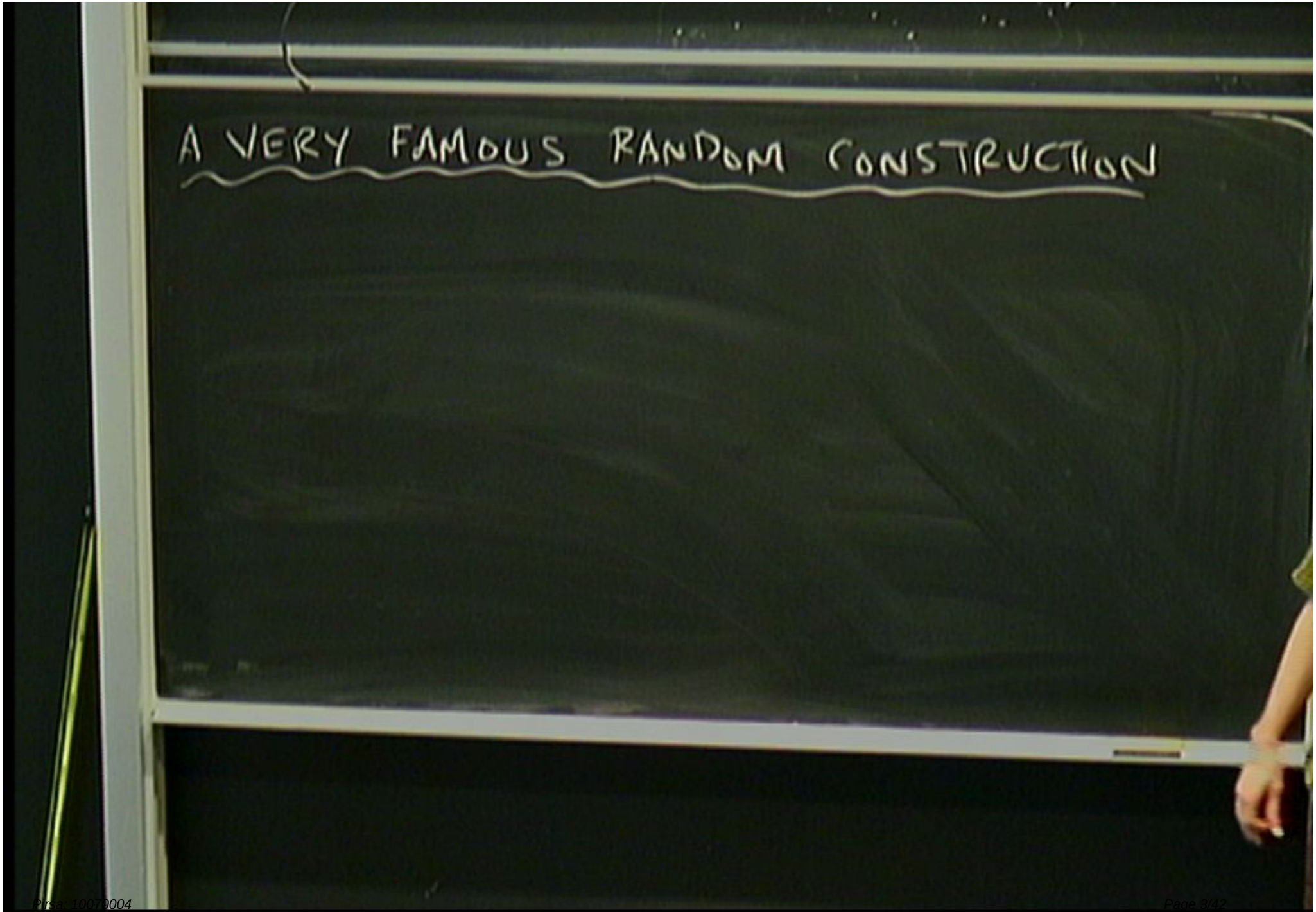
Title: Random constructions in Quantum Information Theory

Date: Jul 04, 2010 09:15 AM

URL: http://pirsa.org/10070004

Abstract: TBA

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS


A VERY FAMOUS RANDOM CONSTRUCTION

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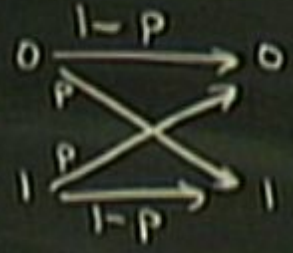
BINARY
SYMMETRIC
CHANNEL



$: p(y|x)$

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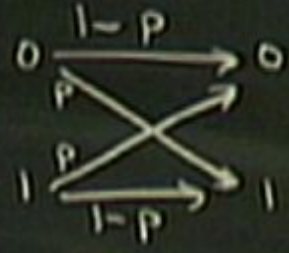
BINARY SYMMETRIC CHANNEL



$: p(y|x)$

MANY USES $p^{\wedge}(y_1 \cdots y_n | x_1 \cdots x_n) = \prod_{i=1}^{n} p(y_i|x_i)$

$y^{\wedge}$ $x^{\wedge}$

$p^{\wedge}$

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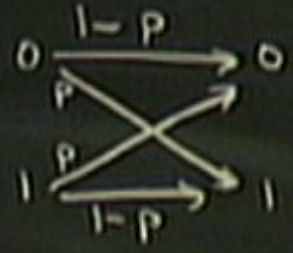
BINARY SYMMETRIC CHANNEL



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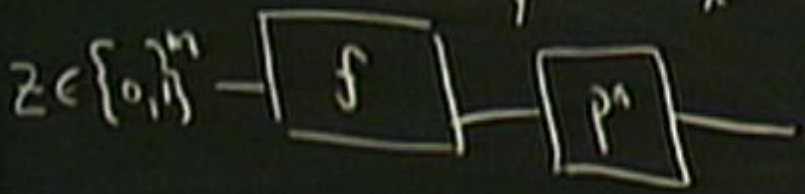

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MANY USES $p^n(y_1 \cdots y_n | x_1 \cdots x_n) = \prod_{i=1}^{n} p(y_i|x_i)$

y^n x^n

$z \in \{0,1\}^m$ — f — p^n — g — $\hat{z}$

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$z \in \{0,1\}^m$ — [f] — [p^n] — [g] — $\hat{z}$ $\quad \Pr\{z \neq \hat{z}\} \xrightarrow{n \to \infty} 0$

HOW TO FIND NEAR-OPTIMAL f?

