

Title: Hausdorff and spectral dimension of random graphs

Date: Jul 04, 2010 10:30 AM

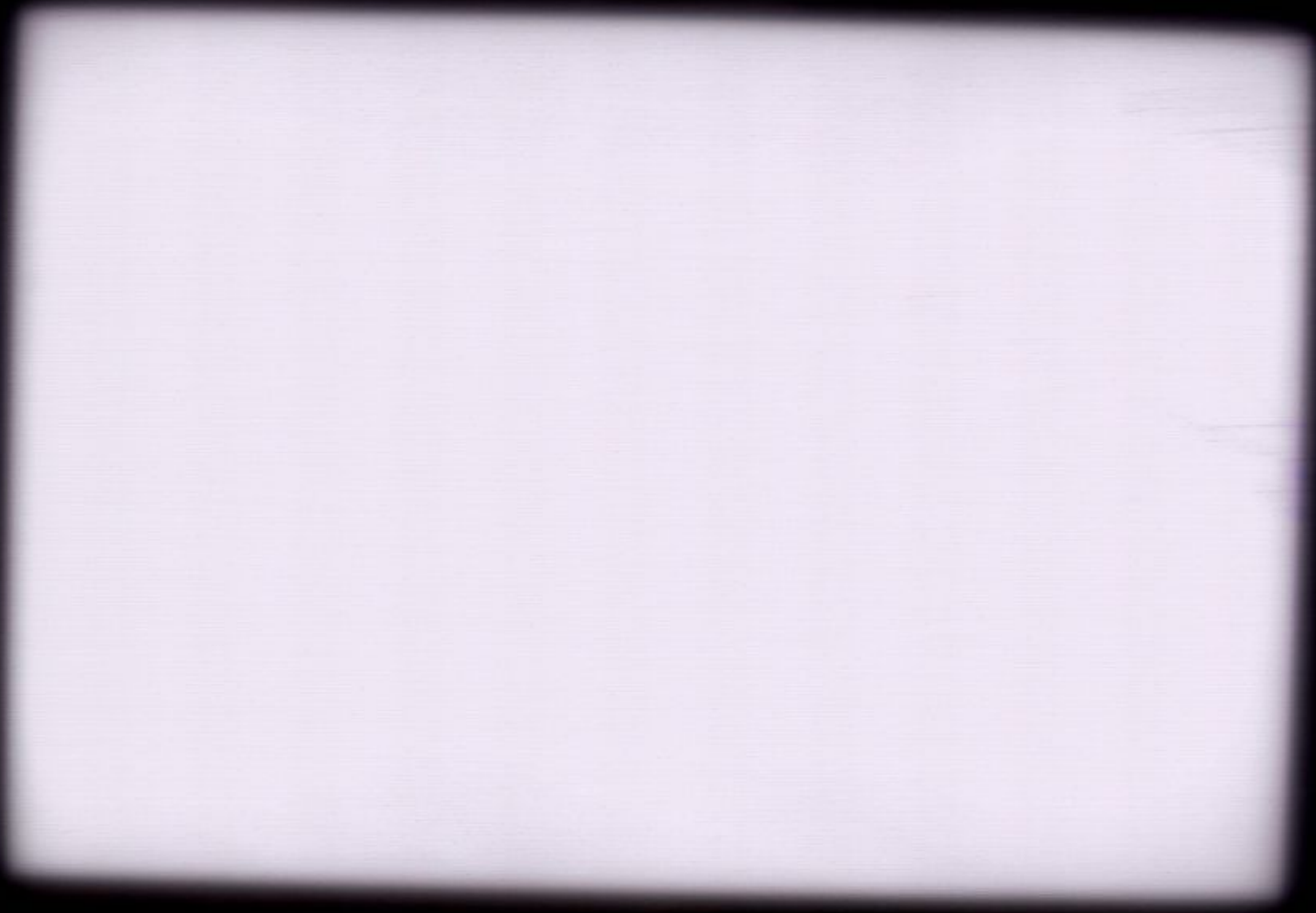
URL: <http://pirsa.org/10070005>

Abstract: We introduce a class of probability spaces whose objects are infinite graphs and whose probability distributions are obtained as limits of distributions for finite graphs. The notions of Hausdorff and spectral dimension for such ensembles are defined and some results on their value in concrete examples, such as random trees, will be described.

HAUSDORFF AND SPECTRAL DIMENSION
OF RANDOM GRAPHS

B. DURHUUS

"Random Matrix Techniques in Quantum Information Theory", Perimeter Institute, July 4, 2010.



Some motivations:

- 1) Random graphs in discretized gravity.
2 dim: Matrix models $\leftrightarrow$ Topological gravity
Intrinsic geometry?

Ref: J. Ambjørn, B.D., T. Jonsson, Quantum
Geometry, Cambridge Univ. Press, 1997.

- 2) Random media. Anomalous diffusion.
3) Percolation on a fixed graph, e.g. $\mathbb{Z}^d$
 $\hookrightarrow$ Random percolation clusters.
4) Structure of networks.

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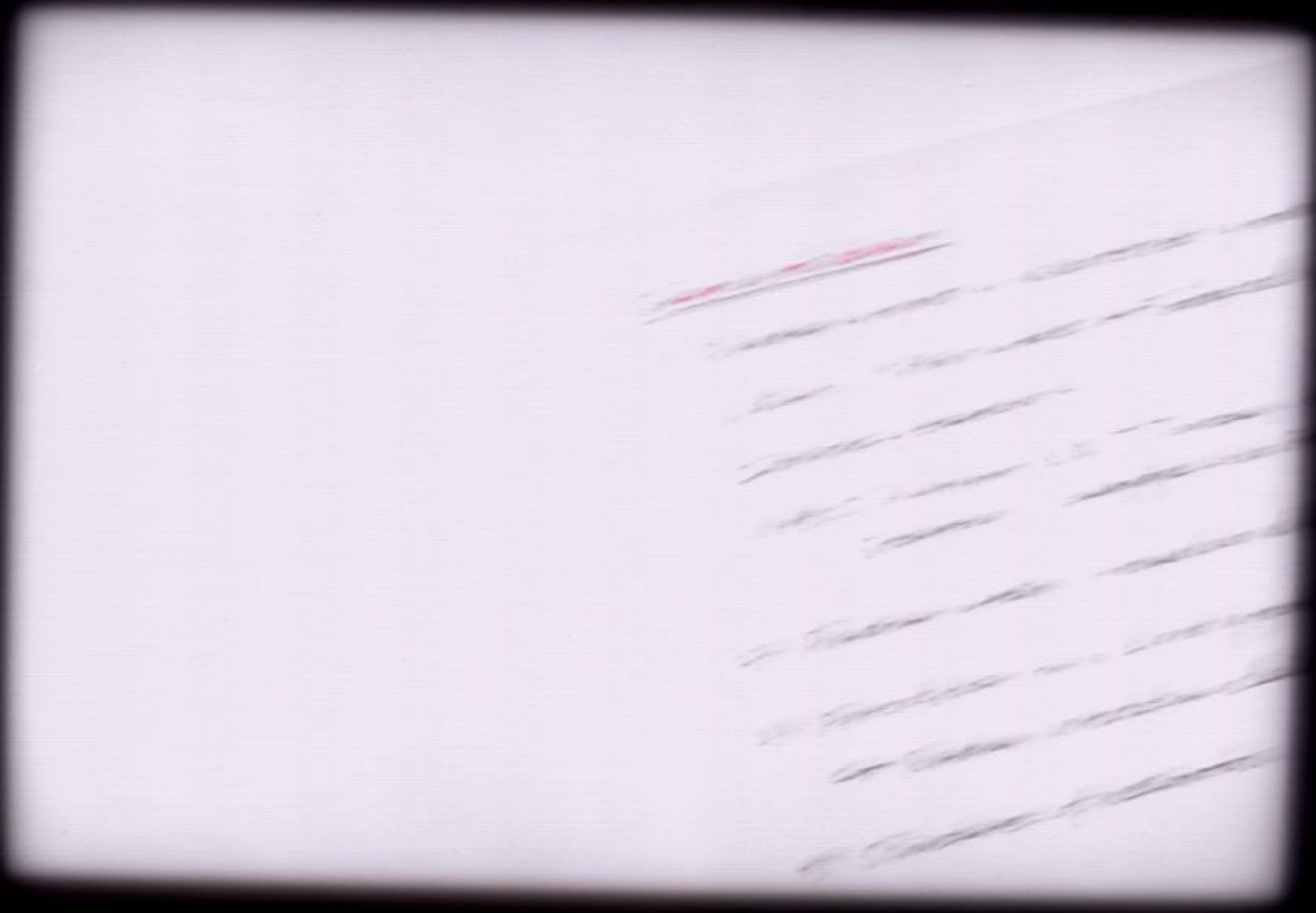
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(2)

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- Graph G . $V(G)$ = vertex set
 $E(G)$ = edge set



d_v = degree of $v < \infty$

- Path in G : $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$
 all v_i different.

Circuit = closed path ($v_0 = v_k$).

Graph distance: $d(v, v') = \# \text{ edges in shortest path from } v \text{ to } v'$

G connected if any pair of vertices can be joined by a path.

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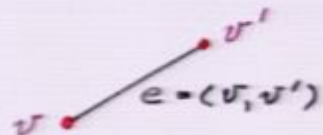
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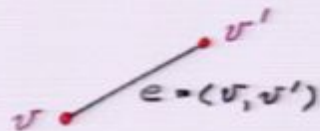
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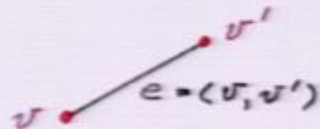
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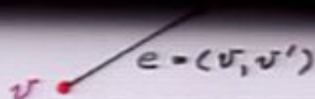
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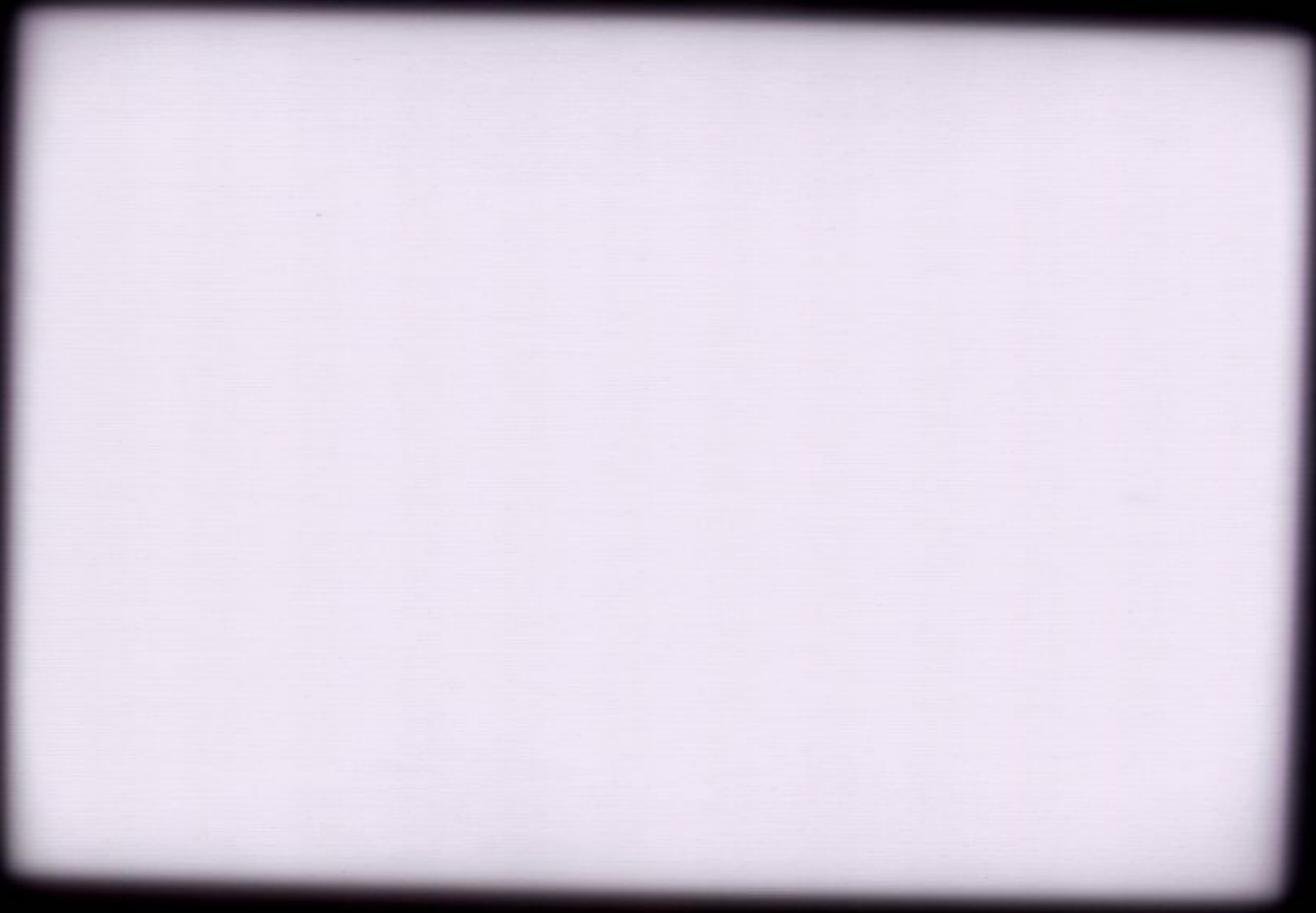
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[Faint, illegible handwritten text, likely bleed-through from the reverse side of the page.]



(3)

- For $v_0 \in V(G)$, $R \geq 0$:

$B_R(G, v_0)$ = ball of radius R centered at v_0 , considered as sub-graph of G .

$$|B_R(G, v_0)| = \# \text{ edges in } B_R(G, v_0)$$

Hausdorff dimension of G connected and infinite:

$$d_h = \lim_{R \rightarrow \infty} \frac{\ln |B_R(G, v_0)|}{\ln R}$$

Indep. of v_0 , but not well-defined for all G .

- A walk in G $w: v_0 \rightarrow v_k$ is a sequence of edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$ with v_i 's not necess. distinct.

$$|w| = k = \text{length of } w.$$

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$$\frac{k-1}{k}$$

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$$\frac{k-1}{k} = 1 - \frac{1}{k}$$

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Set

$$w_k = \frac{k-1}{k} = -1$$

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Set $\mathcal{W}_k$ = set of all walks of length k

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$$p(w) = \prod_{i=0}^{k-1} \sigma_{w(i)}^{-1}$$

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- A walk in G $\omega: v_0 \rightarrow v_k$ is a sequence of edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$ with v_i 's not necess. distinct.

$$|\omega| = k = \text{length of } \omega.$$

Set

$$p(\omega) = \prod_{i=0}^{k-1} d_{\omega(i)}^{-1}$$

where $\omega(i)$ = i 'th vertex in ω .

$B_R(G, v_0)$ = ball of radius R centered at v_0 , considered as subgraph of G .

$$|B_R(G, v_0)| = \# \text{ edges in } B_R(G, v_0)$$

Hausdorff dimension of G connected and infinite:

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Set

$$p(w) = \prod_{i=0}^{k-1} \sigma_{w(i)}^{-1}$$

where $w(i)$ = i 'th vertex in w .

at v_1 , connected to v_2 ,
graph of G .

$$|B_R(\bar{a}, v_2)| = \# \text{ edges in } B_R(\bar{a}, v_2)$$

Hausdorff dimension of G connected
and infinite:

$$d_H = \lim_{R \rightarrow \infty} \frac{\ln |B_R(\bar{a}, v_2)|}{\ln R}$$

Indep. of v_2 , but not well defined
for all $\bar{a}$.

- A walk in G $w: v_1 \rightarrow \dots \rightarrow v_k$ is a sequence
of edges $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$
with v_i 's not necessarily distinct.

$\text{len}(w) = k = \text{length of } w$.

Set

$$p(w) = \prod_{i=1}^{k-1} \frac{1}{d(w_i)}$$

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Set

$$p(\omega) = \prod_{i=0}^{k-1} d_{\omega(i)}^{-1}$$

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p defines a probability distribution on walks of fixed length and starting point:

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Set

$$p(w) = \frac{1}{\prod_{i=0}^{k-1} d(w(i))}^{-1}$$

where $w(i) = i$ 'th vertex in w .

