Title: Multiscale analysis of extreme mass ratio inspirals in Kerr: separatrix crossing

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Abstract: TBA

Modelling extreme mass ratio inspirals

$O(\epsilon^0)$: μ follows a bound geodesic of M's background Kerr spacetime:

- characterized by the constants of motion E, L_z, Q
- form a complete set of first integrals together with the rest mass μ
- can define generalizations of action-angle variables:
  $J_\lambda = (E/\mu,\ L_z/\mu,\ Q/\mu^2,\ M,\ a), \qquad q_\alpha = (q_t, q_r, q_\theta, q_\varphi)$
- tori in phase space,
  fundamental frequencies ω_r, ω_θ (t and φ symmetries)

correction: gravitational self-force effects

- ω_r and ω_θ slowly evolve
- occurence of resonances

Overview

- Theoretical Challenge: model the 2 body problem in the strong field regime with $\epsilon \equiv \mu/M \ll 1$ for a time $t_{\rm inspiral} \sim t_{\rm rr} \sim M/\epsilon$.
- Standard perturbation theory: fails after $t_{\rm dephase} \sim M/\sqrt{\epsilon} \ll t_{\rm rr}$.
- Resolve the difficulties by using two-timescale expansions.
  Two stages: (i) orbital motion ✓ (ii) metric.
- Essence of the method: posit an ansatz for the dependence on ϵ, justify by substitution into Einstein's Eqs.
- Based on the adiabatic evolution ($t_{\rm orb} \ll t_{\rm rr}$),
  augmented by local expansions near separatrices in the phase space (resonances, transition from inspiral to plunge)
- Self-consistent, systematic framework, valid for the entire inspiral

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Equations of motion

Action angle variables $q_\alpha = (q_t, q_r, q_\theta, q_\phi)$, $J_\lambda = (E, L_z, Q, M, a)$

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J_\lambda) + \epsilon g_\alpha^{(1)}(q_r, q_\theta, J_\lambda) + O(\epsilon^2), \\ \frac{dJ_\lambda}{d\tau} &= \epsilon G_\lambda^{(1)}(q_r, q_\theta, J_\nu) + \epsilon^2 G_\lambda^{(2)}(q_r, q_\theta, J_\nu) + O(\epsilon^3) \end{aligned}$$

introduce $\tilde{\tau} = \epsilon\tau$

Ansatz: asymptotic expansion of the solutions at fixed $\tilde{\tau}$:

$$J_\lambda(\tau, \epsilon) = \mathcal{J}_\lambda^{(0)}(\tilde{\tau}) + \sqrt{\epsilon}\mathcal{J}_\lambda^{(1/2)}(\tilde{\tau}) + \epsilon J^{(1)}(\psi_\alpha, \tilde{\tau}) + O(\epsilon^{3/2})$$

$$q_\alpha(\tau, \epsilon) = \psi_\alpha(\tilde{\tau}) + \sqrt{\epsilon} q_\alpha^{(1/2)}(\psi_\beta, \tilde{\tau}) + \epsilon q_\alpha^{(1)}(\psi_\beta, \tilde{\tau}) + O(\epsilon^{3/2})$$

$$\psi_\alpha(\tilde{\tau}, \epsilon) = \frac{1}{\epsilon}\psi_\alpha^{(0)}(\tilde{\tau}) + \frac{1}{\sqrt{\epsilon}}\psi_\alpha^{(1/2)}(\tilde{\tau}) + \psi_\alpha^{(1)}(\tilde{\tau}) + O(\epsilon^{1/2})$$

Adiabatic approximation

$$J_\lambda(\tau,\epsilon) = \mathcal{J}^{(0)}_\lambda(\tilde\tau) + \sqrt{\epsilon}\mathcal{J}^{(1/2)}_\lambda(\tilde\tau) + \epsilon J^{(1)}_\lambda(\psi_\alpha,\tilde\tau) + O(\epsilon^{3/2})$$

$$\psi_\alpha(\tilde\tau,\epsilon) = \frac{1}{\epsilon}\psi^{(0)}_\alpha(\tilde\tau) + \frac{1}{\sqrt{\epsilon}}\psi^{(1/2)}_\alpha(\tilde\tau) + \psi^{(1)}_\alpha(\tilde\tau) + \ldots$$

$$\frac{d\mathcal{J}^{(0)}_\lambda}{d\tilde\tau} = \langle G^{(1)}_\lambda\rangle[\mathcal{J}^{(0)}_\nu(\tilde\tau)], \qquad \psi^{(0)}_\alpha(\tilde\tau) = \int^{\tilde\tau} \omega_\alpha[\mathcal{J}^{(0)}_\lambda(\tilde\tau')]d\tilde\tau'.$$