ie $\frac{m}{n} \longrightarrow 1 - H(p)$, $H(p) = -p \log p - (1-p)\log(1-p)$

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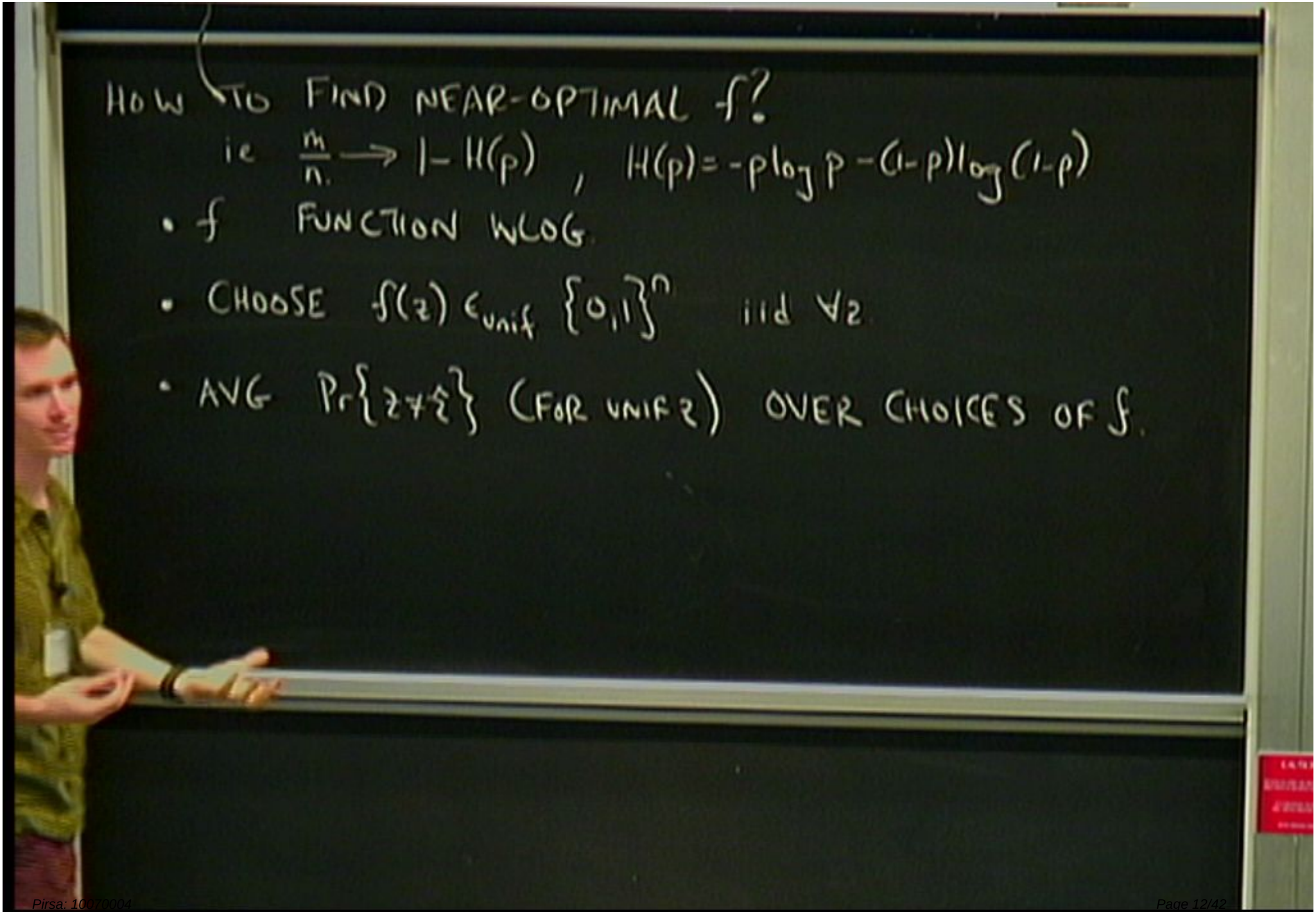
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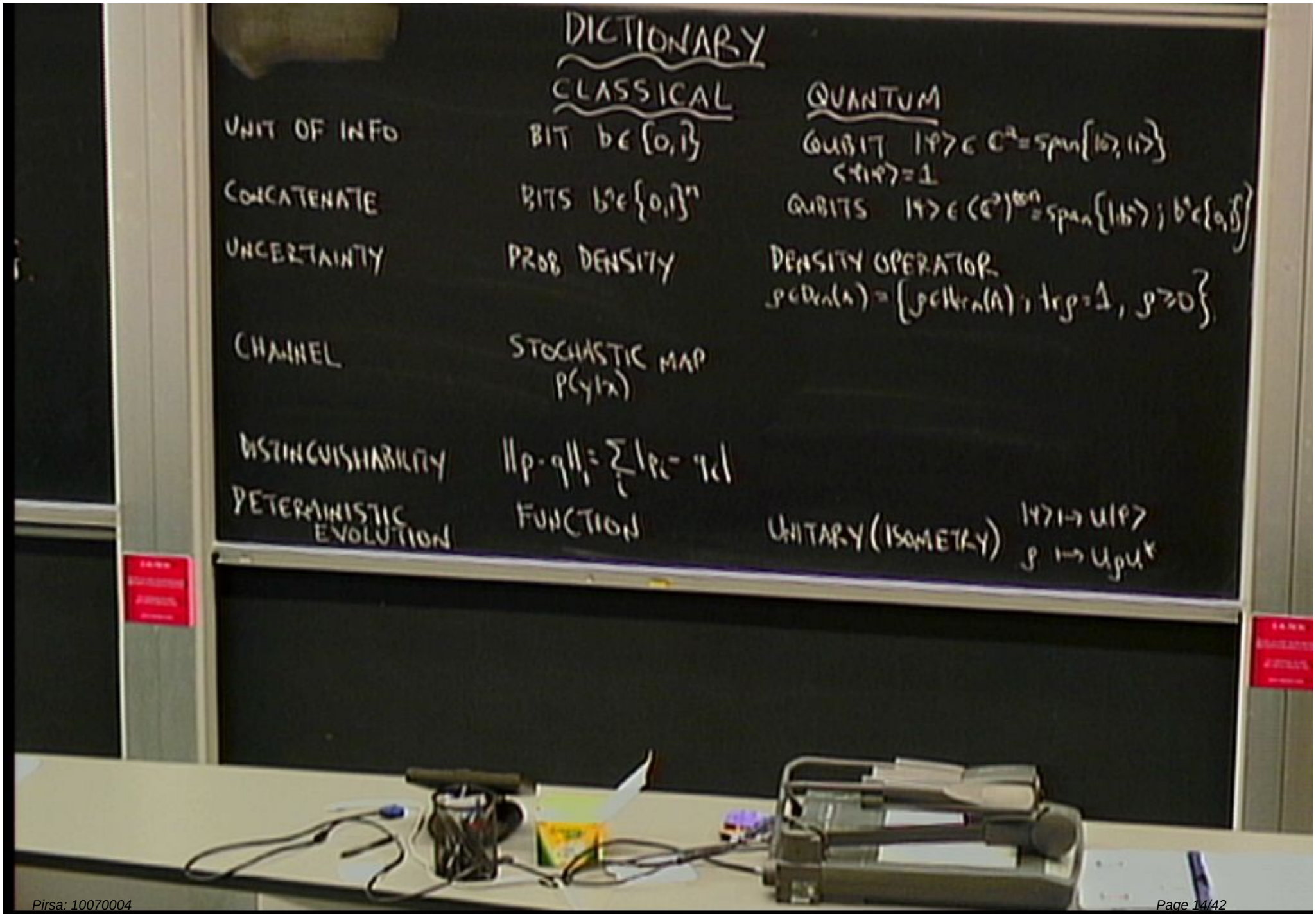
• AVG $\Pr\{z \to \xi\}$ (FOR UNIF z) OVER CHOICES OF f.

