Title: Anderson localization and adiabatic quantum optimization

Date: Jul 04, 2010 11:15 AM

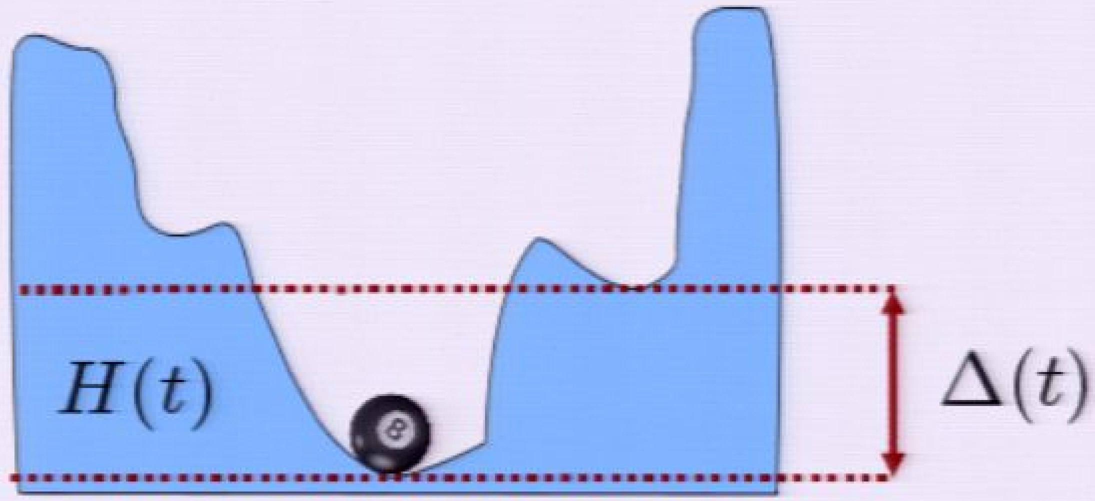
URL: http://pirsa.org/10070006

Abstract: Understanding NP-complete problems is a central topic in computer science. This is why adiabatic quantum optimization has attracted so much attention, as it provided a new approach to tackle NP-complete problems using a quantum computer. The efficiency of this approach is limited by small spectral gaps between the ground and excited states of the quantum computer's Hamiltonian.
We show that the statistics of the gaps can be analyzed in a novel way, borrowed from the study of quantum disordered systems in statistical mechanics. It turns out that due to a phenomenon similar to Anderson localization, exponentially small gaps appear close to the end of the adiabatic algorithm for large random instances of NP-complete problems. We show that this effect makes adiabatic quantum optimization fail, as the system gets trapped in one of the numerous local minima. We will also discuss recent developments including the effect of the exponential number of solutions and Hamiltonian path change.

Joint work with Boris Altshuler and Hari Krovi
Based on arXiv:0908.2782 and arXiv:0912.0746

Adiabatic evolution

NEC Laboratories America
Relentless passion for innovation



Slowly varying $H(t) \rightarrow$ Stays close to ground state

Probability of excitation depends on

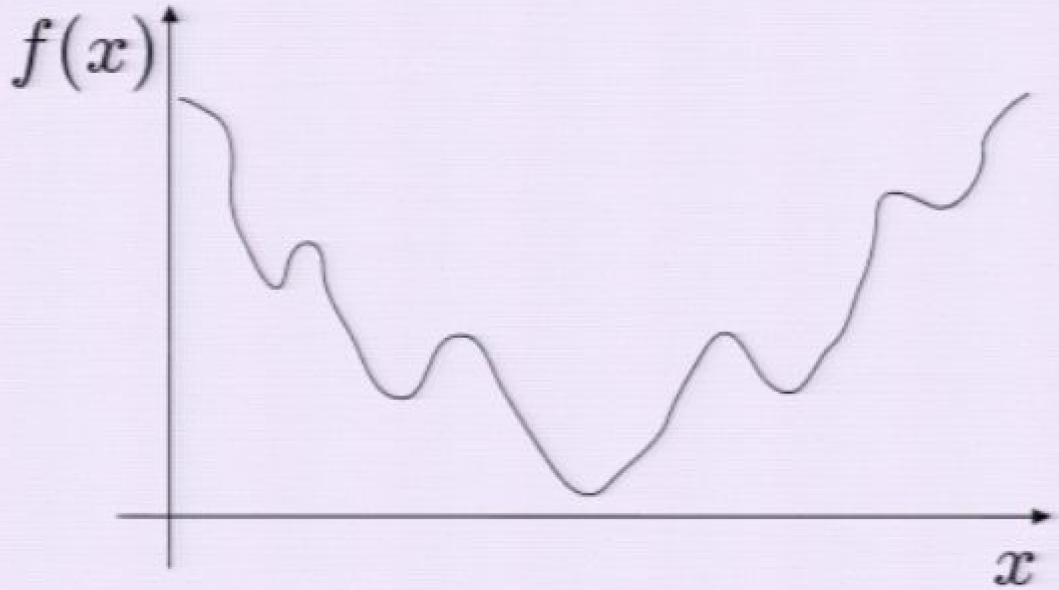
- Total time T (slower is better)
- Gap $\Delta(t)$ (larger gap is better)

Adiabatic quantum optimization

NEC Laboratories America
Relentless passion for innovation

[Farhi *et al.* '00]

- Problem: Find minimum of a function *f(x)*

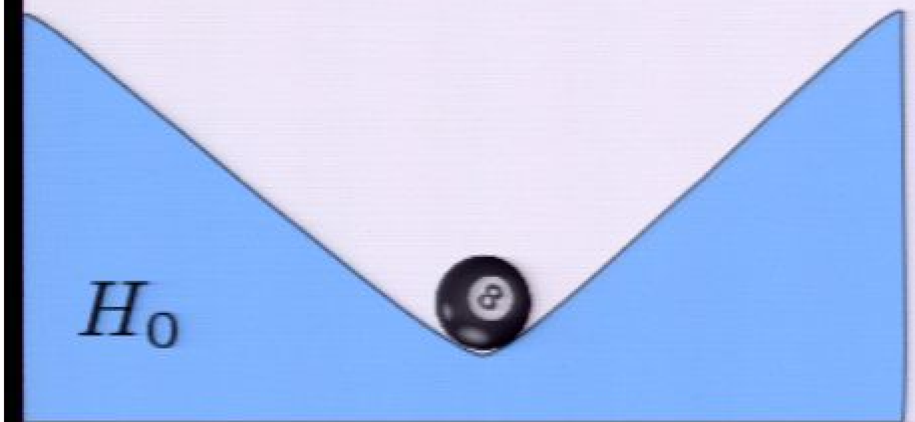


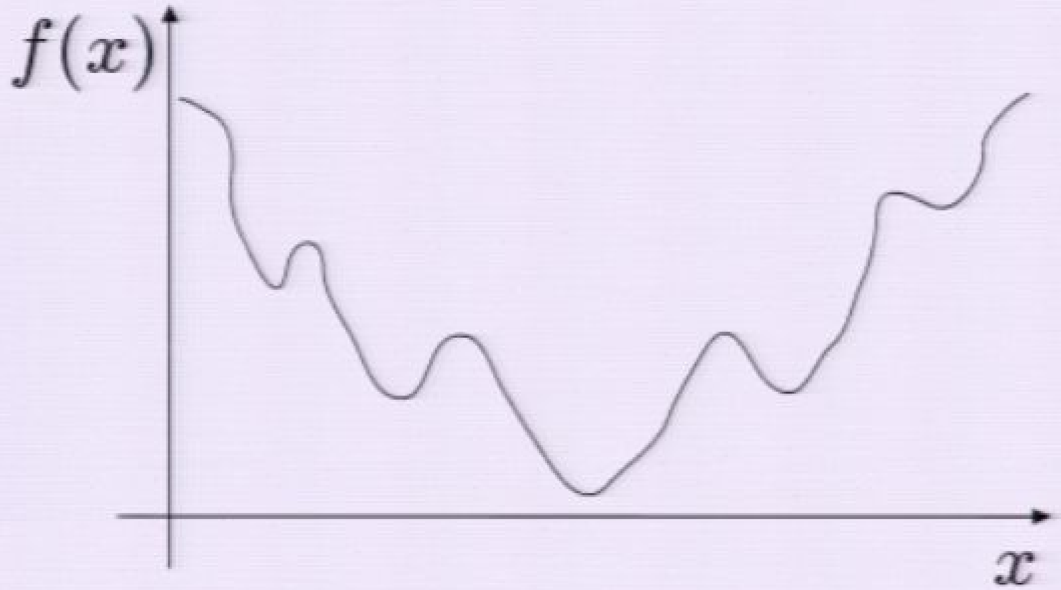
Adiabatic quantum optimization

[Farhi *et al*. '00]

- Problem: Find minimum of a function *f(x)*

    1) Choose initial Hamiltonian H_0 with known ground state





Adiabatic quantum optimization

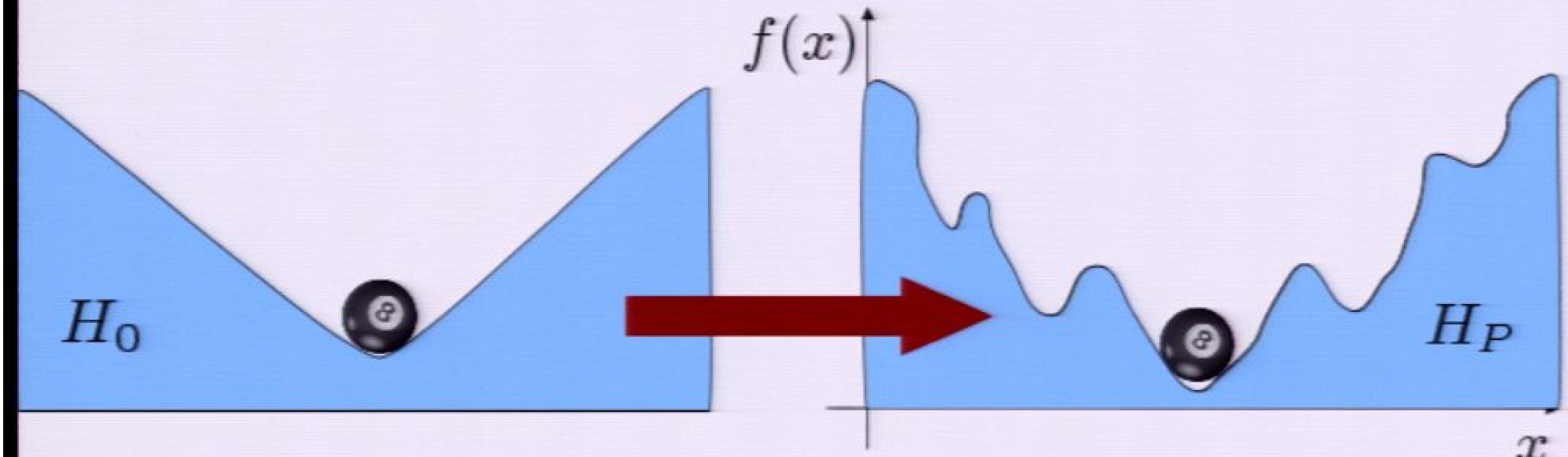
NEC Laboratories America
Relentless passion for innovation

[Farhi *et al.* '00]

- Problem: Find minimum of a function *f(x)*

1) Choose initial Hamiltonian H_0 with known ground state

2) Change Hamiltonian to H_P "matching" *f(x)*



$$H(s) = (1 - s)H_0 + sH_P$$

where $s(t) = t/T$

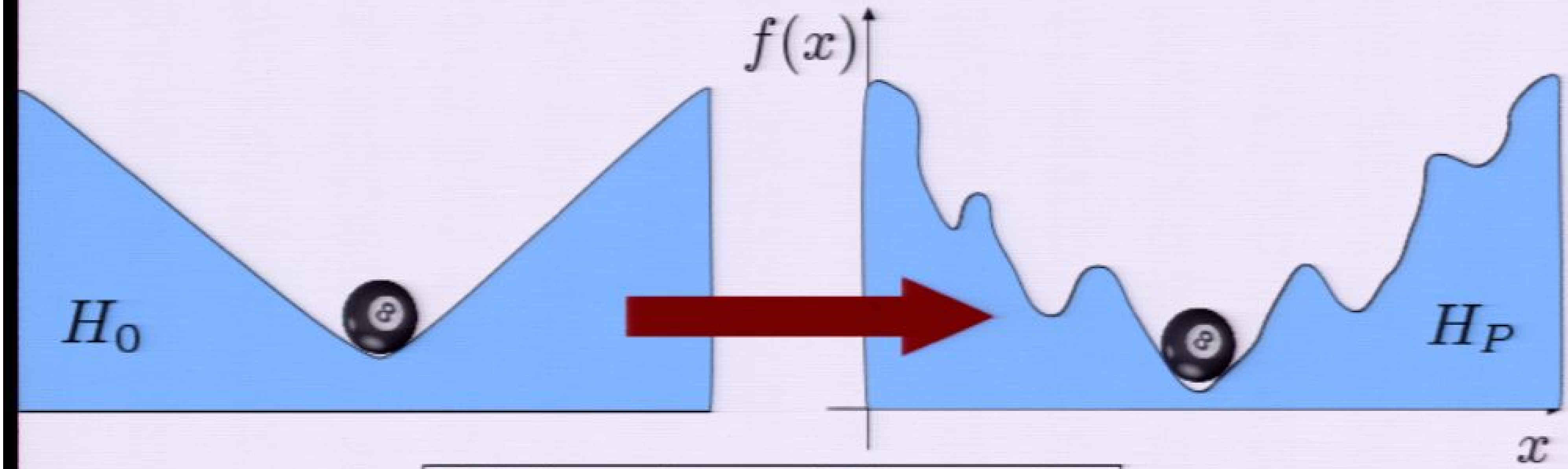
Adiabatic quantum optimization

[Farhi *et al.* '00]

- Problem: Find minimum of a function *f(x)*

  1) Choose initial Hamiltonian H_0 with known ground state

  2) Change Hamiltonian to H_P "matching" *f(x)*



$$H(s) = (1 - s)H_0 + sH_P$$

where $s(t) = t/T$

3) T large enough ⇒ measuring reveals the minimum

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

Adiabatic quantum optimization

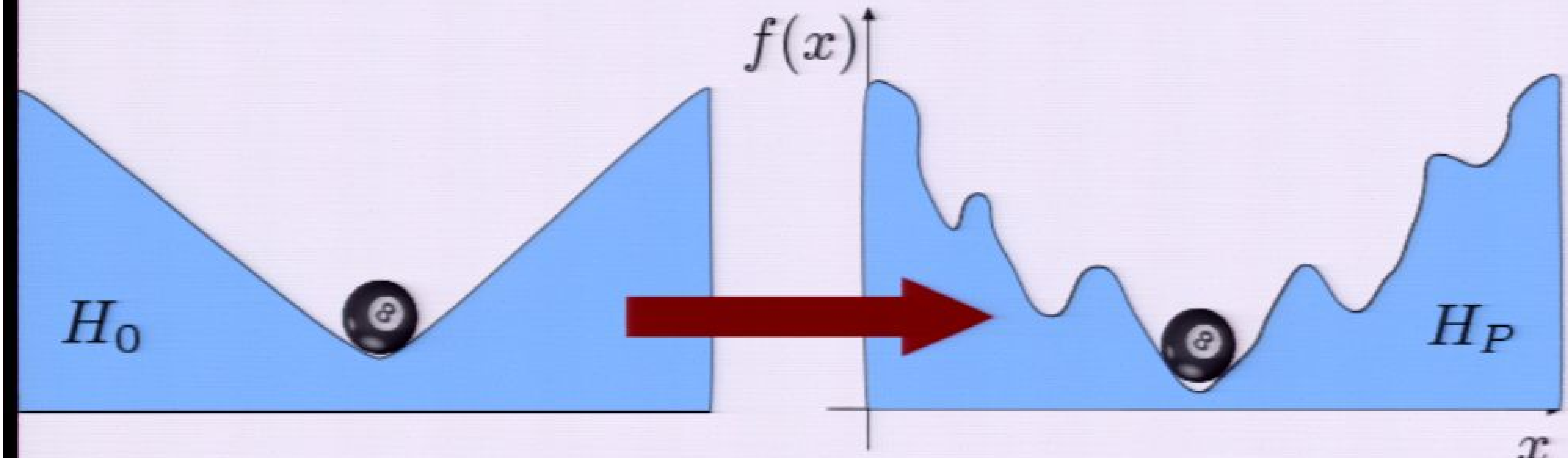
NEC Laboratories America
Relentless passion for innovation

[Farhi *et al.* '00]

- Problem: Find minimum of a function *f(x)*

1) Choose initial Hamiltonian H_0 with known ground state

2) Change Hamiltonian to H_P "matching" *f(x)*



$$H(s) = (1 - s)H_0 + sH_P$$

where $s(t) = t/T$

3) T large enough ⇒ measuring reveals the minimum

NEC Laboratories America
Relentless passion for innovation

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

NEC Laboratories America
Relentless passion for innovation

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

Original motivation: NP-complete problems!

NEC Laboratories America
Relentless passion for innovation

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

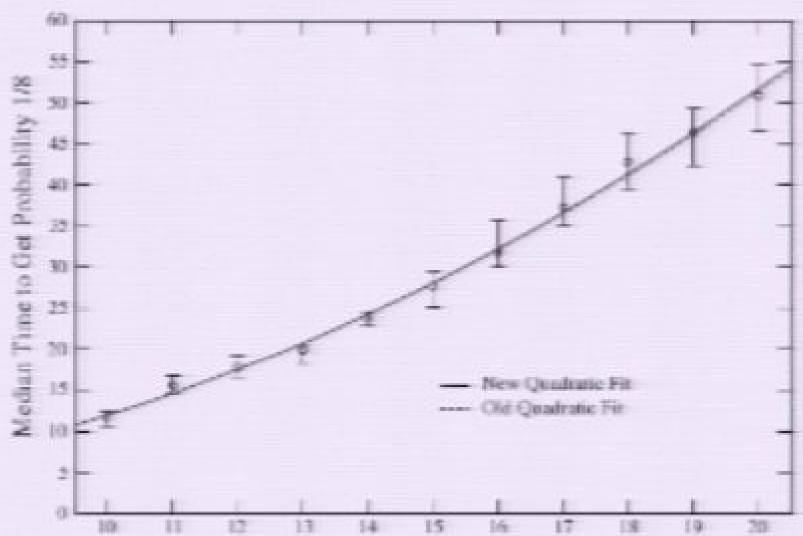
[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

Original motivation: NP-complete problems!

- Numerical simulations: promising scaling

[Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.*'08]



How powerful is it?

NEC Laboratories America
Relentless passion for innovation

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

Original motivation: NP-complete problems!

- Numerical simulations: promising scaling

[Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.*'08]

- But exponentially small gap
  - for bad choice of initial Hamiltonian [Znidaric-Horvat'06,Farhi *et al.*'08]
  - for specifically designed hard instances [vanDam-Vazirani'03,Reichardt'04]

NEC Laboratories America
Relentless passion for innovation

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

Original motivation: NP-complete problems!

- Numerical simulations: promising scaling [Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.*'08]
- But exponentially small gap
  - for bad choice of initial Hamiltonian [Znidaric-Horvat'06,Farhi *et al.*'08]
  - for specifically designed hard instances [vanDam-Vazirani'03,Reichardt'04]

But maybe typical gaps are only polynomial?

Exact-Cover 3 (EC3)

NEC Laboratories America
Relentless passion for innovation

- NP complete problem (similar to 3-SAT)

NEC Laboratories America
Relentless passion for innovation

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

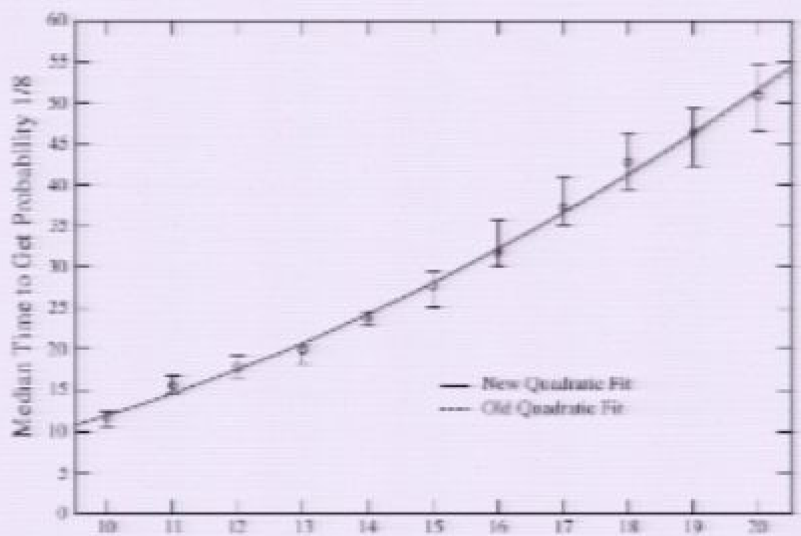
[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

- It is universal for quantum computation [Aharonov *et al.*'05]

Original motivation: NP-complete problems!

- Numerical simulations: promising scaling

[Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.*'08]



Exact-Cover 3 (EC3)

NEC Laboratories America
Relentless passion for innovation

- NP complete problem (similar to 3-SAT)

Exact-Cover 3 (EC3)

NEC Laboratories America
Relentless passion for innovation

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$

$$\Leftrightarrow 100, 010 \text{ or } 001$$

Exact-Cover 3 (EC3)

NEC Laboratories America
Relentless passion for innovation

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$

$$\Leftrightarrow 100, 010 \text{ or } 001$$

- Problem: Find assignment $\vec{x}$ satisfying all clauses

Exact-Cover 3 (EC3)

NEC Laboratories America
Relentless passion for innovation

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$

$$\Leftrightarrow 100, 010 \text{ or } 001$$

- Problem: Find assignment $\vec{x}$ satisfying all clauses

➡ Minimize function $f(\vec{x}) = \sum_C (x_{i_C} + x_{j_C} + x_{k_C} - 1)^2$

NEC Laboratories America
Relentless passion for innovation

Exact-Cover 3 (EC3)

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$
$$\Leftrightarrow 100, 010 \text{ or } 001$$

- Problem: Find assignment $\vec{x}$ satisfying all clauses

➡ Minimize function

$$f(\vec{x}) = \sum_C (x_{i_C} + x_{j_C} + x_{k_C} - 1)^2$$
$$= M - \sum_i B_i x_i + 2 \sum_{i,j} J_{ij} x_i x_j$$

#clauses

#clauses with bit i

#clauses with bits i,j

Random instances

NEC Laboratories America
Relentless passion for innovation

- Pick M clauses uniformly at random

Random instances

NEC Laboratories America
Relentless passion for innovation

- Pick M clauses uniformly at random
- Hardness depends on clauses-to-bit ratio: $\alpha = \dfrac{M}{N}$

Exact-Cover 3 (EC3)

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$

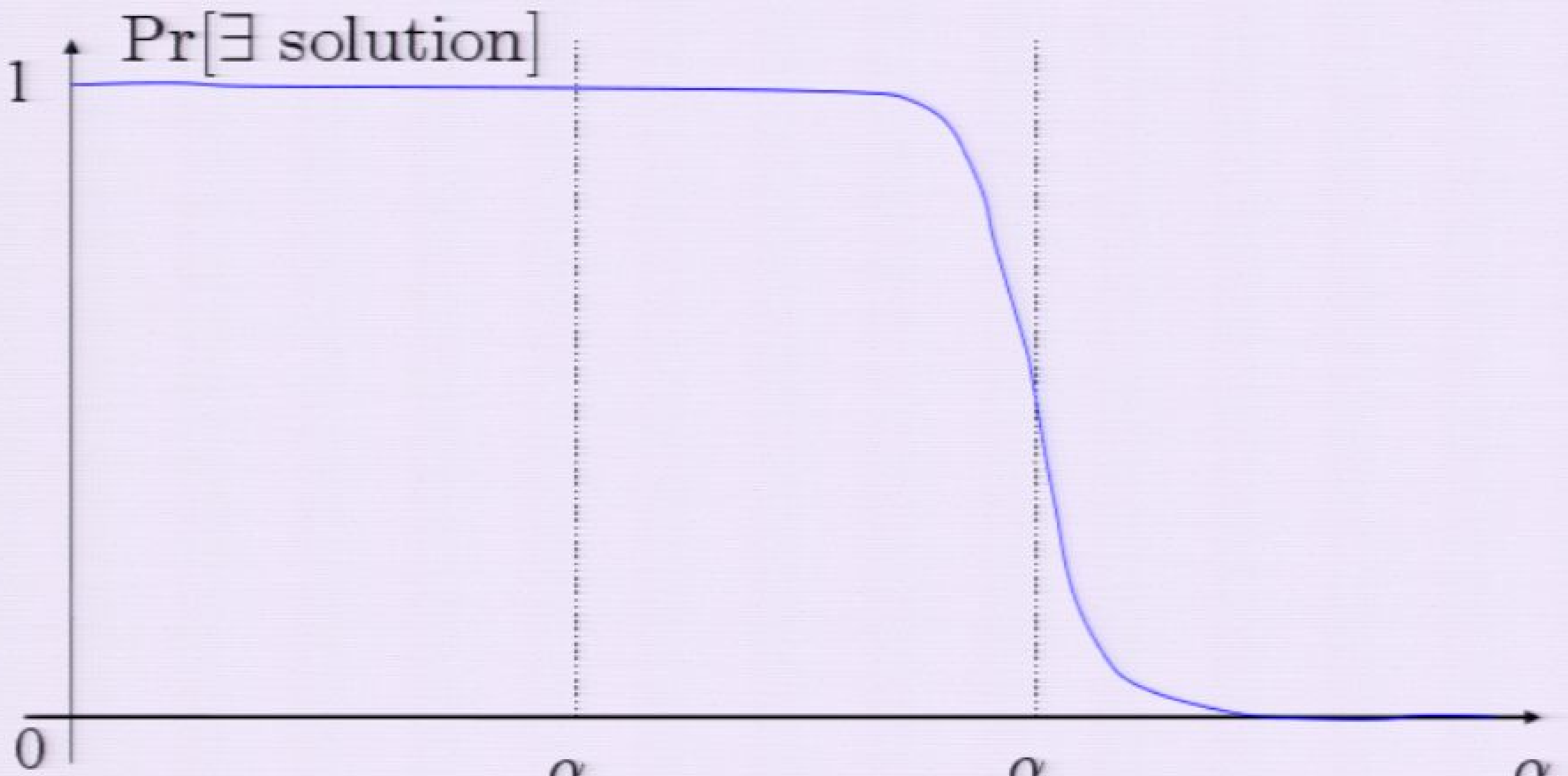
$$\Leftrightarrow 100, 010 \text{ or } 001$$