Nonresonant tori are ergodic:

$$\langle G^{(1)}_\lambda\rangle \equiv \frac{1}{(2\pi)^2}\int_0^{2\pi} dq_r \int_0^{2\pi} dq_\theta\, G^{(1)}_\lambda(q_r, q_\theta, J_\lambda)$$

$$\text{Decompose}: \quad G_\lambda = \epsilon\Big[\underbrace{\langle G^{(1)}_{\lambda,\ \mathrm{diss}}\rangle}_{\text{known}} + \underbrace{\delta G^{(1)}_{\lambda,\ \mathrm{diss}}}_{\text{known}} + \underbrace{G^{(1)}_{\lambda,\ \mathrm{cons}}}_{\text{almost known}}\Big] + \underbrace{O(\epsilon^2)}_{\text{unknown}}$$

All the formulas to compute the adiabatic approximation are in hand

Post-1-Adiabatic corrections

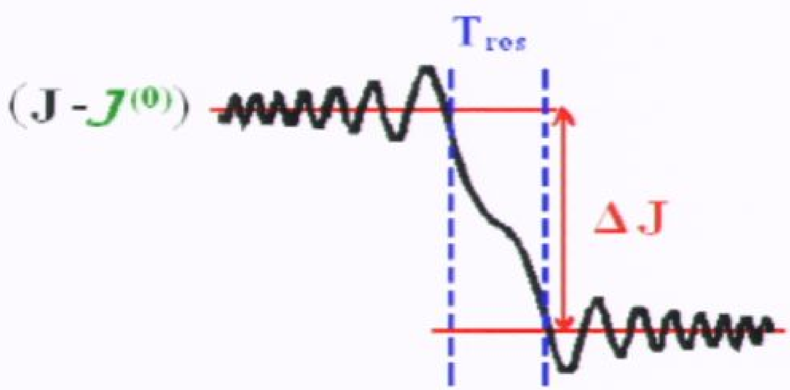
$$J_\lambda(\tau, \epsilon) = \mathcal{J}^{(0)}_\lambda(\tilde{\tau}) + \sqrt{\epsilon}\mathcal{J}^{(1/2)}_\lambda(\tilde{\tau}) + \epsilon J^{(1)}_\lambda(\psi_\alpha, \tilde{\tau}) + O(\epsilon^{3/2})$$

$$\psi_\alpha(\tilde{\tau}, \epsilon) = \frac{1}{\epsilon}\psi^{(0)}_\alpha(\tilde{\tau}) + \frac{1}{\sqrt{\epsilon}}\psi^{(1/2)}_\alpha(\tilde{\tau}) + \psi^{(1)}_\alpha(\tilde{\tau}) + O(\epsilon^{1/2})$$

- require: $g^{(1)}$, $G^{(1)}$ and $\langle G^{(2)} \rangle$
- give rise to phase errors $O(1)$ over an inspiral
- needed for data analysis