DICTIONARY

	CLASSICAL	QUANTUM
UNIT OF INFO	BIT $b \in \{0,1\}$	QUBIT $\|\psi\rangle \in \mathbb{C}^2 = \mathrm{span}\{\|0\rangle, \|1\rangle\}$ $\langle\psi\|\psi\rangle = 1$
CONCATENATE	BITS $b^n \in \{0,1\}^n$	QUBITS $\|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} = \mathrm{span}\{\|b\rangle : b^n \in \{0,1\}^n\}$
UNCERTAINTY	PROB DENSITY	
CHANNEL	STOCHASTIC MAP $P(y\|x)$	
DISTINGUISHABILITY	$\|p - q\|_1 = \sum_i \|p_i - q_i\|$	
DETERMINISTIC EVOLUTION	FUNCTION	UNITARY (ISOMETRY) $\|\psi\rangle \mapsto U\|\psi\rangle$ $\rho \mapsto U\rho U^*$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UNCERTAINTY
PROB DENSITY
DENSITY OPERATOR
$\rho \in \mathrm{Den}(A) = \{\rho \in \mathrm{Herm}(A); \mathrm{tr}\rho = 1, \rho \geq 0\}$
CHANNEL
STOCHASTIC MAP
$P(y|x)$
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DETERMINISTIC EVOLUTION
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UNITARY (ISOMETRY)
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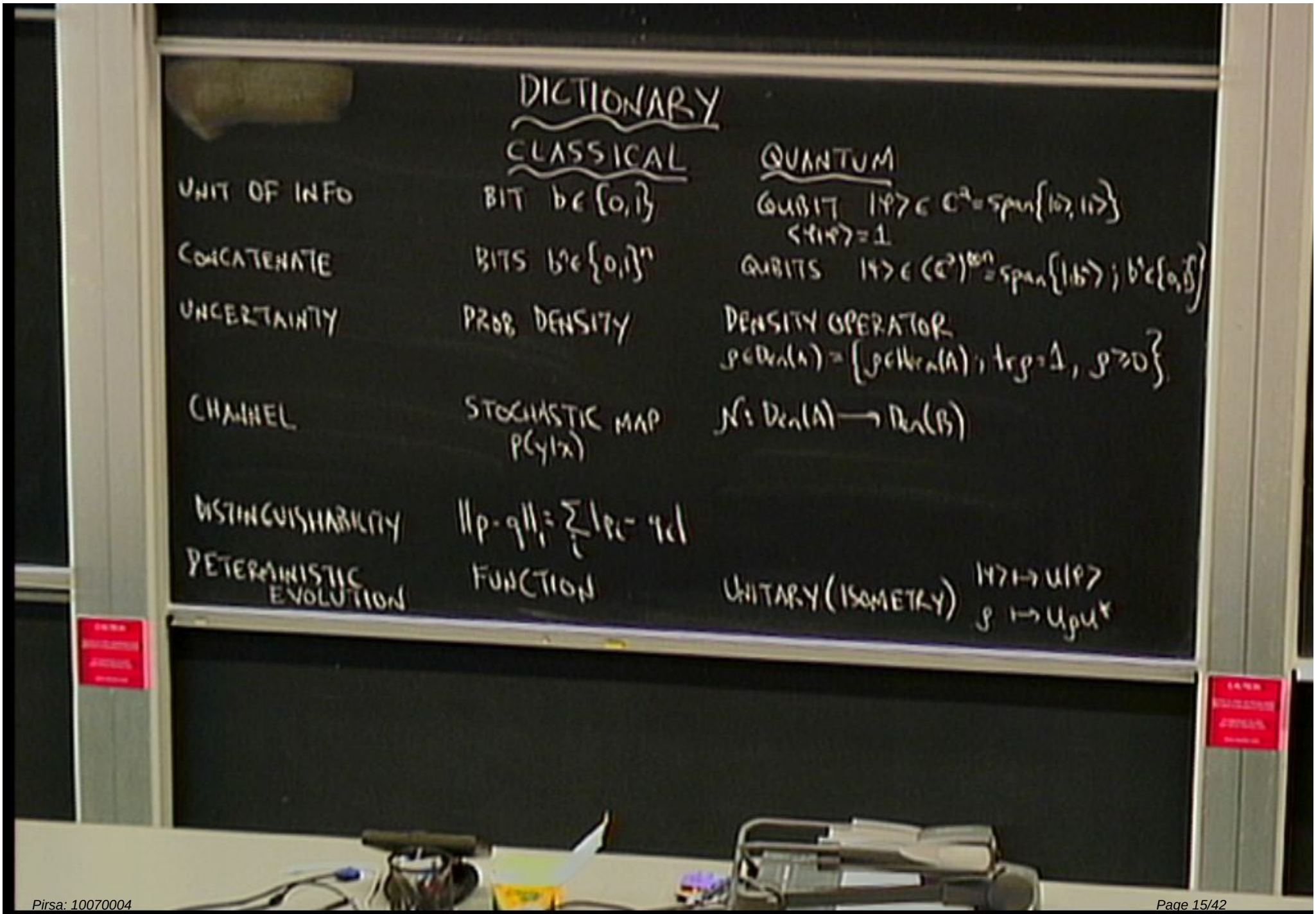
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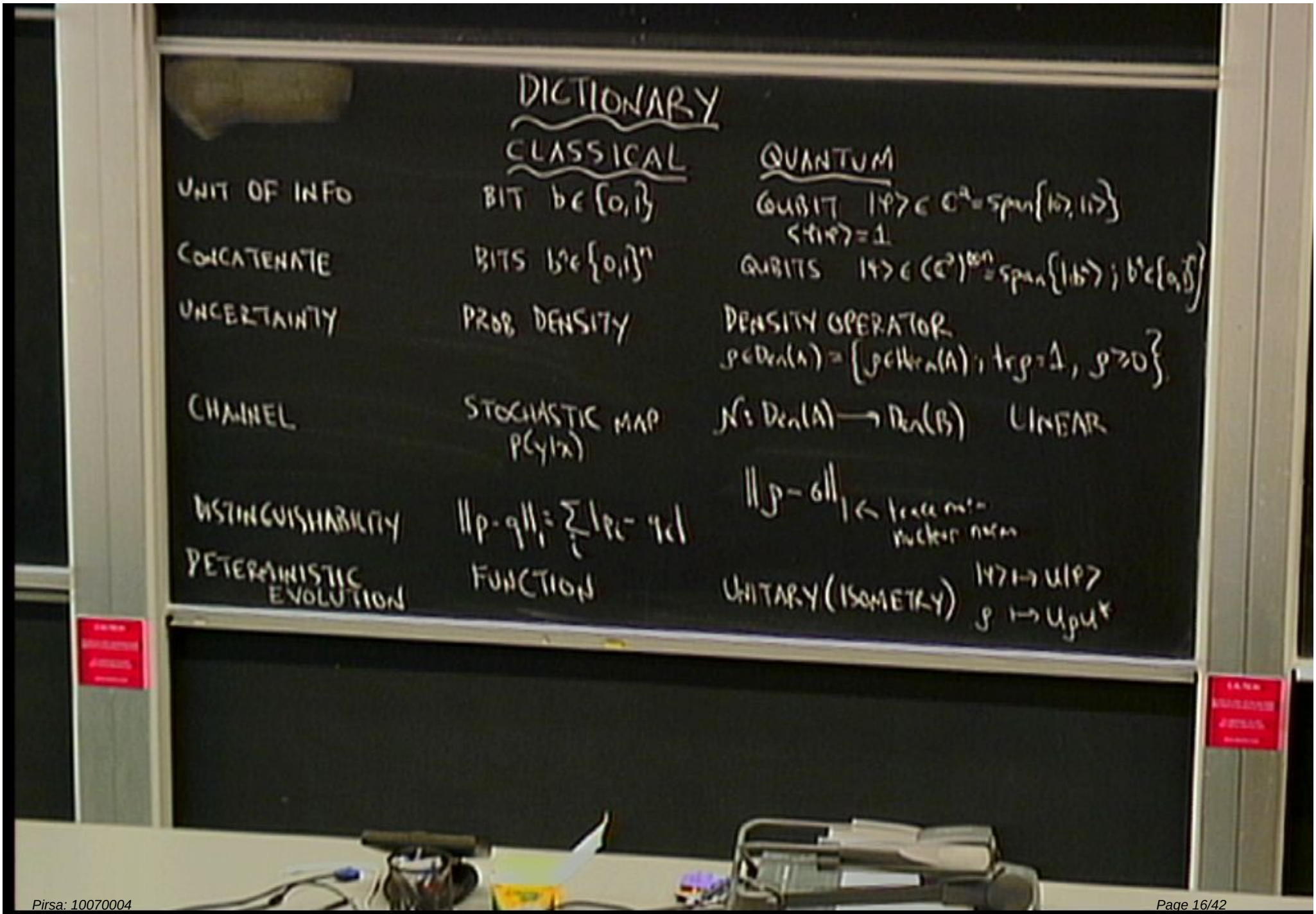
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DISTINGUISHABILITY
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$\|\rho - \sigma\|_1$ ← trace norm
nuclear norm
DETERMINISTIC EVOLUTION
FUNCTION
UNITARY (ISOMETRY)
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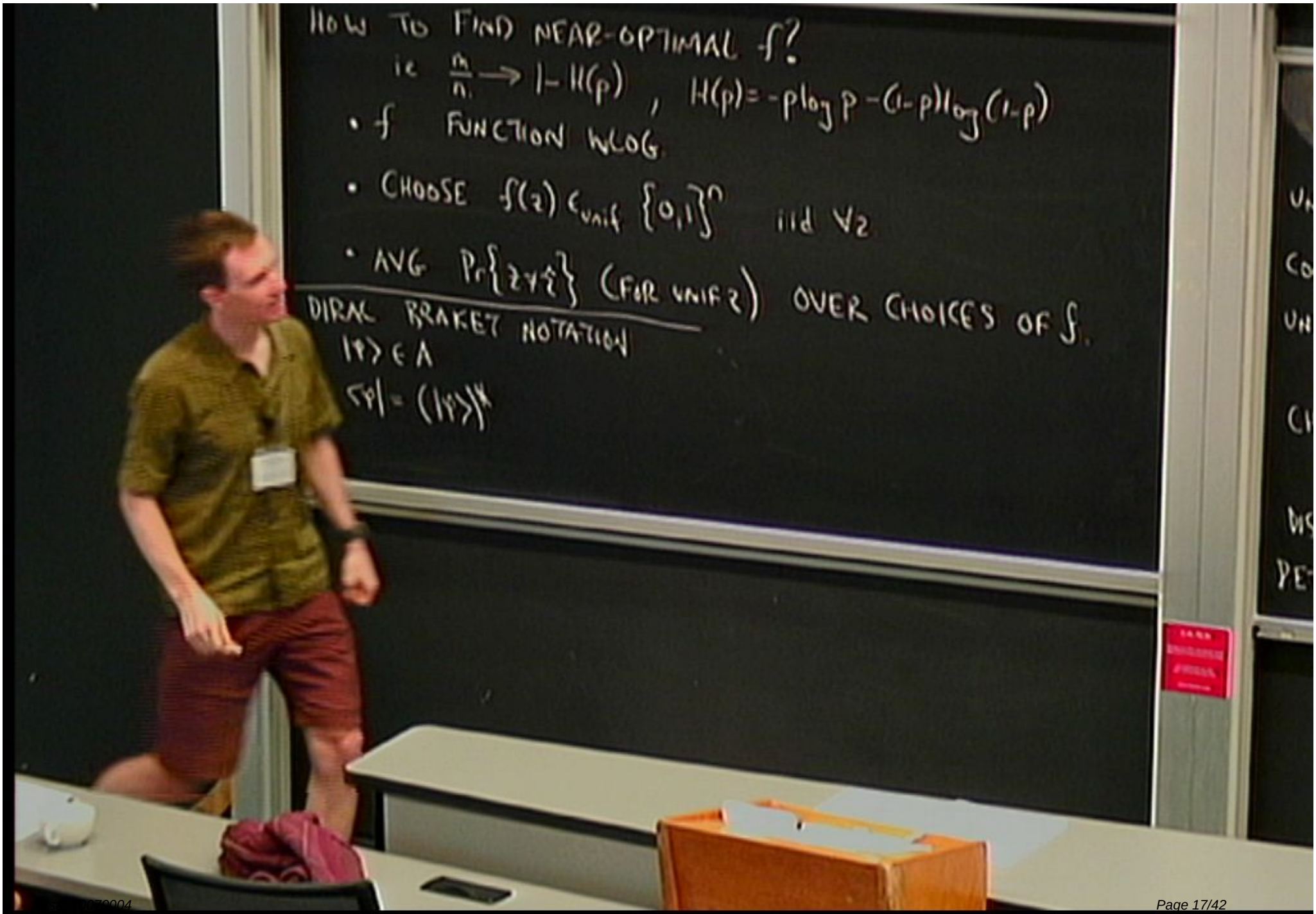
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DIRAC BRAKET NOTATION
$|\psi\rangle \in A$
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DIRAC BRAKET NOTATION