- Problem: Find assignment $\vec{x}$ satisfying all clauses

➡ Minimize function $f(\vec{x}) = \sum_C (x_{i_C} + x_{j_C} + x_{k_C} - 1)^2$

Random instances

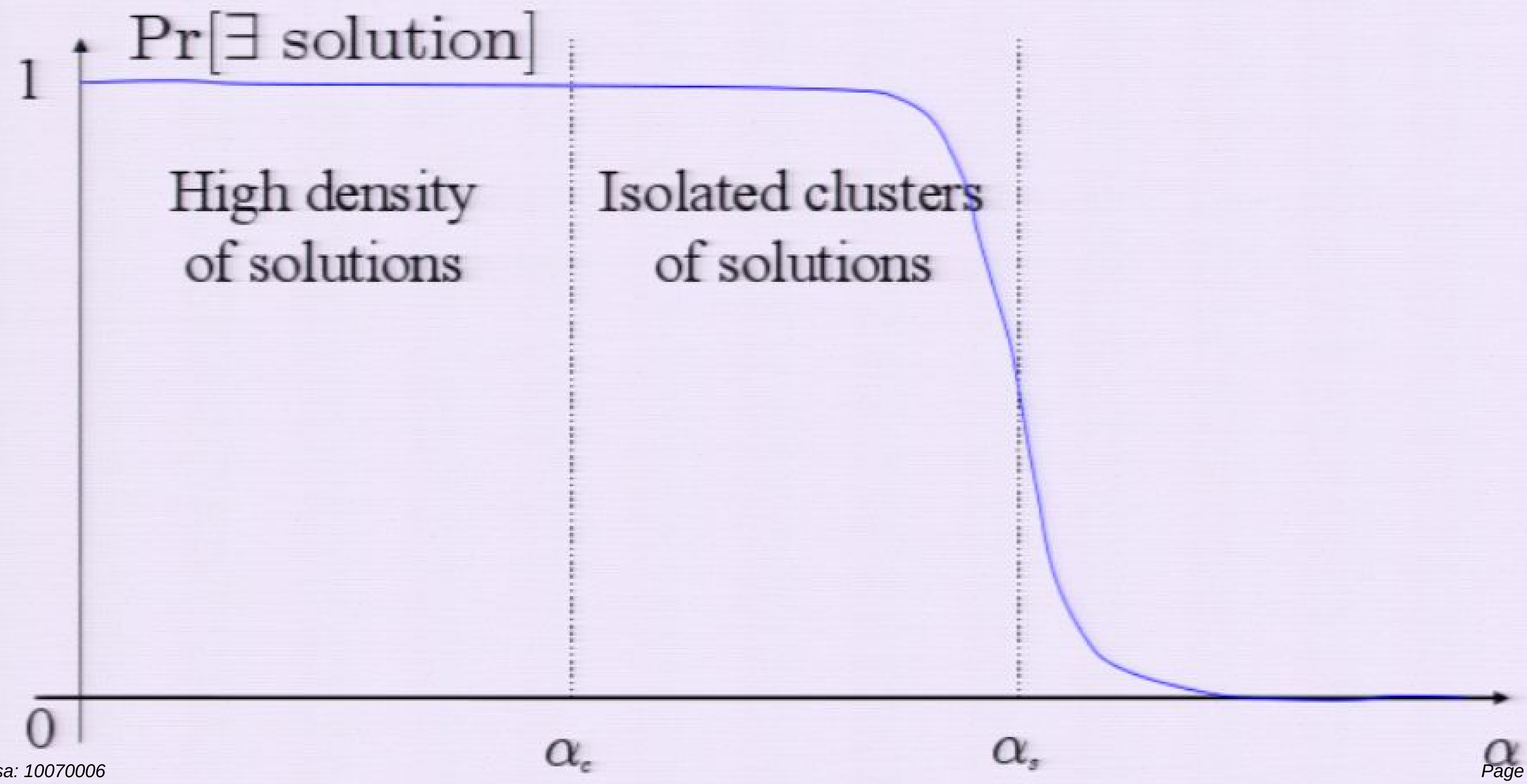
- Pick M clauses uniformly at random
- Hardness depends on clauses-to-bit ratio: $\alpha = \frac{M}{N}$



NEC Laboratories America
Relentless passion for innovation

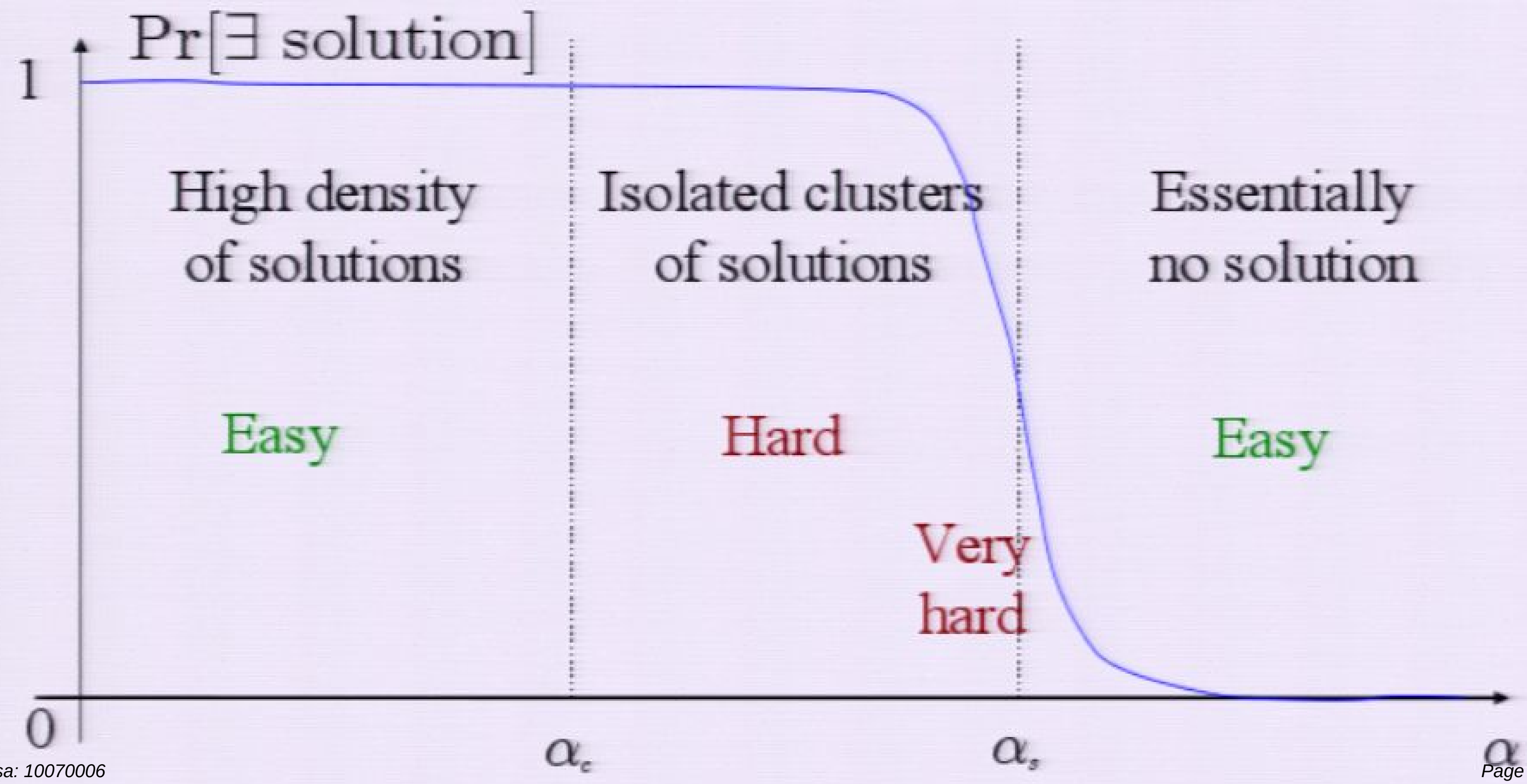
Random instances

- Pick M clauses uniformly at random
- Hardness depends on clauses-to-bit ratio: $\alpha = \dfrac{M}{N}$



Random instances

- Pick M clauses uniformly at random
- Hardness depends on clauses-to-bit ratio: $\alpha = \dfrac{M}{N}$



The adiabatic algorithm

- Problem Hamiltonian $H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}|$

$$= M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

The adiabatic algorithm

NEC Laboratories America
Relentless passion for innovation

- Problem Hamiltonian $$H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}| = M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

- Initial Hamiltonian $$H_0 = -\sum_i \sigma_x^{(i)} = -\sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$

The adiabatic algorithm

NEC Laboratories America
Relentless passion for innovation

- Problem Hamiltonian $H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}|$

$$= M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

- Initial Hamiltonian $H_0 = -\sum_i \sigma_x^{(i)} = -\sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$

- Initial ground state $|\psi_0\rangle = \frac{1}{2^{N/2}}\sum_{\vec{x}} |\vec{x}\rangle$

The adiabatic algorithm

NEC Laboratories America
Relentless passion for innovation

- Problem Hamiltonian $$H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}| = M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

- Initial Hamiltonian $$H_0 = -\sum_i \sigma_x^{(i)} = -\sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$

The adiabatic algorithm

- Problem Hamiltonian $$H_P = \sum_{\vec{x}} f(\vec{x})|\vec{x}\rangle\langle\vec{x}|$$ $$= M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

- Initial Hamiltonian $$H_0 = -\sum_i \sigma_x^{(i)} = -\sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$

- Initial ground state $$|\psi_0\rangle = \frac{1}{2^{N/2}}\sum_{\vec{x}} |\vec{x}\rangle$$

The adiabatic algorithm

NEC Laboratories America
Relentless passion for innovation

- Problem Hamiltonian $$H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}| = M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

- Initial Hamiltonian $$H_0 = -\sum_i \sigma_x^{(i)} = -\sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$

- Initial ground state $$|\psi_0\rangle = \frac{1}{2^{N/2}}\sum_{\vec{x}} |\vec{x}\rangle$$

$$\boxed{H(s) = (1-s)H_0 + sH_P}$$

(where $s(t)=t/T$)

NEC Laboratories America
Relentless passion for innovation

Connection to Anderson localization

To study the spectrum of H(s) close to $s = 1$, we consider

$$H(\lambda) = \frac{H(s)}{s} = H_P + \lambda H_0 \qquad \text{where} \qquad \lambda = \frac{1-s}{s}$$

Connection to Anderson localization

To study the spectrum of H(s) close to $s = 1$, we consider

$$H(\lambda) = \frac{H(s)}{s} = H_P + \lambda H_0 \qquad \text{where} \qquad \lambda = \frac{1-s}{s}$$

$$= \sum_{\vec{x}} f(\vec{x})|\vec{x}\rangle\langle\vec{x}| - \lambda \sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$

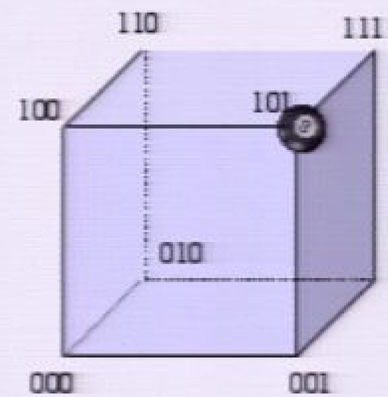
Connection to Anderson localization

NEC Laboratories America
Relentless passion for innovation

To study the spectrum of H(s) close to $s=1$, we consider

$$H(\lambda) = \frac{H(s)}{s} = H_P + \lambda H_0 \qquad \text{where} \qquad \lambda = \frac{1-s}{s}$$

$$= \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}| - \lambda \sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$



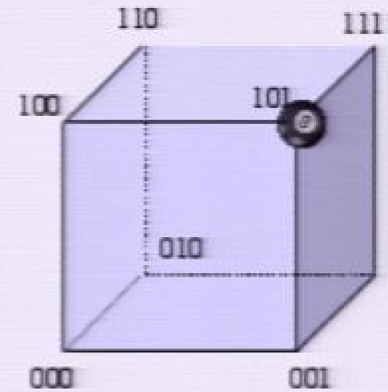
$\Rightarrow$ Particle hopping on a hypercube

Connection to Anderson localization

To study the spectrum of H(s) close to $s = 1$, we consider

$$H(\lambda) = \frac{H(s)}{s} = H_P + \lambda H_0 \qquad \text{where} \qquad \lambda = \frac{1-s}{s}$$

$$= \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}| - \lambda \sum_{d(\vec{x},\vec{y})=1} |\vec{x}\rangle\langle\vec{y}|$$



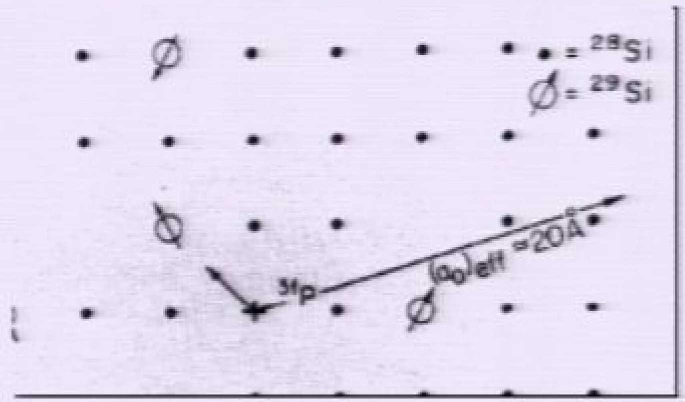
$\Rightarrow$ Particle hopping on a hypercube

$\Rightarrow$ Similar to Anderson's tight binding model

NEC Laboratories America
Relentless passion for innovation

Anderson localization

"Extended states become localized due to disorder"

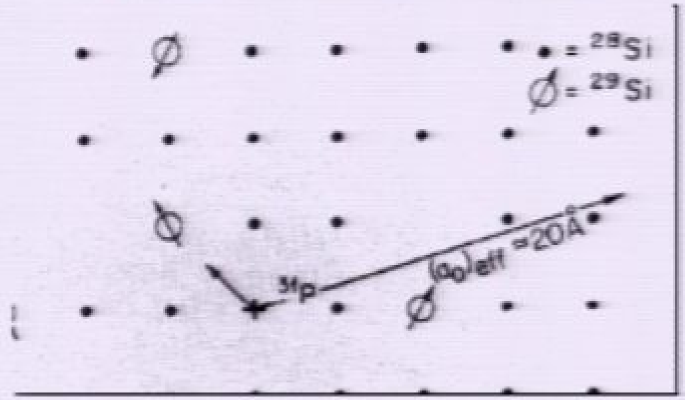


P. Anderson
Nobel Prize
Physics 1977

Anderson localization

"Extended states become localized due to disorder"

Model:
- Grid with coupling λ
- Random energies

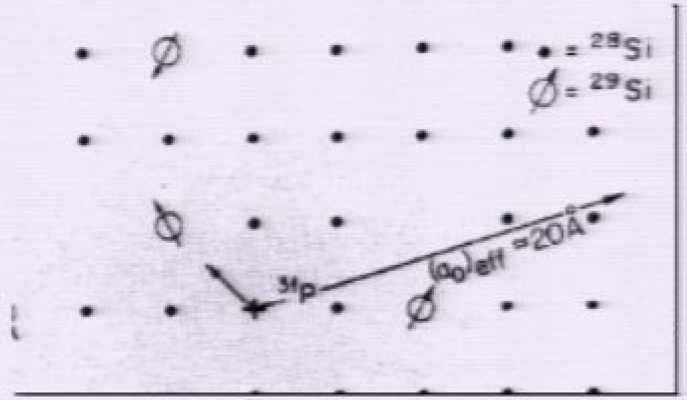


P. Anderson
Nobel Prize
Physics 1977

Anderson localization

NEC Laboratories America
Relentless passion for innovation

"Extended states become localized due to disorder"

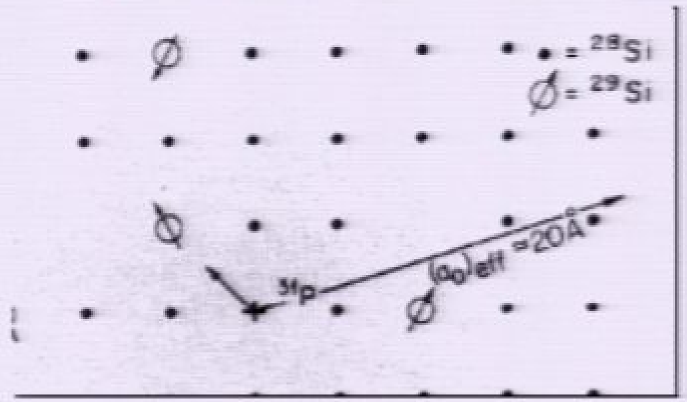


P. Anderson
Nobel Prize
Physics 1977

Anderson localization

"Extended states become localized due to disorder"

Model:
- Grid with coupling λ
- Random energies

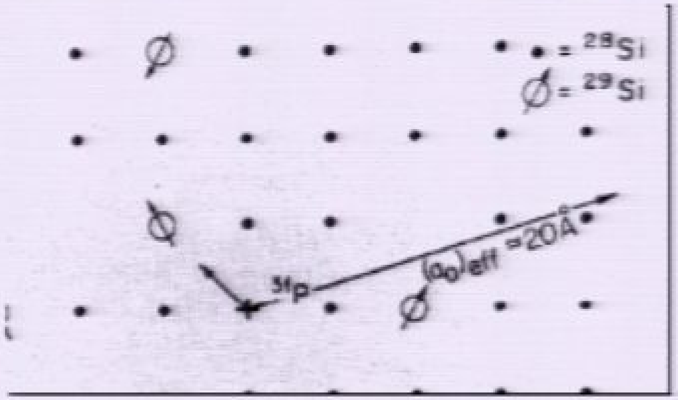


P. Anderson
Nobel Prize
Physics 1977

Anderson localization

NEC Laboratories America
Relentless passion for innovation

"Extended states become localized due to disorder"



P. Anderson
Nobel Prize
Physics 1977

Model:

- Grid with coupling λ
- Random energies

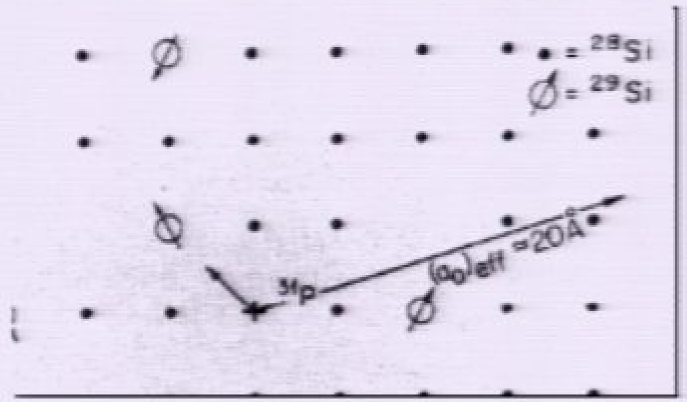
$\lambda > \lambda_c$ → Extended state → Metal

$\lambda < \lambda_c$ → Localized state → Insulator

NEC Laboratories America
Relentless passion for innovation

Anderson localization

"Extended states become localized due to disorder"



Model:
- Grid with coupling λ
- Random energies

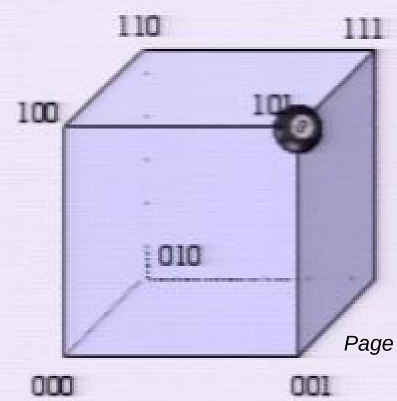
P. Anderson
Nobel Prize
Physics 1977

$\lambda > \lambda_c$ → Extended state → Metal

$\lambda < \lambda_c$ → Localized state → Insulator

In our case:

- Hypercube with coupling λ
- Energies from random Exact-Cover 3



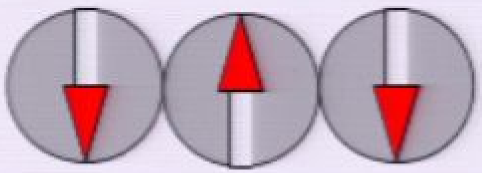
Localized and extended states

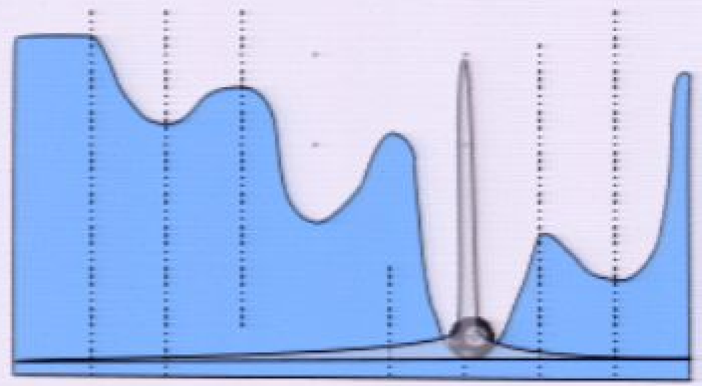
NEC Laboratories America
Relentless passion for innovation

Localized and extended states

$\lambda = 0$

- State is localized



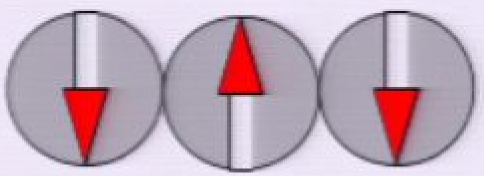


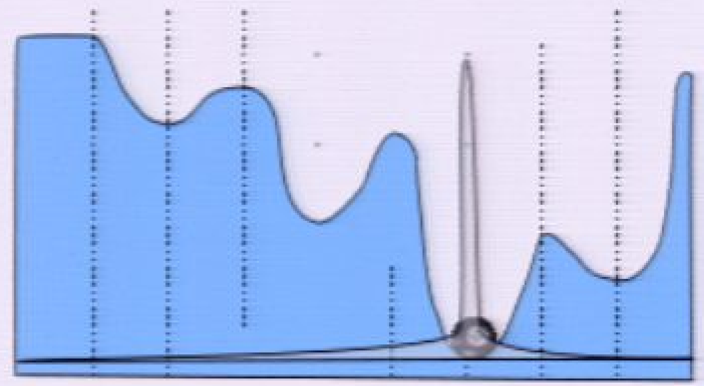
NEC Laboratories America
Relentless passion for innovation