New Feature: Transient Resonances

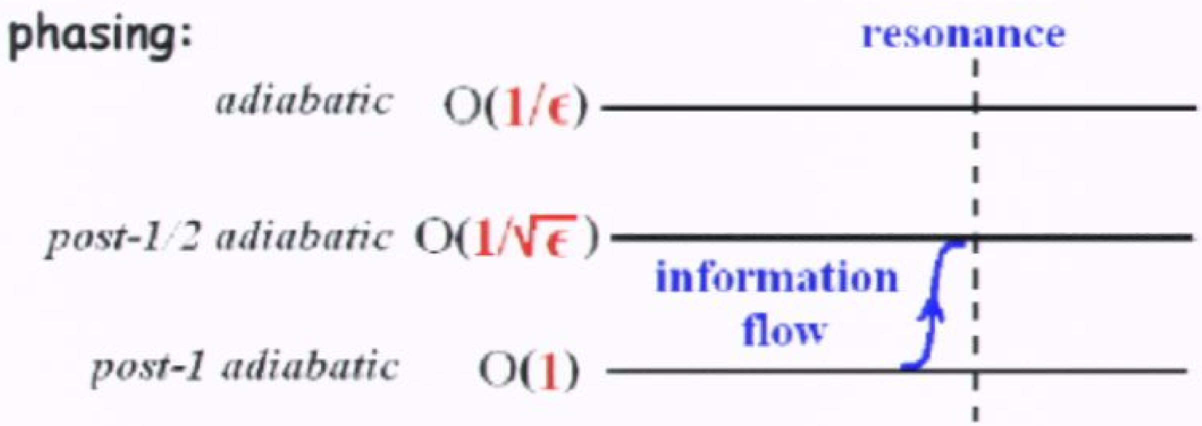
- **Occur** when $\omega_r/\omega_\theta = \mathbf{n}/\mathbf{m}$ with **n**, **m** small integers that have no common factors
- **Timescale:** intermediate between $t_{\rm rr}$ and $t_{\rm orb}$:
  $$T_{\rm res} \sim M/\sqrt{\mathbf{k}\epsilon}, \text{ where } \mathbf{k} = \mathbf{n} + \mathbf{m}$$
- **Effect:**



- **Analytic treatment**: Use matched expansions to compute $\Delta \mathbf{J}_\lambda \sim \sqrt{\epsilon}\, f(\mathbf{Q}_0)$, where $\mathbf{Q}_0 = \mathbf{m} q_r(t_{\rm res}) - \mathbf{n} q_\theta(t_{\rm res})$ is the phase entering the resonance to $O(1)$.

Properties of transient resonances

- The jumps ΔJ_λ give rise to phase errors $\propto\ 1/\sqrt{\epsilon}$ after further inspiral.
- Non-perturbative effects in v/c
- Cause increased sensitivity to initial conditions



- Jumps depend directly on $\delta G^{(1)}_{\lambda,\mathrm{diss}}$,
  but indirectly on $G^{(1)}_{\lambda,\mathrm{cons}}$, $g^{(1)}$, $G^{(2)}_{\lambda,\mathrm{diss}}$ (currently unknown).

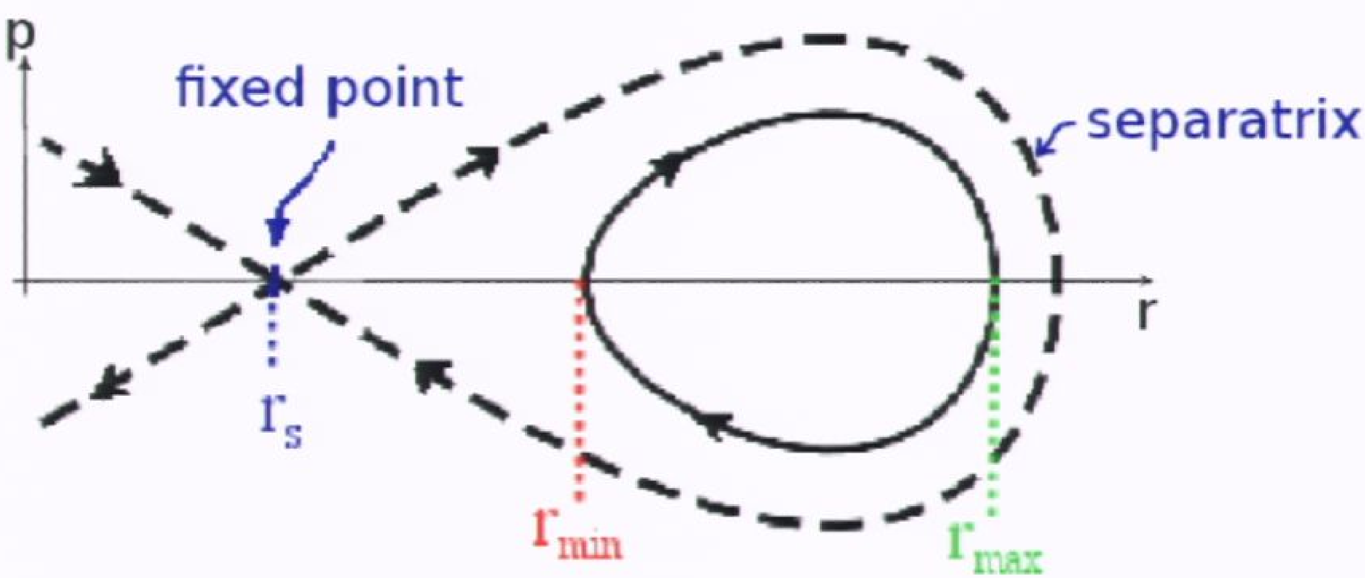
Treatment of the transition from inspiral to plunge