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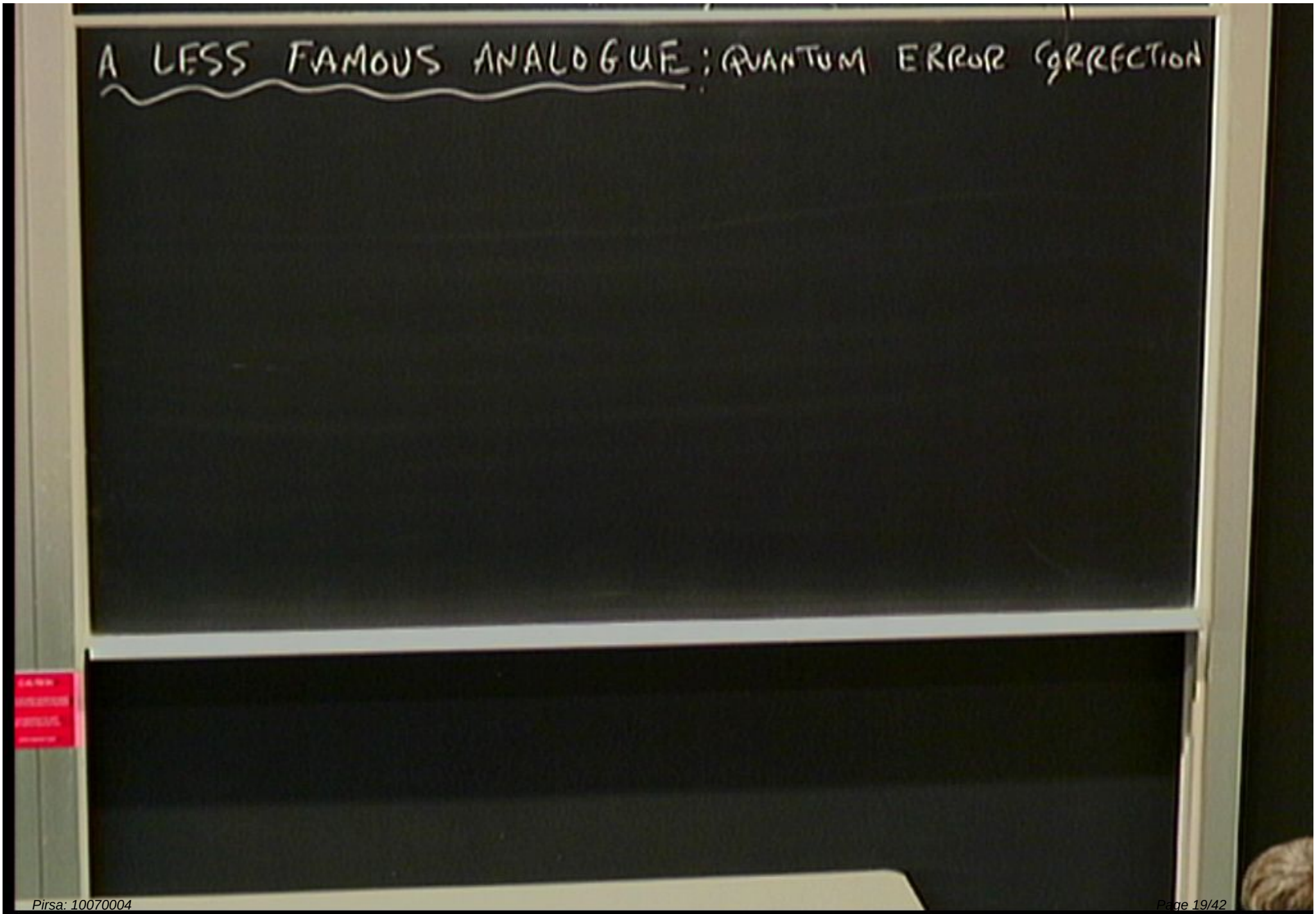
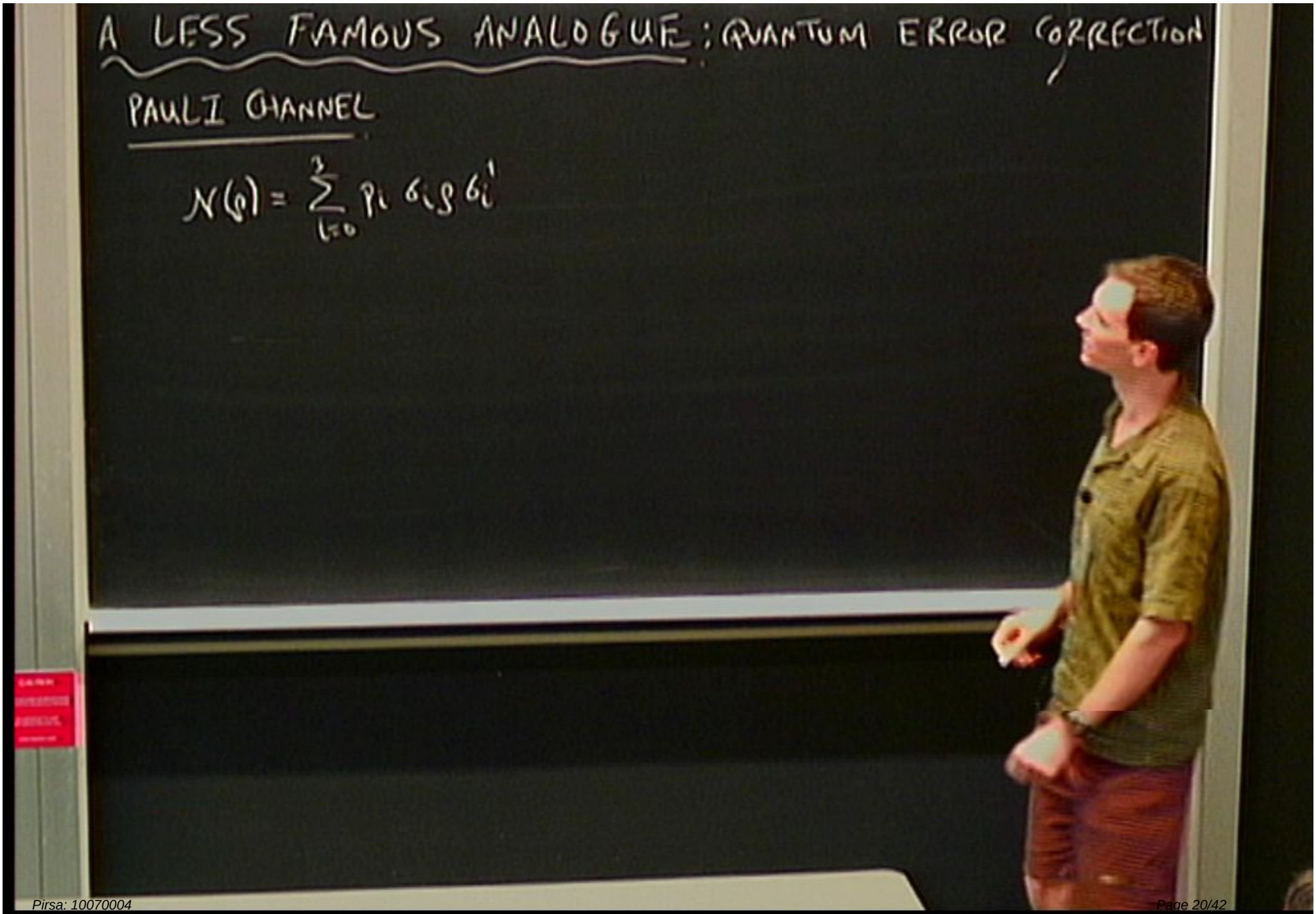
$\langle\varphi| = (|\varphi\rangle)^*$