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p defines a probability distribution on walks of fixed length and starting point:



$$\omega \in \Pi_n(v_0)$$

where $\Pi_n(v_0)$ = set of walks of length n emerging from v_0 . These probability distributions define the simple random walk on G .

For G connected: $p_0(G, v_0)$ = return probability to v_0 after time $t \geq 0$

$$p_t(G, v_0) = \sum_{\substack{\omega: v_0 \rightarrow v_0 \\ |\omega| = t}} p(\omega).$$

Spectral dimension for G connected and infinite:

$$d_s = -2 \lim_{t \rightarrow \infty} \frac{\ln p_t(G, v_0)}{\ln t}$$

Indep. of v_0 but not well-defined for all G .

Note:

$$d_h \geq 1, \quad d_s \geq 1 \quad \text{all } G.$$

$$d_h = d_s = d \quad \text{for } G = \mathbb{Z}^d.$$

where $\mathcal{H}_n(v_0)$ = set of chains of length n emerging from v_0 . These probability distributions define the simple random walk on G .

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Schedule change Monday

Belinschi

Zeitsumme

(cyclic permutation)

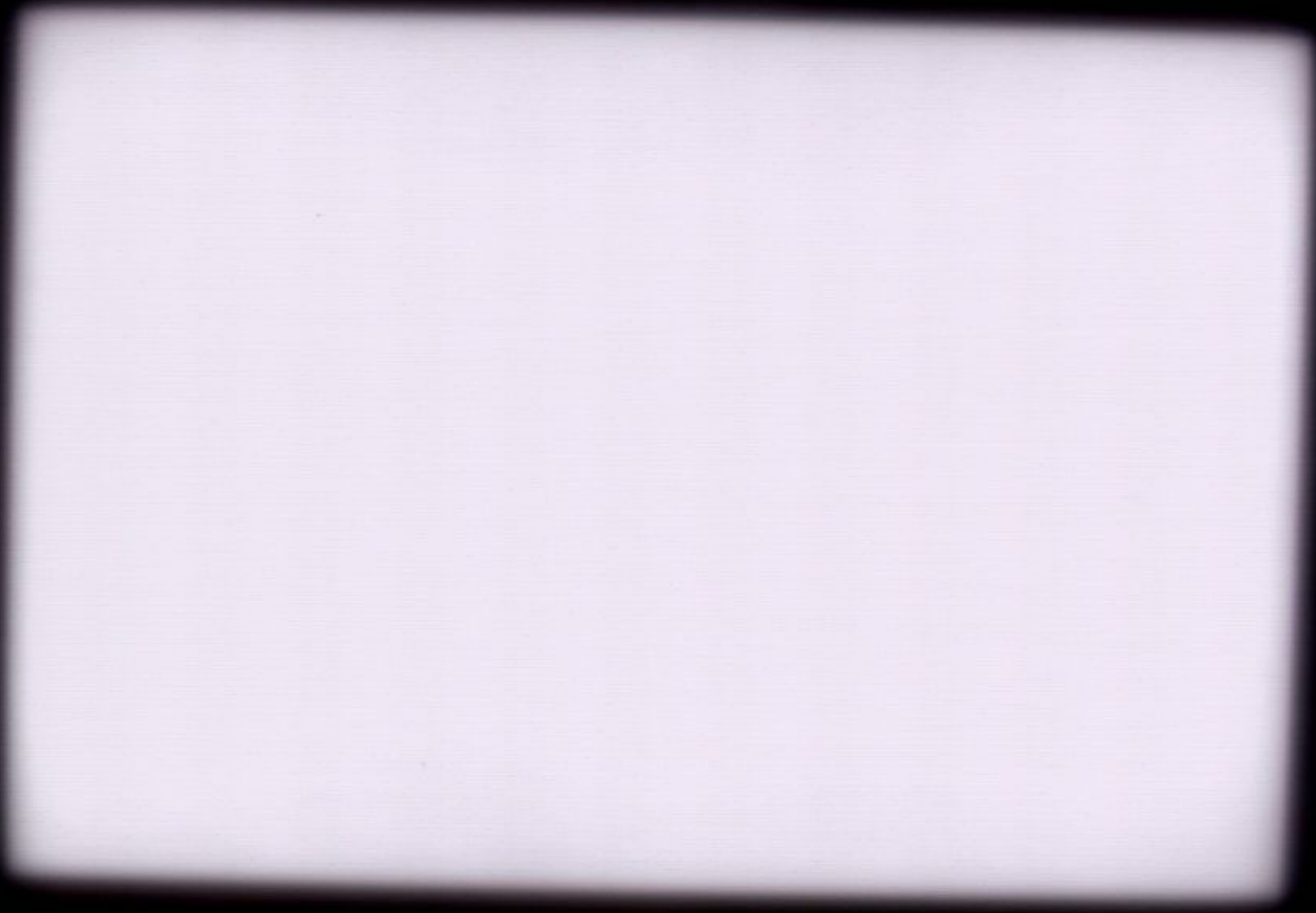
Zyckowski

Nechita

Almbaris

Pellegrini

$$2d_h \geq d_s \geq \frac{2d_h}{1+d_h}$$





- Ensemble of graphs = random graph ⑤
= set of graphs $\mathcal{G}$ with a probability measure μ .

$\langle \cdot \rangle_\mu$ = expectation w.r.t. μ .

Assume graphs are rooted with root vertex r :

$$B_R(G) = B_R(G, r), \quad P_t(G) = P_t(G, r).$$

Hausdorff and spectral dimension of random graph:

$$d_h = \lim_{R \rightarrow \infty} \frac{\ln \langle |B_R| \rangle_\mu}{\ln R}$$

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Almost sure existence of d_h or d_s :

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Almost sure existence of d_h or d_s :

Constant functions on $\mathcal{G}$ except for

$\langle \cdot \rangle_\mu$ = expectation w.r.t. μ .

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Almost sure existence of d_h or d_s :

Constant functions on $\mathcal{G}$ except for a null-set.

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Almost sure existence of d_h or d_s :

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⑥

- Generally obtain μ as a limit of probability distributions μ_N defined on finite sets of finite graphs $\mathcal{G}_N \subseteq \mathcal{G}$.

Consider $\mathcal{G}$ as metric space:

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G infinite, connected, rooted graph.

$0 \leq p \leq 1$ fixed.

For $e \in E(G)$ set

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Given a configuration c define the cluster C_r as the maximal connected subgraph of G containing r such that c assumes value 1 on all edges in C_r .

The case $G = \mathbb{Z}^d$, $d \geq 2$: Choose $r = \underline{0}$.

There exists a critical probability $p_{cr} \in]0, 1[$ such that

$$\mathcal{P}(|C_r| = \infty) \begin{cases} = 0 & \text{if } p < p_{cr} \\ > 0 & \text{if } p > p_{cr}. \end{cases}$$

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$$\mathbb{P}(|C_r| = \infty) \begin{cases} = 0 & \text{if } p < p_{cr} \\ > 0 & \text{if } p > p_{cr}. \end{cases}$$

⑧

For $d=2$ or $d \geq 19$ (but expected to hold for all $d \geq 2$):

$$g_{\text{per}}(|C_r| = \infty) = 0$$

i.e. percolation cluster c_r is almost surely finite at percolation threshold.

For $p < p_{\text{cr}}$

$$g_p(|C_r| = n) \sim e^{-\alpha(p)n}, \quad n \rightarrow \infty,$$

where $\alpha(p) > 0$. Hence mean cluster size

$$\chi(p) = \langle |C_r| \rangle_{g_p}$$

is finite for $p < p_{\text{cr}}$.

power-law scaling

⑧

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Expected scaling

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Expected scaling

$$\chi(p) \sim \text{const} \cdot (p_{cr} - p)^{-\gamma}, \quad p \nearrow p_{cr}$$

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Expected scaling

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For $p < p_{\text{cr}}$

$$S_p(|C_r| = n) \sim e^{-\alpha(p) \cdot n}, \quad n \rightarrow \infty,$$

where $\alpha(p) > 0$. Hence mean cluster size

$$\chi(p) = \langle |C_r| \rangle_{S_p}$$

is finite for $p < p_{\text{cr}}$.

Expected scaling

$$\chi(p) \sim \text{const.} \cdot (p_{\text{cr}} - p)^{-\gamma}, \quad p \nearrow p_{\text{cr}},$$

for some $\gamma > 0$. For clusters of order

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Expected scaling

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Expected scaling

$$\chi(p) \sim \text{const} \cdot (p_{\text{cr}} - p)^{-\gamma}, \quad p \nearrow p_{\text{cr}},$$

for some $\gamma > 0$. I.e. clusters of arbitrarily large size become increasingly frequent as p_{cr} is approached.

Define μ_N to be $S_{p_{\text{cr}}}$ conditioned on clusters of size N , i.e. for $|C| = N$

$$g_{\text{per}}(|C_r| = \infty) = 0$$

i.e. percolation cluster C_r is almost surely finite at percolation threshold.

For $p < p_{\text{per}}$

$$g_p(|C_r| = n) \sim e^{-\alpha(p)n}, \quad n \rightarrow \infty,$$

where $\alpha(p) > 0$. Hence mean cluster size

$$\chi(p) = \langle |C_r| \rangle_{g_p}$$

is finite for $p < p_{\text{per}}$.

Expected scaling

$$\chi(p) \sim \text{const.} \cdot (p_{\text{per}} - p)^{-\gamma}, \quad p \nearrow p_{\text{per}},$$

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Define μ_N to be g_{per} conditioned on clusters of size N , i.e. for $|C| = N$

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Define μ_N to be $g_{p_{\text{cr}}}$ conditioned on cluster size $n \leq N$, i.e. for $|C_r| \leq N$

$$g_{p_{cr}}(|C_r| = \infty) = 0$$

i.e. percolation cluster c_r is almost surely finite at percolation threshold.

For $p < p_{cr}$

$$g_p(|C_r| = n) \sim e^{-\alpha(p)n}, \quad n \rightarrow \infty,$$

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For $p < p_{cr}$

$$g_p(|C_r|=n) \sim e^{-\alpha(p)n}, \quad n \rightarrow \infty,$$

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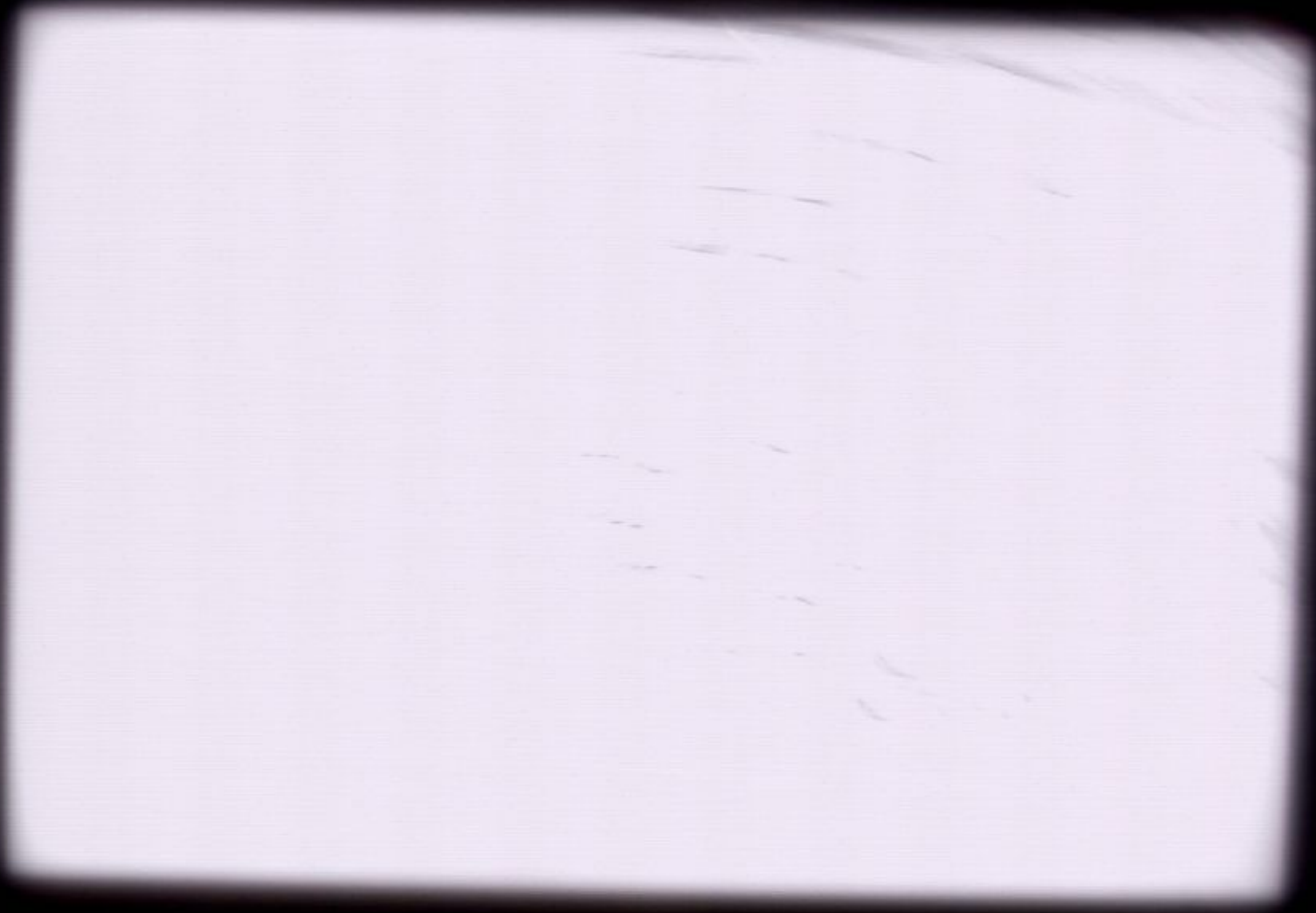
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⑨
 Let $\mathcal{C}$ denote the set of clusters containing o , considered as a set of graphs.
 The incipient infinite cluster = $(\mathcal{C}, \mu)$
 where

$$\mu = \lim_{N \rightarrow \infty} \mu_N$$

provided the limit exists.

$d=2$: Existence shown by H. Kesten '86.

d large: Existence shown by R. van der Hofstad & A.A. Járai 2004.

Alexander & Orbach conjecture: $d_h = \frac{4}{3}$
 for all $d \geq 2$.

Probably only valid for d large.

Proof for $d \geq 19$ by G. Kozma & A. Nachmias 2008.

Not much known about d_h .

Let $\mathcal{C}_l$ denote the set of clusters containing o , considered as a set of graphs. The incipient infinite cluster $= (\mathcal{C}_l, \mu)$ where

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Not much known about d_h .

Let $\mathcal{G}_n$ denote the set of clusters containing G_n , considered as a set of graphs.

The infinite cluster $= (\mathcal{G}_\infty, \mu)$ where

$$\mu = \lim_{n \rightarrow \infty} \mu_n$$

provided the limit exists

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showing ≥ 1 , existence of a non-trivial cluster.

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Proof for $d \geq 19$ by G. Kozma & A. Nachtergaele 2008.

Not much known about d_h .

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Not much known about d_A .