Localized and extended states

$\lambda = 0$

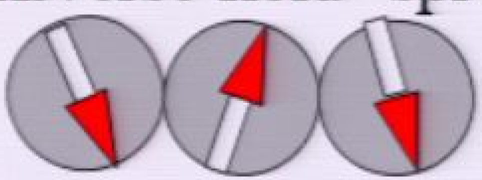
- State is localized

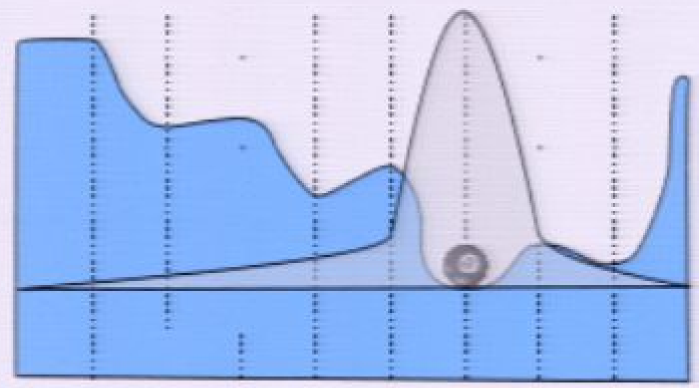




$\lambda > 0$

- Transverse field "spreads" the state

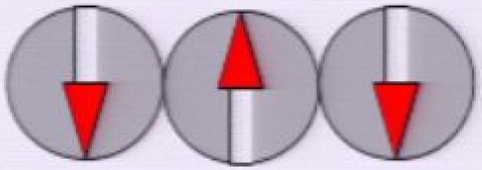


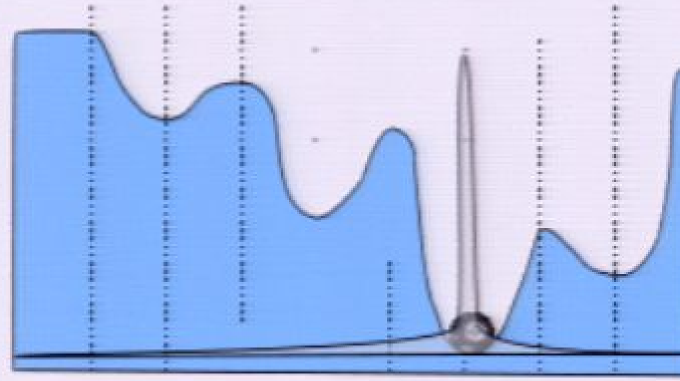


Localized and extended states

$\lambda = 0$

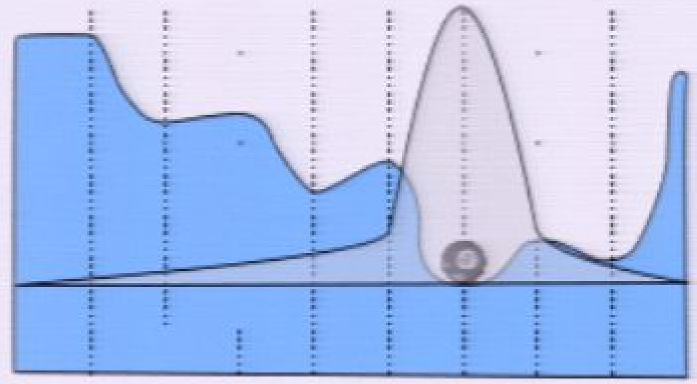
- State is localized





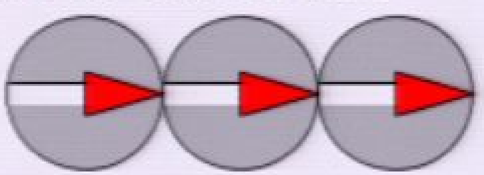
$\lambda > 0$

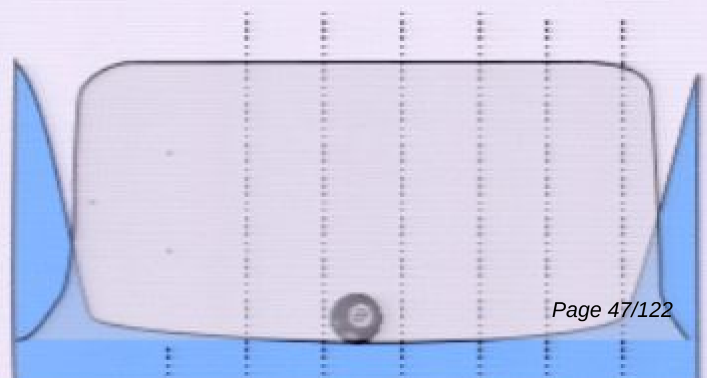
- Transverse field "spreads" the state



$\lambda \gg 1$

- State is extended



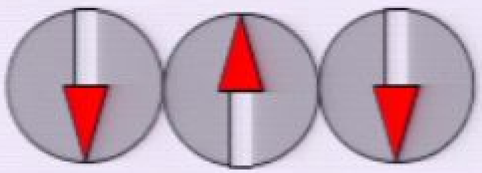


Localized and extended states

NEC Laboratories America
Relentless passion for innovation

$\lambda = 0$

- State is localized

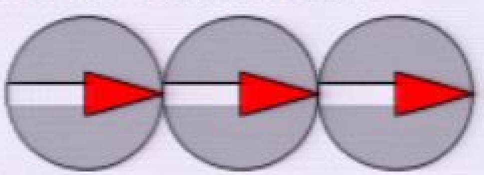


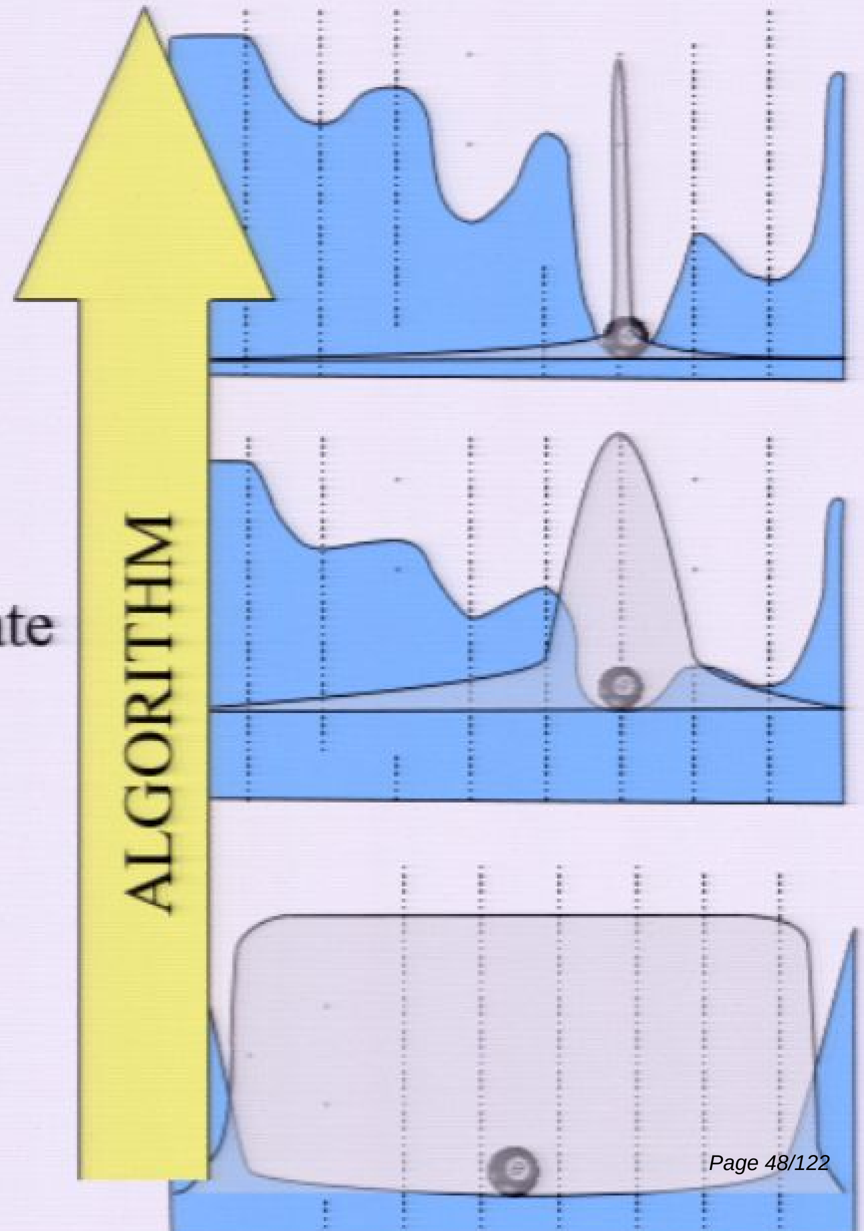
$\lambda > 0$

- Transverse field "spreads" the state

$\lambda \gg 1$

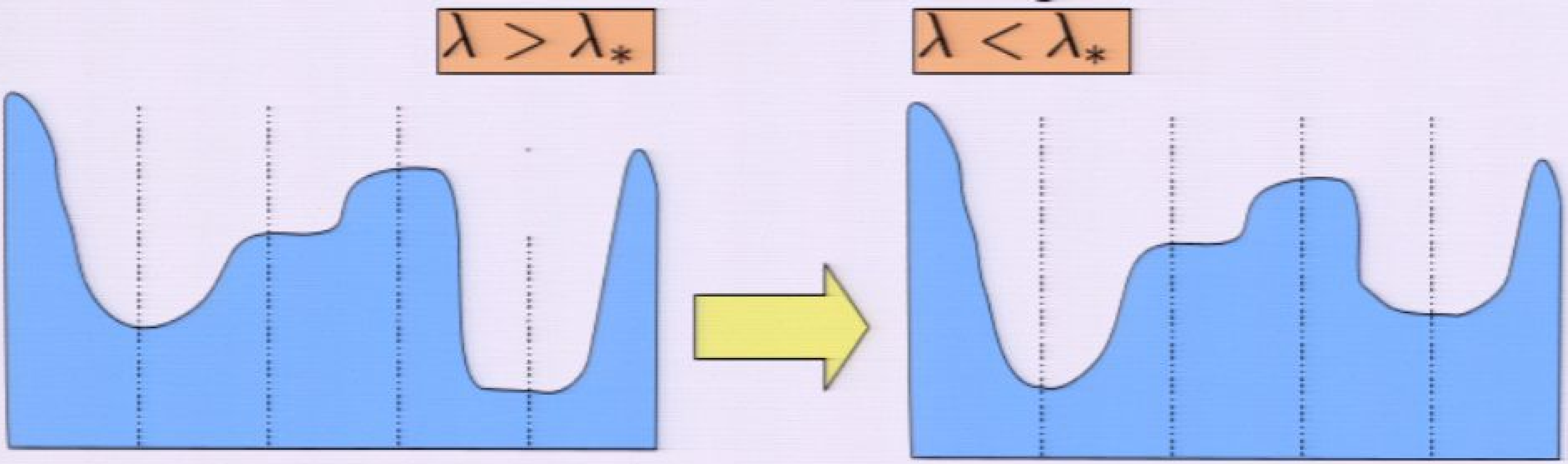
- State is extended

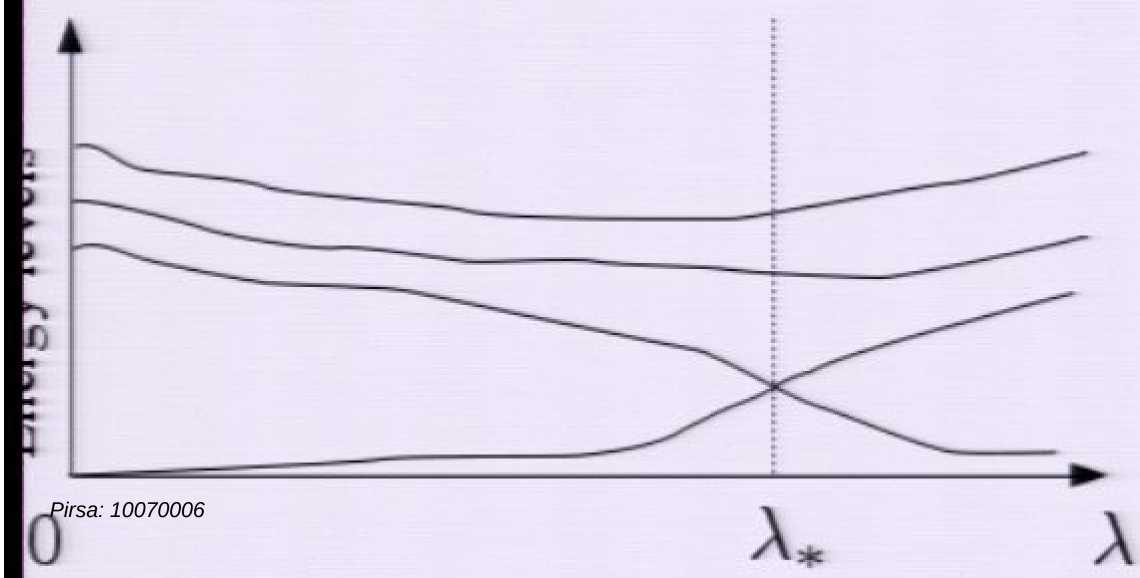




Tunneling: extended state

What if a local minimum later becomes the global minimum?

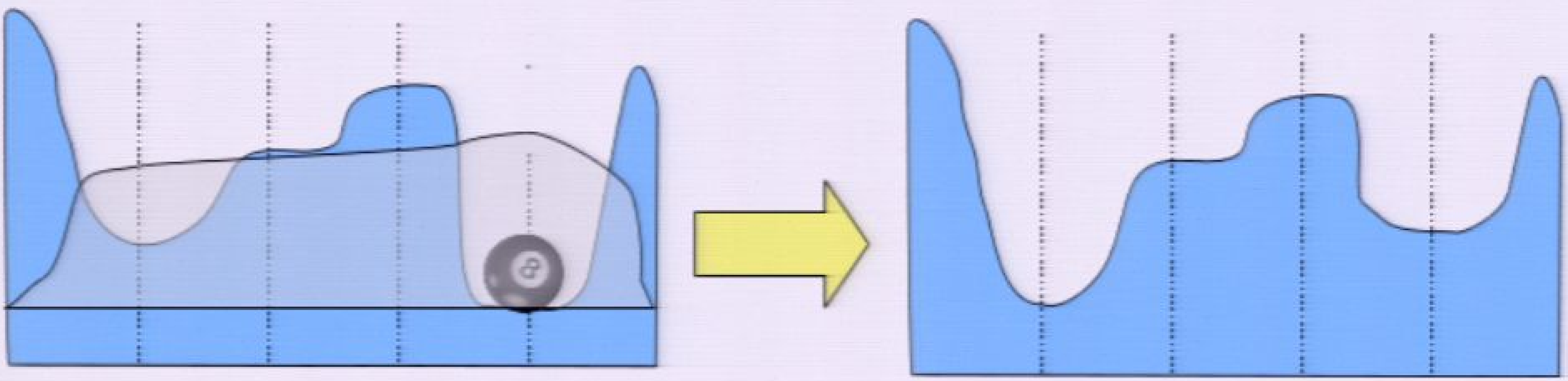


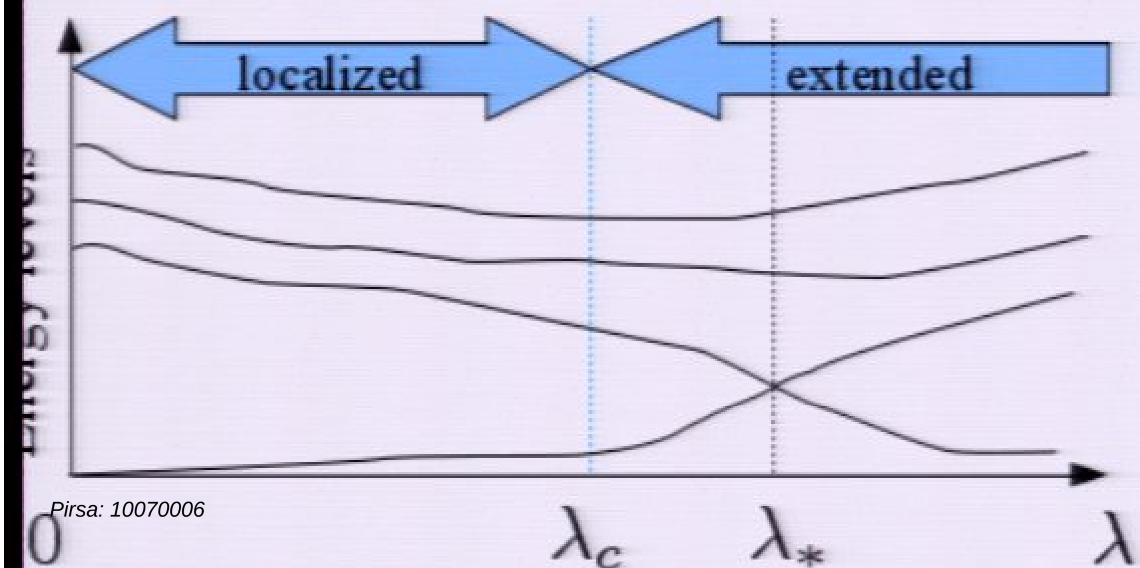


Tunneling: extended state

What if a local minimum later becomes the global minimum?

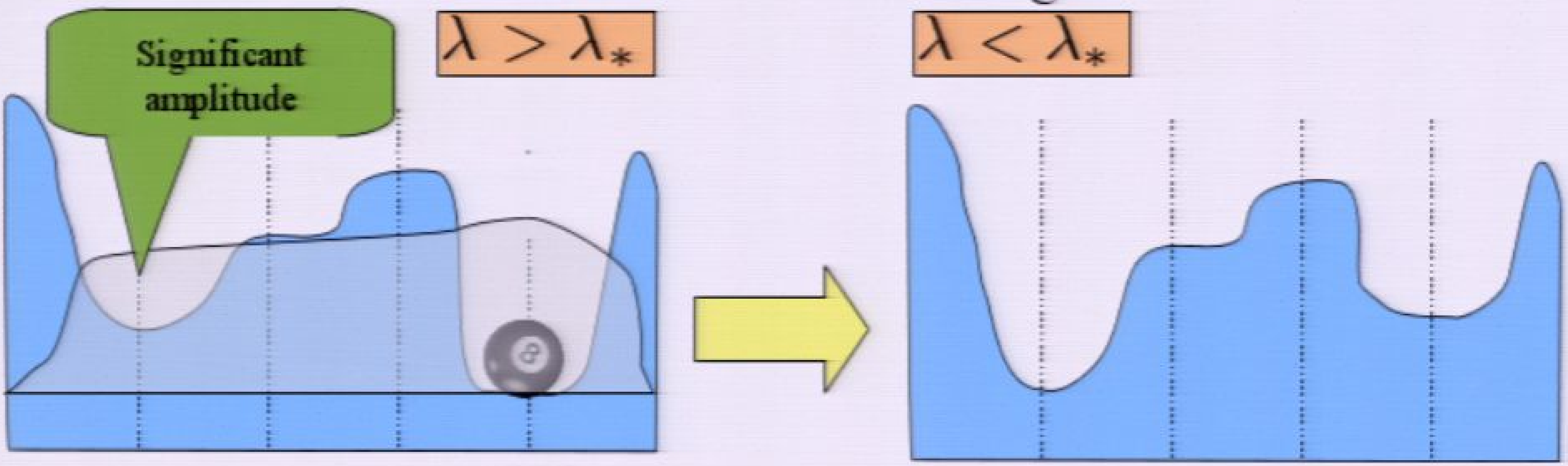
$\lambda > \lambda_*$ $\lambda < \lambda_*$

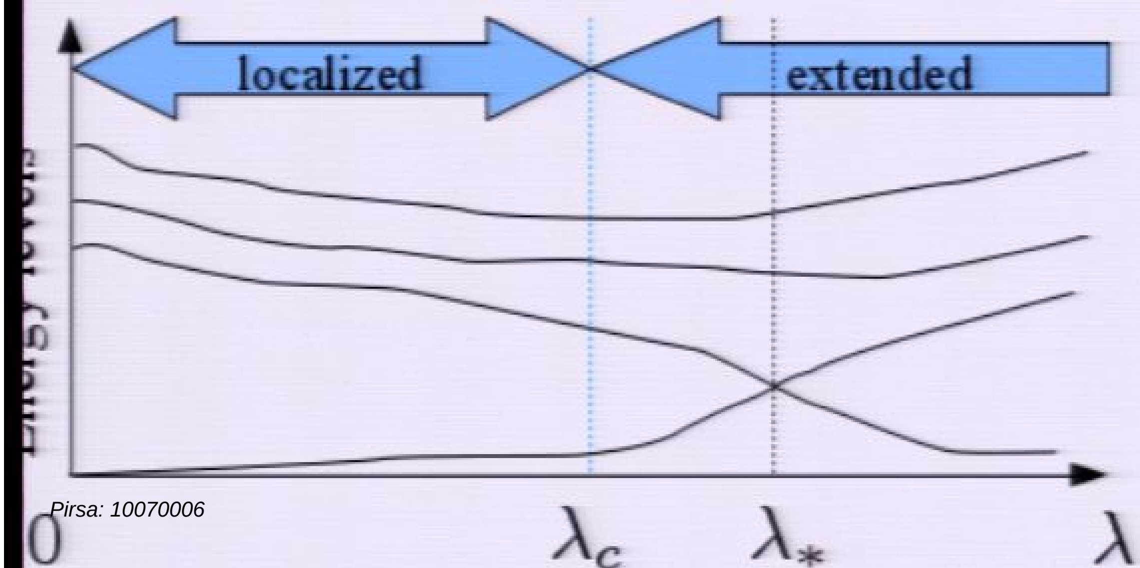




Tunneling: extended state

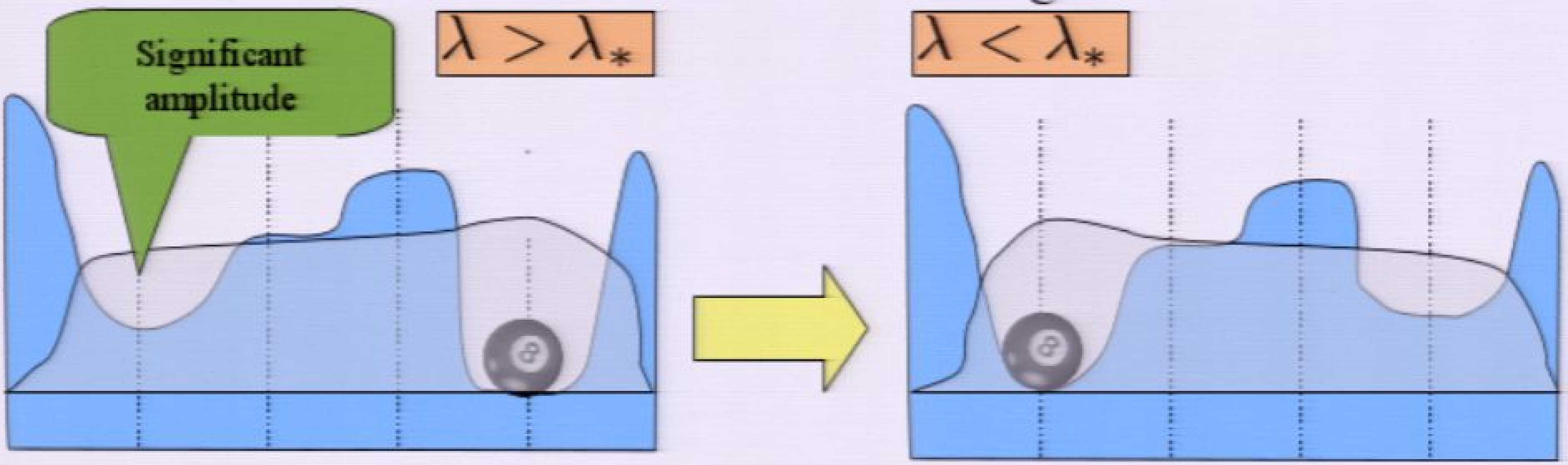
What if a local minimum later becomes the global minimum?