$\langle\varphi|\psi\rangle$ INNER PRODUCT

$|\varphi\rangle\langle\varphi|$


A LESS FAMOUS ANALOGUE: QUANTUM ERROR CORRECTION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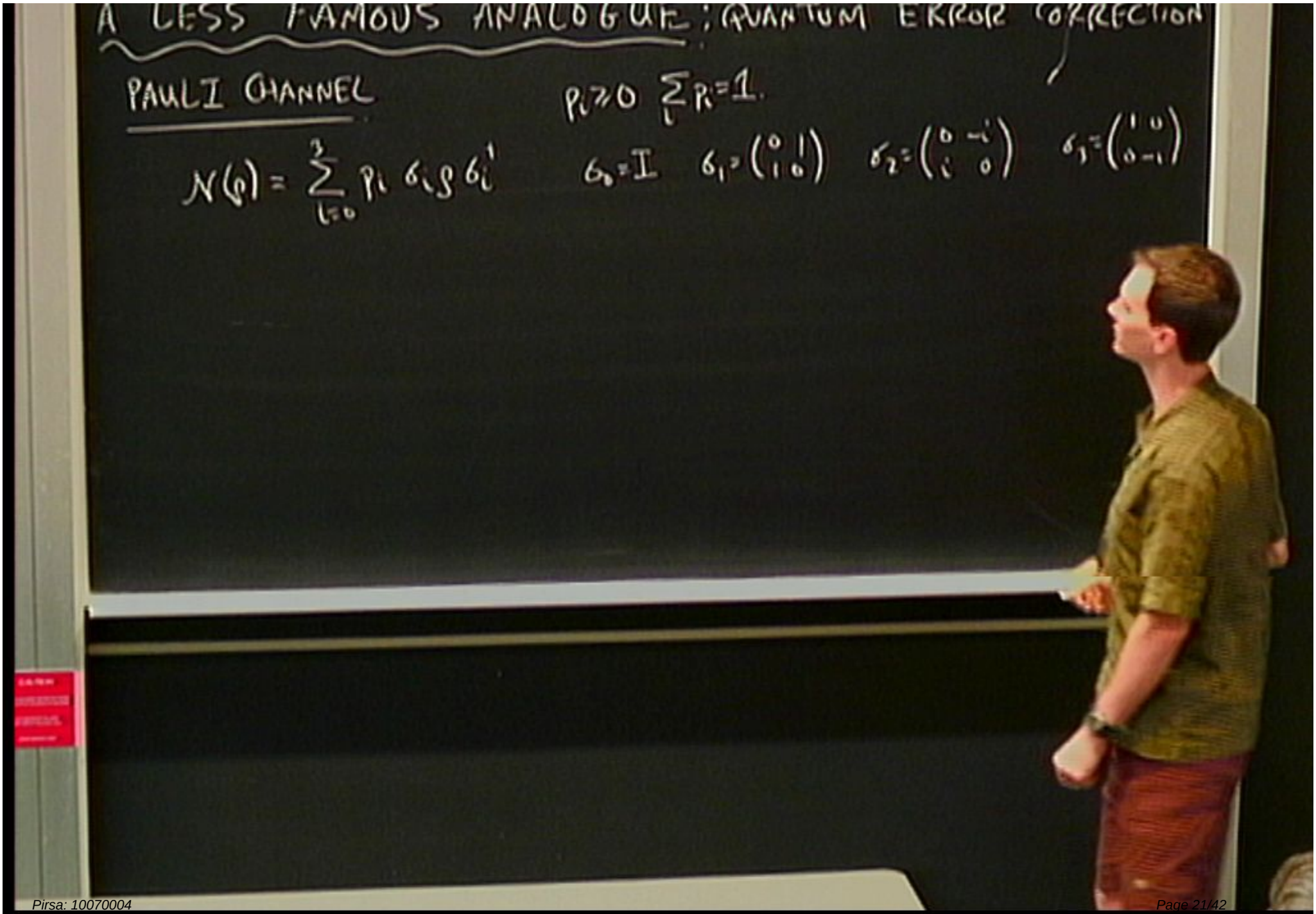
PAULI CHANNEL
$N(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i$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$p_i \geq 0 \quad \sum_i p_i = 1.$
$\mathcal{N}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger$
$\sigma_0 = I \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

A CLASS FAMOUS FOR ACRONYMS: QUANTUM ERROR CORRECTION

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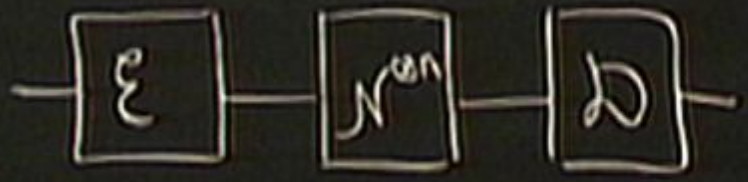

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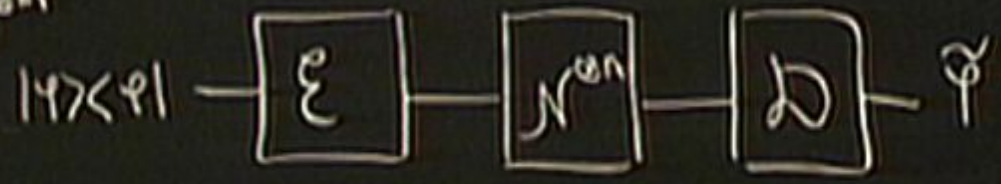
$|\psi\rangle \in (\mathbb{C}^2)^{\otimes m}$



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$\forall |\psi\rangle \quad \| |\psi\rangle\langle\psi| - \mathcal{D} \| < \epsilon$

$\epsilon \xrightarrow{n \to \infty} 0$

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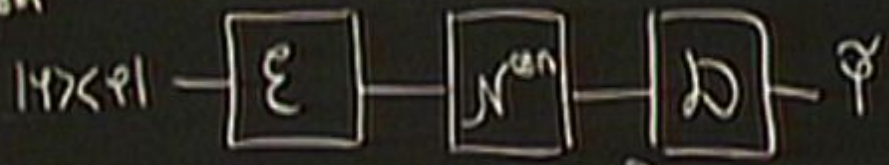

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HOW TO FIND A GOOD $\mathcal{E}$?