References:

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Alexander & Orbach conjecture: $d_1 = \frac{4}{3}$
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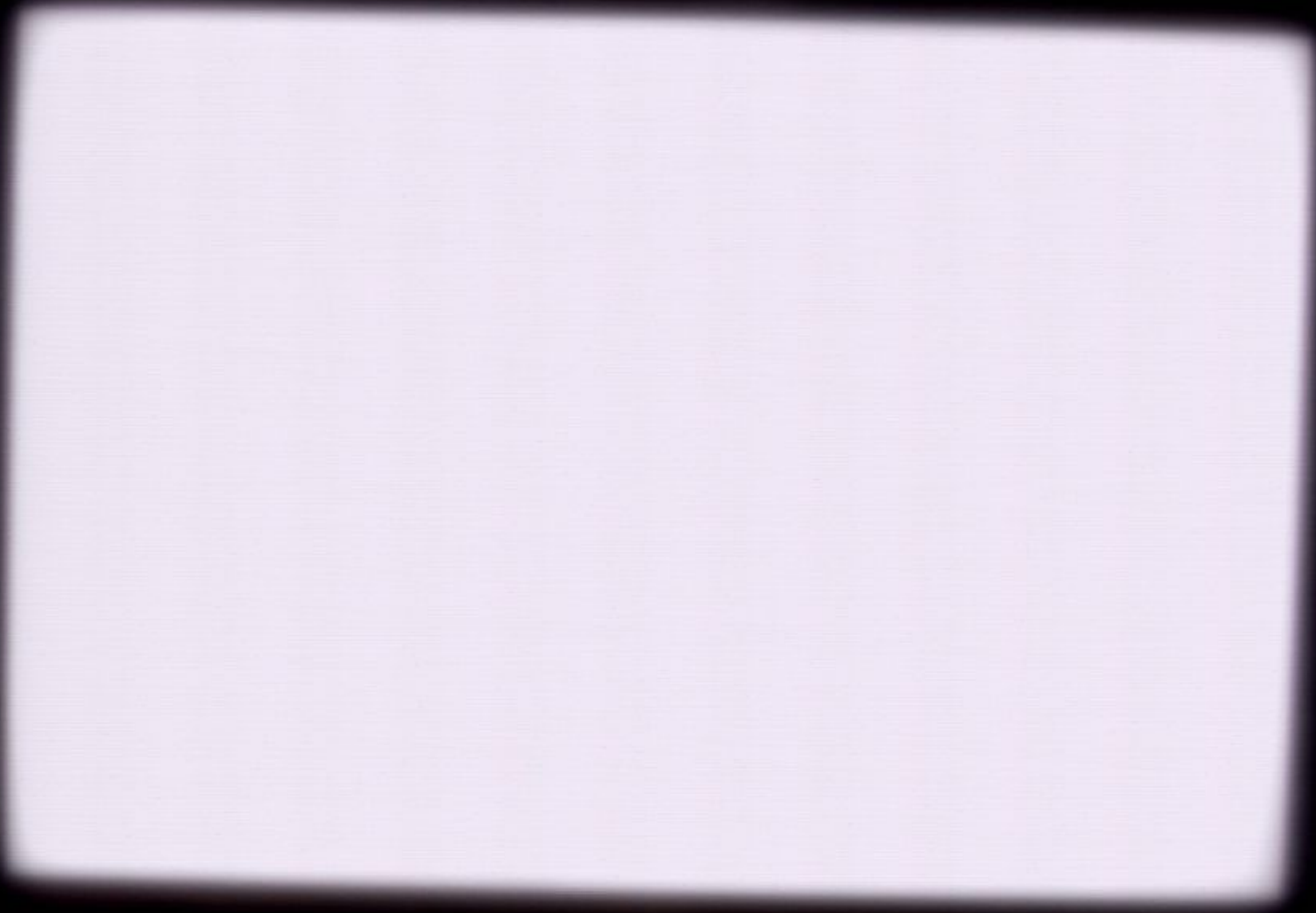
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(10)

4. The case $G = \text{Cayley graph } \Gamma^r$ ($d = \infty$)

Assume vertices of Γ^r have degree n except root of degree 1.



Cayley tree
with $n=3$

For a (non-trivial) rooted subtree T of Γ^r with $1 \leq |T| < \infty$:

$$p_p(C_r = T) = p \prod_{v \in T \setminus r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

$$p(T) = \prod_{v \in T \setminus r} p_{0v-1},$$

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For a (non-trivial) rooted subtree T of Γ with $1 \leq |T| < \infty$:

$$p_p(c_r = T) = p_{\text{root}} \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

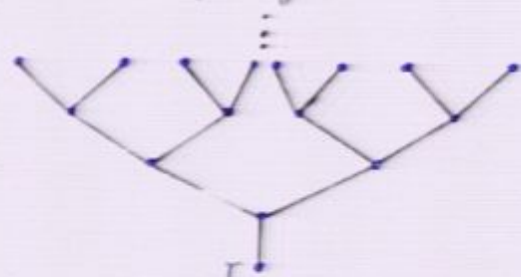
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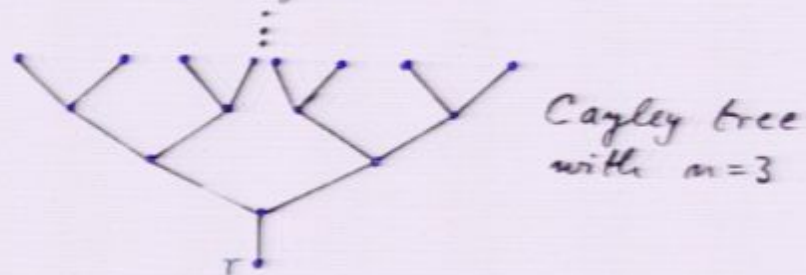
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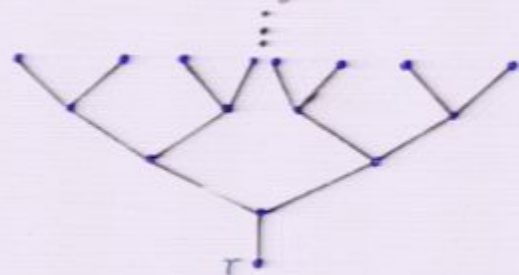
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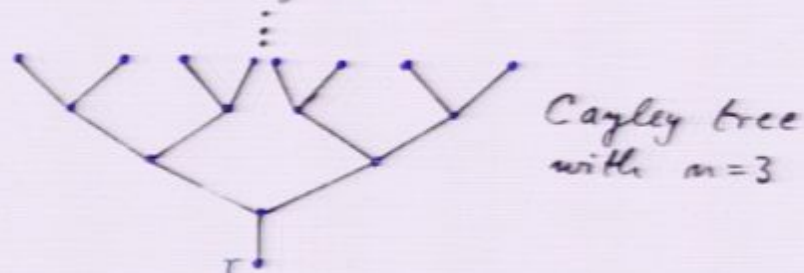
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For a (non-trivial) rooted subtree T of Γ with $1 \leq |T| < \infty$:

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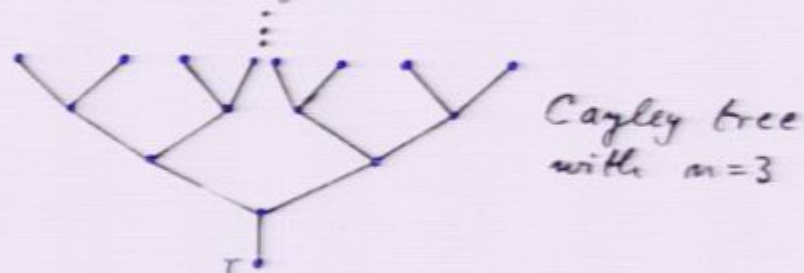
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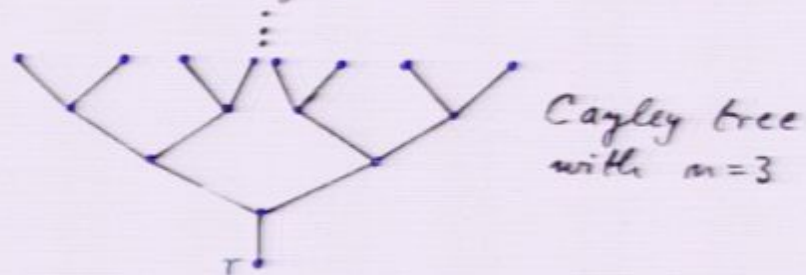
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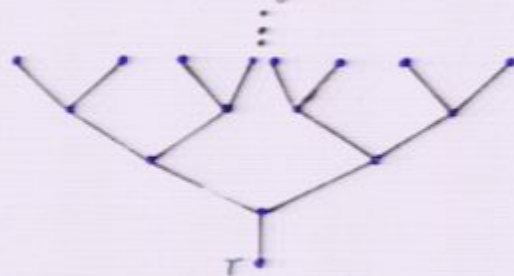
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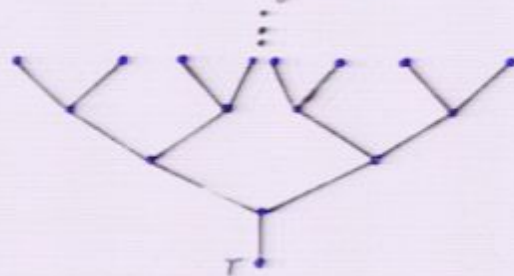
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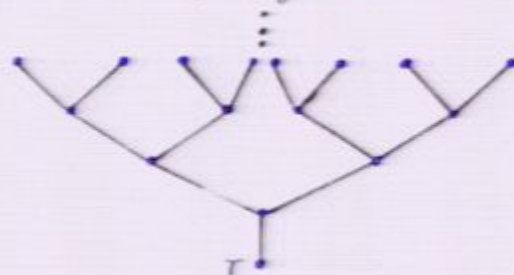
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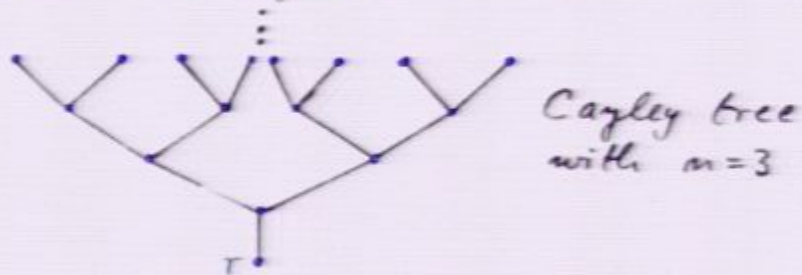
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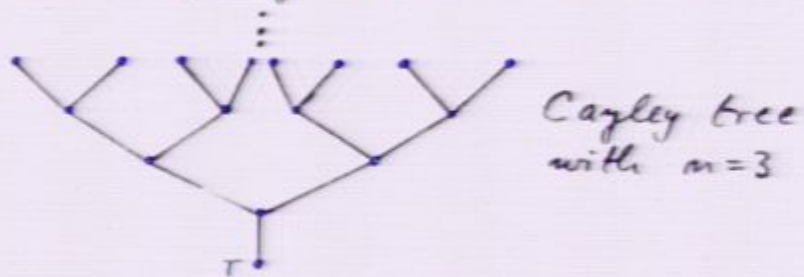
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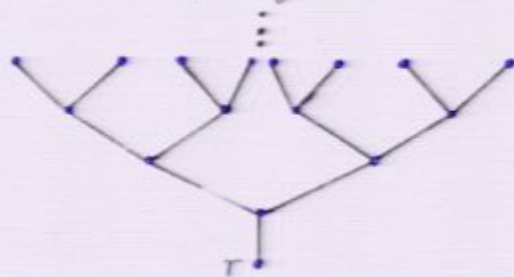
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Assume vertices of Γ have degree n except root of degree 1.



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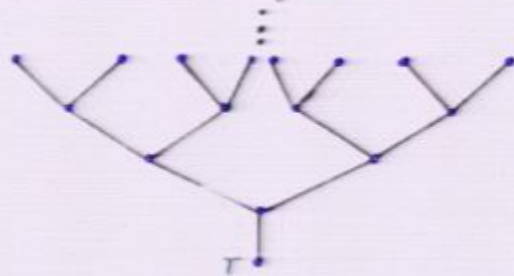
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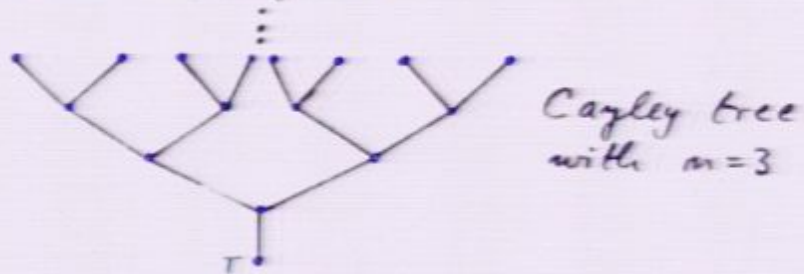
Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

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$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \quad k=0,1,2,\dots,n-1.$$

Assume vertices of Γ have degree n except root of degree 1.



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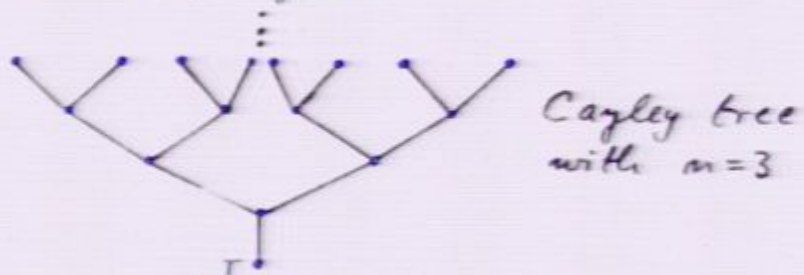
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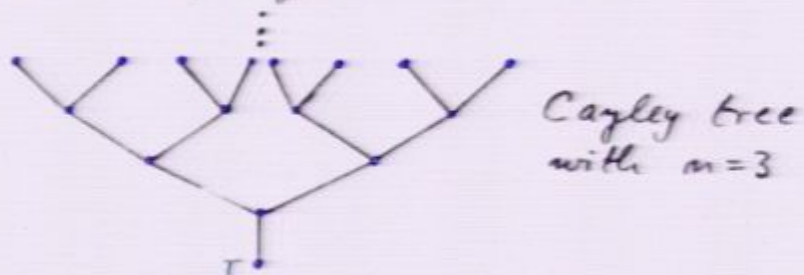
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For a (non-trivial) rooted subtree T of Γ with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p^{\prod_{v \in T, v \neq r} (n-1)} p^{d_r-1} (1-p)^{n-d_r}$$

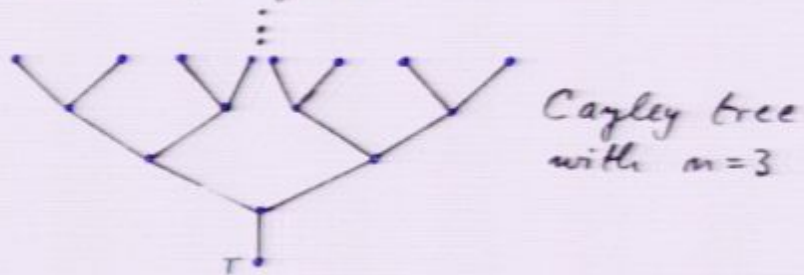
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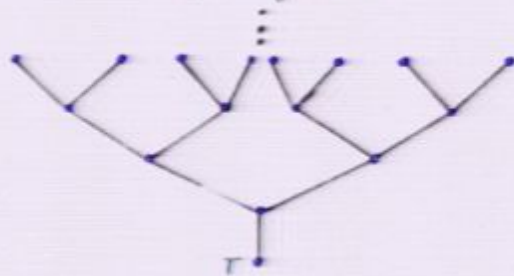
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Assume vertices of Γ have degree n except root of degree 1.



Cayley tree
with $n=3$

For a (non-trivial) rooted subtree T of Γ with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p^{|T|} \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

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where

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Cayley tree

For a (non-trivial) rooted subgraph T with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p \prod_{v \in T, v \neq r} (d_v - 1) p^{d_v - 1}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

$$P(T) = \prod_{v \in T, v \neq r} p_{d_v - 1},$$

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Note:

$$\sum_{k=0}^{\infty} p_k = 1$$

(where $p_k = 0$ for $k \geq n$).



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with $n \geq 3$

For a (non-trivial) rooted subtree T of P with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

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Note:

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(where $p_k \geq 0$ for $k \geq 0$)



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For a (non-trivial) rooted subtree T of P with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p \prod_{v \in T \setminus r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

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$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \quad k=0, 1, 2, \dots, n-1.$$

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with $n=5$

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10/10/1953

Let π be a non-degenerate subalgebra of π^0 with $\pi \neq \pi^0$.

$$E(\pi) = \pi^0 - \pi = \sum_{k=1}^n \binom{n}{k} p_k^{n-k} p^{n-k} \text{ with } p = 1 - \pi$$

Having determined a configuration with each edge of cube π we give the distribution of vertices of type

$$E(\pi) = \sum_{k=1}^n p_k^{n-k} p^{n-k}$$

where

$$p_k = \binom{n}{k} p_k^{n-k} p^{n-k} \text{ for } k=1, 2, \dots, n$$

where

$$\sum_{k=1}^n p_k = 1$$

$$p_1 = p, p_2 = 1 - p$$

For a (non-trivial) rooted subtree T of Π with $1 \leq |T| < \infty$:

$$\mathbb{P}_p(C_r = T) = p^{|T|} \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

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Note:

$$\sum_{k=0}^{\infty} p_k = 1$$

(where $p_k = 0$ for $k \geq n$).