Radial momentum: $\Delta^2 p_r^2 = V_r(r, J_\lambda)$

V_r has an unstable fixed point: $\partial V_r / \partial r(r_s) = 0$

For generic orbits: $-\partial^2 V_r / \partial r^2(r_s) \gg O(\epsilon^{1/2})$

Features of the radial phase plane

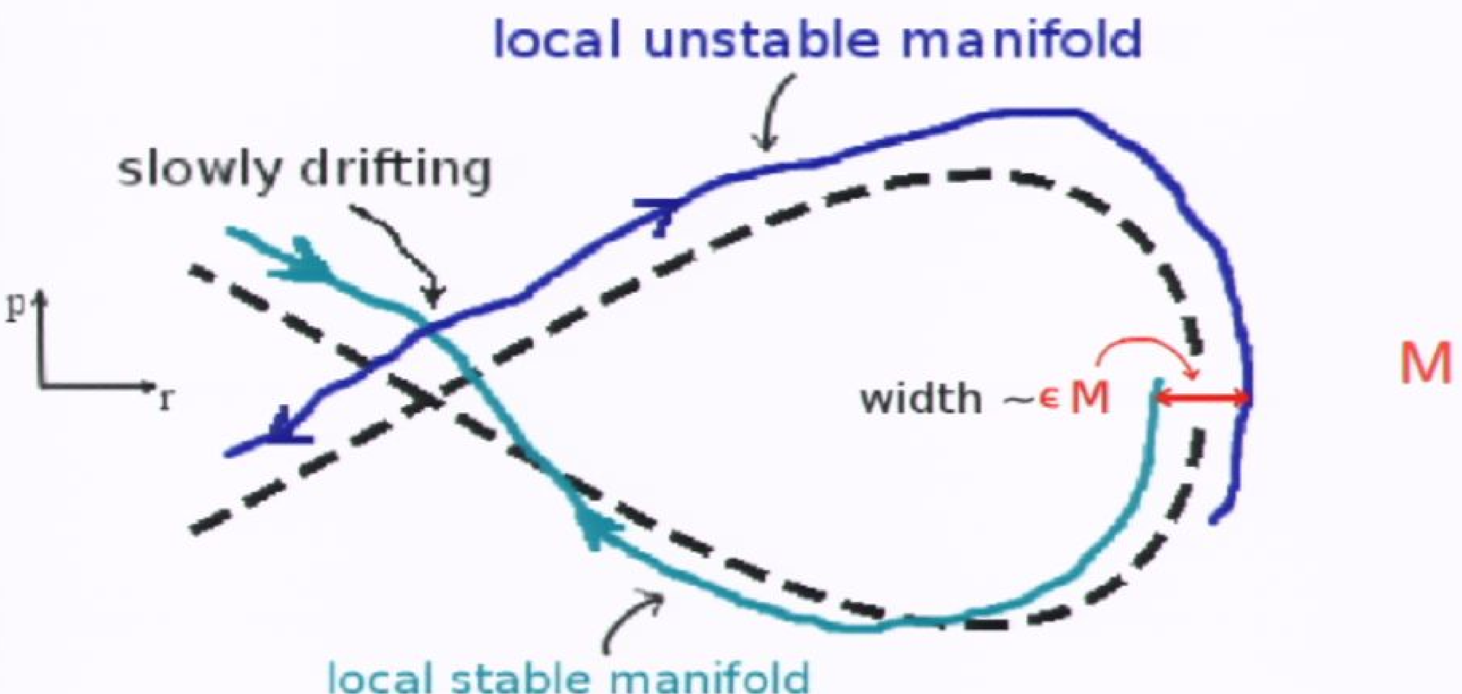


Qualitative features of the phase space

Unperturbed case:
the stable and unstable sets of the fixed pt coincide along the separatrix