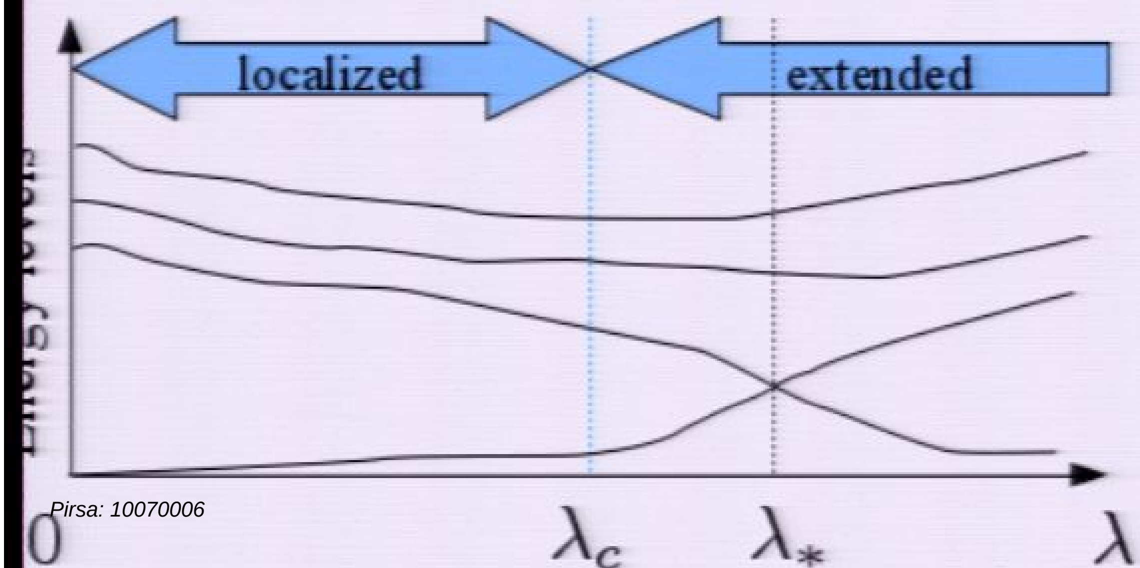




Tunneling: extended state

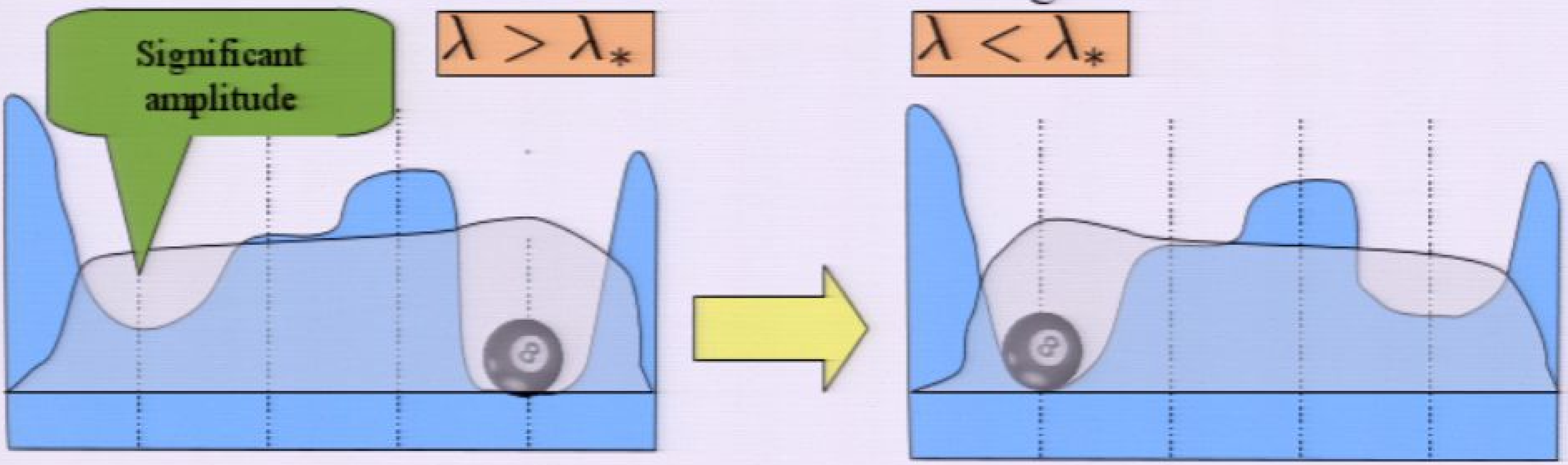
What if a local minimum later becomes the global minimum?

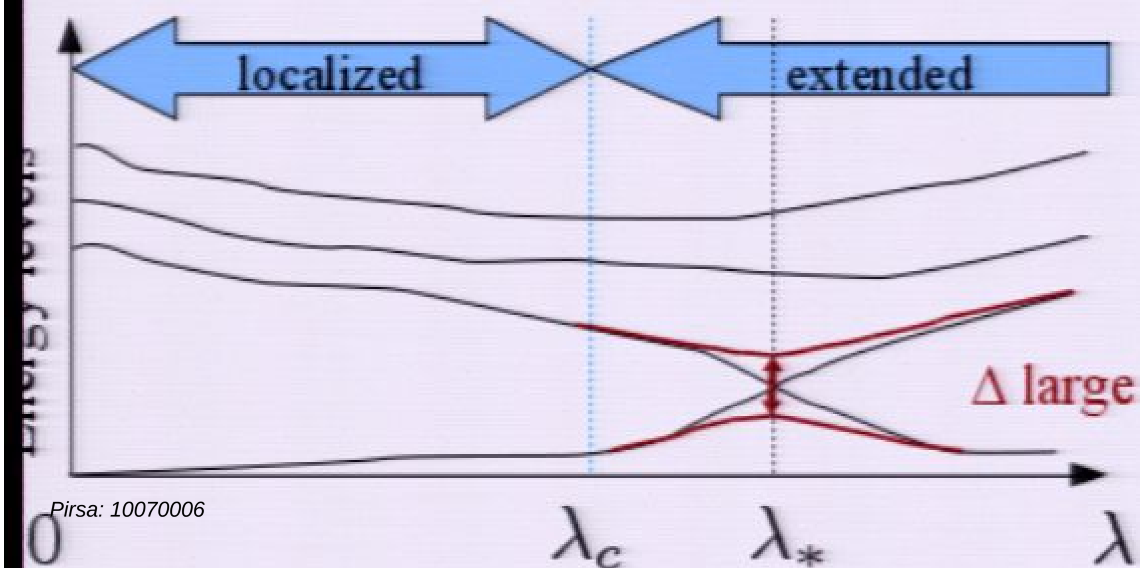




Tunneling: extended state

What if a local minimum later becomes the global minimum?

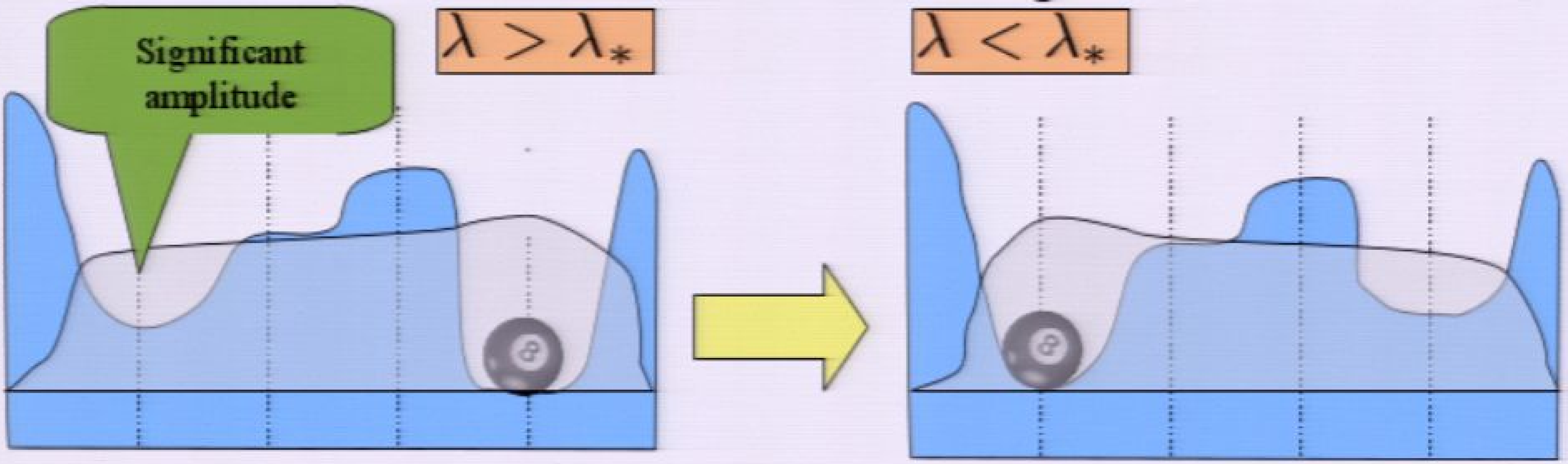


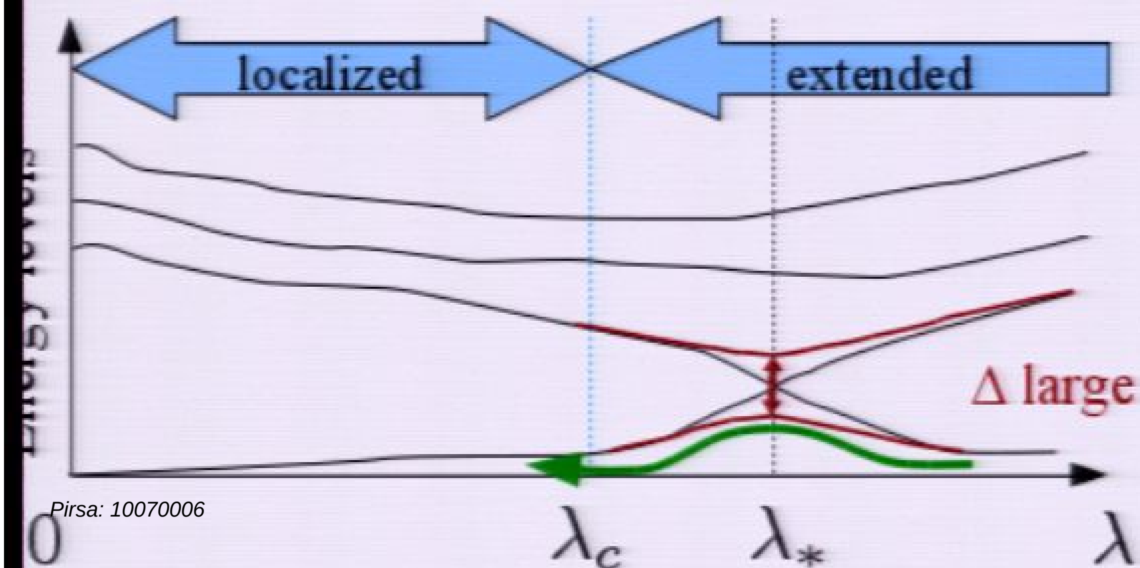


Large anti-crossing gap

Tunneling: extended state

What if a local minimum later becomes the global minimum?





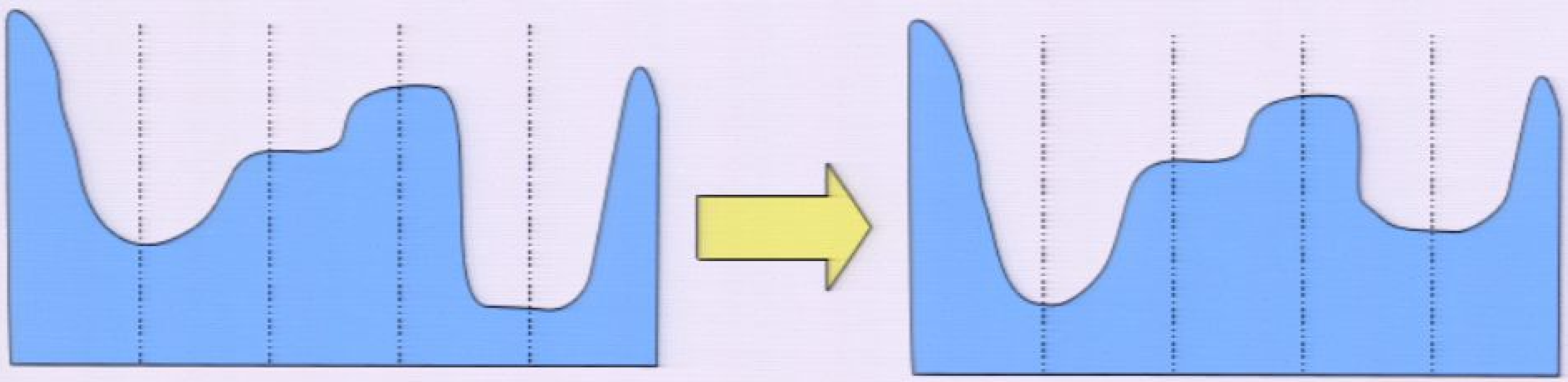
Large anti-crossing gap

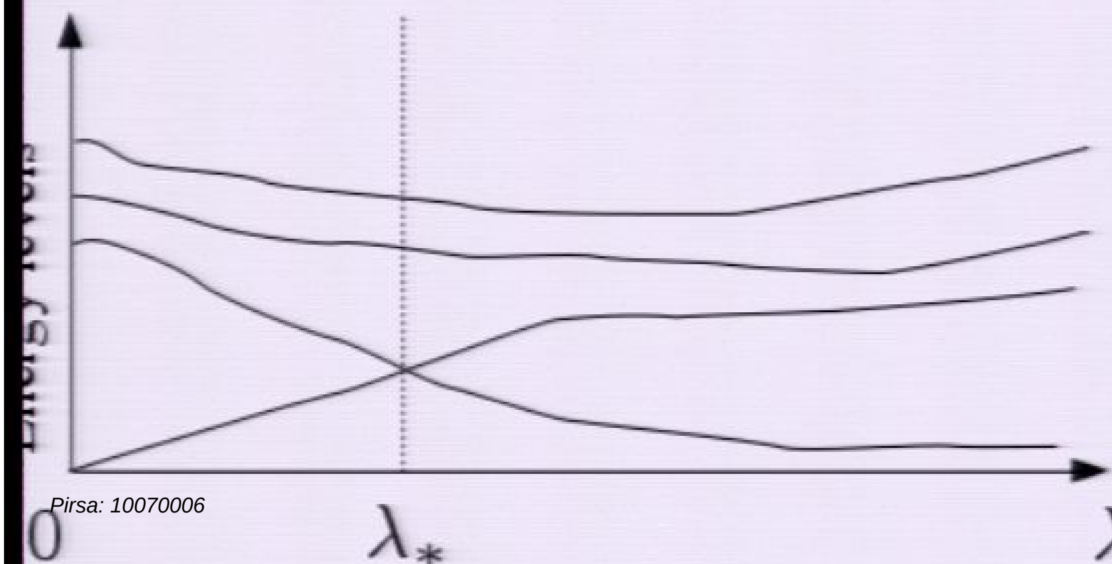
→ Tunneling

Tunneling: localized state

What if a local minimum later becomes the global minimum?

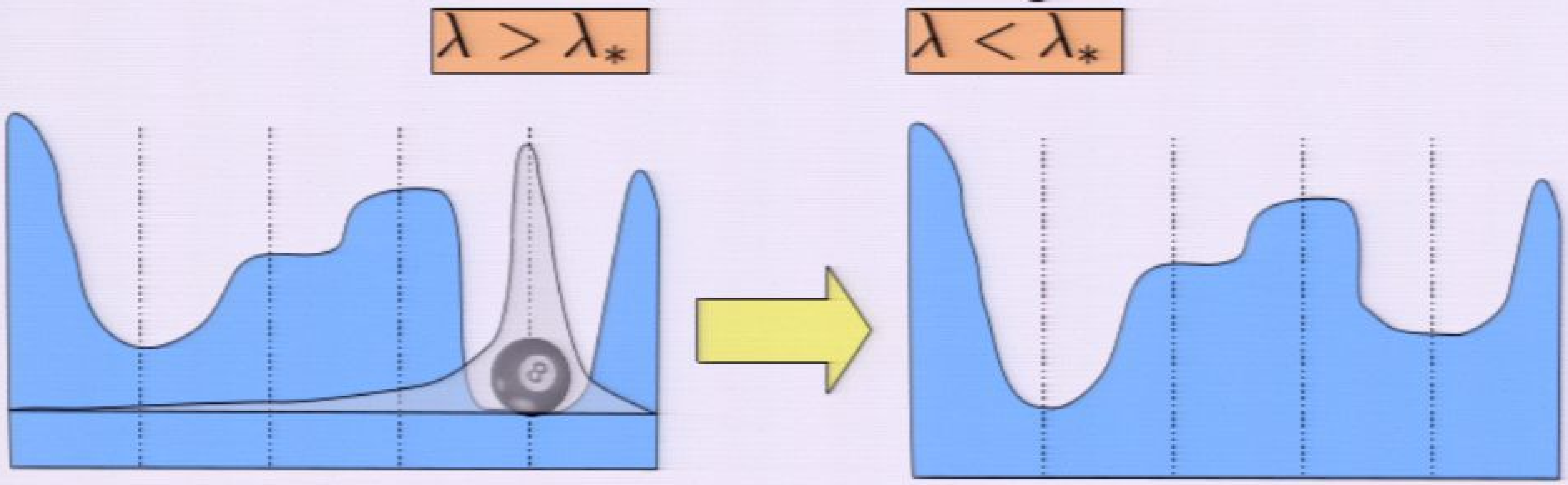
$\lambda > \lambda_*$ $\lambda < \lambda_*$

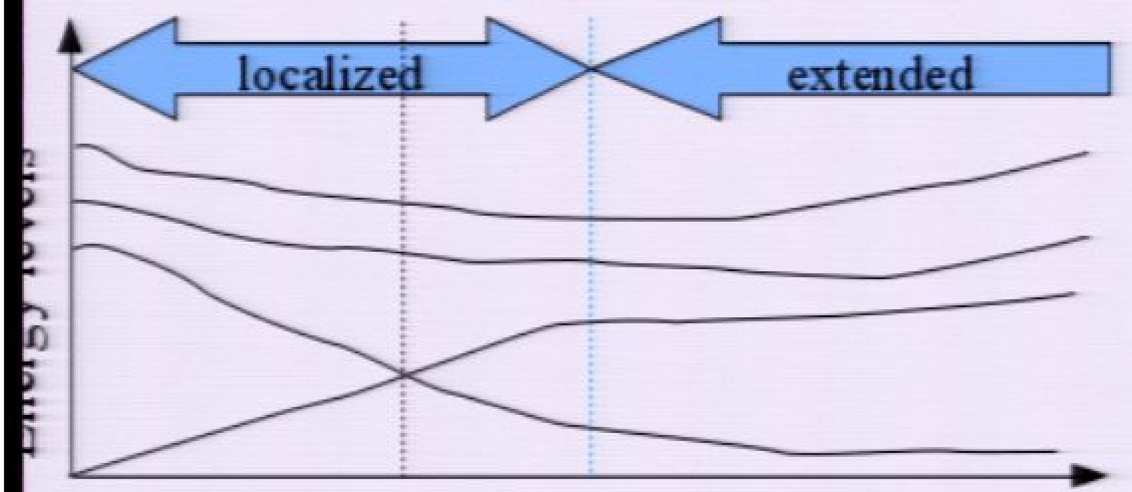




Tunneling: localized state

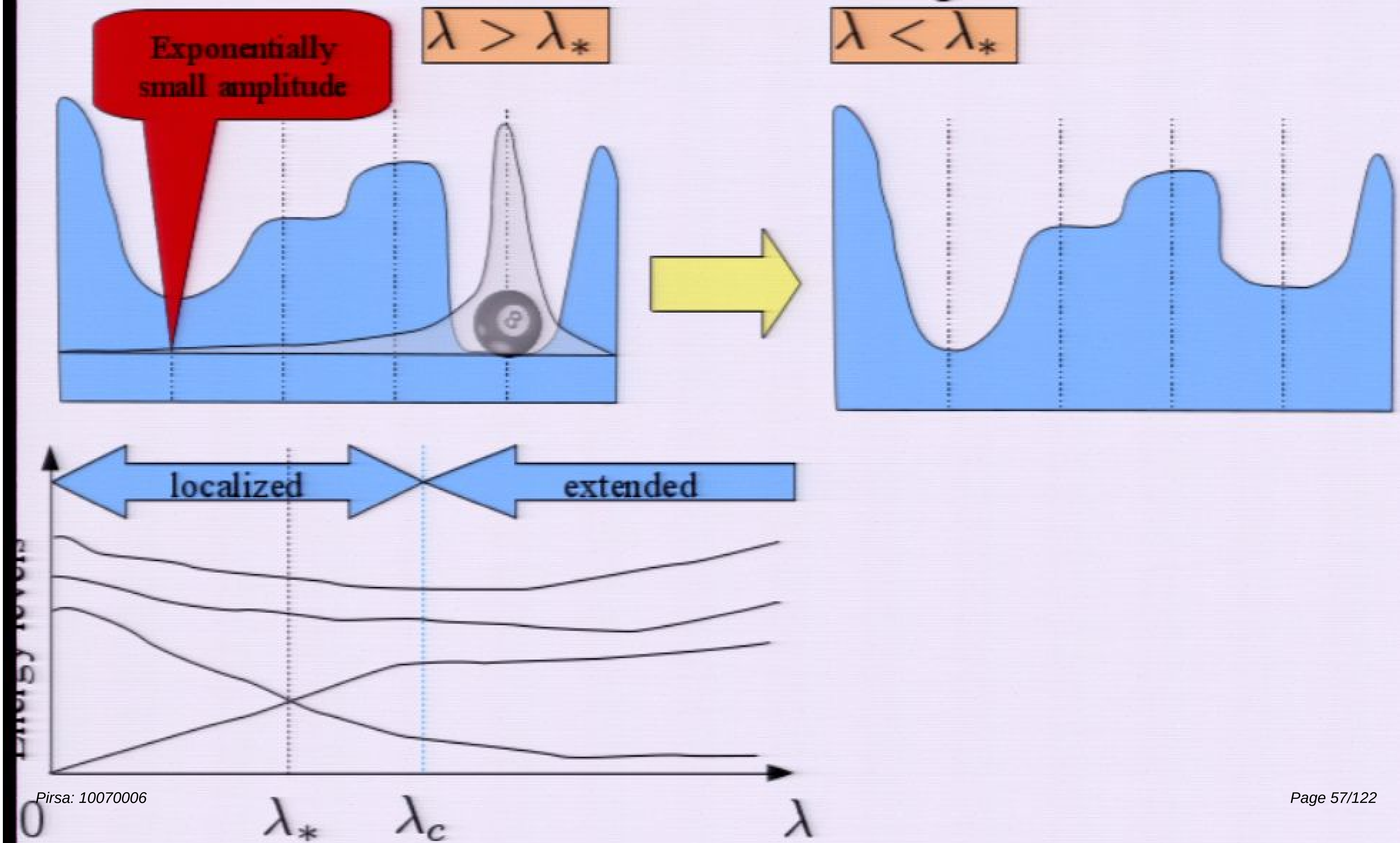
What if a local minimum later becomes the global minimum?





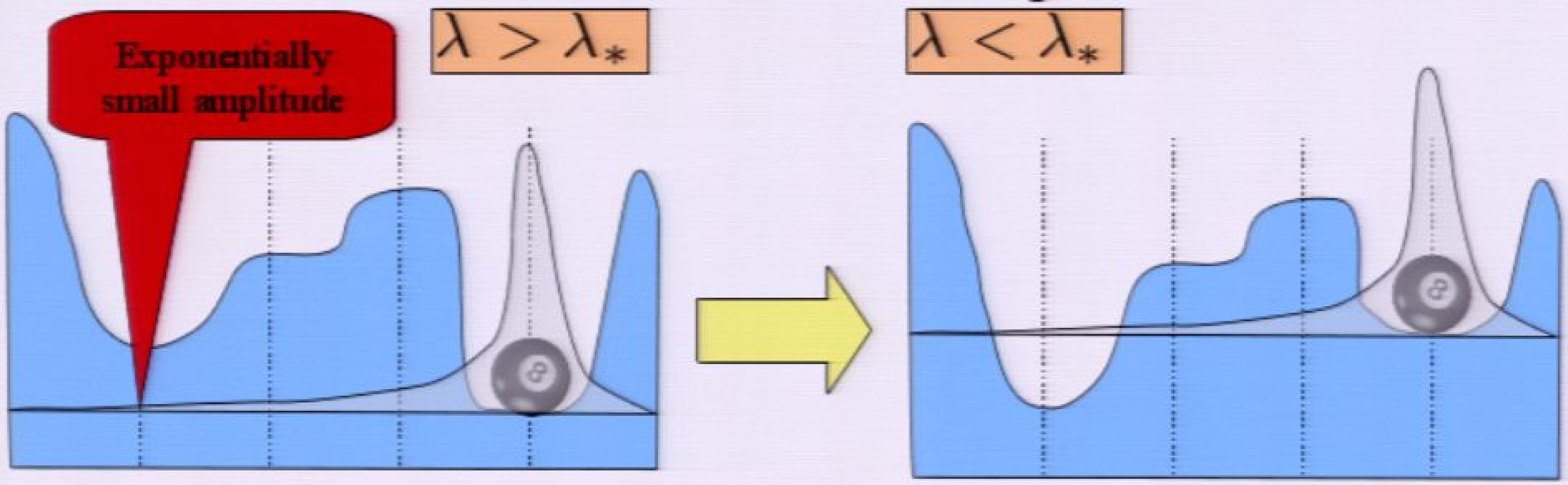
Tunneling: localized state

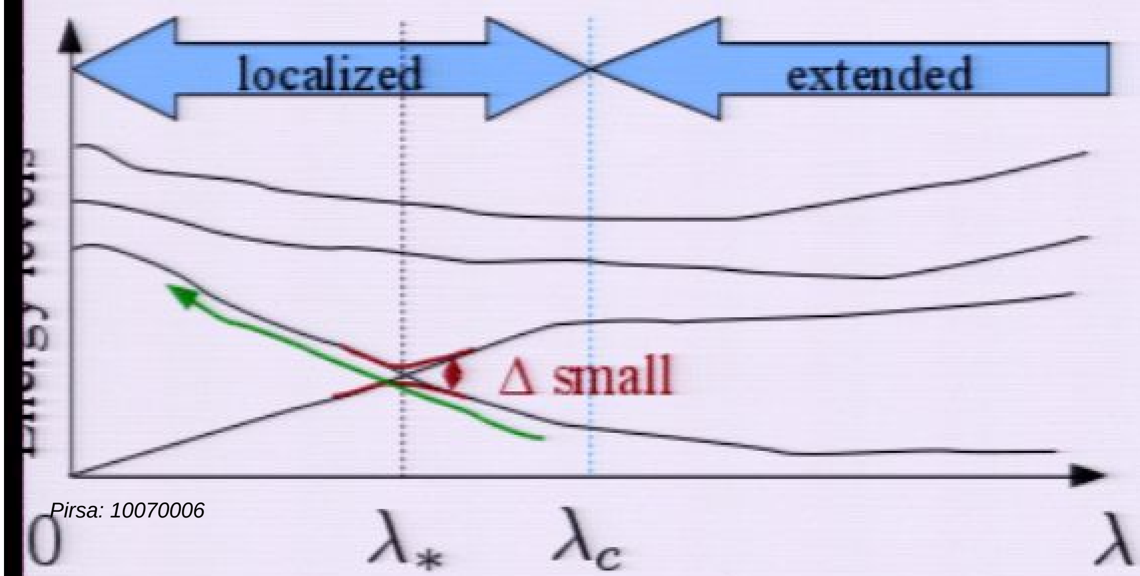
What if a local minimum later becomes the global minimum?