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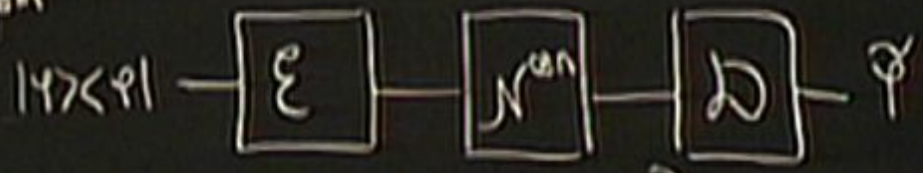

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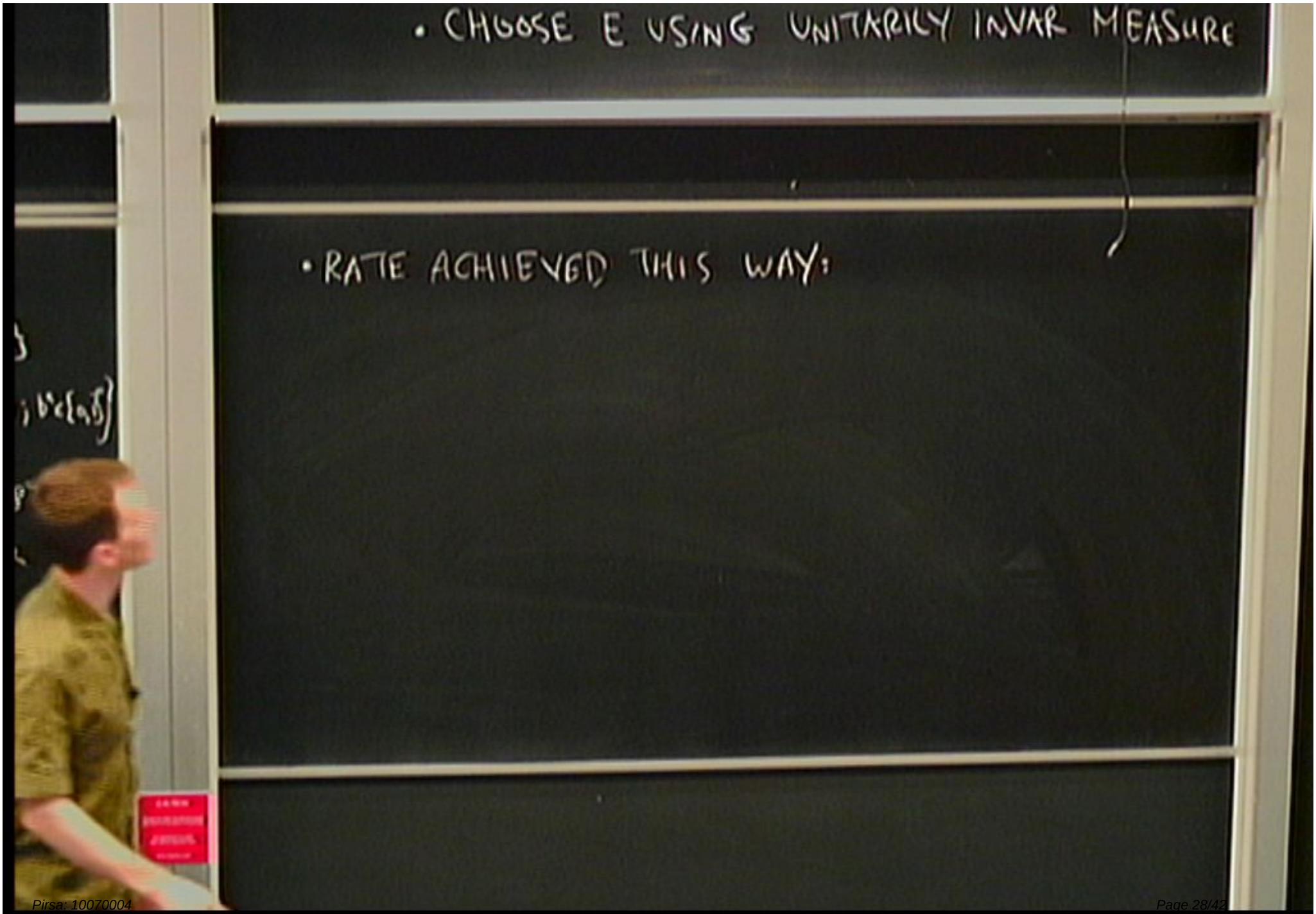
HOW TO FIND A GOOD $\mathcal{E}$?

- USE ISOMETRY $\mathcal{E}(\cdot) = E \cdot E^\dagger$
- CHOOSE E USING UNITARILY INVAR MEASURE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$$\frac{m}{n} \longrightarrow 1 - H(p) \qquad H(p) = -\sum_i p_i \log p_i$$

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$$\frac{m}{n} \longrightarrow 1 - H(p) \qquad H(p) := -\sum_i p_i \log p_i$$

ANALYSIS BY RANDOM MATRICES.

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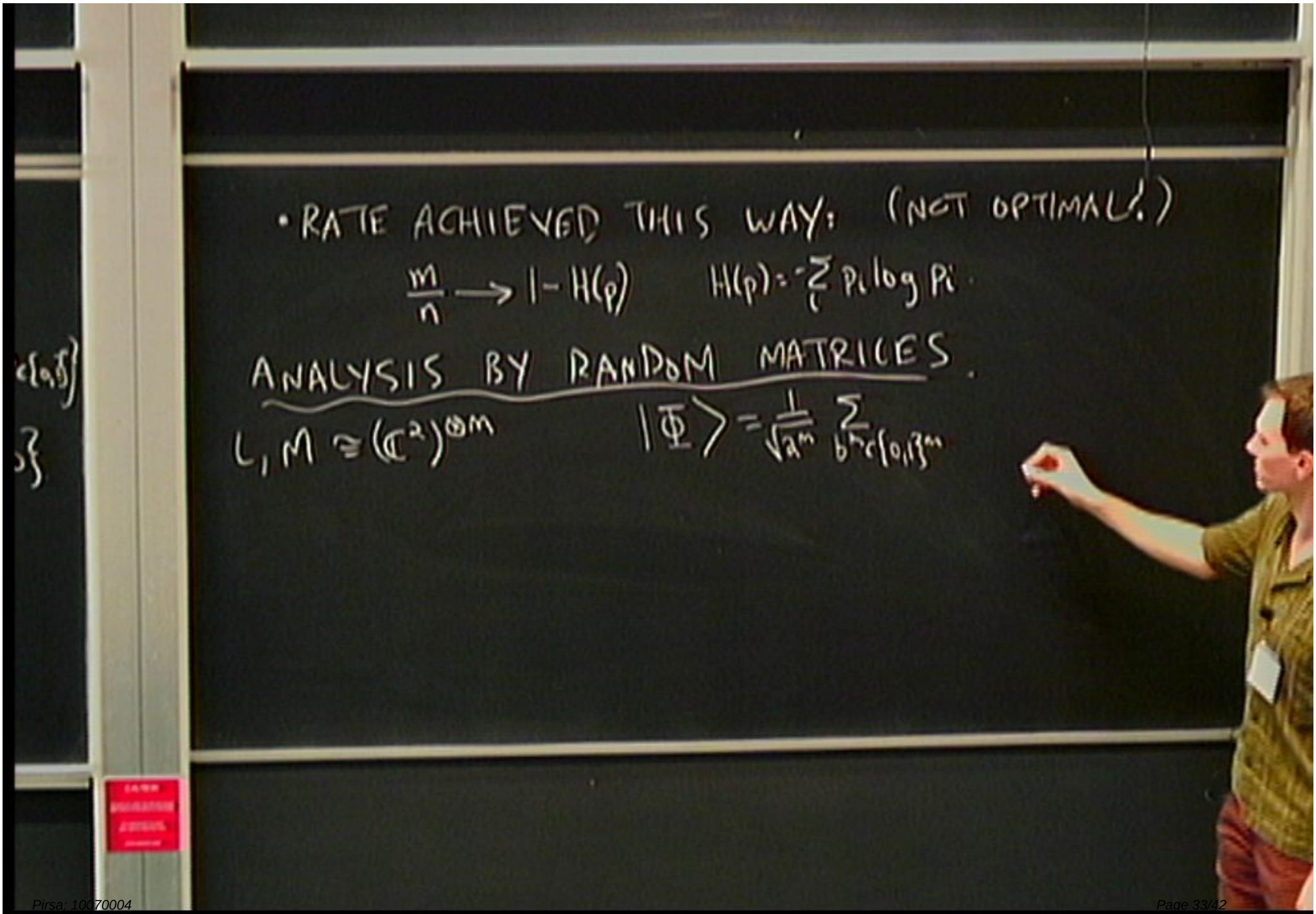
$L, M \cong (\mathbb{C}^2)^{\otimes m}$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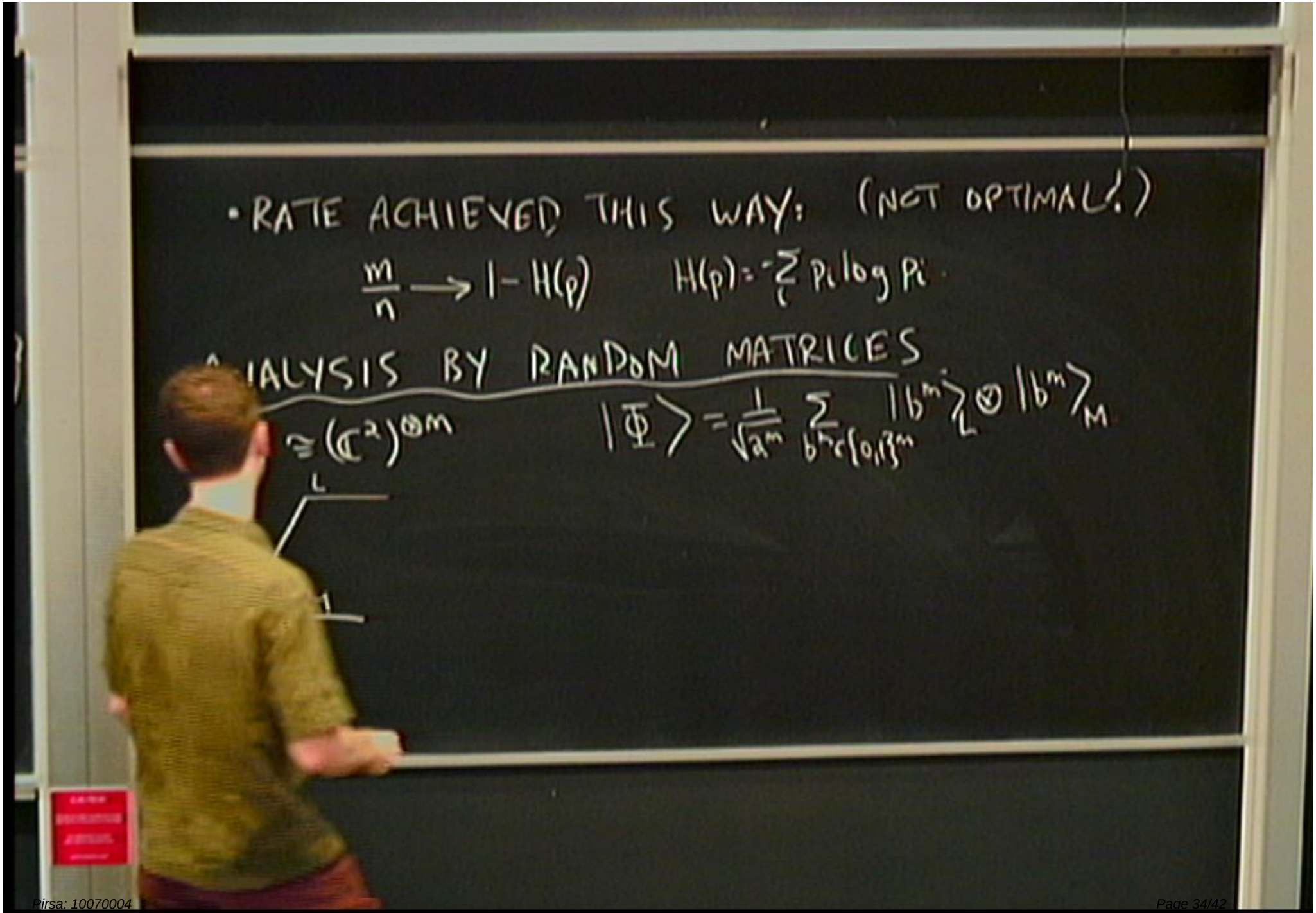
ANALYSIS BY RANDOM MATRICES
$L, M \cong (\mathbb{C}^2)^{\otimes m}$ $|\Phi\rangle = \frac{1}{\sqrt{2^m}} \sum_{b_1 \ldots b_m \in \{0,1\}^m}$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$\cong (\mathbb{C}^2)^{\otimes m}$
$|\Phi\rangle = \frac{1}{\sqrt{a^m}} \sum_{b^m \in \{0,1\}^m} |b^m\rangle_L \otimes |b^m\rangle_M$

