Title: Comparisons between post-Newtonian and self-force calculations

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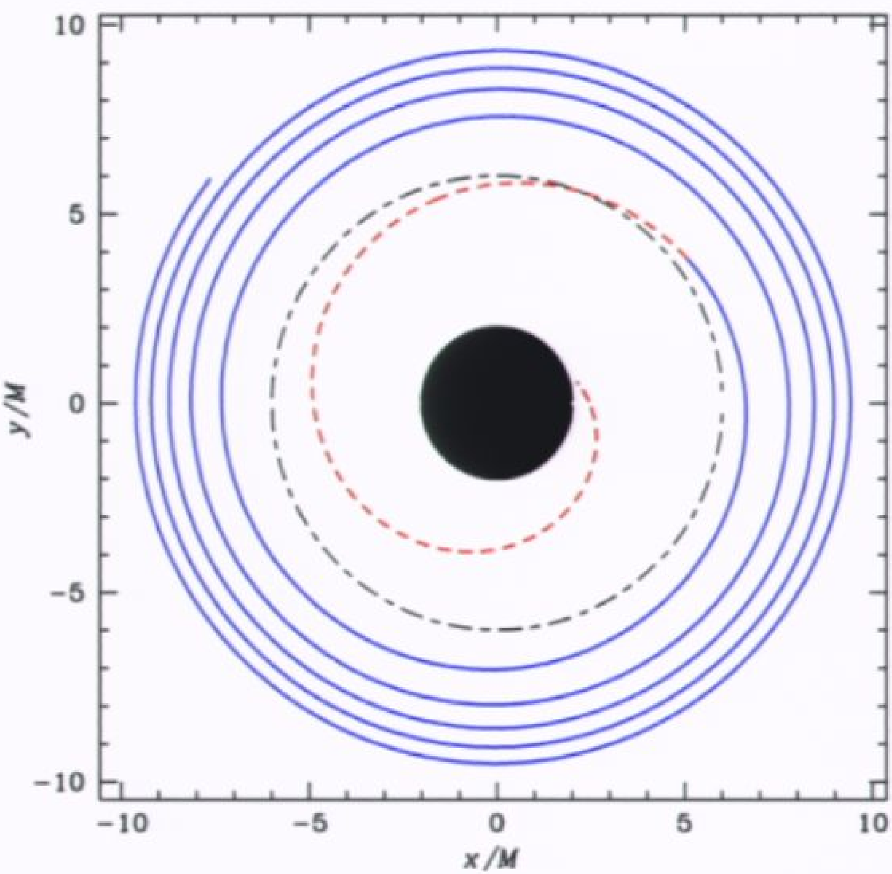
Abstract: TBA

# Comparisons between post-Newtonian and self-force ISCO calculations





Marc Favata

JPL/Caltech

# onservative correction to the ISCO:

ecently, Barack & Sago have computed the self-force along eccentric geodesics in hwarzschild [see Barack & Sago, PRL '09; Barack & Sago, arXiv:1002.2386]

he self-force can be split into dissipative and conservative pieces

- **dissipative piece:** radiation-reaction forces; secular losses in energy, angular momentum
- **conservative piece:** test-mass corrections to periastron precession rate, ISCO location

## Barack-Sago conservative GSF ISCO shift:

$$r_{\rm isco}^{\rm BS} = 6M[1 - 1.545\eta + O(\eta^2)]$$

$$M\Omega_{\rm isco}^{\rm BS} = 6^{-3/2}[1 + 1.251\eta + O(\eta^2)]$$

$$M = m_1 + m_2$$

$$\eta = \frac{m_1 m_2}{M^2} = \frac{q}{(1+q)}$$

- a milestone in self-force computations

## ote: scaling of dissipative corrections to the ISCO

i-Thorne computed dissipative corrections to the ISCO.
hwarzschild ISCO get's "blurred" by amounts:

$$\frac{\Delta r^{\text{diss.}}}{m_2} \approx 18 q^{2/5} \quad \text{and} \quad \frac{\Delta \Omega^{\text{diss.}}}{\Omega_{\text{isco}}} \approx 4.4 q^{2/5}$$

$$\frac{\Delta \Omega^{\text{conserv.}}_{\text{isco}}}{\Delta \Omega^{\text{diss.}}_{\text{isco}}} \approx 0.1 q^{3/5} \sim 10^{-2}\text{–}10^{-5}, \quad \text{for} \quad q \sim 10^{-2}\text{–}10^{-7}$$

**onservative GSF ISCO shift not likely to be observable.**

ather, its utility relies on the fact that it is a readily understood and uge-invariant quantity that can be compared among GSF codes and th PN/EOB approaches (but probably not with *full* NR evolutions).