Tunneling: localized state

What if a local minimum later becomes the global minimum?

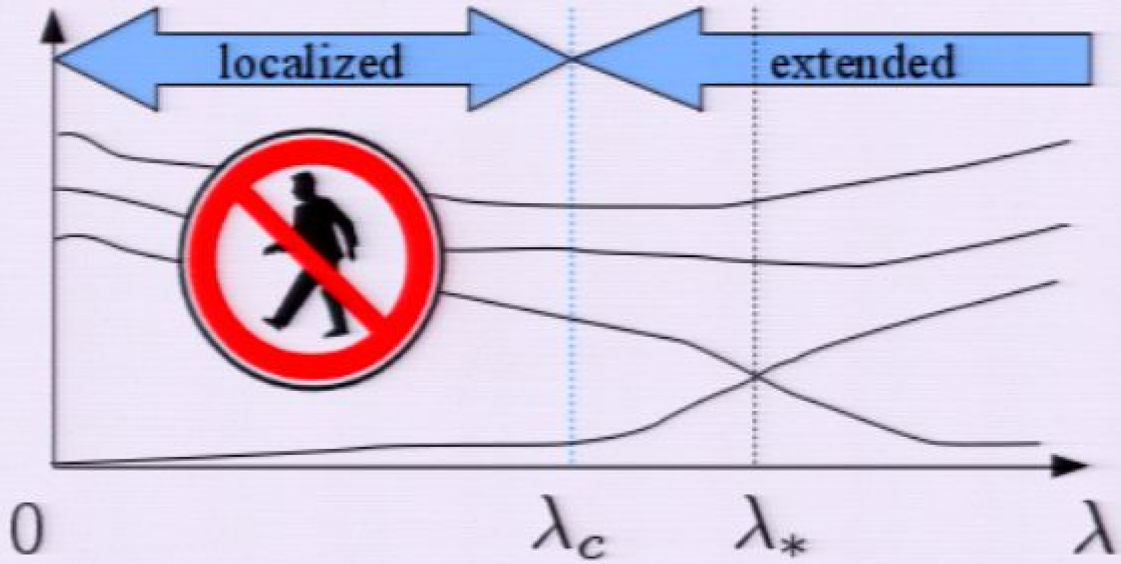




Small anti-crossing gap

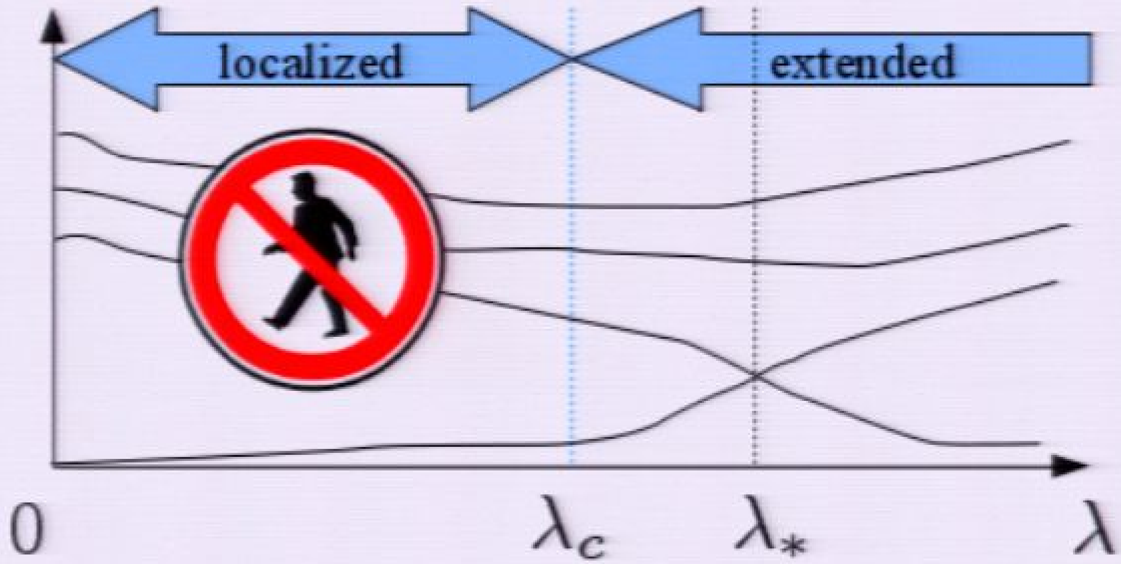
→ Landau-Zener transition

Our result



As the size of the problem N increases

Our result

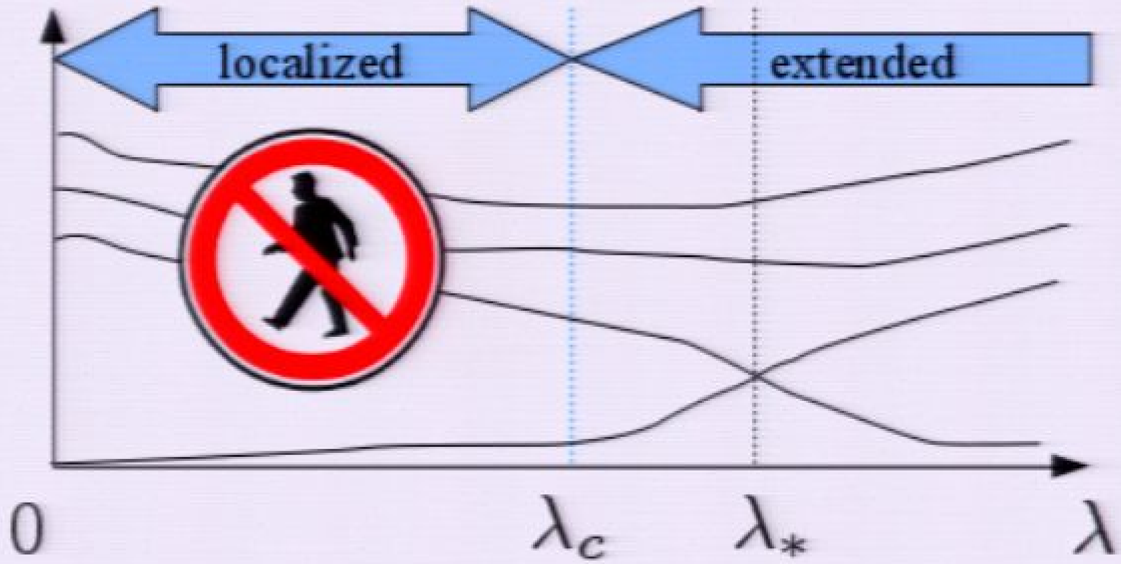


As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$

Our result

NEC Laboratories America
Relentless passion for innovation

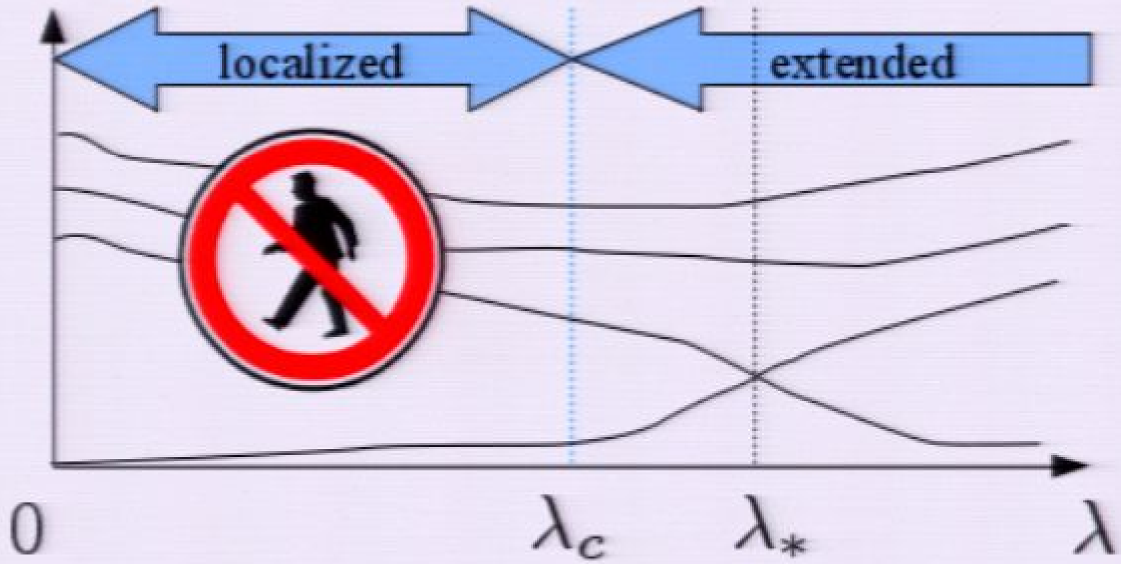


As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$
2) Anti-crossings for smaller and smaller $\lambda_* = (CN)^{-1/8}$

NEC Laboratories America
Relentless passion for innovation

Our result



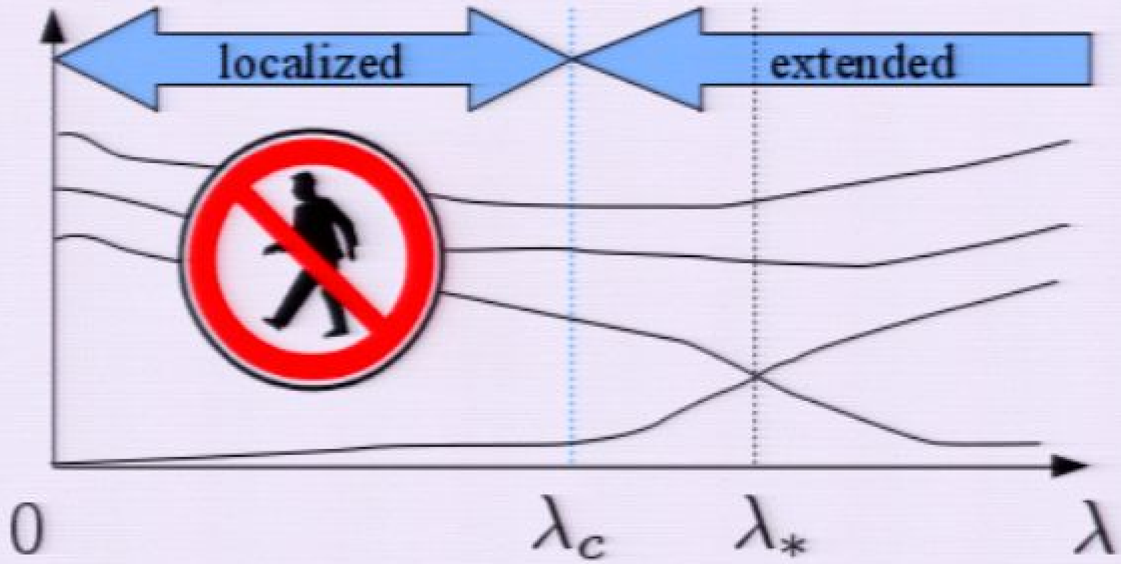
As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$
2) Anti-crossings for smaller and smaller $\lambda_* = (CN)^{-1/8}$

For $N > \dfrac{1}{C\lambda_c^8}$ we have $\lambda_* < \lambda_c$

NEC Laboratories America
Relentless passion for innovation

Our result



As the size of the problem N increases

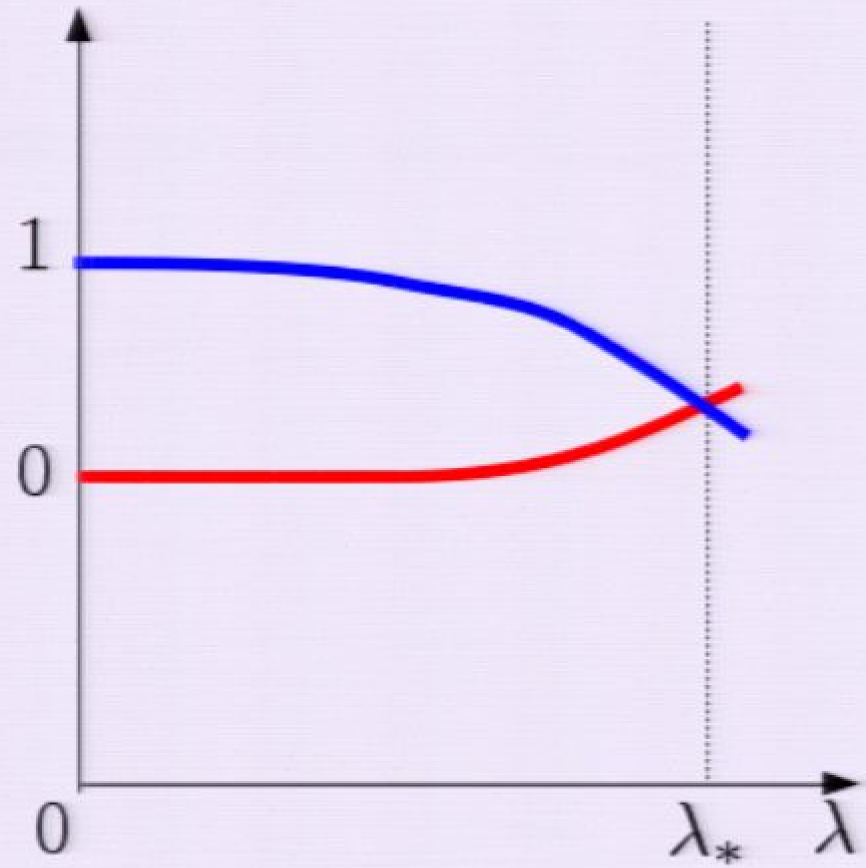
1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$
2) Anti-crossings for smaller and smaller $\lambda_* = (CN)^{-1/8}$

For $N > \frac{1}{C\lambda_c^8}$ we have $\lambda_* < \lambda_c$

The algorithm fails (stuck in a local minimum)

Level anti-crossings

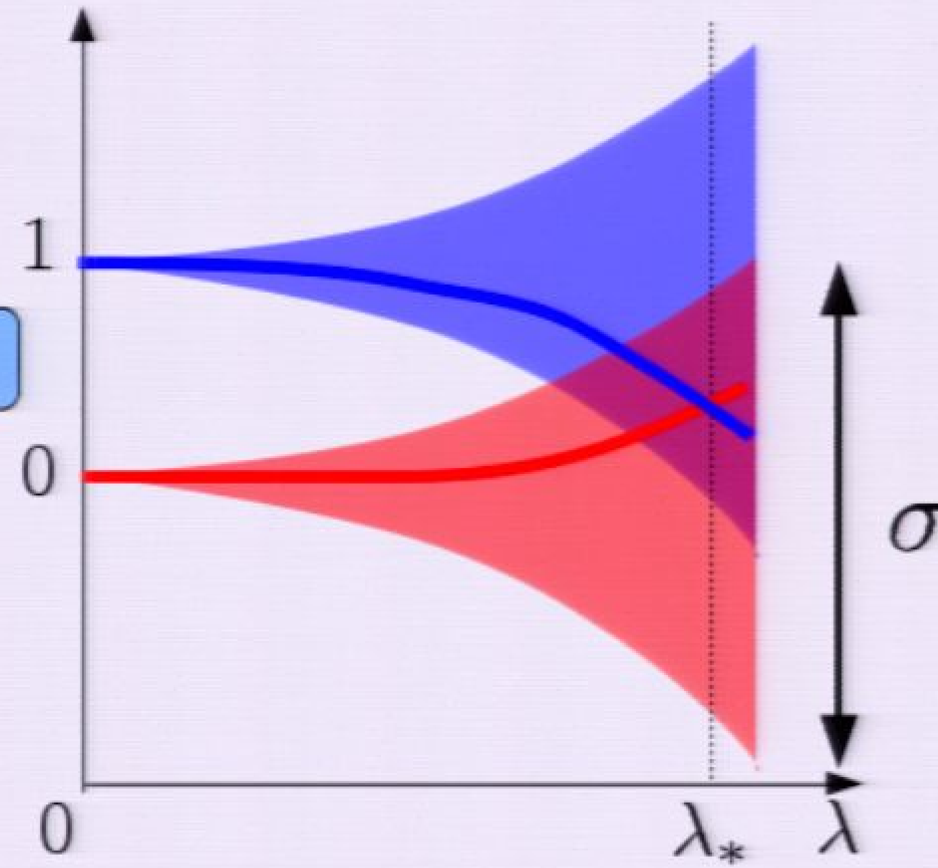
Study slopes by perturbation theory



Level anti-crossings

Study slopes by perturbation theory

Random instances ⇒ random slopes!

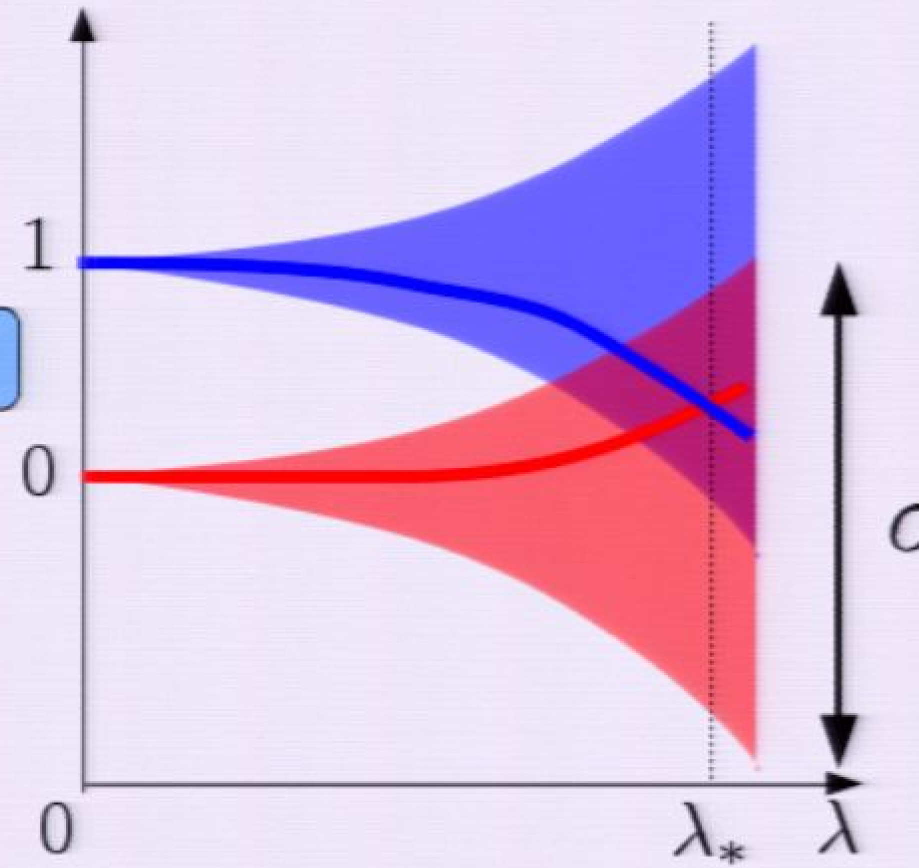


NEC Laboratories America
Relentless passion for innovation

Level anti-crossings

Study slopes by perturbation theory

Random instances ⇒ random slopes!



As N increases,

- Standard deviation σ increases
- Position of anti-crossing λ_* goes to 0

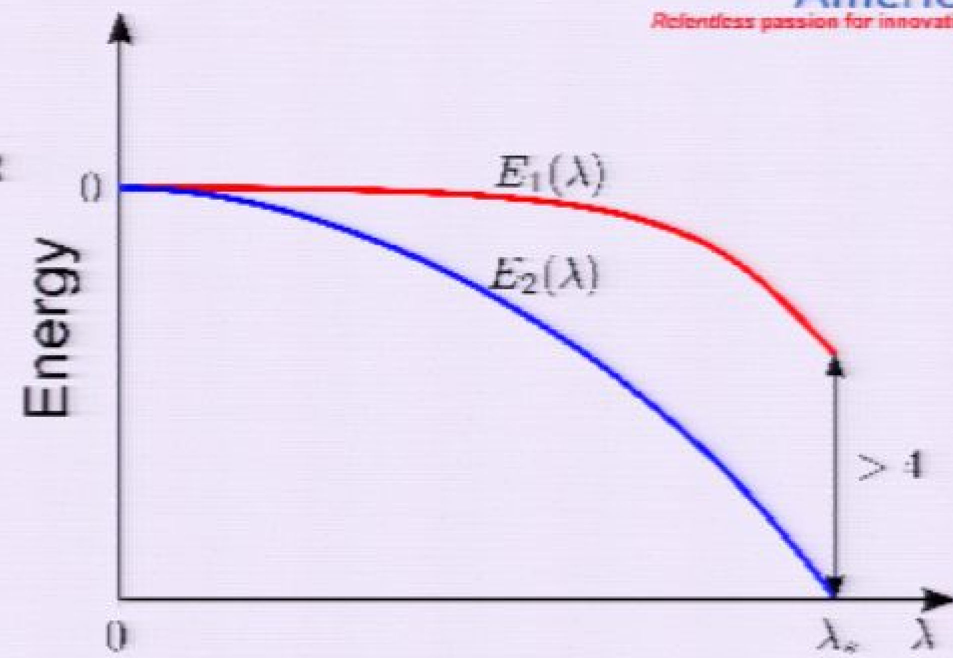
Level anti-crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

Suppose

$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$

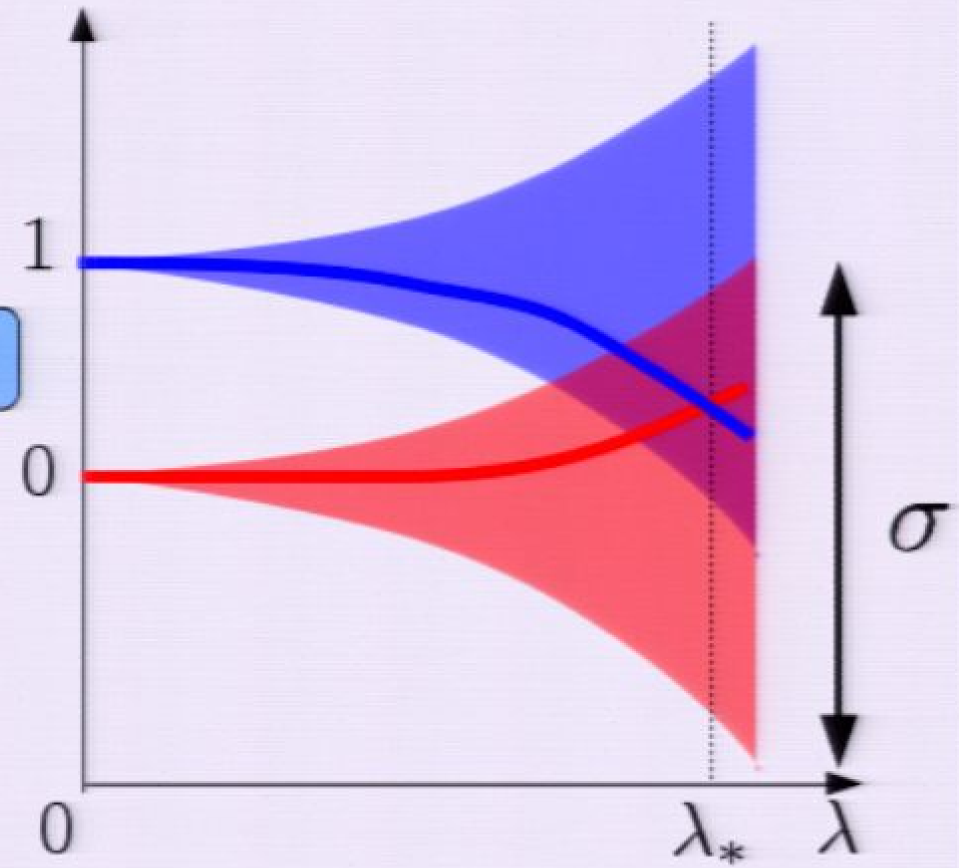


Level anti-crossings

NEC Laboratories America
Relentless passion for innovation

Study slopes by perturbation theory

Random instances ⇒ random slopes!