For a (non-trivial) rooted subtree T of $\mathcal{T}$ with $1 \leq |T| < \infty$:

$$\mathbb{P}_p(C_r = T) = p \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

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T

For a (non-trivial) rooted subtree T of Γ with $1 \leq |T| < \infty$:

$$p_p(c_r = T) = p \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

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$$\mathbb{P}_p(C_r = T) = p^{|T|} \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

$$\mathbb{Q}(T) = \prod_{v \in T, v \neq r} p_{d_v-1},$$

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For a (non-trivial) rooted subtree T of P with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p^{|T|} \prod_{v \in T, v \neq r} \binom{n-1}{d_v-1} p^{d_v-1} (1-p)^{n-d_v}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

$$Q(T) = \prod_{v \in T, v \neq r} p_{d_v-1},$$

where

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \quad k=0, 1, 2, \dots, n-1.$$

Note:

$$\sum_{k=0}^{\infty} p_k = 1$$

(where $p_k = 0$ for $k \geq n$).



For a (non-trivial) rooted subtree T of P with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p^{\prod_{v \in T, v \neq r} (d_v - 1)} \binom{n-1}{d_r - 1} p^{d_r - 1} (1-p)^{n-d_r}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

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For a (non-trivial) rooted subtree T of P with $1 \leq |T| < \infty$:

$$P_p(C_r = T) = p^{\prod_{v \in T, v \neq r} (d_v - 1)} p^{d_r - 1} (1-p)^{n - d_r}$$

Hence, conditioning on configurations with root edge of value 1 we get the distribution of Galton-Watson type

$$P(T) = \prod_{v \in T, v \neq r} p_{d_v - 1},$$

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$$\sum_{k=0}^{\infty} p_k = 1$$

(where $p_k = 0$ for $k \geq n$).



②

Determine for:

Set

$$Z_N = \sum_{T: |T|=N} p(T) = p(|T|=N).$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

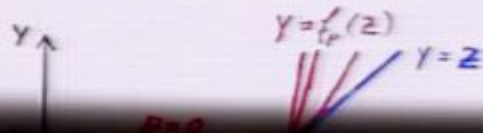
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



(11)

Determine for:

Set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

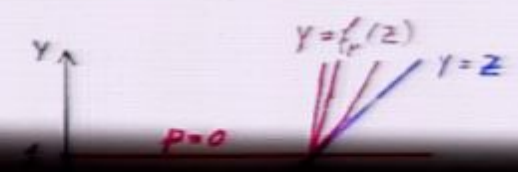
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine p_{cr} :

Set

$$Z_N = \sum_{T: |T| \geq N} \ell(T) = \ell(|T| \geq N) .$$

Generating functions

$$Z(t) = \sum_{n=1}^{\infty} Z_n t^n$$

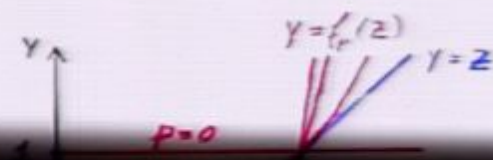
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for:

Set

$$Z_N = \sum_{T: |T| \geq N} \ell(T) = \ell(|T| \geq N).$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

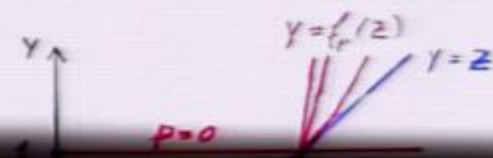
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine p_{cr} :

Set

$$Z_N = \sum_{T: |T| \geq N} \rho(T) = \rho(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

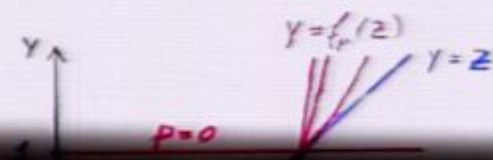
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for :

Set

$$Z_N = \sum_{T: |T| \geq N} \ell(T) = \ell(|T| \geq N) .$$

Generating functions

$$Z(t) = \sum_{n=1}^{\infty} Z_n t^n$$

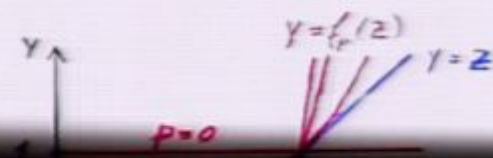
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} \mathcal{P}(T) = \mathcal{P}(|T| \geq N)$$

generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

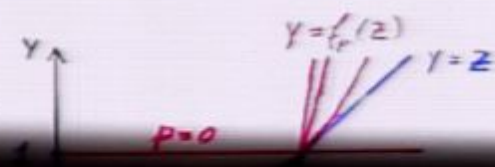
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For r : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for:

set

$$Z_N = \sum_{T: |T|=N} \ell(T) = \ell(|T|=N).$$

Generating functions

$$Z(t) = \sum_{n=0}^{\infty} Z_n t^n$$

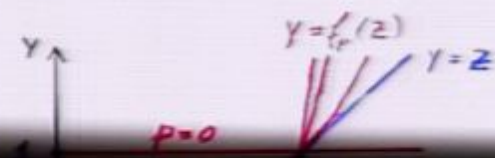
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} \ell(T) = \ell(|T| \geq N).$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

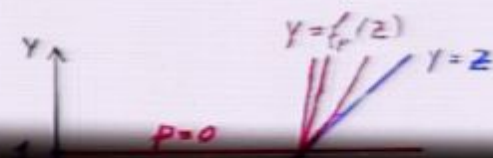
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



②

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} \ell(T) = \ell(|T| \geq N)$$

generating functions

$$Z(t) = \sum_{n=1}^{\infty} Z_n t^n$$

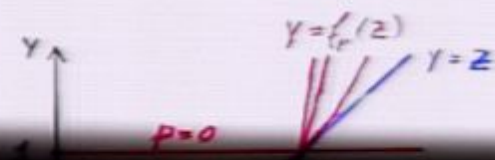
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\boxed{Z(t) = t f(Z(t))}$$

For r : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



(11)

Determine per:

Set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N).$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

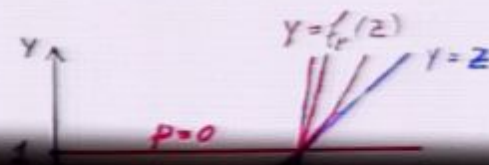
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



②

Determine for:

Set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{N=0}^{\infty} Z_N t^N$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P :

$$f(t) = (1-p+tp)^{n-1} = f_p(t)$$

$y \uparrow$

$$y = f_p(z)$$

$$y = z$$

②

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{N=0}^{\infty} Z_N t^N$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P :

$$f(t) = (1-p+tp)^{n-1} = f_p(t)$$

$y \uparrow$

$$y = f_p(z) \\ // \quad // \quad y = z$$

②

Determine for:

Set

$$Z_N = \sum_{T: |T| > N} p(T) = p(|T| > N)$$

Generating functions

$$Z(t) = \sum_{N=0}^{\infty} Z_N t^N$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



(11)

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{n=0}^{\infty} Z_n t^n$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$

 $\uparrow$
 Y

$$Y = f_p(Z)$$

$$Y = Z$$

II

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} P(T) = P(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{N=0}^{\infty} Z_N t^N$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$

$Y \uparrow$

$Y = f_p(Z)$
 $Y = Z$

(11)

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} P(T) = P(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{n=0}^{\infty} Z_n t^n$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$

 $\uparrow$
 Y

$$Y = f_p(Z)$$

$$Y = Z$$

(11)

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

Generating functions

$$Z(t) = \sum_{n=0}^{\infty} Z_n t^n$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$

 $\uparrow$
 Y

$$Y = f_p'(Z)$$

$$Y = Z$$

②

Determine for:

set

$$Z_N = \sum_{T: |T| \geq N} p(T) = p(|T| \geq N)$$

generating functions

$$Z(t) = \sum_{N=0}^{\infty} Z_N t^N$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$

$Y \uparrow$

$Y = f_p'(Z)$
 $Y = Z$



(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

Diagram 1: A circle with vertical lines inside, connected to a vertical line segment.

Diagram 2: A circle with vertical lines inside, connected to a vertical line segment, which then branches into multiple lines, each leading to a smaller circle with vertical lines inside.

(11')


$$= t \sum_{k=0}^{\infty} p_k$$


(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

Diagram 1: A circle with vertical lines inside, a horizontal line across the middle, and a vertical line extending downwards from the center.

Diagram 2: A tree structure with a root node at the bottom, branching upwards into several nodes, each represented by a circle with vertical lines. A horizontal line is drawn above the topmost node, with the letter 'k' written above it.

(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

The diagram on the left is a circle with vertical hatching and a vertical line extending downwards from its center. The diagram on the right is a tree structure with a root node at the bottom, indicated by a vertical line and a dot. From the root, several branches lead to circular nodes, each containing vertical hatching. One of these nodes is further expanded into a larger tree structure, with a label k and an arrow pointing to it.

(11')

$$\begin{array}{c} \text{Diagram 1: A circle with vertical lines inside, connected by a vertical line to a point below it.} \end{array} = t \sum_{k=0}^{\infty} p_k \begin{array}{c} \text{Diagram 2: A tree structure with a root node at the bottom, branching into multiple nodes above it. The root node is connected to a point below it. The nodes are labeled with 't' and 'p_k'."} \end{array}$$

(11')

$$\begin{array}{c} \text{Diagram 1: A circle with a vertical line segment extending downwards from its center. Inside the circle, there are several small dots and some faint, illegible markings. } \end{array} = t \sum_{k=0}^{\infty} p_k \begin{array}{c} \text{Diagram 2: A tree-like structure. At the bottom is a single dot. From this dot, several lines branch out upwards. Each branch leads to a small circle. The topmost branch is labeled 'k'. The circles contain some faint, illegible markings. } \end{array}$$

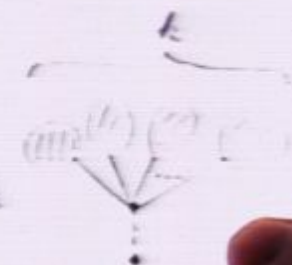
(11')


$$= t \sum_{k=0}^{\infty} p_k$$


(11')

$$\begin{array}{c} \text{Diagram 1: A circle with diagonal lines and a vertical line below it.} \end{array} = t \sum_{k=0}^{\infty} p_k \begin{array}{c} \text{Diagram 2: A tree structure with a root node and multiple branches, labeled with } p_k. \end{array}$$

(11)


$$= t \sum_{k=0}^{\infty} p_k$$


(11)

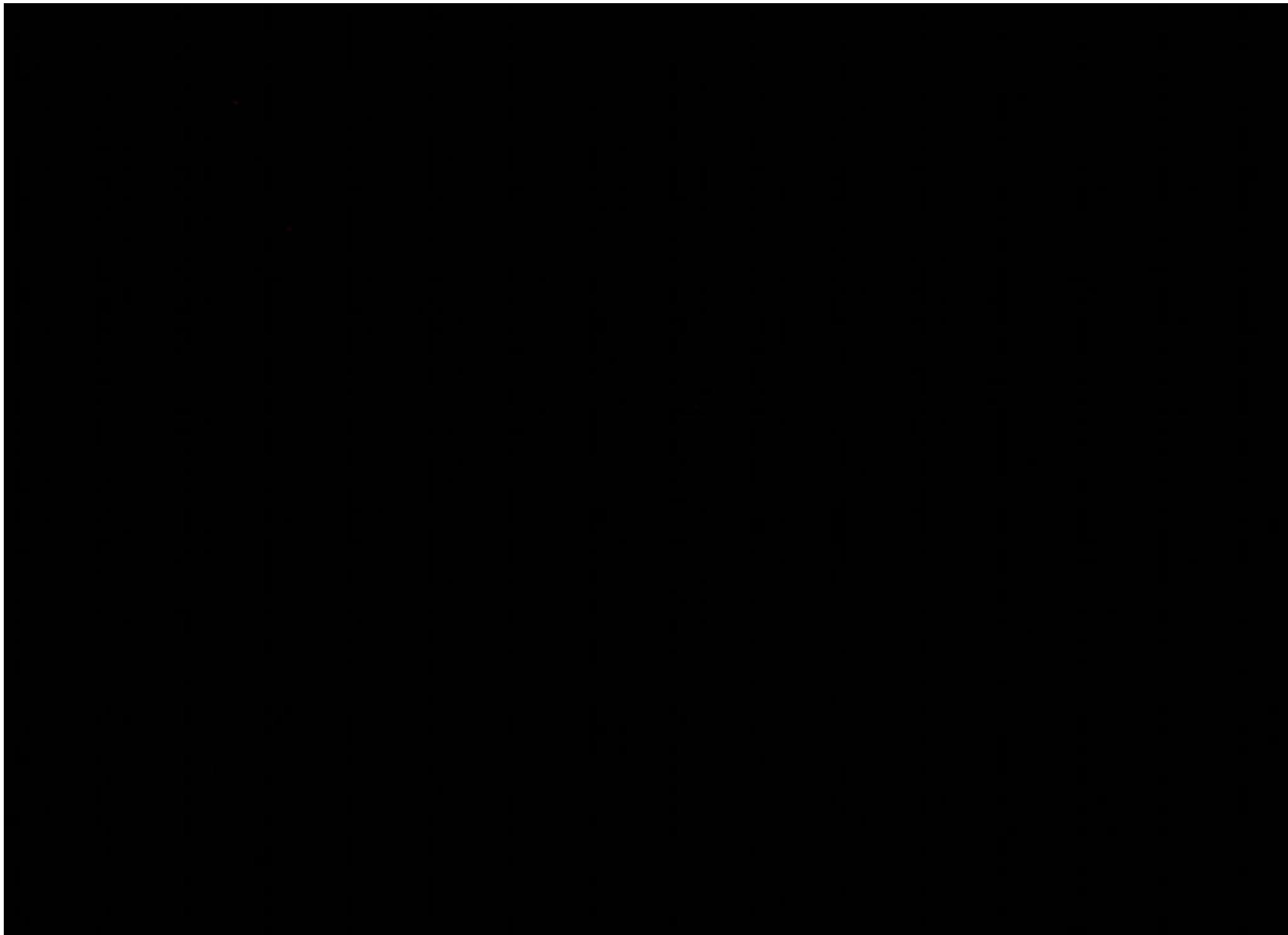
$$\begin{array}{c} \text{Diagram 1: A circle with a vertical line segment below it. Inside the circle, there are several small circles arranged in a horizontal row. The entire diagram is labeled with a '1' below it. \end{array} = t \sum_{k=0}^{\infty} p_k \begin{array}{c} \text{Diagram 2: A tree structure. The root node is labeled 'k'. It has four children, each labeled with a circled '1'. Each of these children has a vertical line segment below it. The entire diagram is labeled with a '2' below it. \end{array}$$

(11)

$$\begin{array}{c} \text{Diagram 1: A circle with a vertical line extending downwards from its center. Inside the circle, there are several small, faint, illegible markings. } \end{array} = t \sum_{k=0}^{\infty} p_k \begin{array}{c} \text{Diagram 2: A tree-like structure. At the bottom is a single point. From this point, several lines branch out upwards. Each branch terminates in a small circle. Above the circles, there are some faint, illegible markings. } \end{array}$$

(11')


$$= t \sum_{k=0}^{\infty} p_k$$

(10)

$$1 = t \sum_{k=0}^{\infty} p_k \underbrace{\quad}_{\vdots}$$

(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

Diagram 1: A single circle with diagonal hatching, connected to a point below it by a vertical line.

Diagram 2: A tree structure with a root node connected to k child nodes, each represented by a hatched circle. A bracket above the child nodes is labeled k . Ellipses indicate that there can be more than four child nodes.