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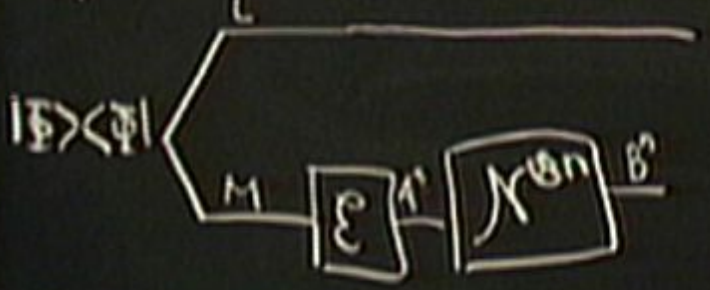
ANALYSIS BY RANDOM MATRICES

$L, M \cong (\mathbb{C}^2)^{\otimes m}$ $\quad |\Phi\rangle_{LM} = \frac{1}{\sqrt{2^m}} \sum_{b^m \in \{0,1\}^m} |b^m\rangle_L \otimes |b^m\rangle_M$



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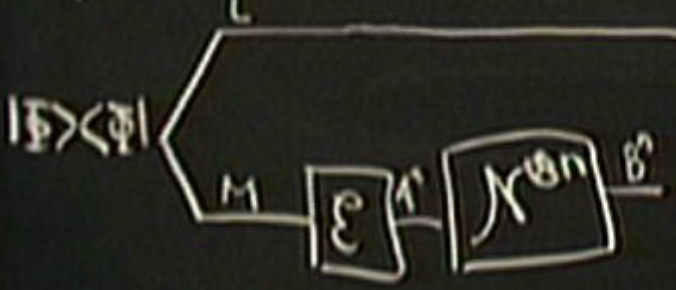
NOTE: $\mathcal{N}(\rho) = \mathrm{tr}_C U_{\mathcal{N}} \rho U_{\mathcal{N}}^*$



$U : A \longrightarrow B \otimes C$

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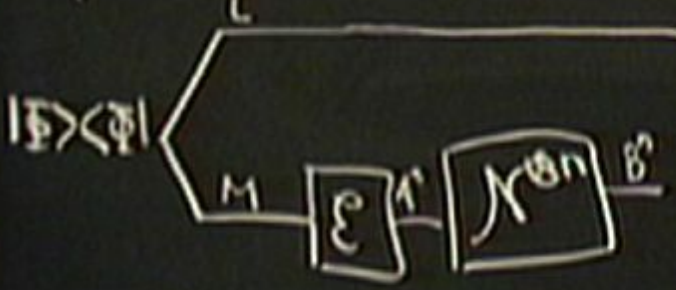

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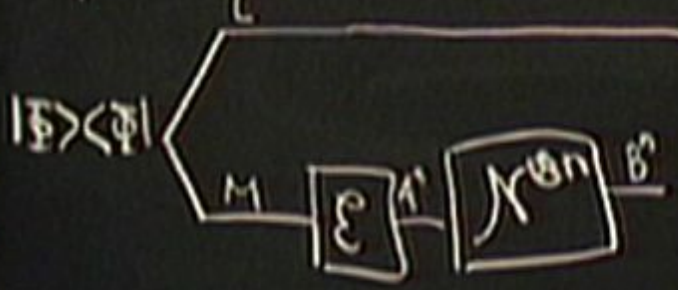
ANALYSIS BY RANDOM MATRICES

$L, M \cong (\mathbb{C}^2)^{\otimes m}$ $\quad |\Phi\rangle_M = \frac{1}{\sqrt{2^m}} \sum_{b^m \in \{0,1\}^m} |b^m\rangle_L \otimes |b^m\rangle_M$



NOTE: $\mathcal{N}(\rho) = \mathrm{tr}_C U_{\mathcal{N}} \rho U_{\mathcal{N}}^\dagger$

$U_{\mathcal{N}} : A \to B \otimes C$ $\quad U_{\mathcal{N}} |\psi\rangle = \sum_{i=0}^{2} \sqrt{p_i}\, \sigma_i |\psi\rangle_B |i\rangle_C$

• RATE ACHIEVED THIS WAY: (NOT OPTIMAL!)