As N increases,

- Standard deviation σ increases
- Position of anti-crossing λ_* goes to 0

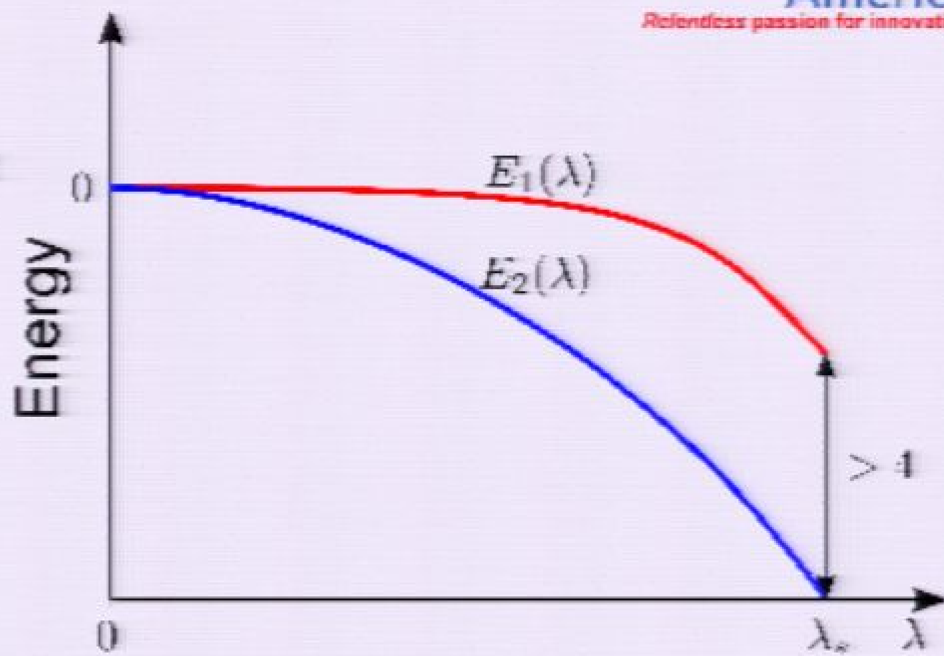
Level anti-crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

Suppose

$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$



Level anti-crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

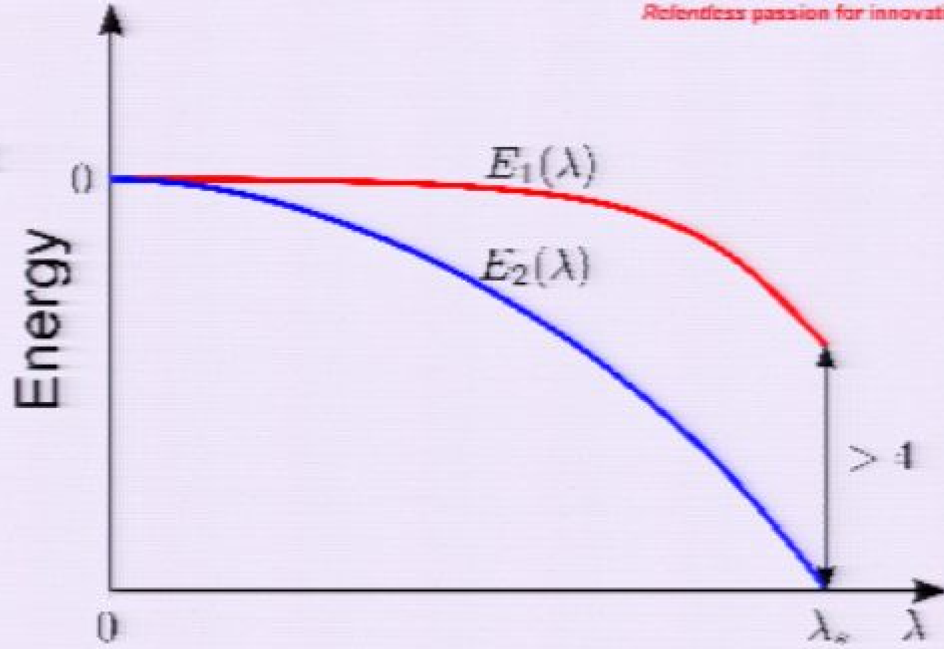
$$E_1(0) = E_2(0) = 0$$

Suppose

$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$

Add a clause

- satisfied by $\vec{x}_1$ $\quad \tilde{E}_1(0) = 0$
- violated by $\vec{x}_2$ $\quad \tilde{E}_2(0) = 4$



Level anti-crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

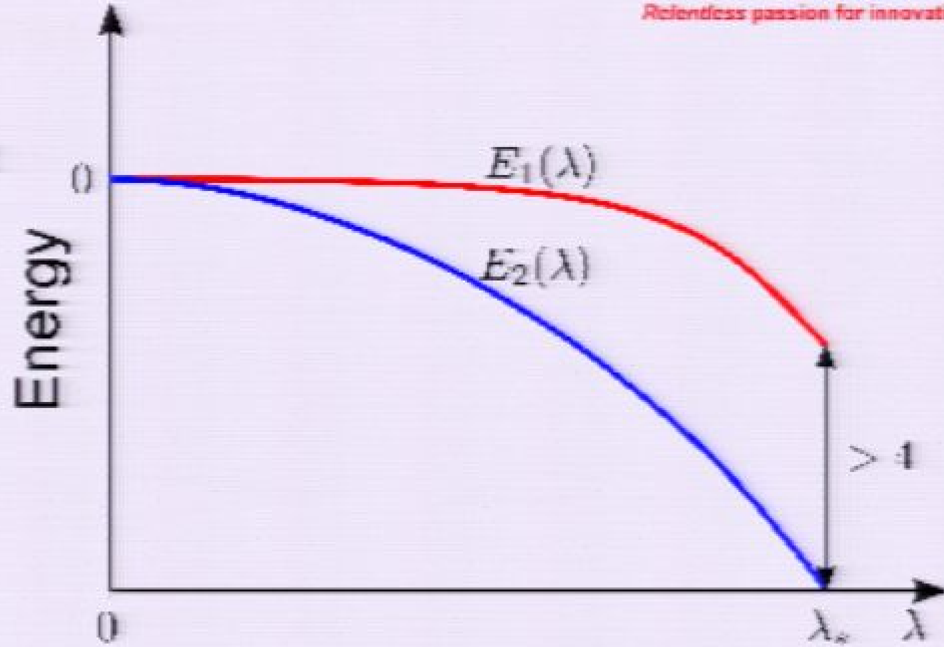
Suppose

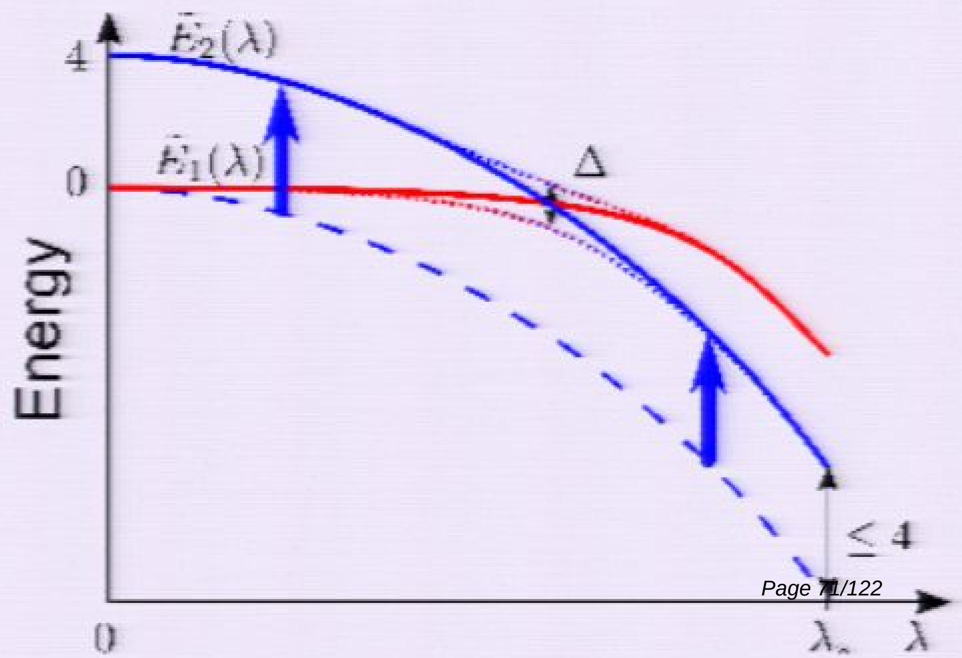
$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$

Add a clause

- satisfied by $\vec{x}_1$ $\quad \tilde{E}_1(0) = 0$
- violated by $\vec{x}_2$ $\quad \tilde{E}_2(0) = 4$

anti-crossing





Perturbation theory

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

Perturbation theory

NEC Laboratories America
Relentless passion for innovation

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

We prove: $\boxed{F_{\vec{x}}^{(m)} = O(N)\ \forall m}$

Proof based on statistical properties of random instances

Perturbation theory

NEC Laboratories America
Relentless passion for innovation

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

We prove: $$\boxed{F_{\vec{x}}^{(m)} = O(N)\ \forall m}$$

For 2 solutions, the difference has zero mean, so

$$\boxed{(F_1^{(m)} - F_2^{(m)})^2 = O(N)\ \forall m}$$

NEC Laboratories America
Relentless passion for innovation

Numerical simulations

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory

Leading order because:
- Odd orders are zero
- Order 2 is solution-independent for EC3

Perturbation theory

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

We prove:

$$\boxed{F_{\vec{x}}^{(m)} = O(N) \ \forall m}$$

For 2 solutions, the difference has zero mean, so

$$\boxed{(F_1^{(m)} - F_2^{(m)})^2 = O(N) \ \forall m}$$

NEC Laboratories America
Relentless passion for innovation

Numerical simulations

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory

Leading order because:
- Odd orders are zero
- Order 2 is solution-independent for EC3

NEC Laboratories America
Relentless passion for innovation

Perturbation theory

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

We prove: $$\boxed{F_{\vec{x}}^{(m)} = O(N) \ \forall m}$$

For 2 solutions, the difference has zero mean, so

$$\boxed{(F_1^{(m)} - F_2^{(m)})^2 = O(N) \ \forall m}$$

NEC Laboratories America
Relentless passion for innovation

Numerical simulations

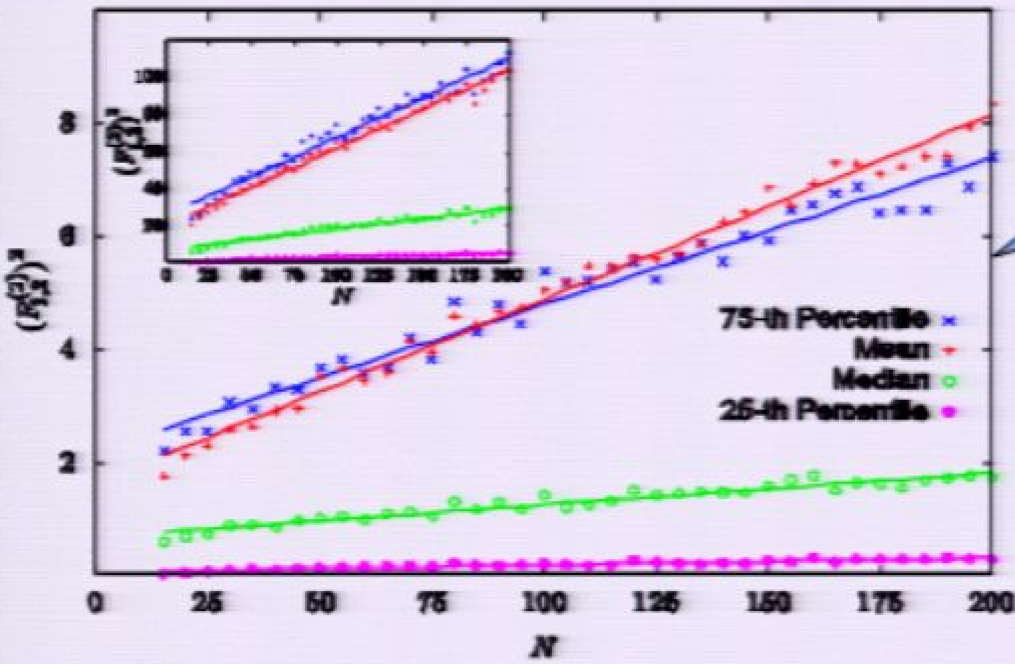
- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory

Leading order because:
- Odd orders are zero
- Order 2 is solution-independent for EC3

NEC Laboratories America
Relentless passion for innovation

Numerical simulations

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory



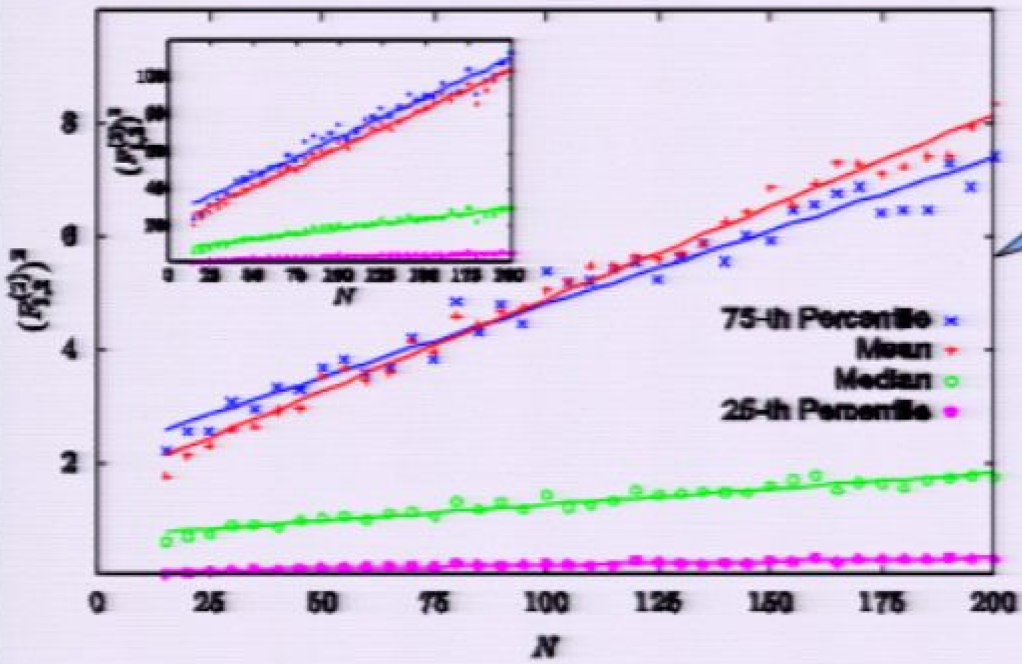
Each data point computed from 2500 instances

$$(E_1(\lambda) - E_2(\lambda))^2 \approx CN\lambda^8$$

NEC Laboratories America
Relentless passion for innovation

Numerical simulations

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory



Each data point computed from 2500 instances

$$(E_1(\lambda) - E_2(\lambda))^2 \approx C N \lambda^8$$

We have $E_1(\lambda) - E_2(\lambda) > 4$ for $\lambda > \sqrt{2}(CN)^{-1/8}$

NEC Laboratories America
Relentless passion for innovation

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Proof by reduction to the "Agree" problem:
2-bit clauses $(x_{i_C} = x_{j_C})$

NEC Laboratories America
Relentless passion for innovation

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

NEC Laboratories America
Relentless passion for innovation

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

2) typical distance between solutions is $n = \Theta(N)$

NEC Laboratories America
Relentless passion for innovation

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

2) typical distance between solutions is $n = \Theta(N)$

We have: $\boxed{\Delta = O(\exp(-N\log N))}$

Can we trust perturbation theory?

NEC Laboratories America
Relentless passion for innovation

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

NEC Laboratories America
Relentless passion for innovation

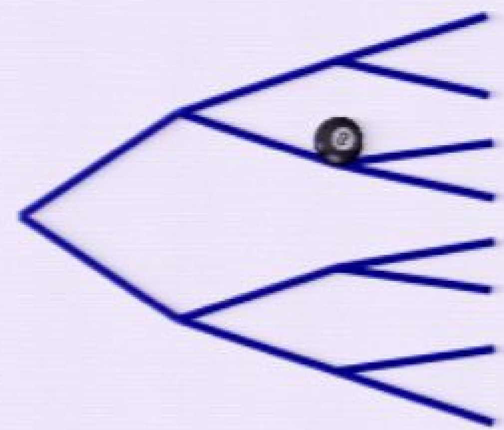
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



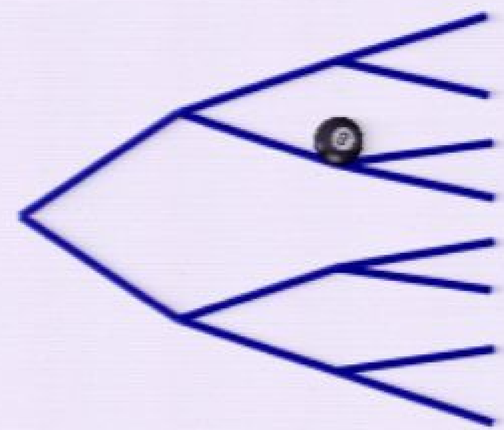
Can we trust perturbation theory?

NEC Laboratories America
Relentless passion for innovation

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:



$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$

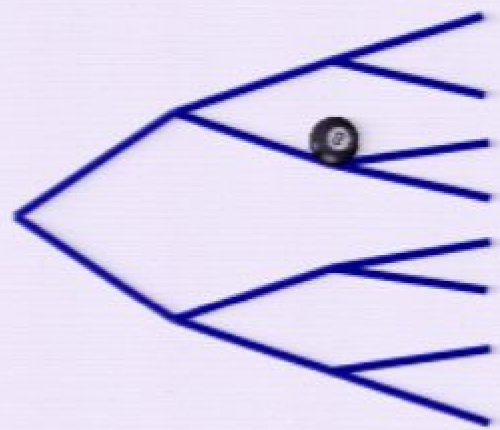
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



Here: Energy E and degree K are $\Theta(N)$, which would imply

$$\lambda_c = \Theta((\log N)^{-1}) \gg \Theta(N^{-1/8}) = \lambda_*$$

NEC Laboratories America
Relentless passion for innovation

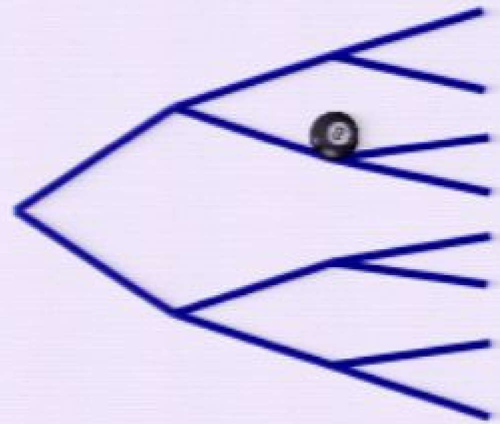
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K :

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



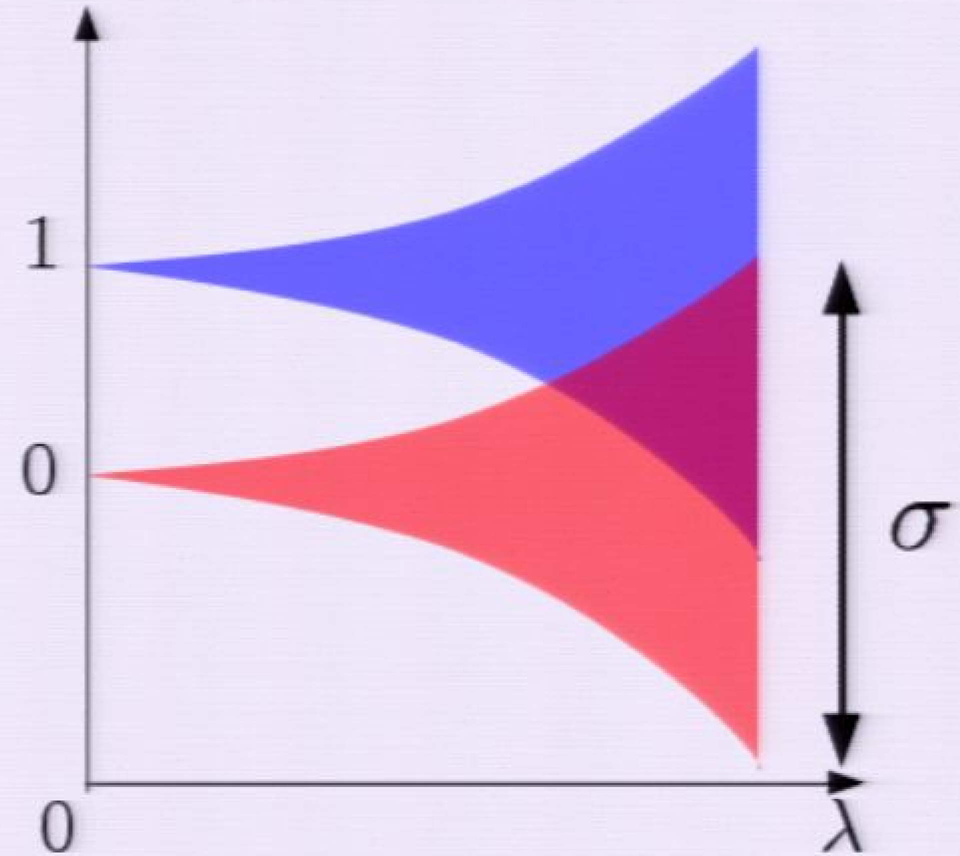
Here: Energy E and degree K are $\Theta(N)$, which would imply

$$\lambda_c = \Theta((\log N)^{-1}) \gg \Theta(N^{-1/8}) = \lambda_*$$

However, $F_{\vec{x}}^{(m)} = O(N) \;\; \forall m$ suggests $\lambda_c = \Theta(1)$

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$



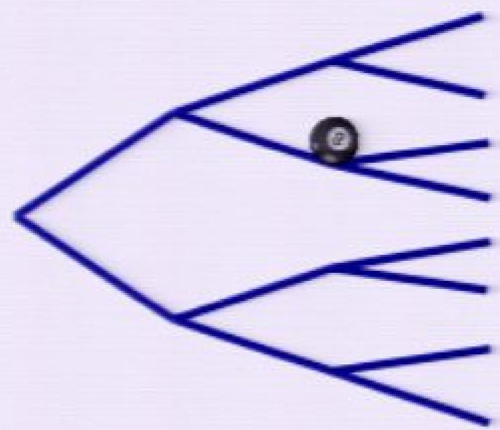
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



Here: Energy E and degree K are $\Theta(N)$, which would imply

$$\lambda_c = \Theta((\log N)^{-1}) \gg \Theta(N^{-1/8}) = \lambda_*$$

However, $F_{\vec{x}}^{(m)} = O(N) \;\; \forall m$ suggests $\lambda_c = \Theta(1)$

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$

- Ground state is degenerate

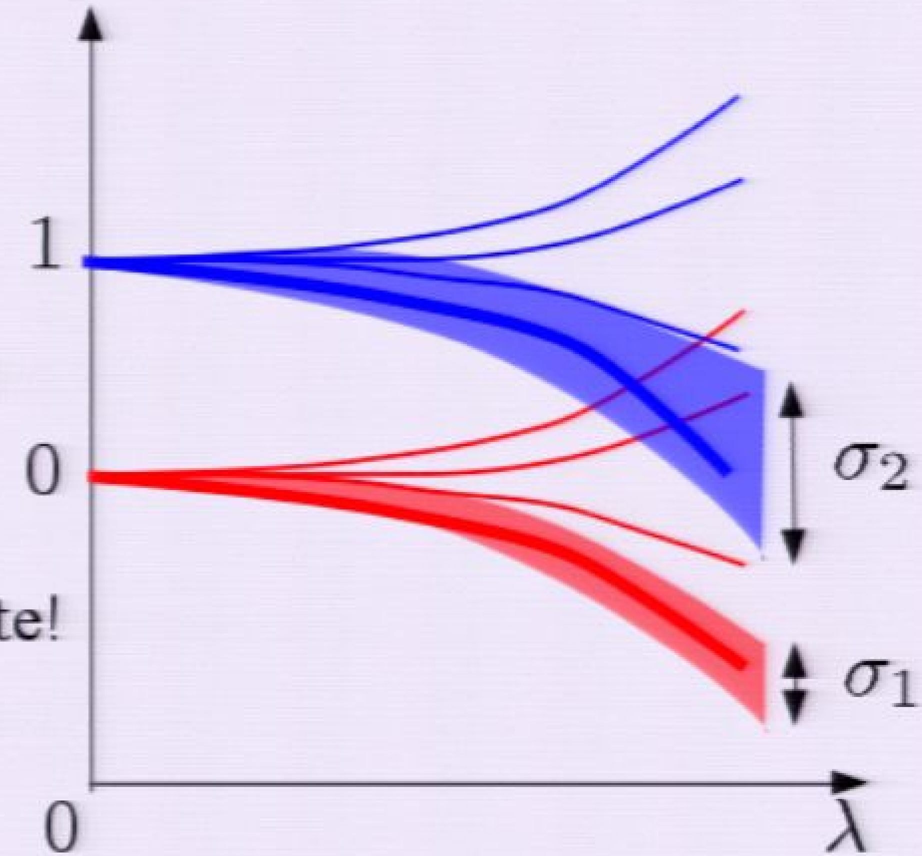
  $S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$

  [Knysh-Smelyanskiy'10]

  BUT

- First excited state is more degenerate!