# evious PN + NR comparison studies:

After NR successes in '05 & '06, a variety of PN comparison studies in q ~1 regime were performed

- Important to quantify errors associated with PN methods
- determine reliability of PN templates for LIGO searches
- Calibration of extra PN "fitting" parameters

ISCO comparison studies by Blanchet '02, and Damour, Gourgoulhon, Grandclement '02, compared newly computed equal-mass ISCO/ICO calculations (using quasi-circular initial data sequences) with the (then) newly-computed 3PN and EOB results.

Conservative GSF calculations provide exact, fully-relativistic result in the q<<1 regime to further compare with PN techniques

- Compare various PN/EOB approaches, comment on their relative performance
- Further calibrate "extra" parameters to match test-mass limit

# ther GSF + PN comparison studies:

**Poisson, Tagoshi, + collaborators (1990s):** comparison of BH perturbation theory w/ PN expressions.

**Detweiler '08:** computed gauge-invariant redshift function $u^t(\Omega)$ along circular orbits in Schw. w/ GSF code and compared w/ 2PN result.

**Blanchet, Detweiler, Le Tiec, Whiting '10a:** extended calculations to 3PN order.

**Blanchet, Detweiler, Le Tiec, Whiting '10b:** extended calculation to 4PN & 5PN logarithmic terms (as well as $2^{nd}$ order PN self-force terms).

- Excellent agreement w/ known PN terms
- higher-order PN terms can be fit.

**Damour '10:** discussed how various GSF calculations can constrain higher-

# omparison with post-Newtonian ISCO: procedure

## How to compute the ISCO/ICO in PN?

- Examine all known approaches and compare with the Barack-Sago result.
- There are about 10 – 15 distinct methods.
- They can be broadly separated into ***non-resummed*** and ***resummed*** approaches…

# ost-Newtonian techniques: to resum or not to resum?

## Un-resummed ("ordinary") methods:

– **(E3PN)** Minimum of orbital binding energy $E^{PN}(\Omega)$

$$\frac{E^{\rm PN}(\Omega)}{\eta M} = -\frac{1}{2}x\left\{1 + x\left(-\frac{3}{4} - \frac{\eta}{12}\right) + x^2\left(-\frac{27}{8} + \frac{19}{8}\eta - \frac{\eta^2}{24}\right)\right.$$

$$\left. + x^3\left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\eta - \frac{155}{96}\eta^2 - \frac{35}{5184}\eta^3\right]\right\}$$

$$x \equiv (M\Omega)^{2/3}$$

- Need to go to 4PN order to get Schwarzschild result ($x$=1/6) to within 12%

– Stability analysis of the PN eqs of motion (Blanchet & Iyer '03):

$$\dot{\boldsymbol{v}} = -\frac{M}{r^2}[(1 + \mathcal{A}(r, \dot{r}, \dot{\varphi}))\boldsymbol{n} + \mathcal{B}(r, \dot{r}, \dot{\varphi})\boldsymbol{v}]$$

Several ways to do this:

- Numerically solve system of Eqs. for ISCO ($r$,$\Omega$)
- PN expansion directly in terms of harmonic $r$ or ADM $R$
- **($C_0$3PN)** Gauge-invariant condition in terms of $x=(M\,\Omega)^{2/3}$

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## Other ingredients:

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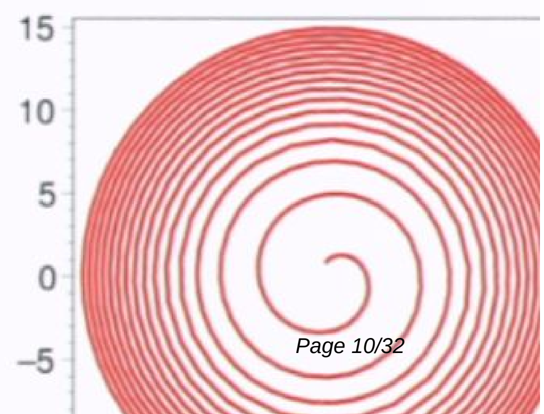

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– Stability analysis of Hamiltonian formulation of PN ...