$$\frac{m}{n} \longrightarrow 1 - H(p) \qquad H(p) = -\sum_i p_i \log p_i$$

ANALYSIS BY RANDOM MATRICES

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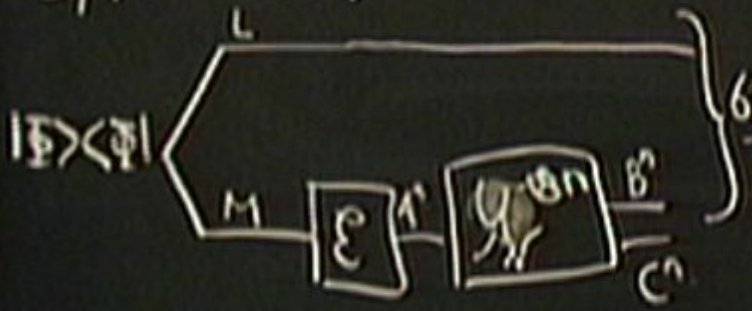
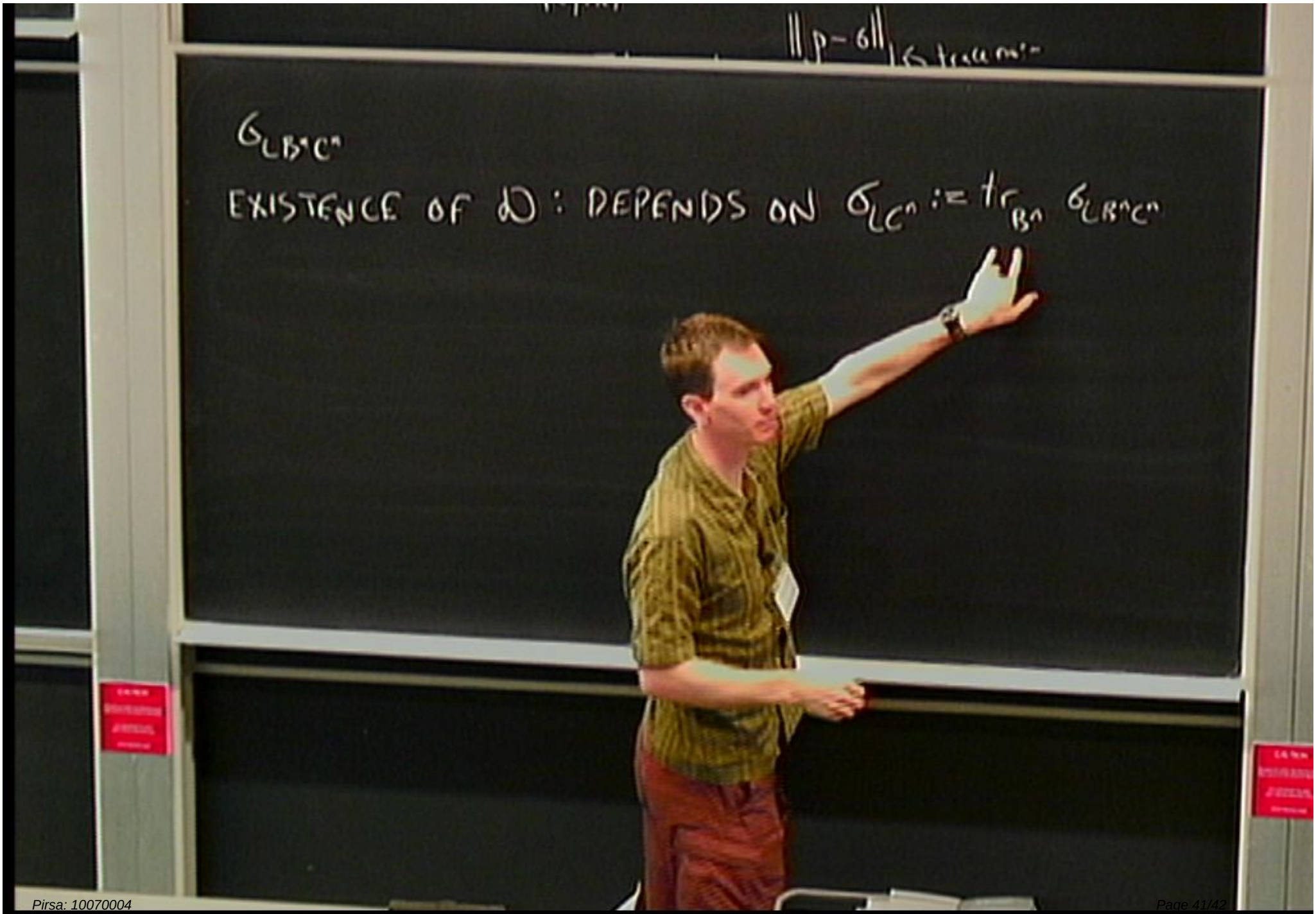


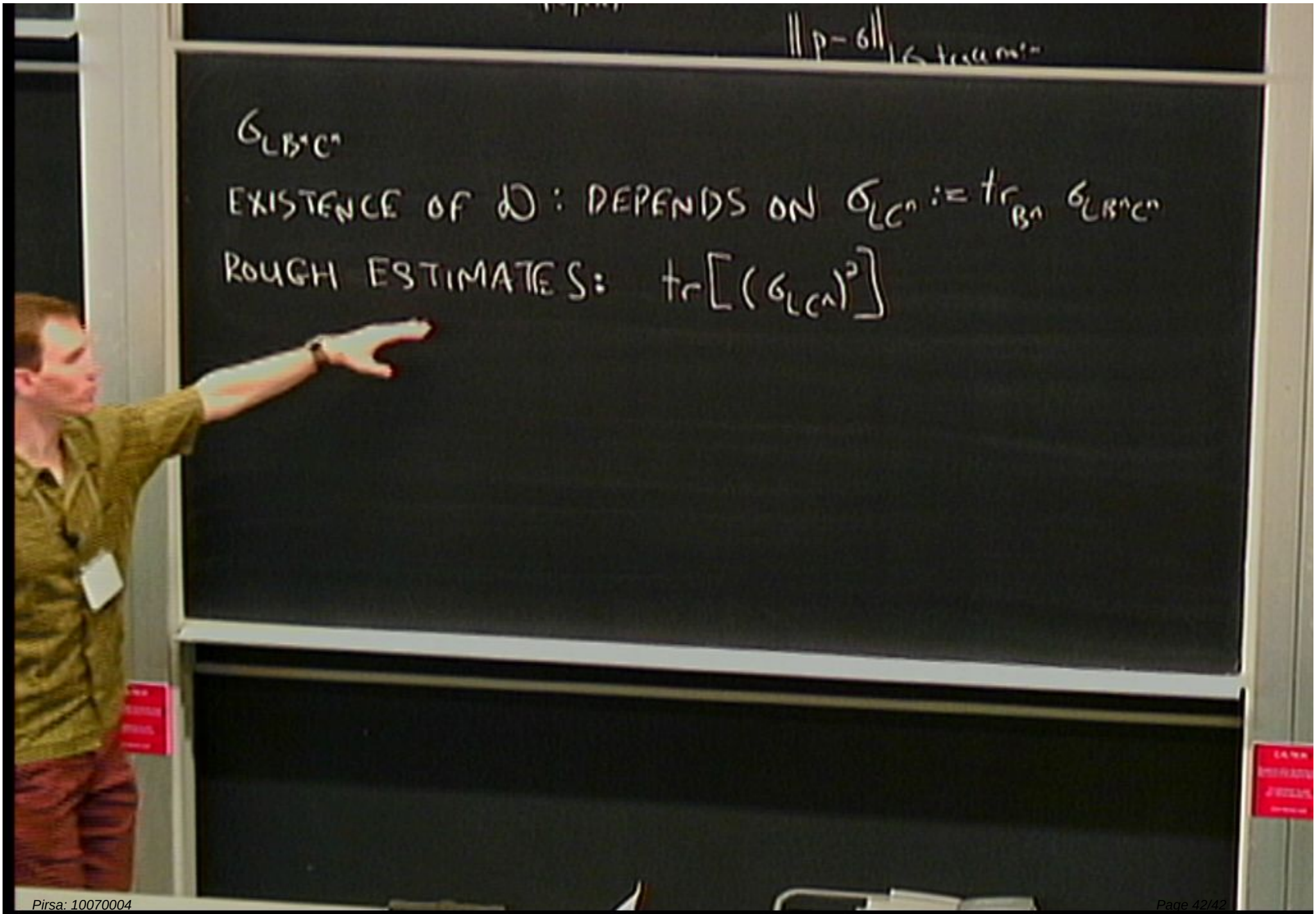
$|\Phi\rangle\langle\Phi|$ $\qquad$ block-wise $U_x : A \to B \otimes C$ $\qquad$ NOTE: $\chi(\rho) =$ $\qquad$ $U_x|\psi\rangle = \sum_{i=0}^{2} \sqrt{p_i}$

$\|p - 6\|$

GLB'C"

EXISTENCE OF $\mathcal{D}$


$\sigma_{LB'C''}$
EXISTENCE OF $\mathcal{D}$: DEPENDS ON $\sigma_{LC''} := tr_{B'} \sigma_{LB'C''}$


$G_{LB^nC^n}$
EXISTENCE OF ∂ : DEPENDS ON $\sigma_{LC^n} := tr_{B^n} G_{LB^nC^n}$
ROUGH ESTIMATES: $tr[(G_{LC^n})^2]$