  $S_2 \sim N\, e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$

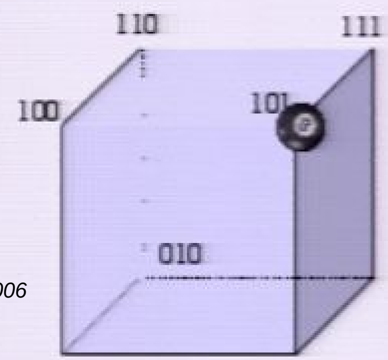


- Also: $\eta \to 0$ as $\alpha \to \alpha_s$

  $\Rightarrow$ Effect of degeneracy only appears for large N

Conclusion

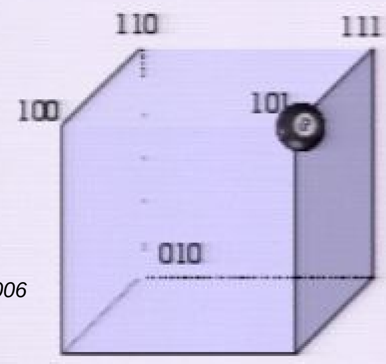
- Anderson localization causes exponentially small gaps in adiabatic quantum optimization



NEC Laboratories America
Relentless passion for innovation

Conclusion

- Anderson localization causes exponentially small gaps in adiabatic quantum optimization
- Does not depend on the particular problem (same for 3SAT)



NEC Laboratories America
Relentless passion for innovation

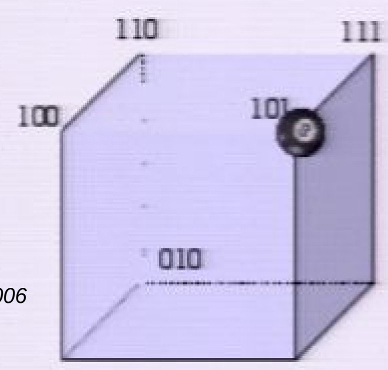
Conclusion

- Anderson localization causes exponentially small gaps in adiabatic quantum optimization

- Does not depend on the particular problem (same for 3SAT)

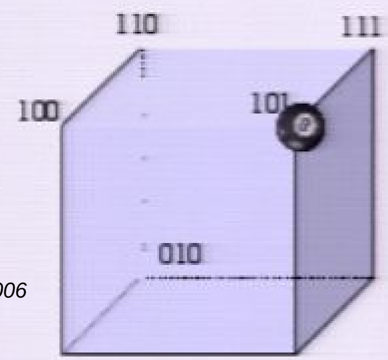
- Does not depend on the particular path $H(s)$ either

(as long as $H(s)$ is local)



Conclusion

- Anderson localization causes exponentially small gaps in adiabatic quantum optimization
- Does not depend on the particular problem (same for 3SAT)
- Does not depend on the particular path $H(s)$ either (as long as $H(s)$ is local)
- Important assumption: Localization on the hypercube



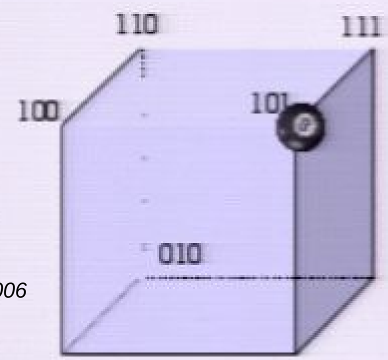
⇒ should be studied more closely


Thank you!
www.nec-labs.com

Conclusion

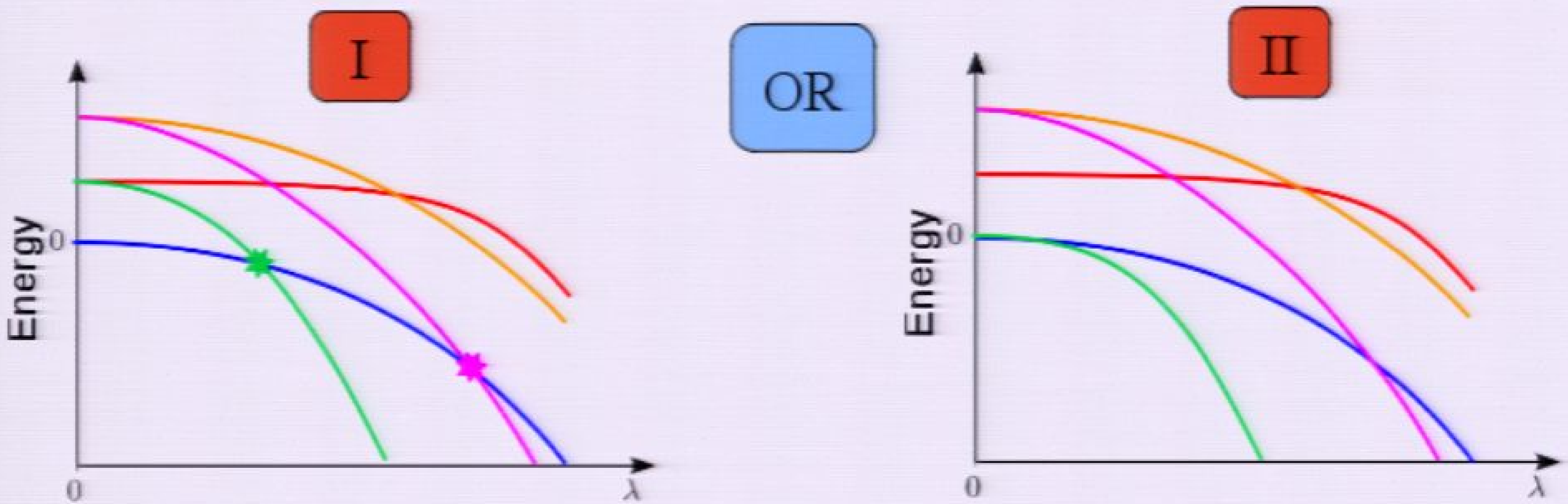
NEC Laboratories America
Relentless passion for innovation

- Anderson localization causes exponentially small gaps in adiabatic quantum optimization



NEC Laboratories America
Relentless passion for innovation

Effect of path change

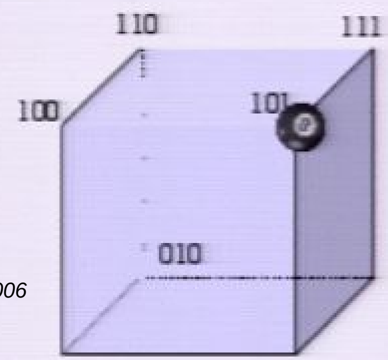


- Idea: Pick random $H(s)$ ("path change") to obtain case II [Farhi *et al.* '09]
- Avoid 1 crossing: $Pr[\text{"II"}] = \text{constant}$
- Avoid poly # of crossings: $Pr[\text{"II"}] = 1/\text{poly}$
- Estimated # of crossings: $\exp(N/\log^7 N)$

Conclusion

NEC Laboratories America
Relentless passion for innovation

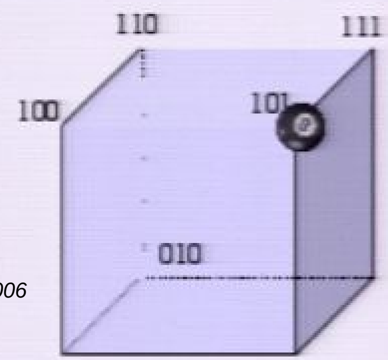
- Anderson localization causes exponentially small gaps in adiabatic quantum optimization



Conclusion

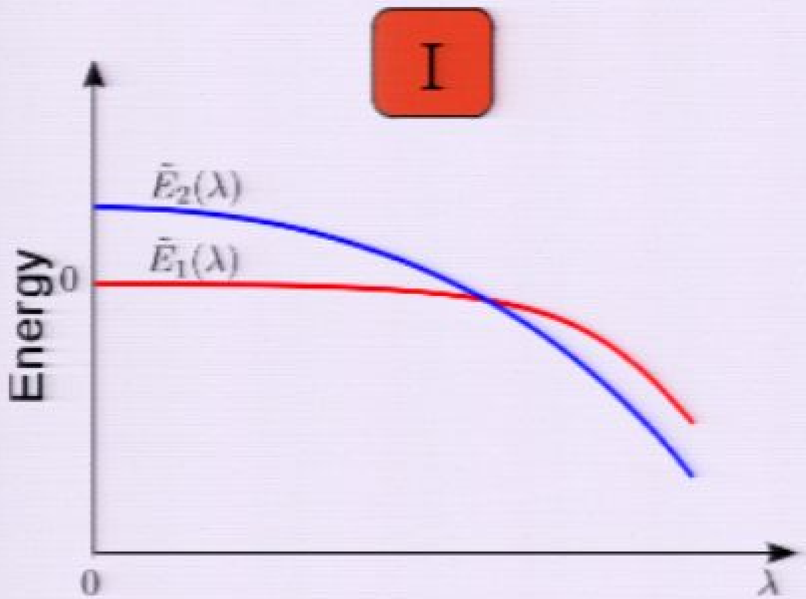
NEC Laboratories America
Relentless passion for innovation

- Anderson localization causes exponentially small gaps in adiabatic quantum optimization

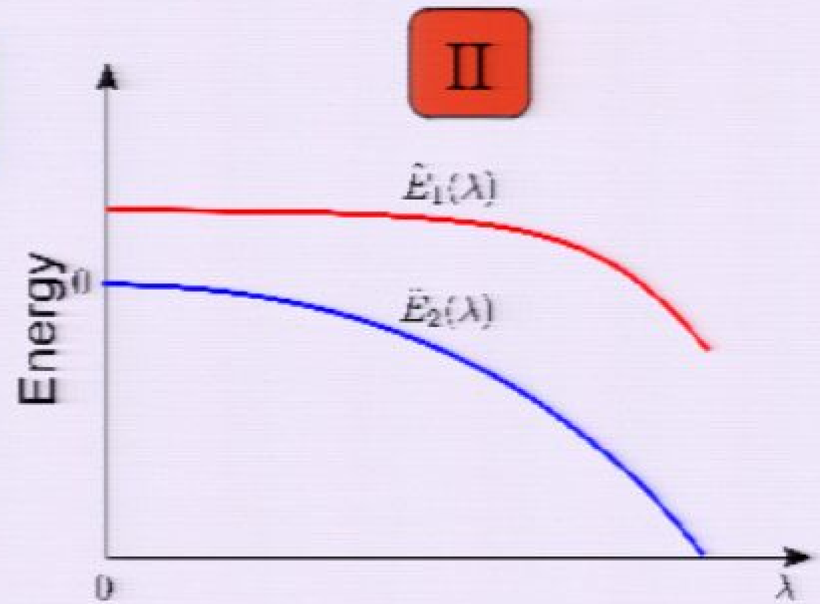


NEC Laboratories America
Relentless passion for innovation

Effect of path change



OR



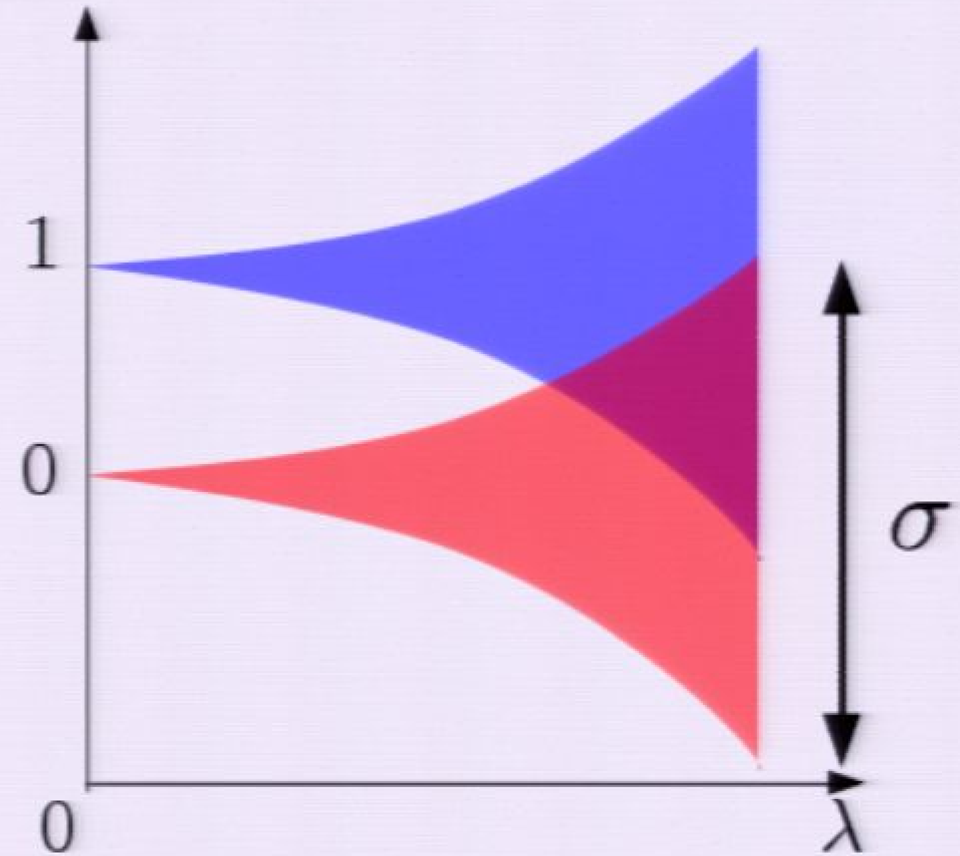
- Idea: Pick random $H(s)$ ("path change") to obtain case II

[Farhi *et al.* '09]

- Avoid 1 crossing: $Pr[\text{"II"}] = \text{constant}$

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$



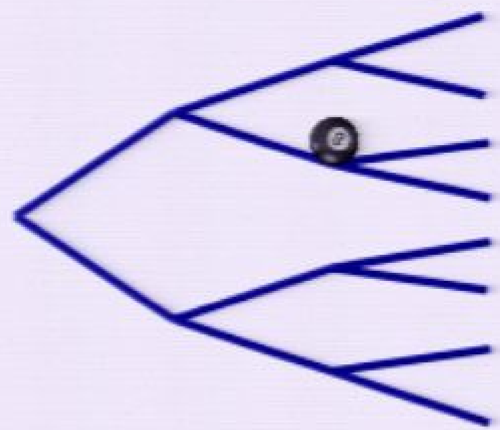
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



Here: Energy E and degree K are $\Theta(N)$, which would imply

$$\lambda_c = \Theta((\log N)^{-1}) \gg \Theta(N^{-1/8}) = \lambda_*$$

However, $F_{\vec{x}}^{(m)} = O(N)\ \ \forall m$ suggests $\lambda_c = \Theta(1)$

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

Can we trust perturbation theory?

NEC Laboratories America
Relentless passion for innovation

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

NEC Laboratories America
Relentless passion for innovation

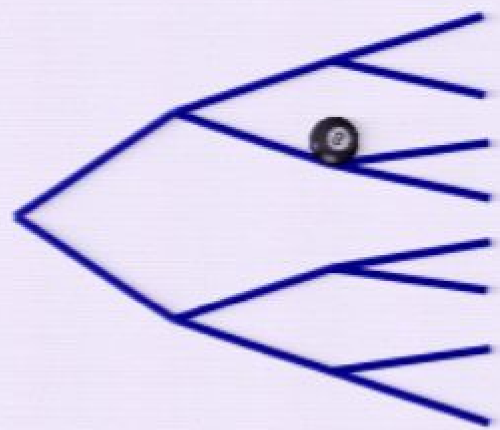
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K:

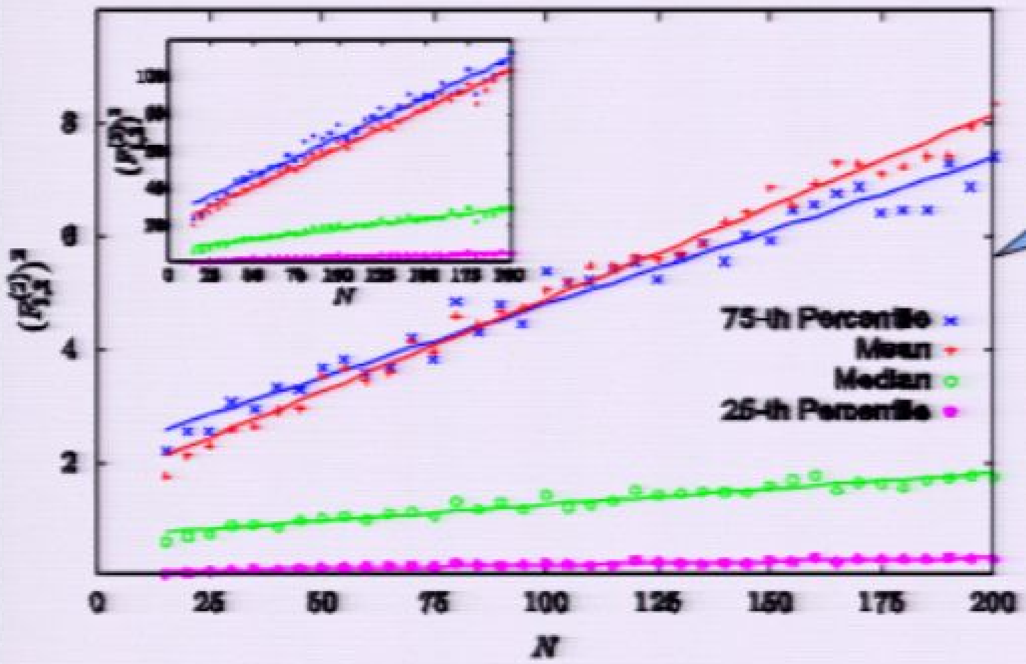
$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



NEC Laboratories America
Relentless passion for innovation

How small is the gap?

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory



Each data point computed from 2500 instances

$$(E_1(\lambda) - E_2(\lambda))^2 \approx CN\lambda^8$$

We have $E_1(\lambda) - E_2(\lambda) > 4$ for $\lambda > \sqrt{2}(CN)^{-1/8}$

NEC Laboratories America
Relentless passion for innovation

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

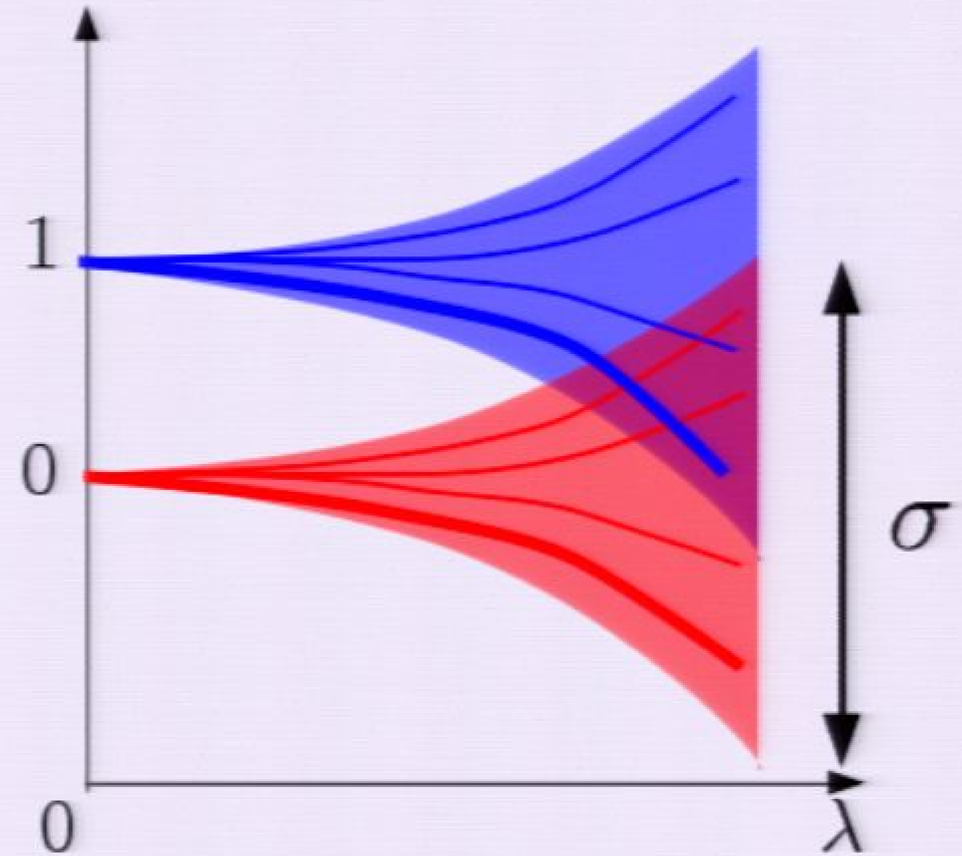
Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

2) typical distance between solutions is $n = \Theta(N)$

NEC Laboratories America
Relentless passion for innovation

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

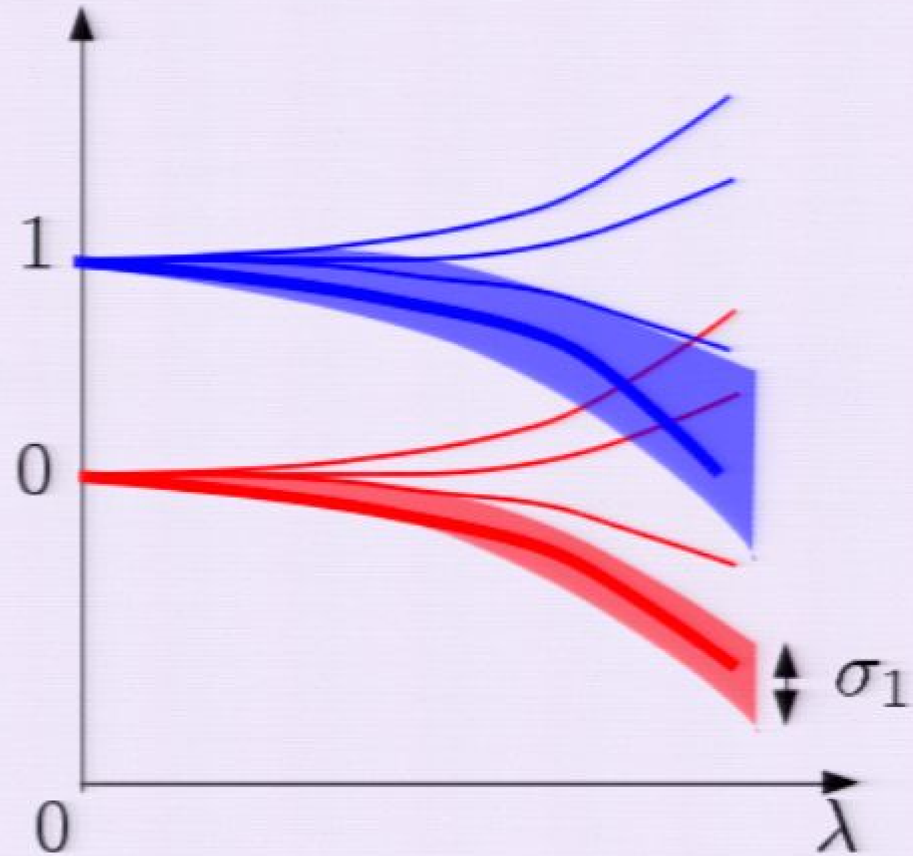


Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]



