$

→ Permutations of copies correspond to product states relative to decomposition $\mathcal{H}_{M_t} = \mathcal{H}_{I_t}^{\otimes n}$:

$$|\sigma^{(n)}\rangle\rangle = \left(\sum_{i_1\ldots i_t=0,1} |i_1\ldots i_t\rangle\langle i_{\sigma(1)}\ldots i_{\sigma(t)}|\right)^{\otimes n} \equiv \left(|\sigma\rangle\rangle\right)^{\otimes n}$$

- For any LMG Hamiltonian corresponding to a (reversible) random quantum circuit on n qubits, the ground-state manifold is exactly spanned by (degenerate) factorized eigenstates.

→ Highly non generic: related to physics of 'ground-state factorization'...

Kurmann et al. Physica A 1982; Giampaolo et al. Phys. Rev. B 2009

Low-lying excitations

- Lowest excitation energy in the large-n limit may determined by a variational mean-field Ansatz.
  - → For the totally symmetric irrep, start from realizing $\mathcal{U}(d)$ in terms of Schwinger bosons:

$$B_{\alpha\beta}=a_\alpha^\dagger a_\beta,\ \ \alpha,\beta=1,\ldots,d \quad \Rightarrow \quad M_t[\mu]=\frac{1}{n(n-1)}\sum_{\alpha\beta\gamma\delta=1}^{d} c_{\alpha\beta\gamma\delta}\, a_\alpha^\dagger a_\gamma^\dagger a_\beta a_\delta$$

  - → Regard the boson mode that corresponds to a(ny) ground state $\left(|\sigma\rangle\rangle\right)^{\otimes n}$ as 'frozen' in the vacuum for a generalized Holstein-Primakoff transformation:

$$\theta(n)\equiv\left(n-\sum_{\alpha\neq\sigma}a_\alpha^\dagger a_\alpha\right)^{1/2}, \quad a_\sigma\rightarrow\theta(n), \quad a_\sigma^\dagger\rightarrow\theta(n)$$

Okubo, J. Math. Phys. 1975.

Only terms up to the leading order in $1/n$ need to be kept in square root $\Rightarrow$

$$M_t[\mu]=1-\frac{1}{n}\sum_{\alpha\beta=1}^{d}E_{\alpha\beta}\,a_\alpha^\dagger a_\beta+O\left(\frac{1}{n^2}\right), \quad E_{\alpha\beta}=2\left(\delta_{\alpha\beta}-\langle\langle\sigma\alpha|m_t|\sigma\beta\rangle\rangle-\langle\langle\sigma\alpha|m_t|\beta\sigma\rangle\rangle\right)$$

- Large-n expansion: to order $1/n$, lowest excitations correspond to single-boson excitations

$$|n_\sigma=n-1, n_\alpha=1\rangle\rangle=\frac{1}{\sqrt{n}}\left(|\sigma\ldots\sigma\alpha\rangle\rangle+\ldots+|\alpha\sigma\ldots\sigma\rangle\rangle\right),$$

with excitation energies determined by the eigenvalues of the d x d matrix $E_{\alpha\beta}$.

Spectral gap scaling

- Spectral gap is determined to leading order by $a_1 \equiv \min\{\mathrm{eig}[E_{\alpha\beta}]\}$ $\Rightarrow$ Claim 1 is established,

$$\Delta_t = \sum_{p=1}^{\infty} \frac{a_p}{n^p} = \frac{a_1}{n} + O\left(\frac{1}{n^2}\right),$$

provided that:

(1) Leading-order coefficient is non-vanishing – Formal proof that $a_1 > 0$ follows from invariance properties of bi-local moment superoperator m_t.

http://link.aps.org/supplemental/10.1103/PhysRevLett.104.250501

(2) Mean-field expansion captures all low-lying excitations – Rigorous proof not available (?)...

→ Mean-field Ansatz extremely well supported (analytically and numerically) for LMG models.

Ortiz et al, Nucl. Phys. B 2005; Dusuel & Vidal, Phys. Rev. B 2005;
Leyraz & Heiss, Phys. Rev. Lett. 2005; Ribeiro et al, ibid. 2007...

→ Rigorous proof *might* be possible for LMG models supporting ground-state factorization (?)

- Under the additional assumption of local invariance, Claim 2 follows upon showing that matrix elements determining the spectral gap attain their maximum at $t = 2$...

http://link.aps.org/supplemental/10.1103/PhysRevLett.104.250501