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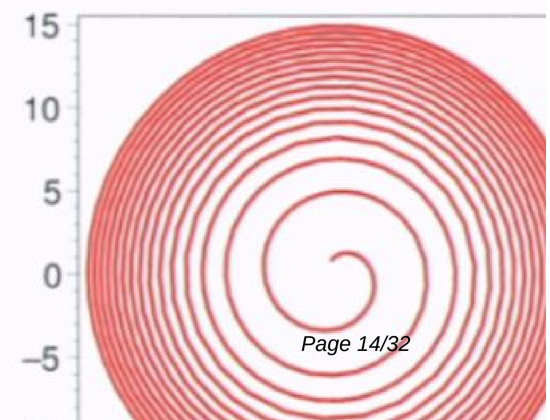

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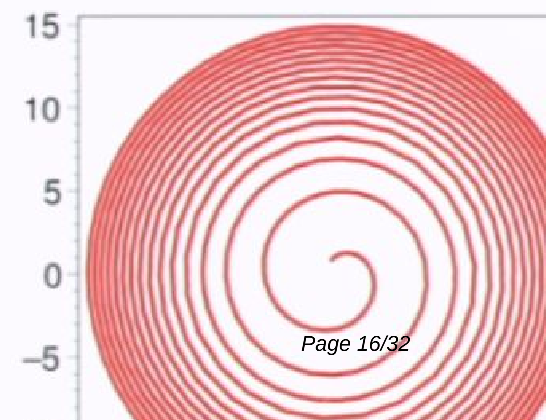

# ISCO comparison: results

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| N-P | -0.1667 | -1.133 |
| 'W-2PN | -1.542 | -2.232 |
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| 'W-3PN | 4.851 | 2.877 |
| [illegible] | [illegible] | [illegible] |

$$M_{\rm tot}\Omega^{\rm PN}_{\rm isco} = 6^{-3/2}[1 + c_\Omega^{\rm PN}\eta + O(\eta^2)]$$

$$c_\Omega^{\rm BS} = 1.251$$

4PN EOB, fit $a^{4PN}(\eta)$ to Caltech/Cornell NR

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Gauge-invariant stability condition for un-resummed 3PN eqns of motion

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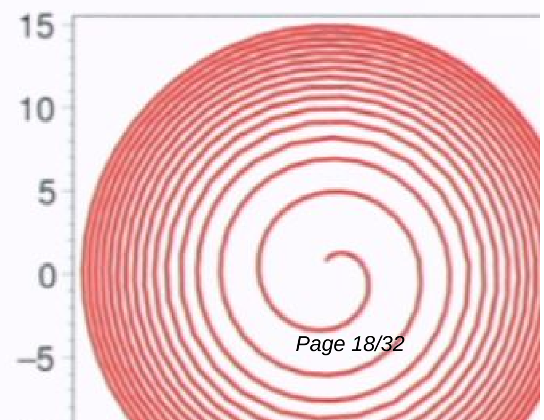

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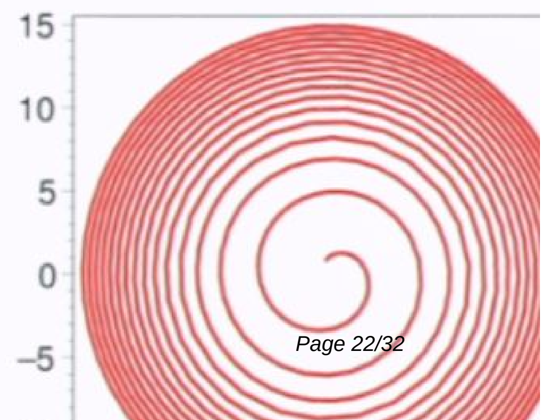