Degeneracy of the ground state

NEC Laboratories America
Relentless passion for innovation

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

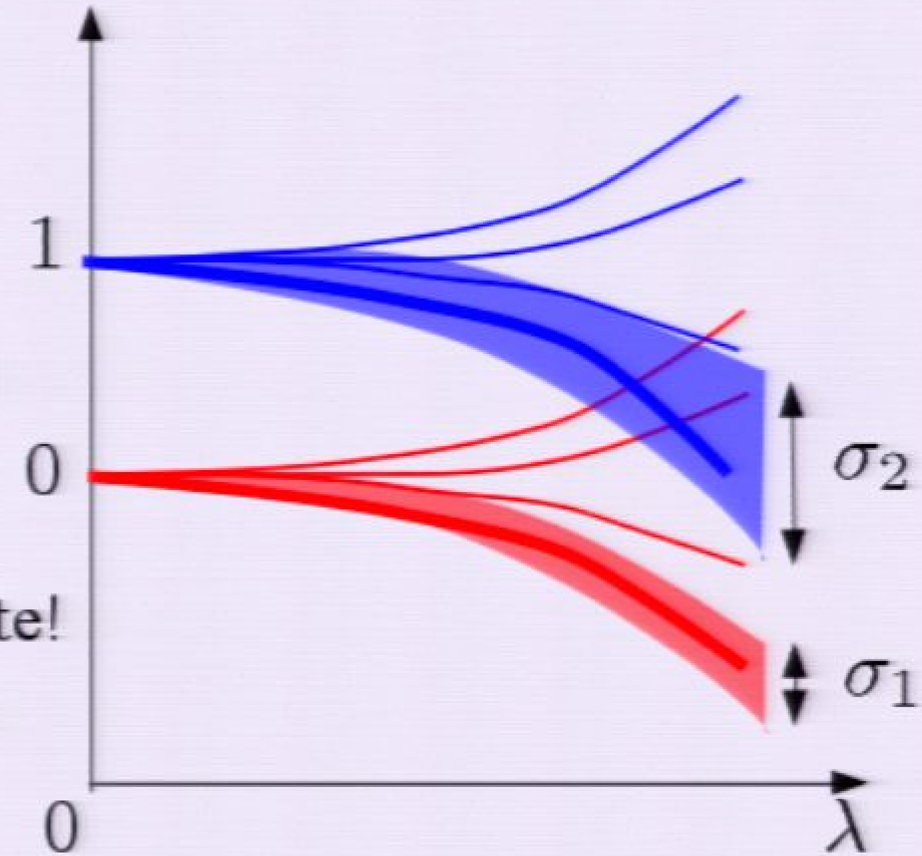
$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]

BUT

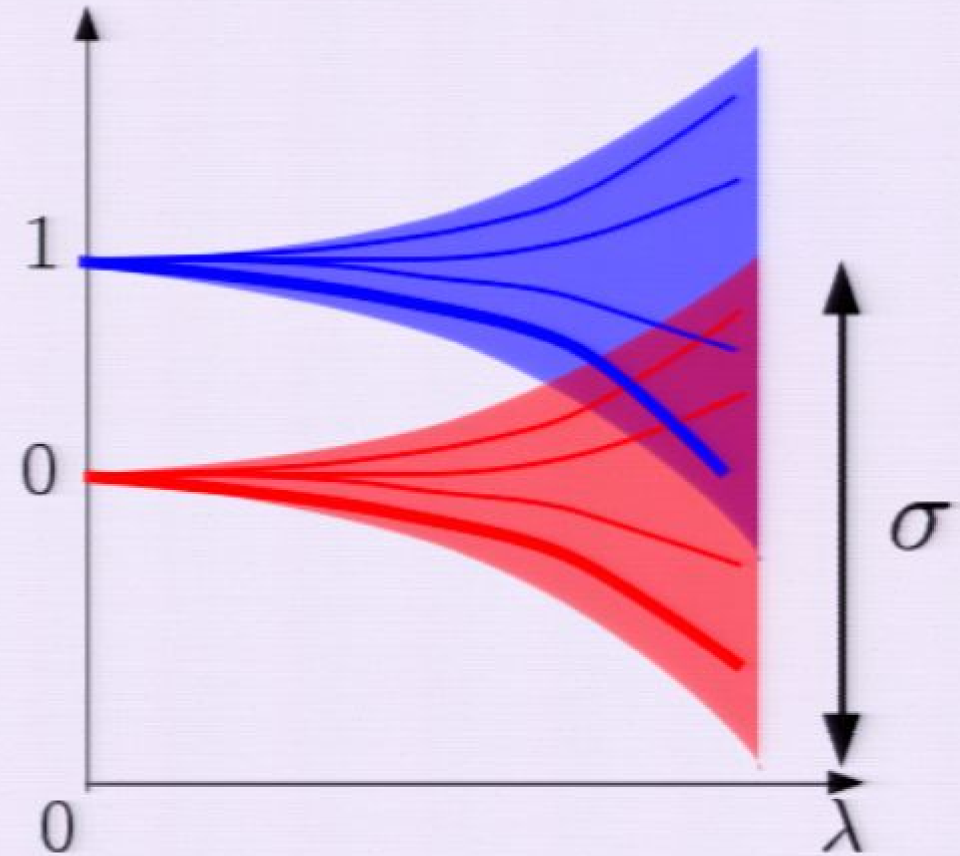
- First excited state is more degenerate!

$$S_2 \sim N\ e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$$



Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate



NEC Laboratories America
Relentless passion for innovation

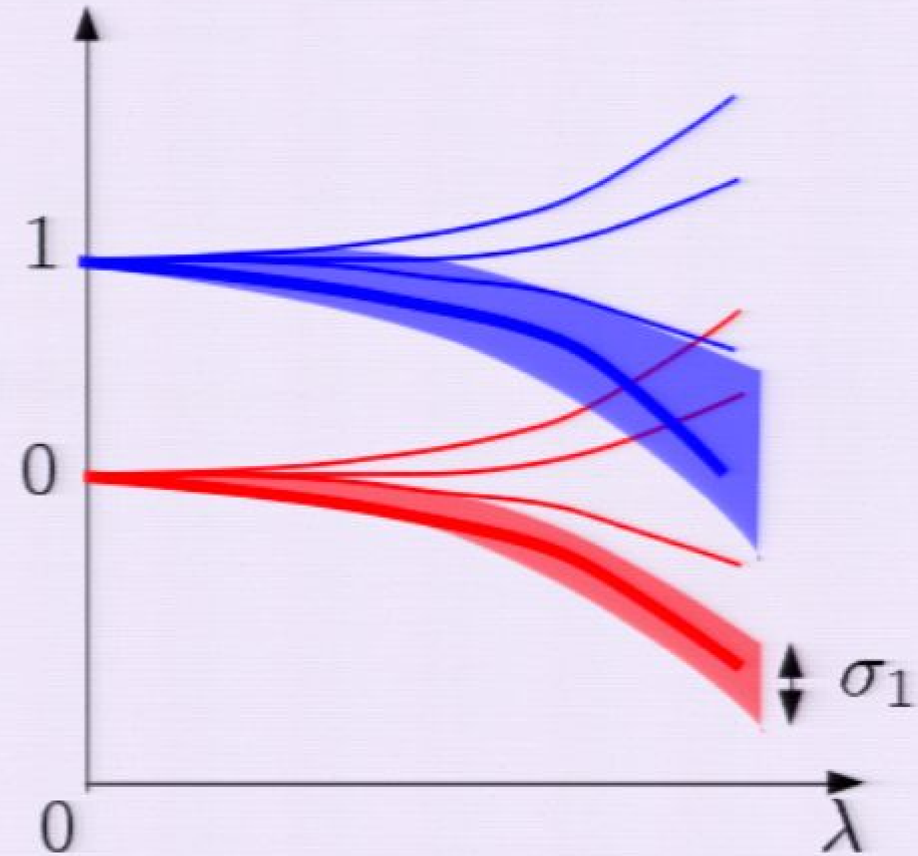
Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$

- Ground state is degenerate

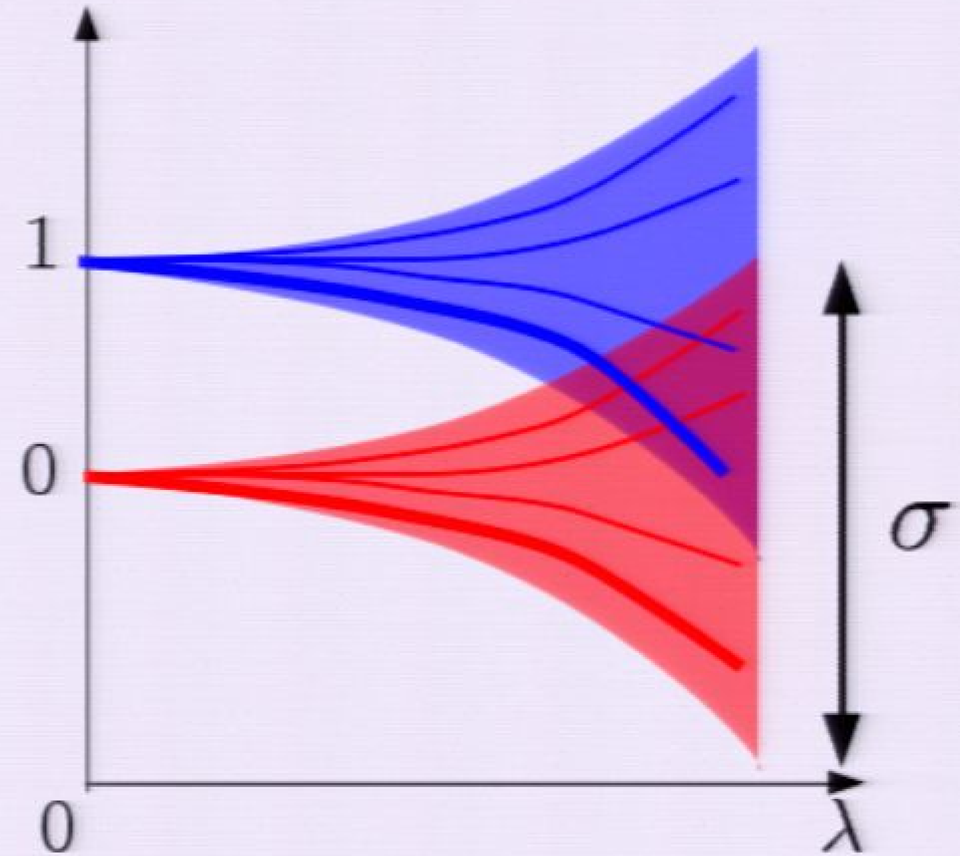
  $S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$

  [Knysh-Smelyanskiy'10]



Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate



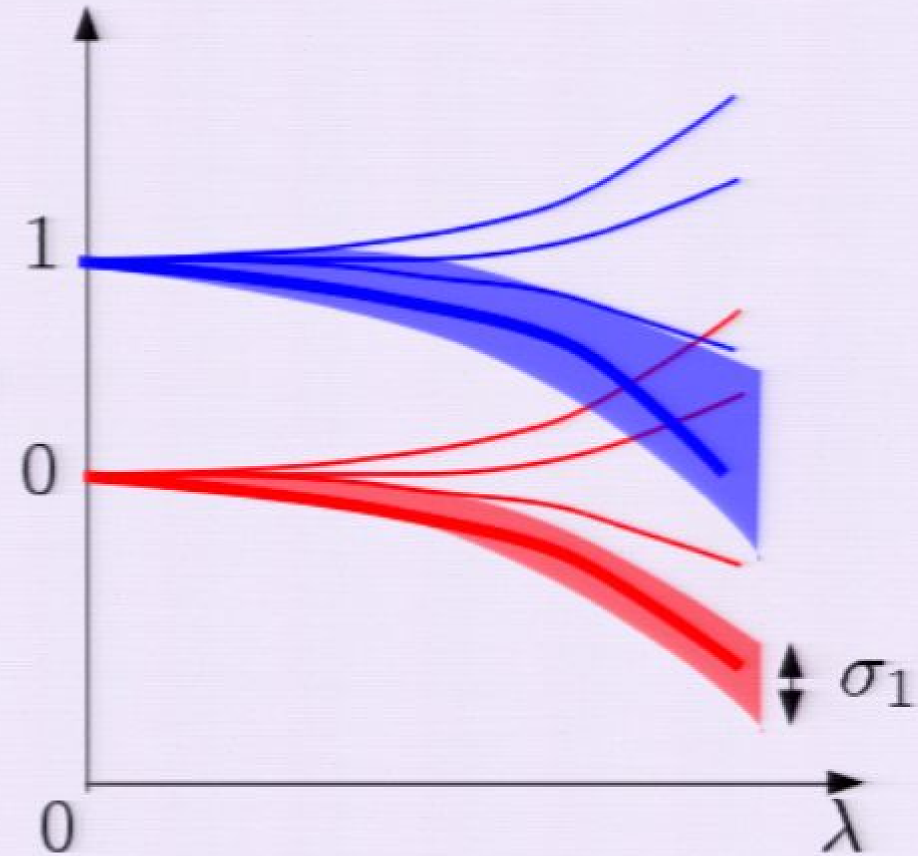
Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$

- Ground state is degenerate

$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]



Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

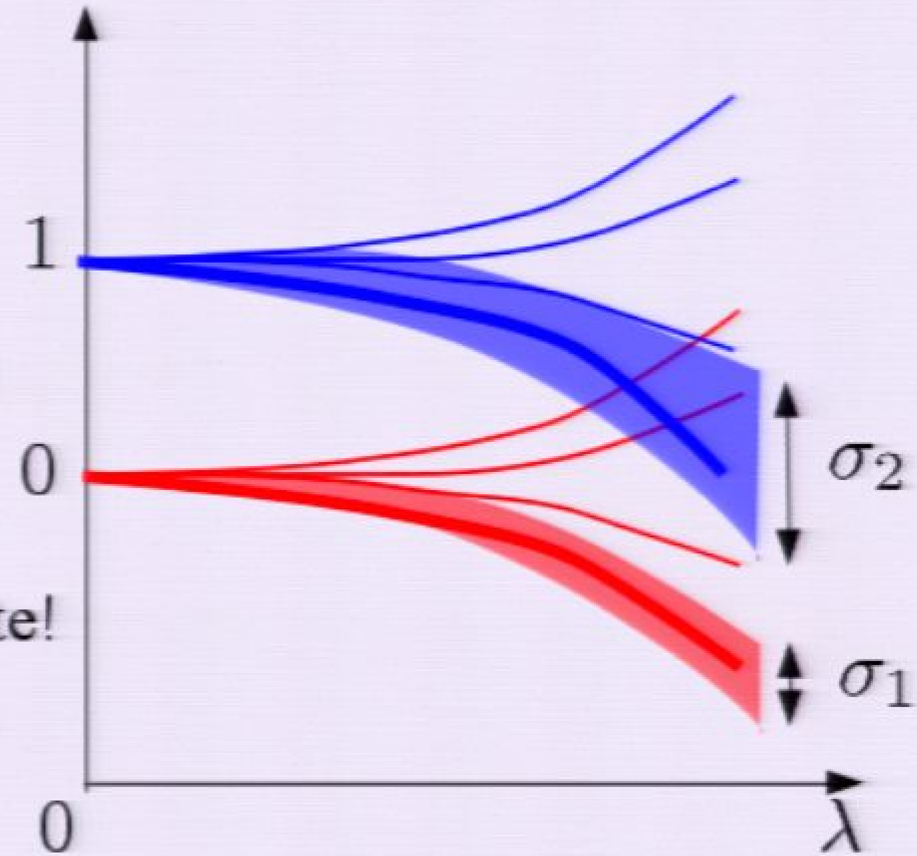
$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]

BUT

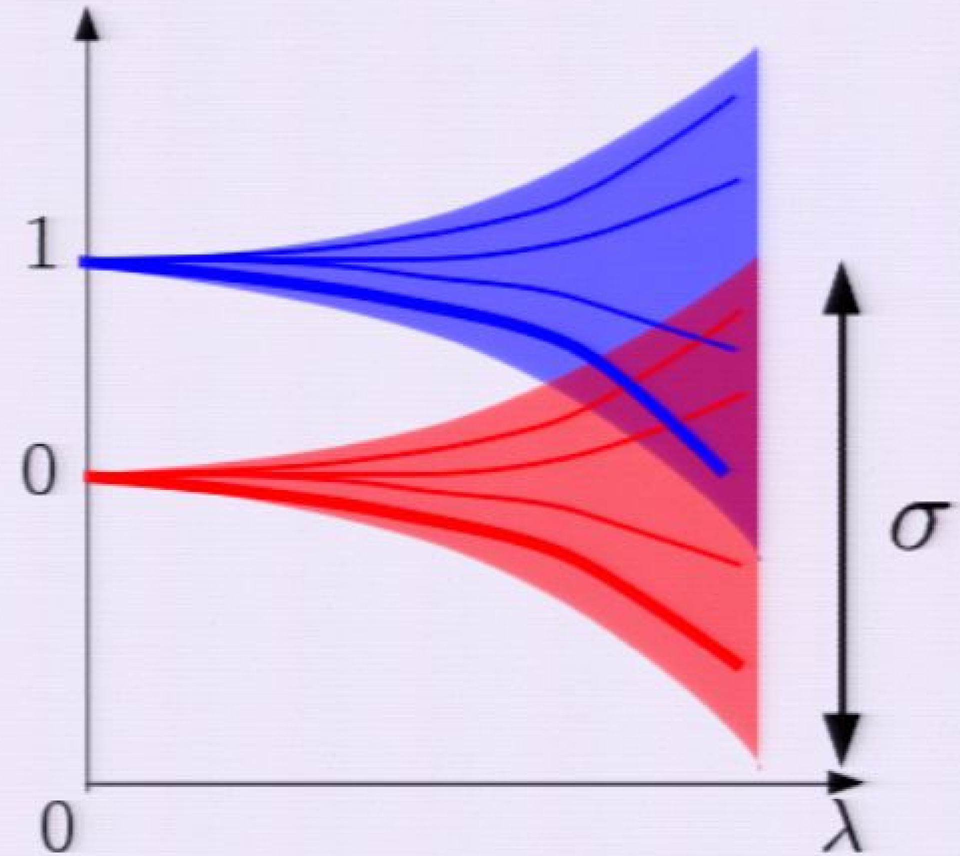
- First excited state is more degenerate!

$$S_2 \sim N\ e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$$



Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate



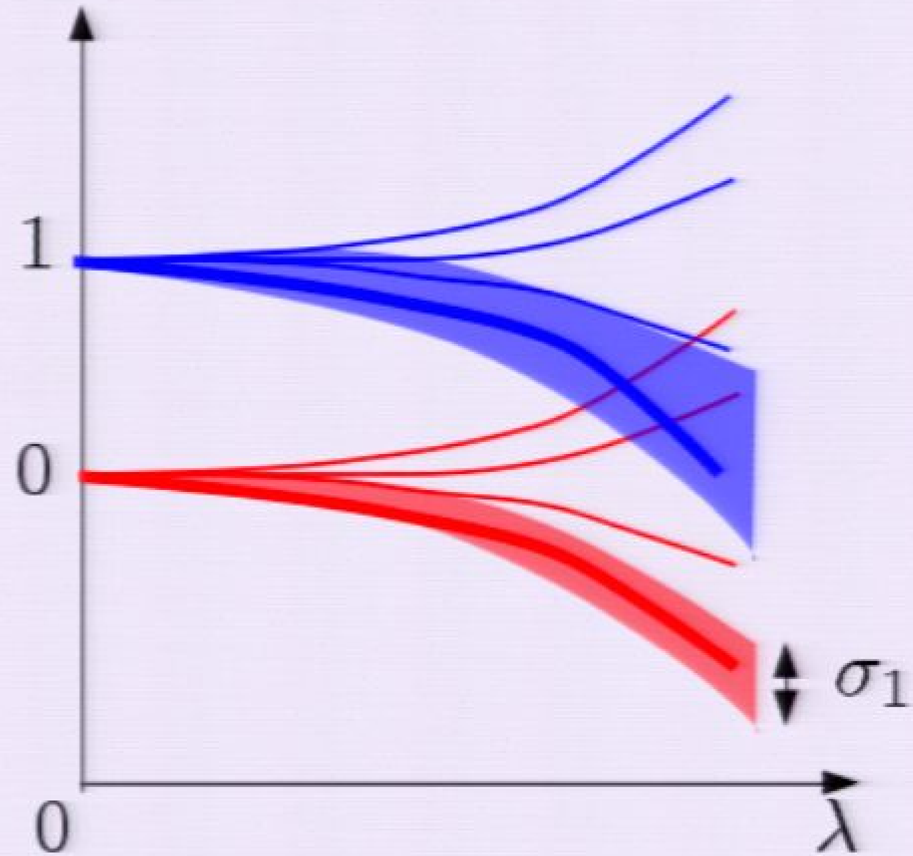
Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$

- Ground state is degenerate

$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]



NEC Laboratories America
Relentless passion for innovation

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

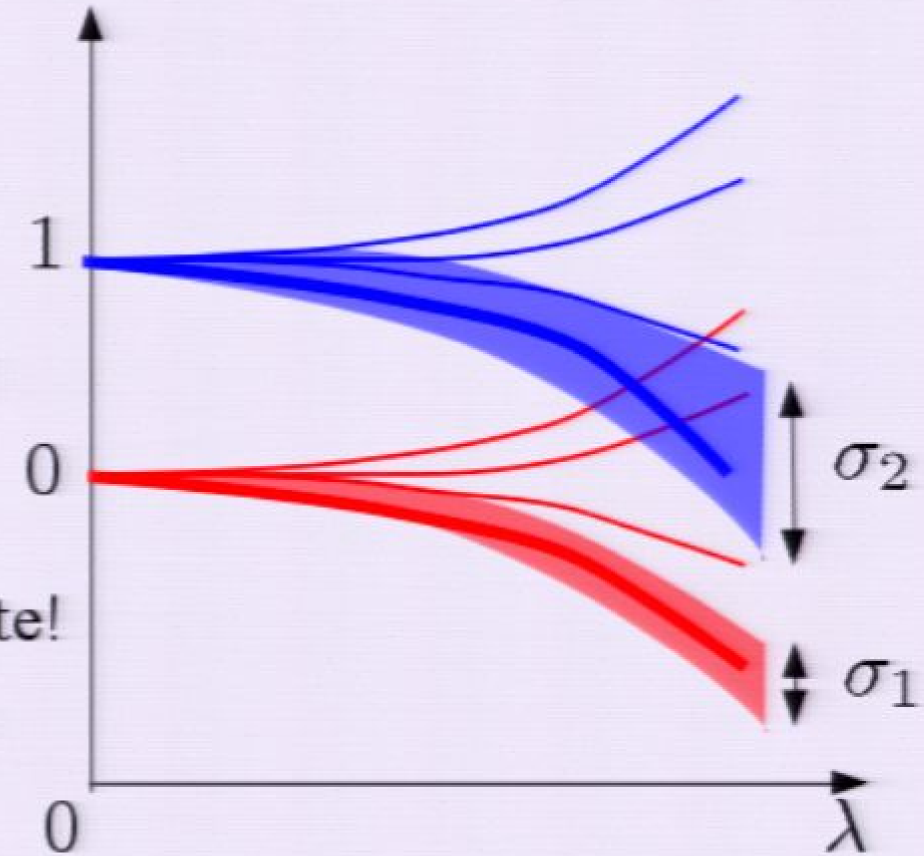
$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]

BUT

- First excited state is more degenerate!

$$S_2 \sim N\ e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$$



Degeneracy of the ground state

NEC Laboratories America
Relentless passion for innovation

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$
- Ground state is degenerate

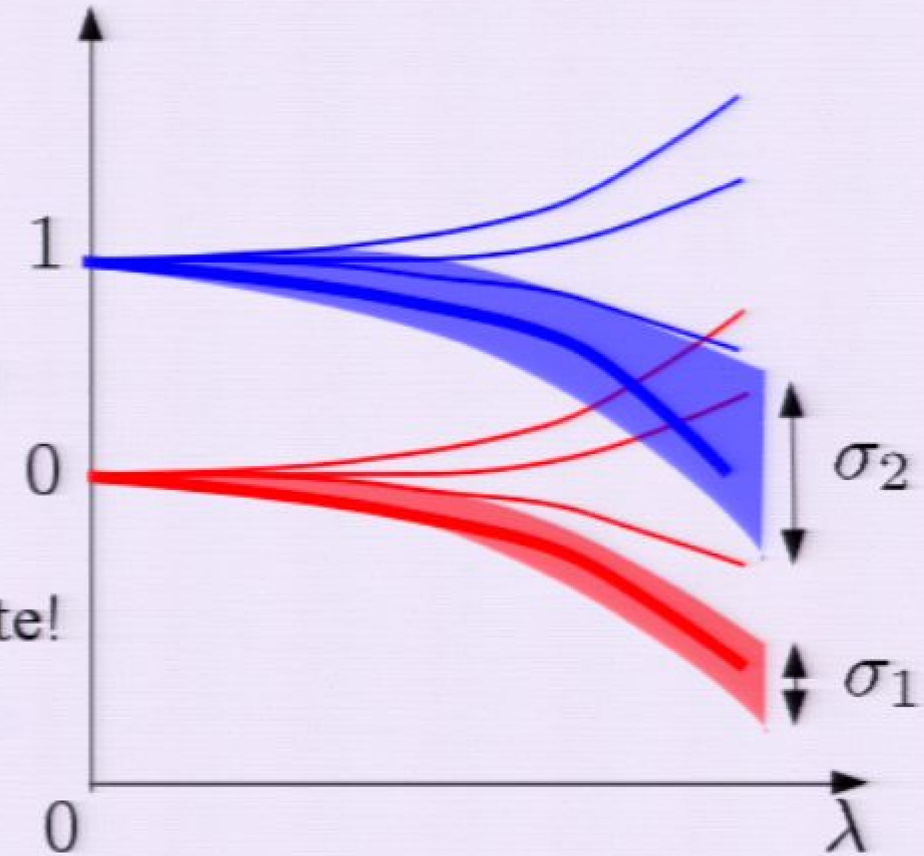
$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$

[Knysh-Smelyanskiy'10]

BUT

- First excited state is more degenerate!

$S_2 \sim N\ e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$



- Also: $\eta \to 0$ as $\alpha \to \alpha_s$

$\Rightarrow$ Effect of degeneracy only appears for large N