(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

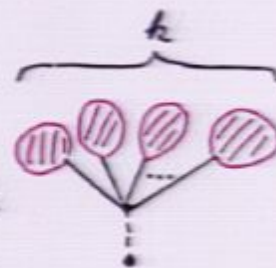
Diagram 1: A single circle with diagonal hatching, connected to a point below it by a vertical line.

Diagram 2: A tree structure with a root node connected to k child nodes, each represented by a hatched circle. A bracket above the child nodes is labeled k . Ellipses indicate that there can be more than four child nodes.

(11')



$$= t \sum_{k=0}^{\infty} p_k$$



(11')

$$\text{Diagram 1} = t \sum_{k=0}^{\infty} p_k \text{Diagram 2}$$

Diagram 1: A single circle with diagonal hatching, connected to a dot below it.

Diagram 2: A tree structure with a root dot at the bottom. It branches into k circles, each with diagonal hatching. A bracket above the circles is labeled k . Ellipses indicate the continuation of the branches.

(11')

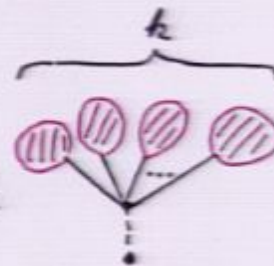
The diagram shows a single circle with diagonal hatching and a short vertical line (tail) extending downwards from its center. This is followed by an equals sign, then a summation symbol with a superscript ∞ and a subscript $k=0$. To the right of the summation is the symbol p_k . This is followed by a diagram of k smaller hatched circles, each with a tail, all connected by lines to a single point. A bracket above this group of circles is labeled k . A vertical ellipsis is shown below the connection point.

$$\text{shaded circle with tail} = t \sum_{k=0}^{\infty} p_k \text{ (group of } k \text{ shaded circles with tails)}$$

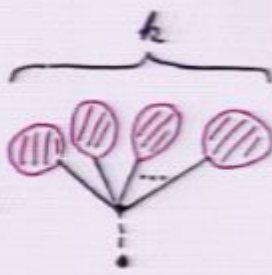
(11')



$$= t \sum_{k=0}^{\infty} p_k$$



(11')


$$= t \sum_{k=0}^{\infty} p_k$$


(11')



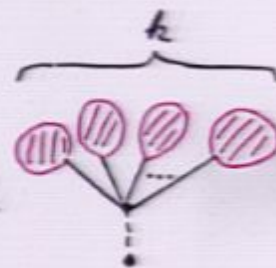
$$= t \sum_{k=0}^{\infty}$$



(11')



$$= t \sum_{k=0}^{\infty} p_k$$



(11')

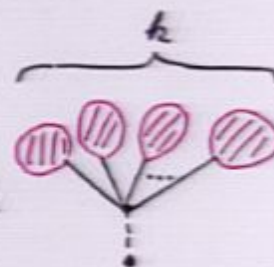
The diagram shows a large circle with diagonal hatching, representing a propagator, with a single line extending downwards from its center. This is equated to a sum over k from 0 to infinity of $t p_k$ multiplied by a diagram of k smaller hatched circles connected to a common vertex by lines. A bracket above the smaller circles is labeled k . The vertex has a vertical dashed line extending downwards.

$$\text{Diagram} = t \sum_{k=0}^{\infty} p_k \text{Diagram}_k$$

(11')



$$= t \sum_{k=0}^{\infty} p_k$$



Generating Functions

$$Z(t) = \sum_{n=0}^{\infty} z_n t^n$$

and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

$$p_0 = 1 \quad f(t) = (1 - p + tp)^{n+1} = f_p(t)$$

Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

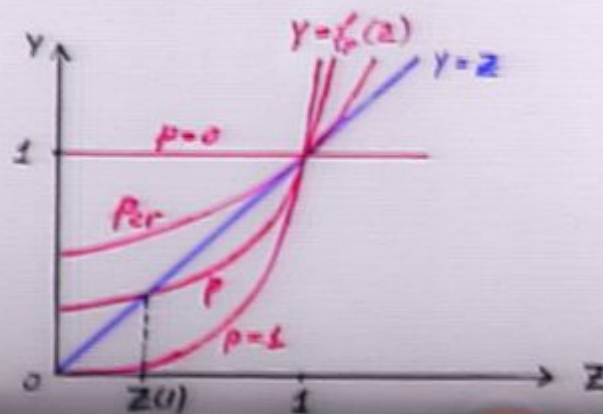
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For r : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



Generating functions

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

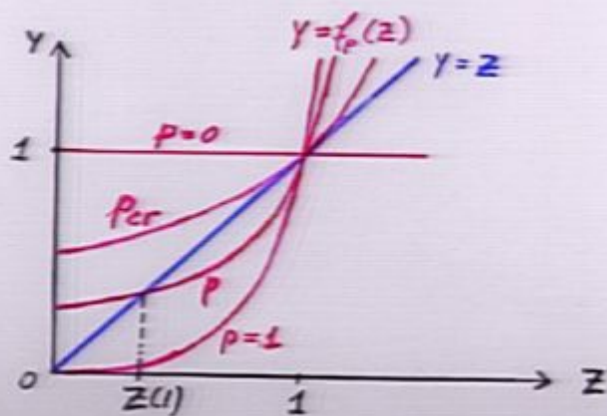
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For P : $f(t) = (1-p+tp)^{n-1} = f_p(t)$



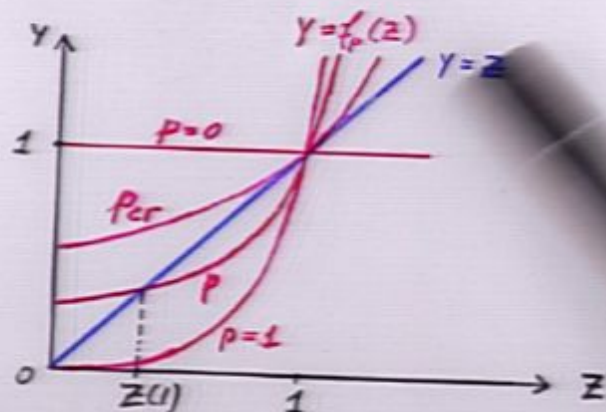
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



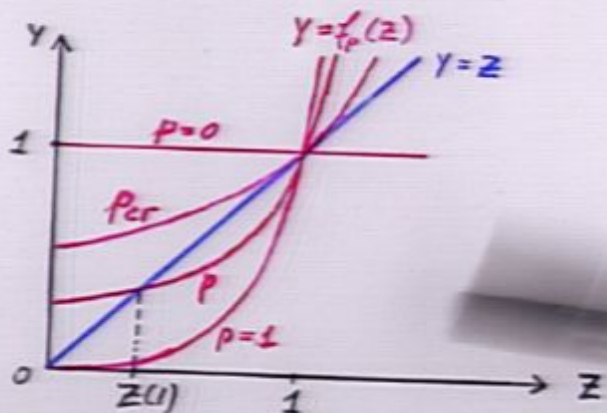
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



$$\bar{z}(t) = \sum_{N=1}^{\infty} \bar{z}_N t^N$$

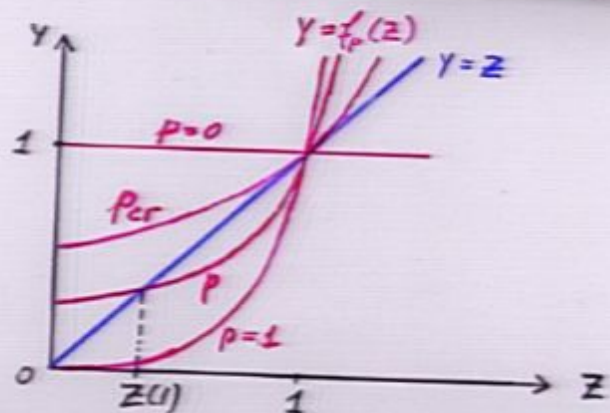
and

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$\bar{z}(t) = t f(\bar{z}(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



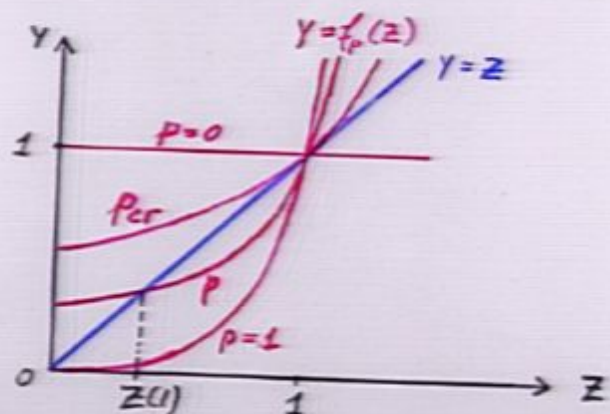
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



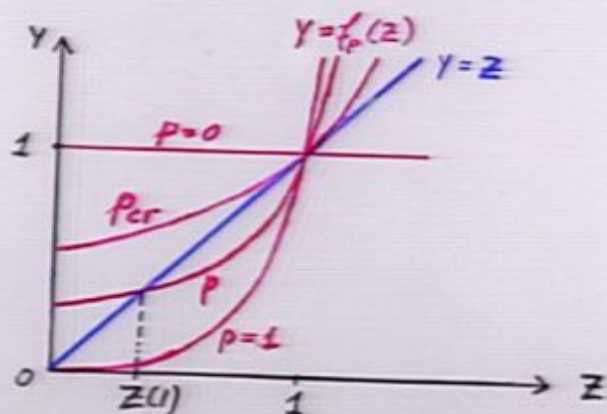
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



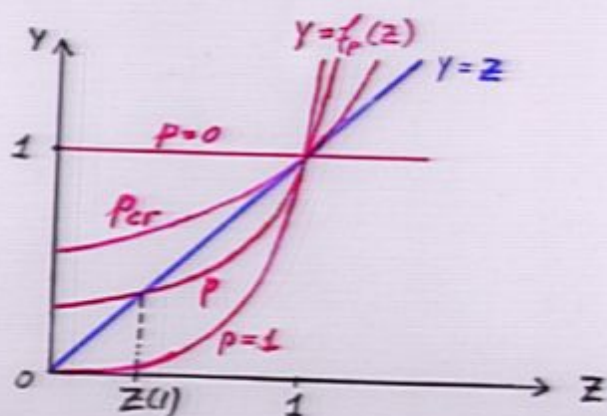
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



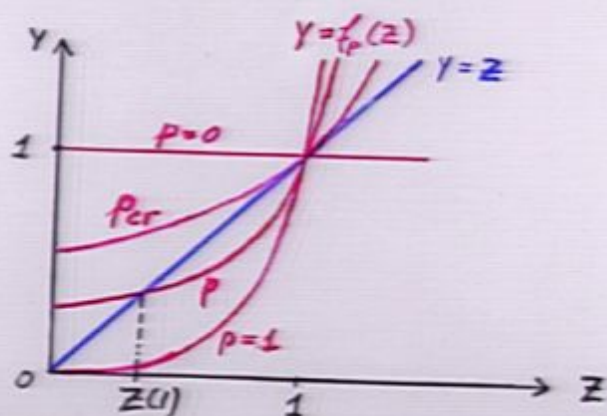
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

are related by

$$Z(t) = t f(Z(t))$$

For Γ : $f(t) = (1 - p + tp)^{n-1} = f_p(t)$



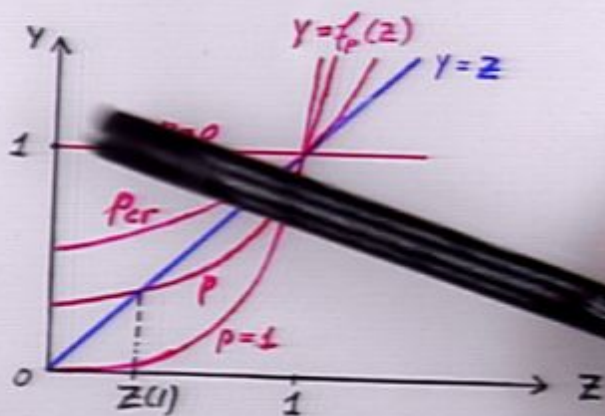
$$\text{and } Z(t) = \sum_{N=1}^{\infty} Z_N t^N$$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