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| PN | 0.1287 | 0.073 |
| PN-P | 0.1340 | 0.12 |
| PN-$P_C$ | 0.1036 | -0.14 |
| PN | 0.1371 | 0.14 |
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| og4PN | 0.08999 | -0.25 |
| PN-P | 0.08850 | -0.26 |
| PN-P | 0.08822 | -0.26 |
| 4PN | 0.1567 | 0.31 |
| 2PN | 0.0809 | -0.33 |
| PN-P | 0.07980 | -0.33 |
| og3PN | 0.07754 | -0.35 |
| 3PN | 0.07698 | -0.36 |
| PN-T | 0.07340 | -0.39 |
| PN-P | 0.07312 | -0.39 |
| og2PN | 0.07056 | -0.41 |
| 2PN | 0.06959 | -0.42 |
| 1PN | 0.06779 | -0.44 |
| -S | 0.06721 | -0.44 |
| PN-P | 0.06530 | -0.46 |

- 3PN energy function $E^{PN}(\Omega)$ (rows: 0.1287, 0.073)
- 4PN EOB, fit $a^{4PN}(\eta)$ to Caltech/Cornell NR (rows: 0.1004, -0.16; 0.09807, -0.18)
- 3PN EOB, no free parameters (row: 0.08822, -0.26)
- 4PN EOB, fit to Barack-Sago ISCO shift (row: 0.1036, -0.14)

$$M\Omega^{NR}_{isco}(1/4) \approx 0.122$$

NR value from Caudilll et. al '06; improves on previous studies by Pfeiffer, Cook, Baumgart, Grandclement, Gourgoulhon, & others

# ummary of tables:

## Best method is EOB with pseudo-4PN term calibrated to Caltech/Cornell waveform (10%)

– Especially remarkable that fits to an equal-mass simulation help to fix a small-mass-ratio quantity.

## EOB without calibration (3PN) is 3 times worse (28% error)

Best ***uncalibrated*** method uses a gauge-invariant ISCO stability criterion derived from the un-resummed 3PN equations of motion

– Surprise: this method **exactly** reproduces the test-mass ISCO **without resummation!**

(Kidder-Will-Wiseman hybrid equations extended to 3PN order don't work well.)



In the equal-mass case, the standard 3PN energy function performs

# tension of gauge-invariant ISCO criterion to spinning binary

st, consider the ISCO in the Kerr spacetime, given by the condition dE/dr =0 , where

$$\frac{E}{m_1} = \frac{1-2w+\chi_2 w^{3/2}}{\sqrt{1-3w+2\chi_2 w^{3/2}}} \qquad w = \frac{m_2}{r} \qquad \chi_2 = \frac{|\boldsymbol{S}_2|}{m_2^2}$$

is condition for the Kerr ISCO is equivalent to solving the following equation:

$$\hat{C}_0^{\rm kerr} \equiv 1 - \frac{X_0}{\beta^{2/3}}\left[6 - \chi_2 \frac{X_0^{1/2}}{\beta^{1/3}}\left(8 - 3\chi_2 \frac{X_0^{1/2}}{\beta^{1/3}}\right)\right] = 0,$$

$$\text{with} \qquad \beta \equiv 1 - \chi_2 X_0^{3/2} \qquad \text{and} \qquad X_0 \equiv |m_2 \Omega_0|^{2/3}.$$

r small spin, this condition reduces to:

$$C_0^{\rm kerr} = 1 - 6X_0 + \chi_2\left(8X_0^{3/2} - 4X_0^{5/2}\right) + \chi_2^2\left(-3X_0^2 + 8X_0^3 - 10X_0^4/3\right) + O\left(\chi_2^3\right)$$

# ...tension of gauge-invariant ISCO criterion to spinning binary

## ...w, derive an analogous condition for the PN equations of motion w/ spin.

...r non-precessing binaries, the PN Eqs. of motion can be put in the following form:

$$\dot{r} = u$$

$$\dot{u} = -\frac{m}{r}[1 + \mathcal{A}^{\rm tot}(r,u,\omega)] + r\omega^2$$

$$\dot{\omega} = -\frac{1}{r}\left[\frac{m}{r^2}\mathcal{B}^{\rm tot}(r,u,\omega) + 2u\omega\right]$$

where

$$\mathcal{A}^{\rm tot} = \mathcal{A}^{\rm NS} + \mathcal{A}^{\rm SO}_{\rm 1.5PN} + \mathcal{A}^{\rm SO}_{\rm 2.5PN} + \mathcal{A}^{\rm SS+QM}_{\rm 2PN}$$

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Perturb about a circular orbit...

$$r = r_0 + \epsilon\delta r$$

$$u = 0 + \epsilon\delta u$$

...and linearize the above equations of motion...