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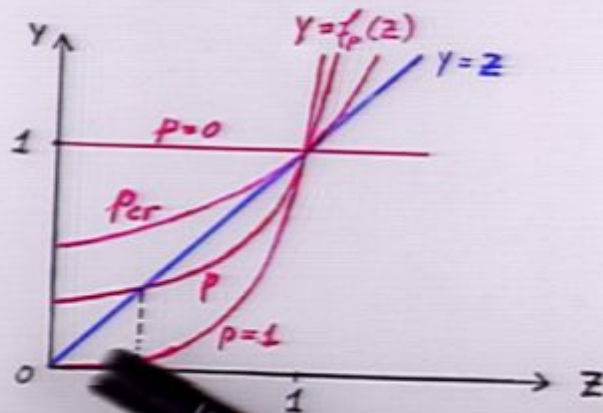
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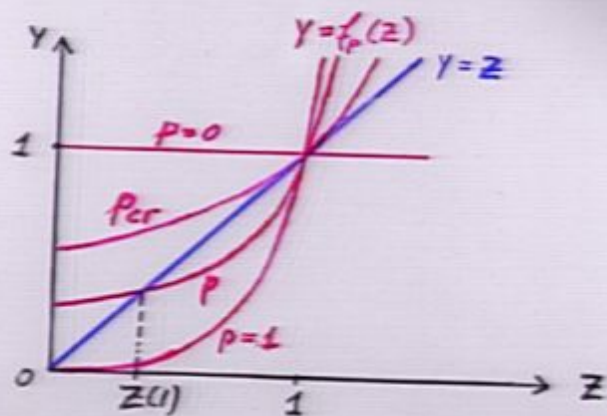
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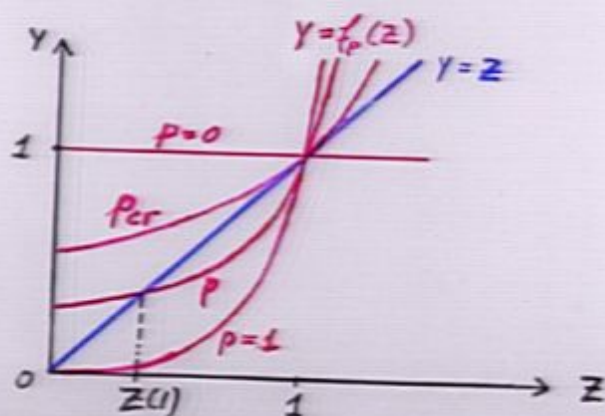
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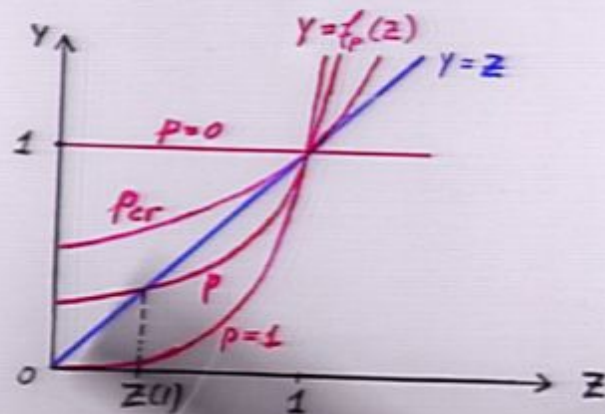
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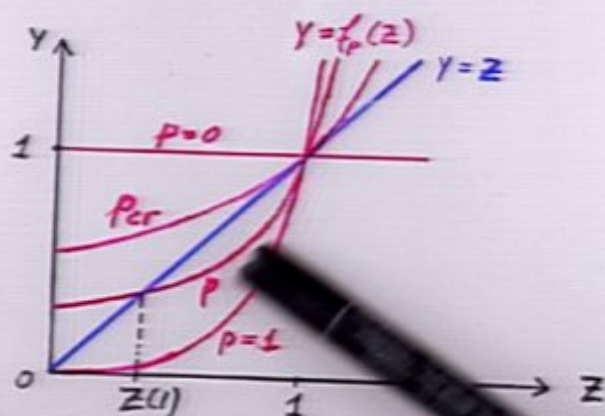
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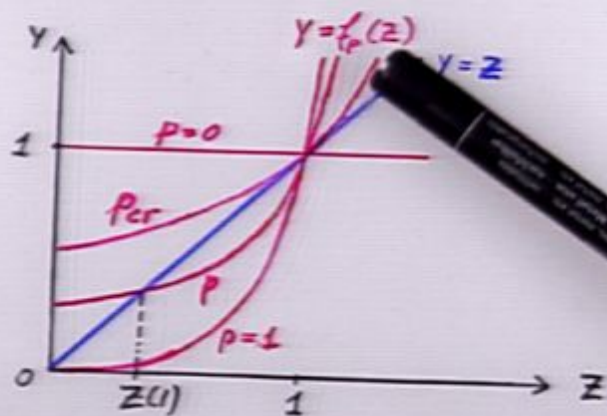
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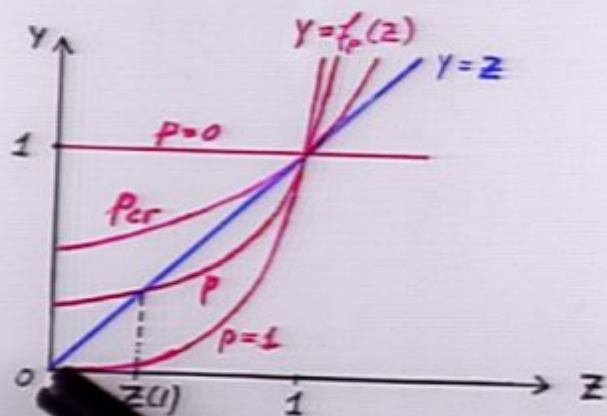
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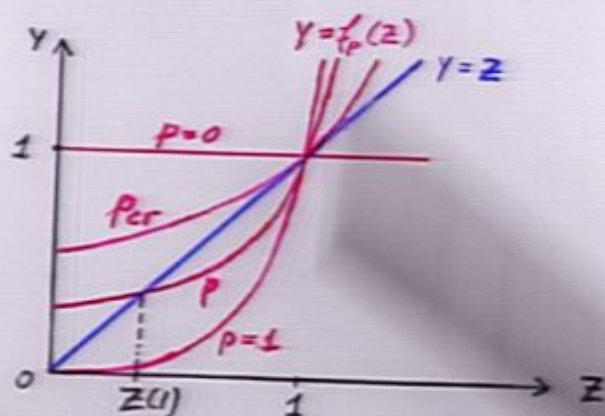
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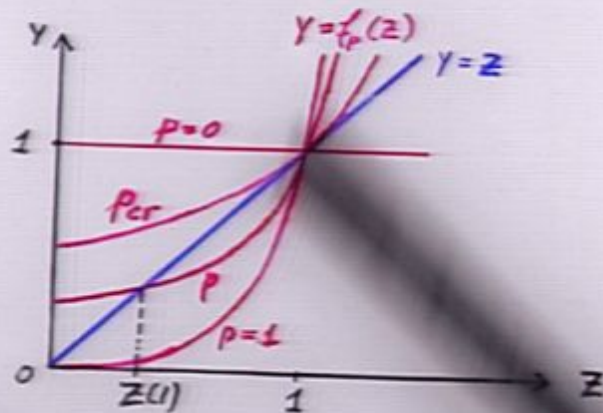
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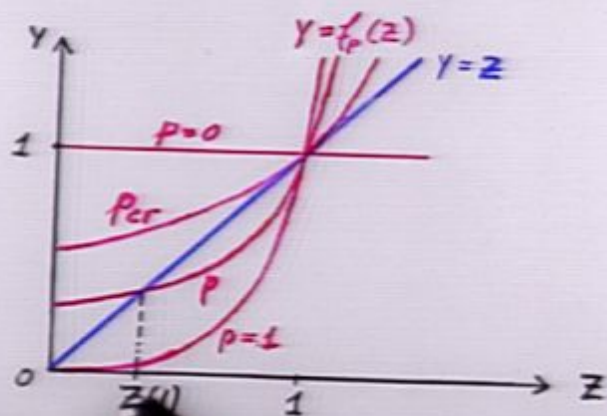
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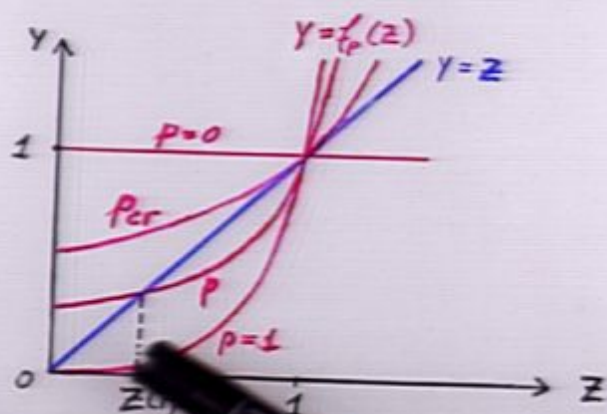
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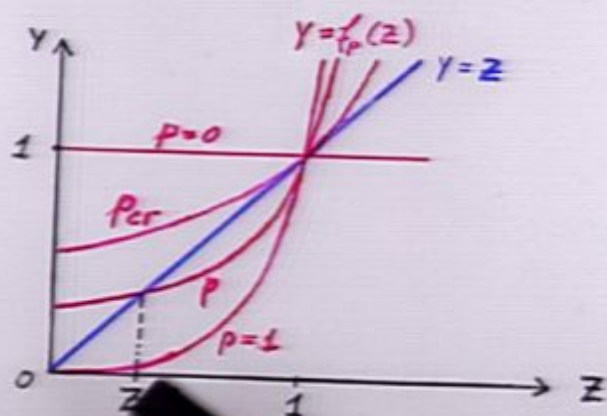
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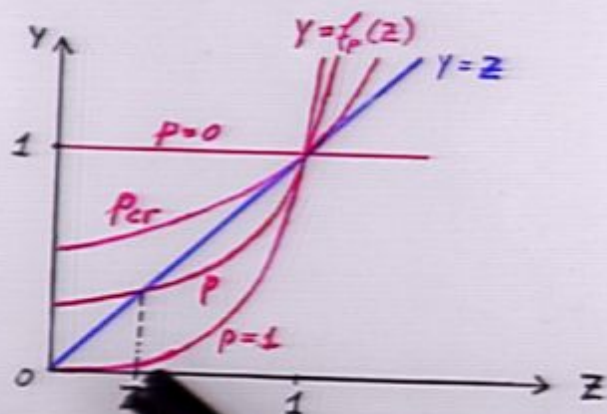
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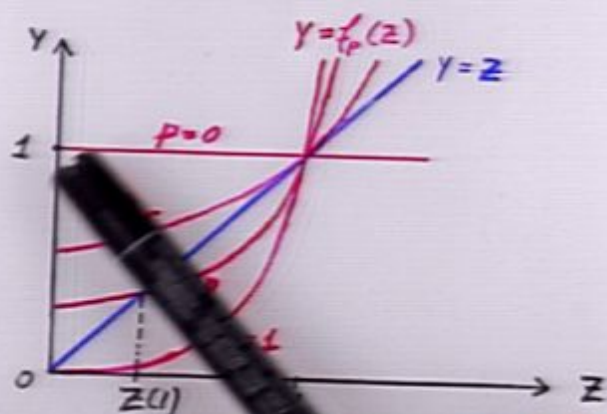
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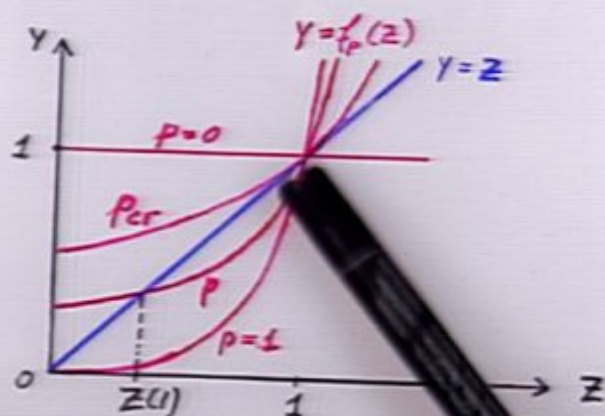
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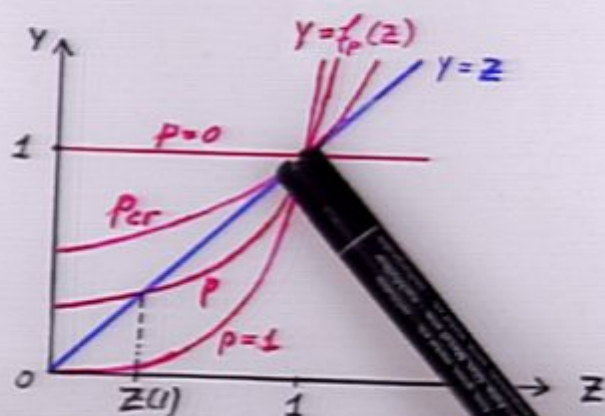
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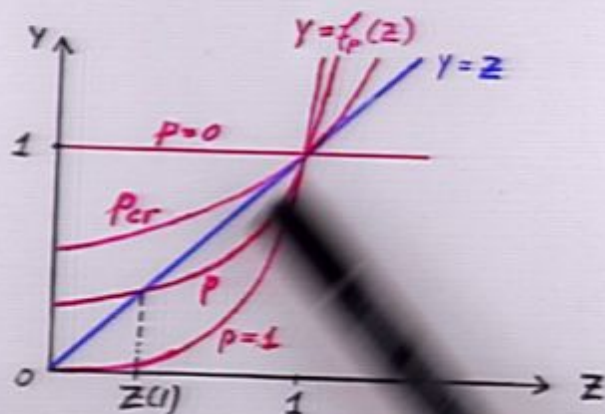
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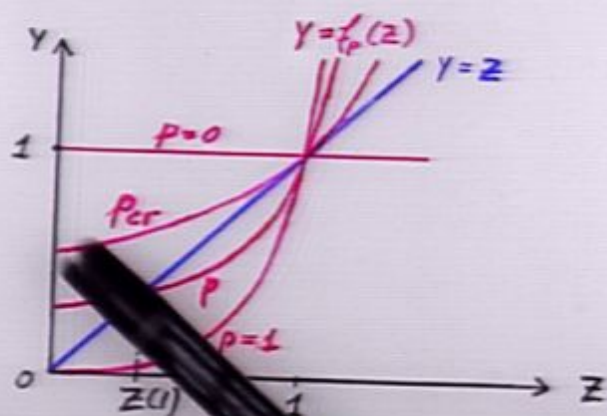
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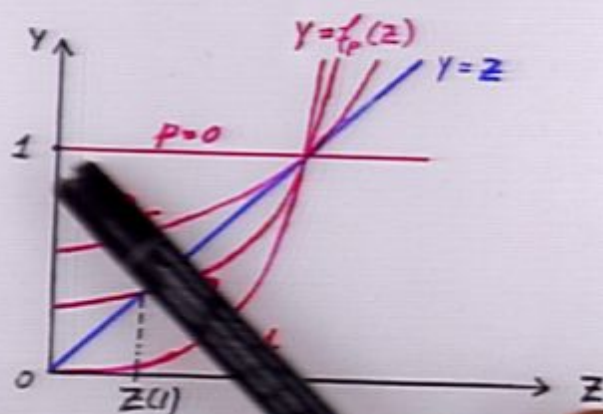
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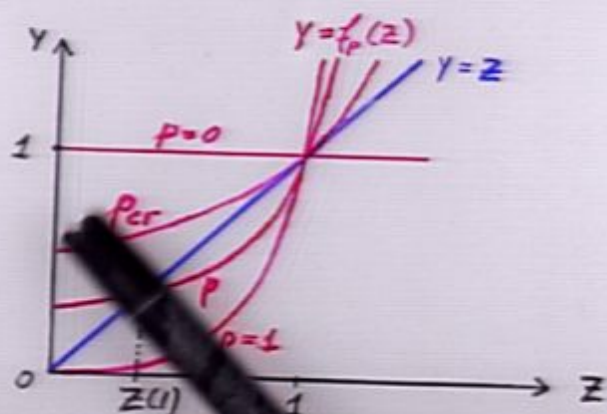
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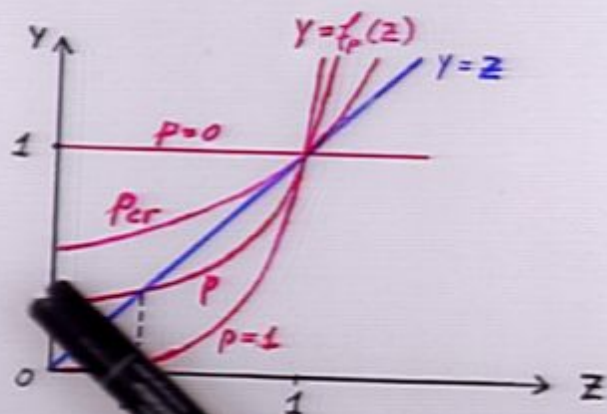
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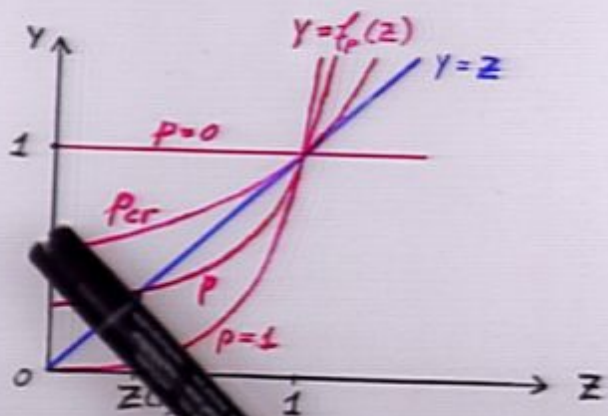
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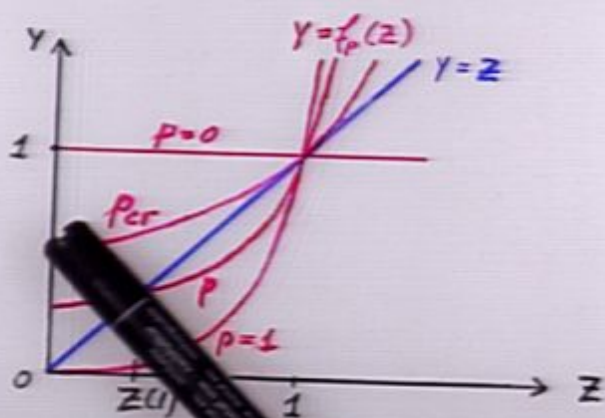
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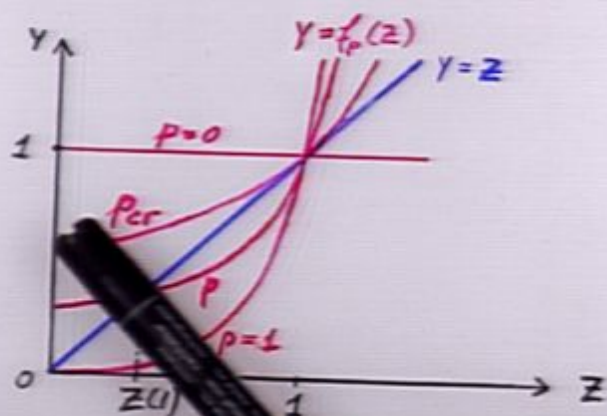
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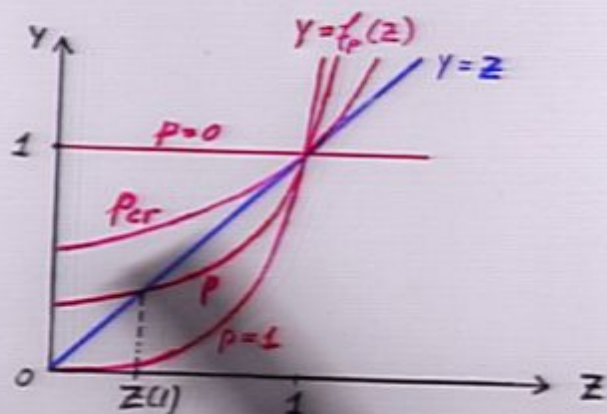
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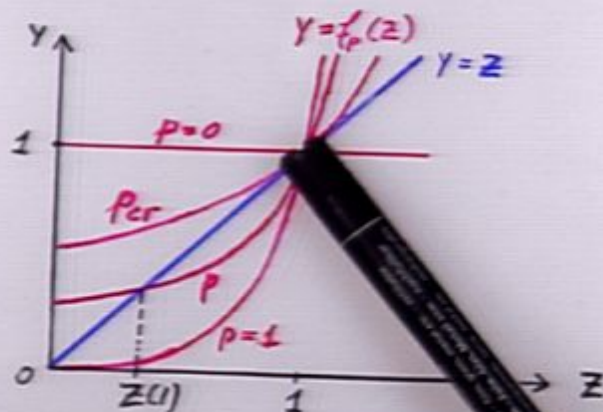