# tension of gauge-invariant ISCO criterion to spinning binary

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rom the linearized Eqs, one can derive a gauge-invariant stability condition for the istence of stable, circular orbits [Blanchet-Iyer derived this in the non-spinning case]

$$\equiv 1 - 6x + x^{3/2}\left(14\frac{S_\ell}{m^2} + 6\frac{\delta m}{m}\frac{\Sigma_\ell}{m^2}\right) + x^2\left[14\eta - 3\left(\frac{S_{0,\ell}}{m^2}\right)^2\right]$$
$$+ x^{5/2}\left[-\frac{S_\ell}{m^2}(13+30\eta) - \frac{\delta m}{m}\frac{\Sigma_\ell}{m^2}(9+14\eta)\right] + x^3\left[\left(\frac{397}{2} - \frac{123}{16}\pi^2\right)\eta - 14\eta^2\right.$$

In the limit of a test-particle around a spinning BH, this expression reduces to:

$$\hat{C}_0 = 1 - 6x + 8\chi_2 x^{3/2} - 3\chi_2^2 x^2 - 4\chi_2 x^{5/2} + O(x^3)$$

ompare with the exact Kerr result:

$$^{\rm kerr}_{0} = 1 - 6X_0 + \chi_2\left(8X_0^{3/2} - 4X_0^{5/2}\right) + \chi_2^2\left(-3X_0^2 + 8X_0^3 - 10X_0^4/3\right) + O\left(\chi_2^3\right)$$

# int GSF/NR/PN constraints on NR/PN quantities

F conservative ISCO shift can also:

ix pseudo-4PN parameter in EOB effective metric A(r)

ix one parameter in the phenomenological IMR templates of Ajith et al

F combined with NR quasi-circular initial data [eg., Caudill et al '06]:

an constrain undetermined functions in PN circular orbit binding energy at N and 5PN orders [Blanchet et al '10]

$$\frac{^{\mathrm{PN}}(\Omega)}{\eta M} = -\frac{1}{2}x\left\{1 + x\left(-\frac{3}{4} - \frac{\eta}{12}\right) + x^2\left(-\frac{27}{8} + \frac{19}{8}\eta - \frac{\eta^2}{24}\right) + x^3\,[(\text{known 3PN coeff.})]\right.$$

$$x^4\left(-\frac{3969}{128} + \eta e_4(\eta) + \frac{448}{15}\eta\ln x\right) + x^5\left(-\frac{45927}{512} + \eta e_5(\eta) + \left[-\frac{4988}{35} - \frac{1904}{15}\eta\right]\eta\ln x\right)$$

SF + NR quasi-circular initial data ISCO/ICO allows first two terms in $e_4(\eta)$ pansion to be estimated. NR initial data calculations at several mass ratios or

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# ke-home messages:

While the EOB approach was designed to incorporate exact test-mass results…it does not (automatically) incorporate small-deviations from the test-mass limit.

- A warning on using EOB to compute 2nd-order GSF effects
- But EOB is amazingly adept at fitting NR (and EMRI) waveforms (so far…)

The standard PN equations have some interesting cards up their sleeves:

- Standard 3PN conservative ISCO shift best matches Barack-Sago conservative GSF calculation (if we ignore calibrated approaches)
- It also reproduces the test-mass ISCO (in Schwarzschild, and up to our current knowledge of PN spin terms, in Kerr); why?

GSF calculations (possibly in combination w/ NR calculations) can constrain parameters in PN or EOB functions; this will be useful in modeling both comparable mass and extreme-mass-ratio waveforms.