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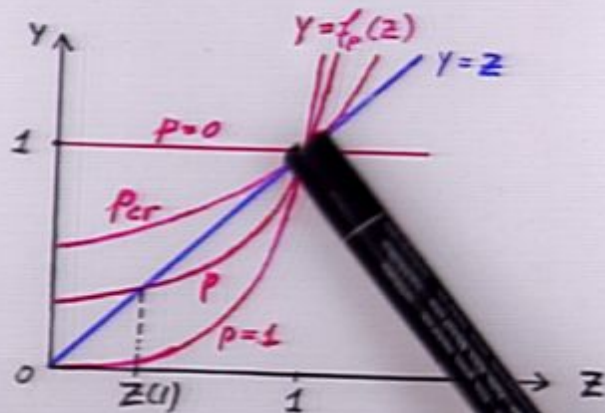
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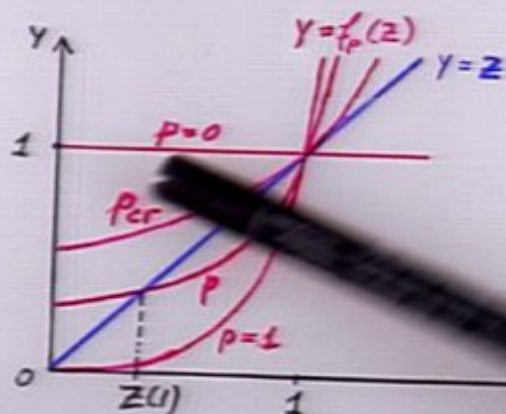
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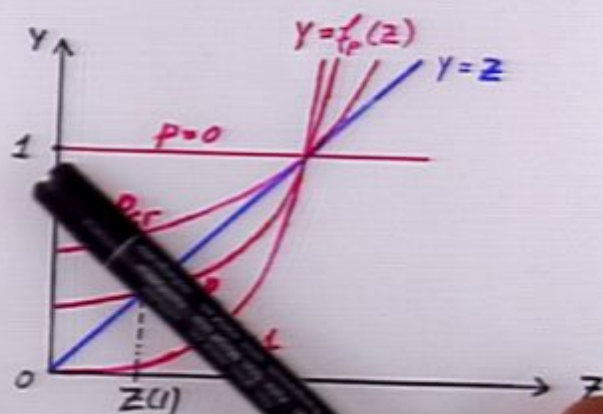
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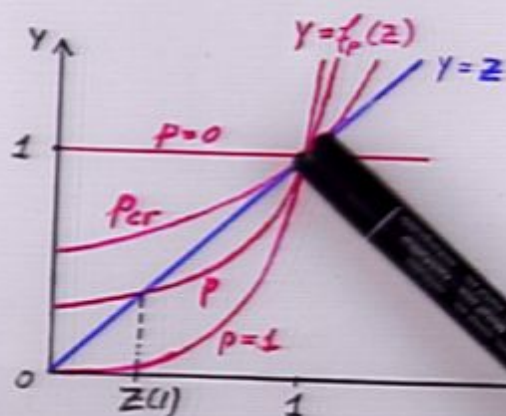
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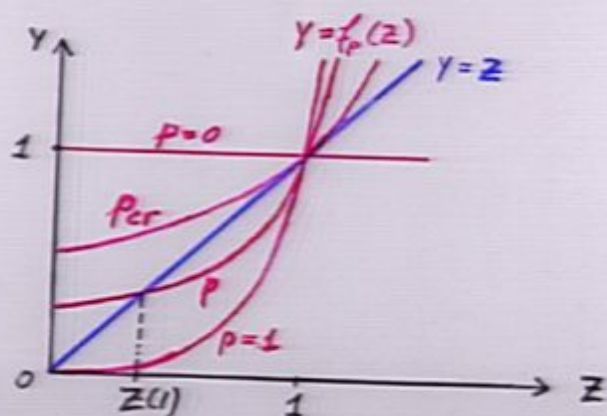
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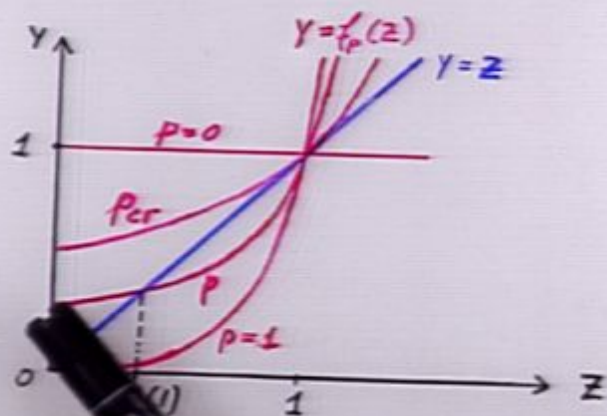
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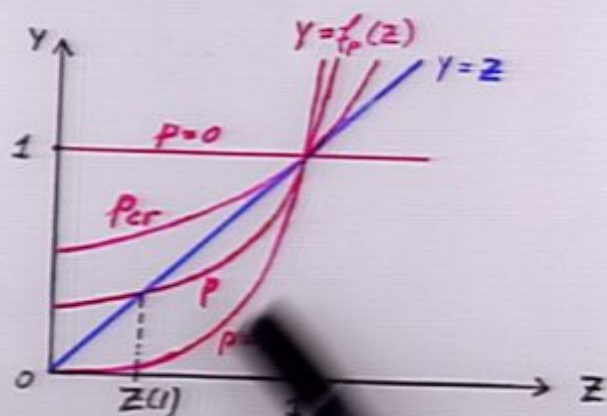
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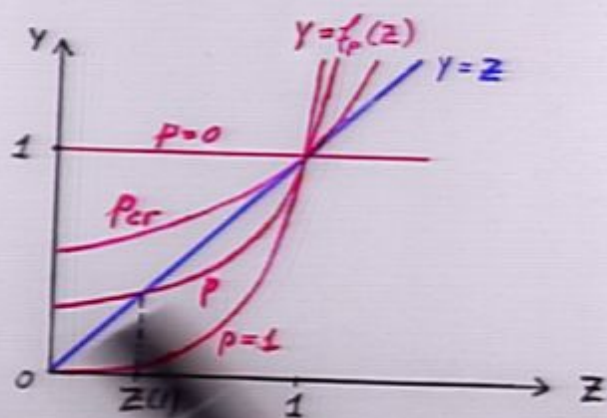
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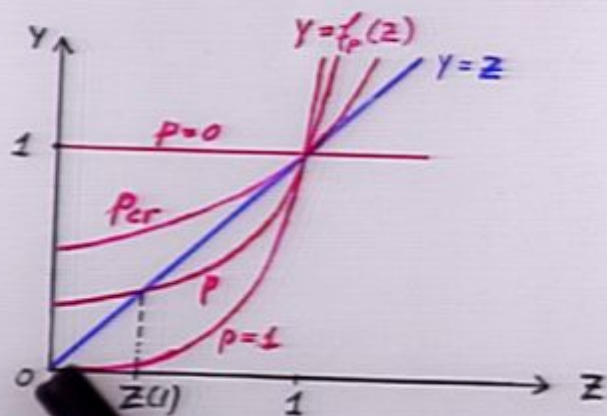
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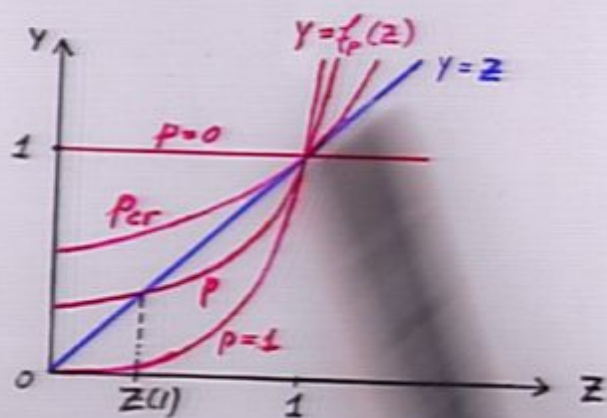
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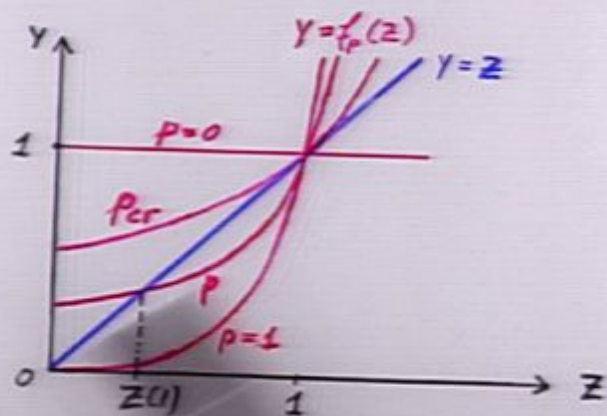
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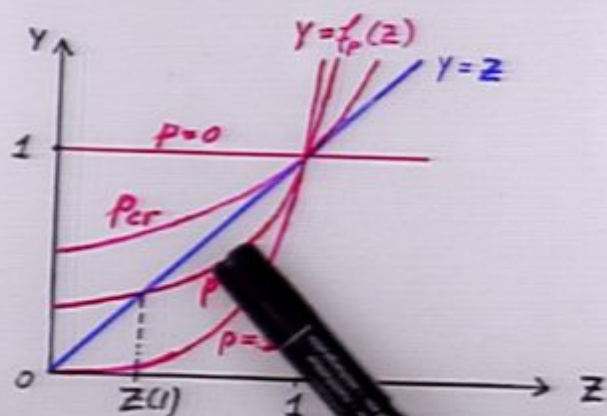
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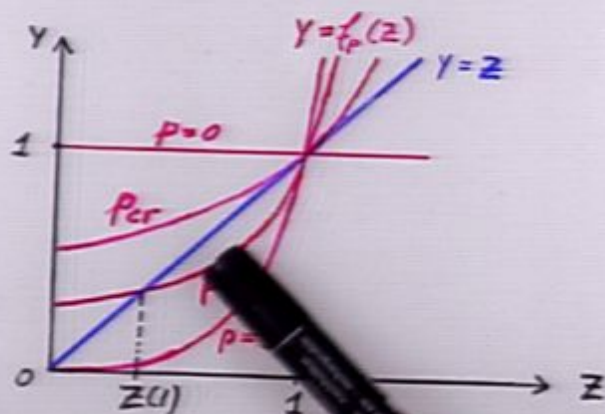
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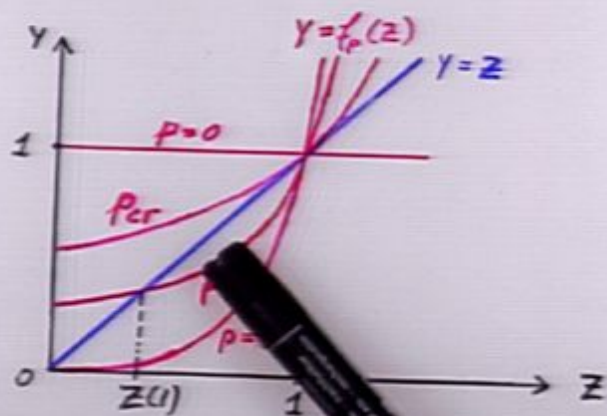
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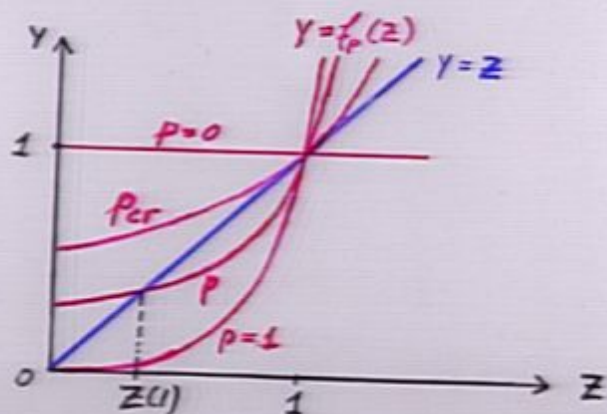
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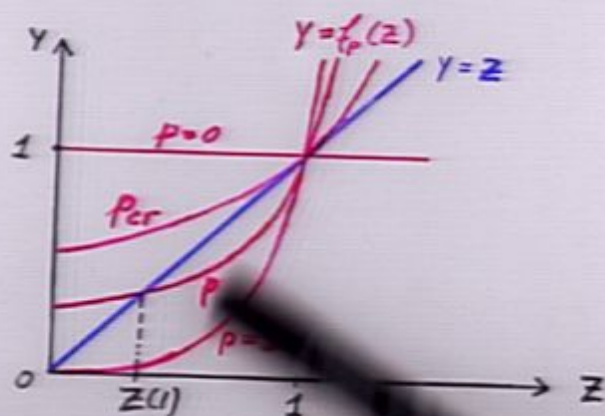
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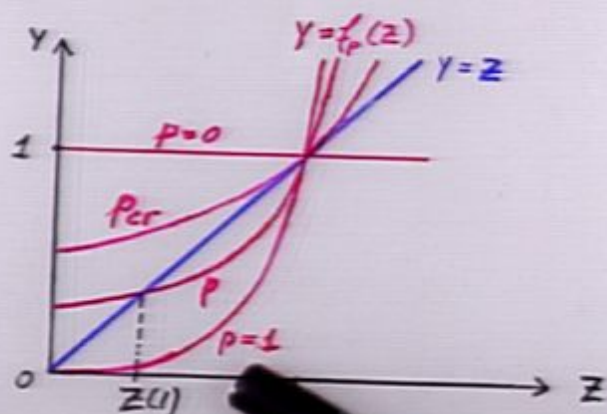
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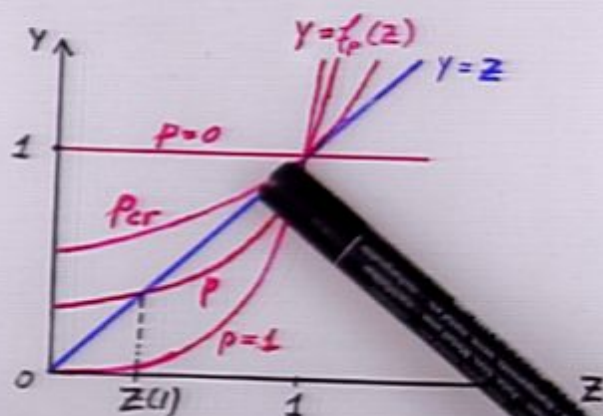
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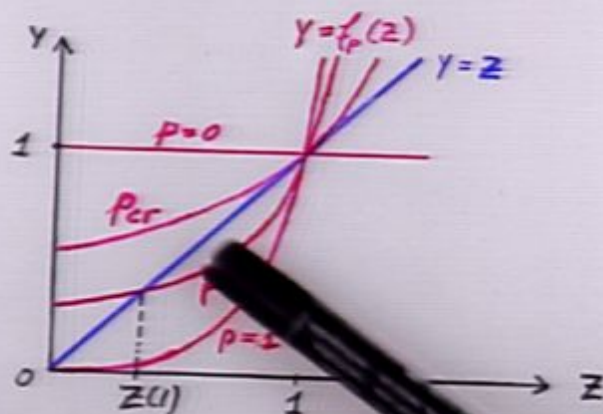
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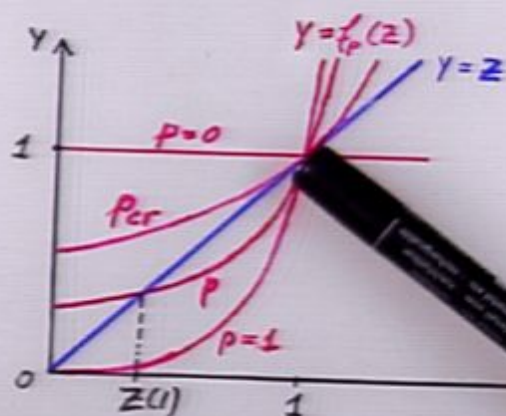
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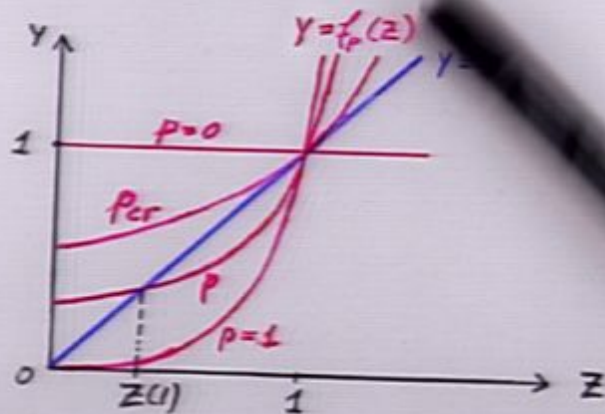
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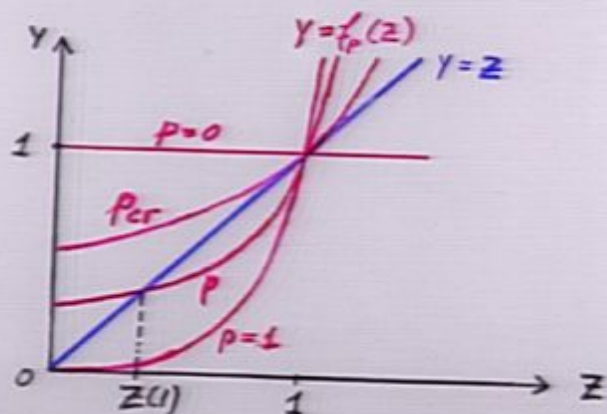
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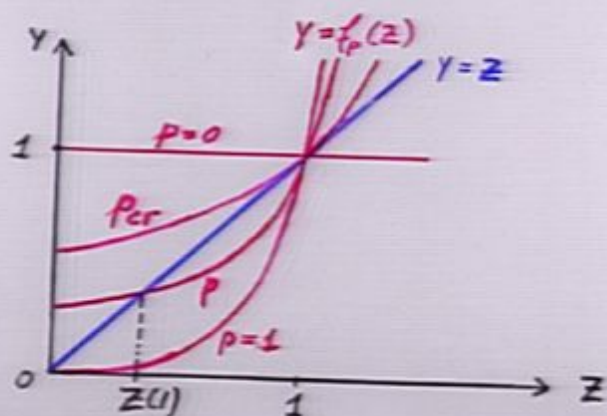
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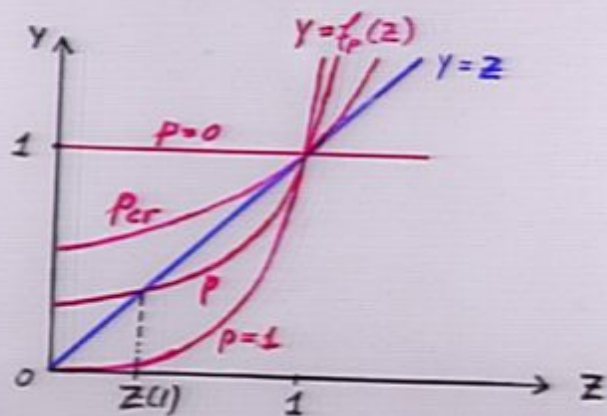
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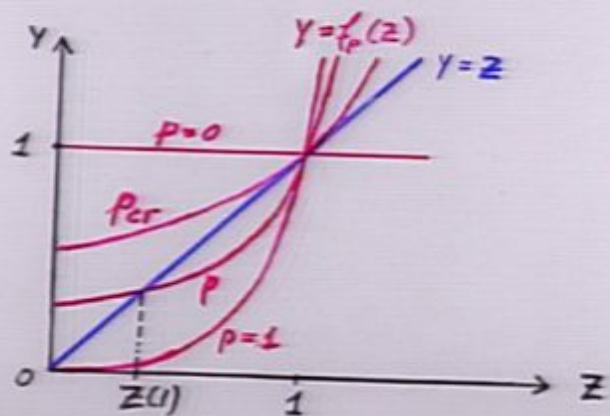
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Probability that T is finite:

$$P(T < \infty) = \sum_{n=1}^{\infty} P(T=n) = Z(1).$$

From figure: $Z(1) = 1$ if $p = p_{cr}$, where
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Conditioning on trees of fixed size N
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Assume off-spring probabilities $(p_k)_{k \geq 0}$ satisfy

$$\sum_{k=0}^{\infty} p_k = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k p_k = 1.$$

The corresponding critical Galton-Watson process is **generic** if generating function

$$f(t) = \sum_{k=0}^{\infty} p_k t^k$$

is analytic in a neighborhood of the unit circle, i.e. has radius of convergence > 1 .

Note: f is a polynomial for percolation on a Cayley tree, hence generic.

Define probability distribution ρ on finite rooted trees by

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on $\mathcal{T}_N$, where $Z_N = g(\mathcal{T}_N)$.

(14)

From functional relation

$$Z(t) = t f(Z(t))$$

follows

$$Z(t) \sim 1 - \text{cst} \cdot \sqrt{1-t}$$

for $t \rightarrow 1$. Hence by genericity

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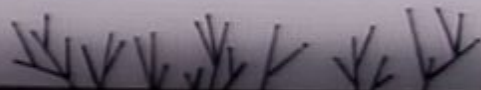
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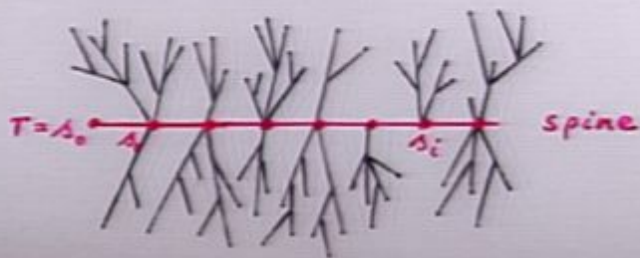


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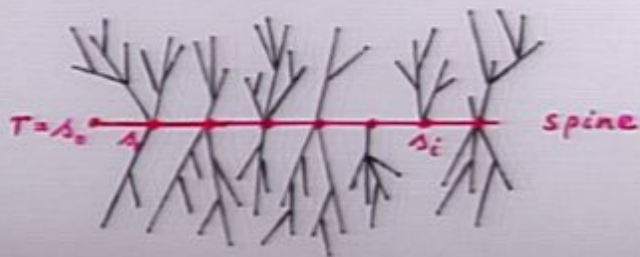


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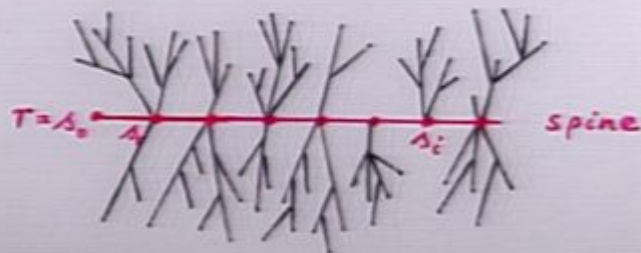


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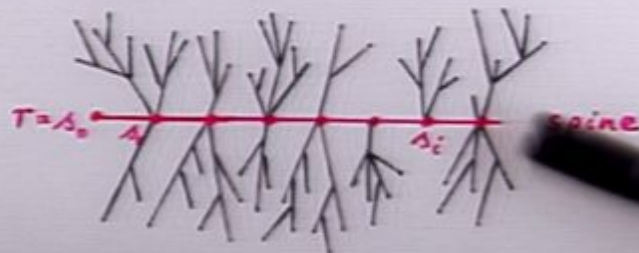


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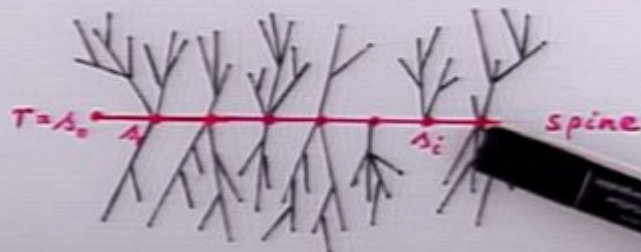


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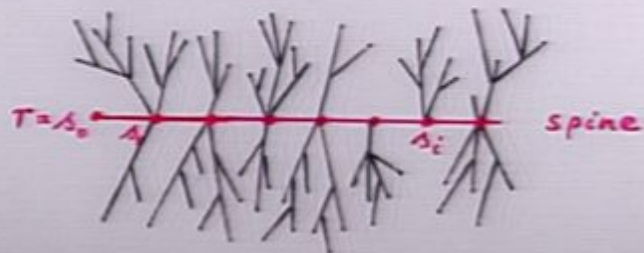


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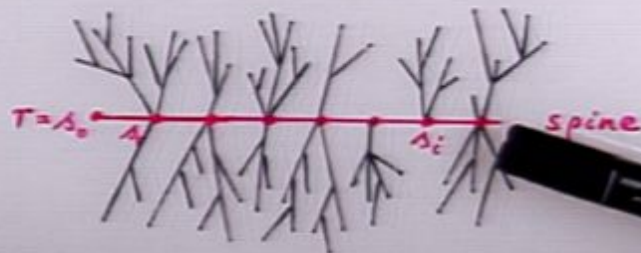


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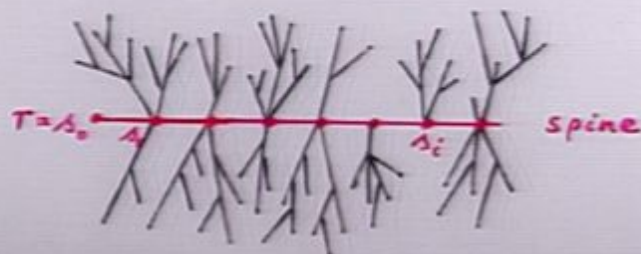


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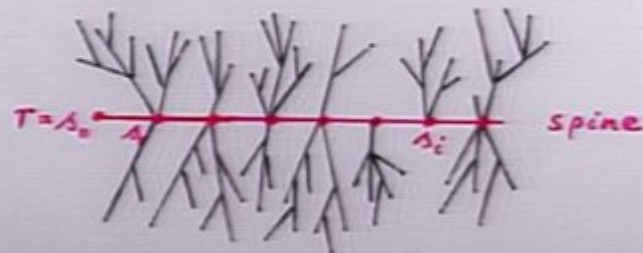


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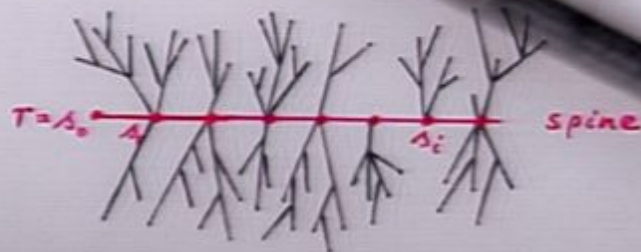


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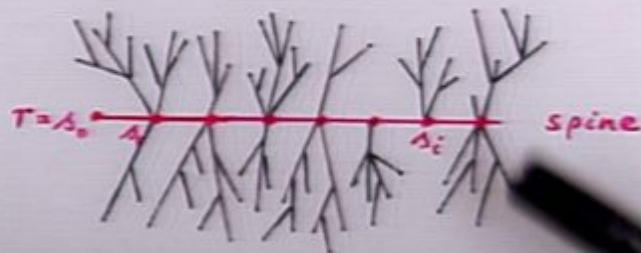


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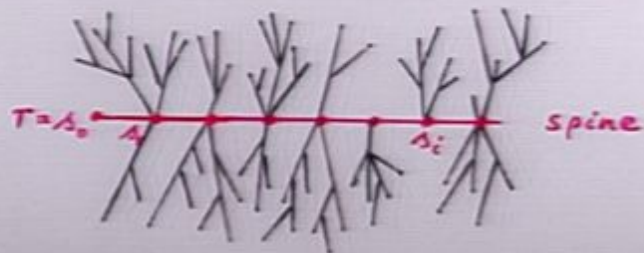


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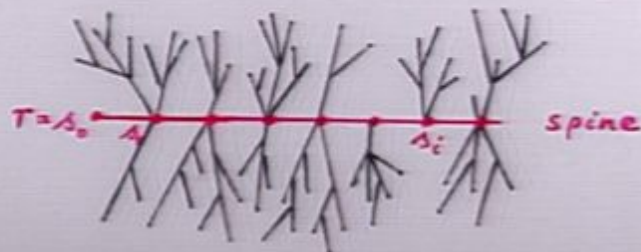


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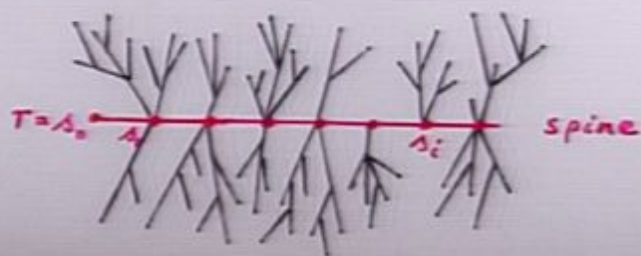


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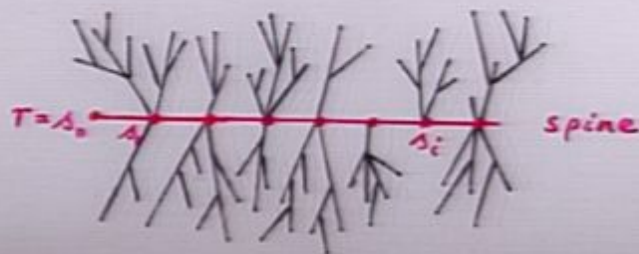


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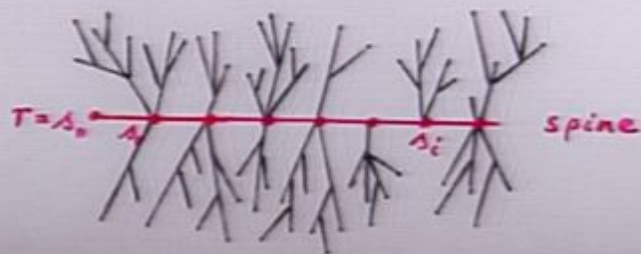


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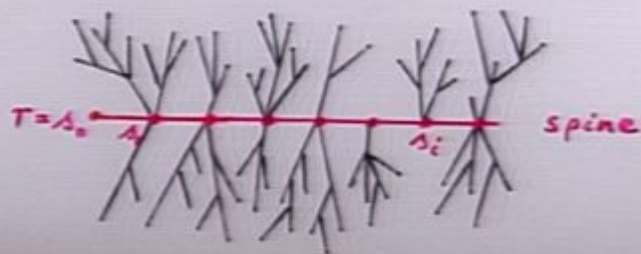


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(15)

Theorem 2 For a generic tree ensemble $(\mathcal{T}, \mu)$ we have

$$d_h = 2 \quad \text{and} \quad d_s = \frac{4}{3}.$$

Almost sure statements also hold, in particular

$$c'(\ln R)^{-2} R^2 \leq |B_R| \leq c R^2 \ln R$$

almost surely.

- $d_h = 2$ follows from previous theorem and

$$\langle |B_R| \rangle_g \sim R$$

Average number of individuals in a Galton-Watson tree is constant.

For d_s one uses, among other results, Kolmogorov '32: Probability that more than R generations survive $\sim \frac{cst}{R}$ for $R \rightarrow \infty$.

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almost surely.

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For d_s one uses, among other results, Kolmogorov '32: Probability that more than R generations survive $\sim \frac{cst}{R}$ for $R \rightarrow \infty$.

(15)

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