Effect of perturbations: (with dissipation)

⋆ the invariant manifolds persist locally but the degeneracy breaks



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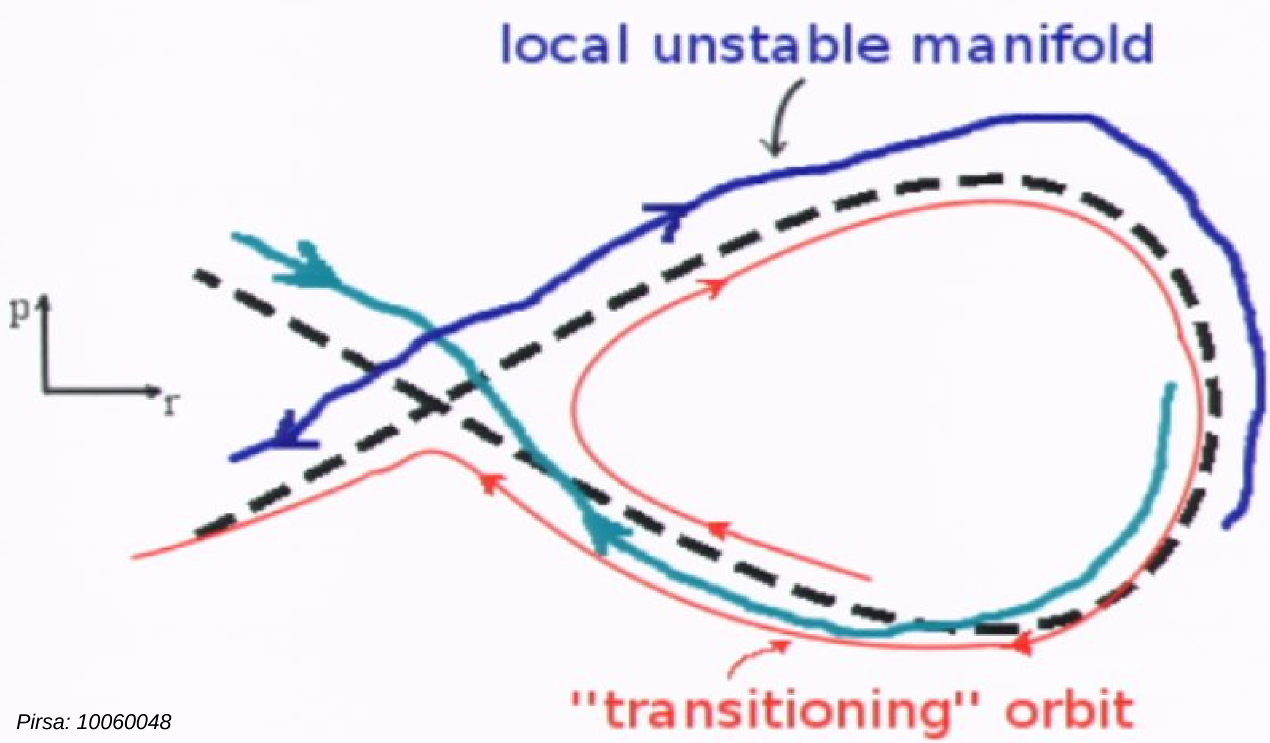

Modelling the transition region near the separatrix

Timescale: $\omega_\theta^{-1} \ll T_{\rm trans} \sim O(\log \epsilon) \ll t_{\rm rr}$

Analytic treatment: local analysis, matched asymptotic expansions

⋆ resulting jumps: $\Delta J_\lambda^{(\rm trans)} \sim A\, \epsilon \log(\epsilon\, M) b_0 + \epsilon f(b_0)$,

where b_0 is the phase $\psi_r(t_0)$ at the crossing to $O(1)$

The composite asymptotic expansion at fixed $\tilde{\tau}$ is:

$$J_\lambda(\tau, \epsilon) = \mathcal{J}_\lambda^{(0)}(\tilde{\tau}) + \sqrt{\epsilon}\mathcal{J}_\lambda^{(1/2)}(\tilde{\tau}) + \epsilon \ln \epsilon \mathcal{J}^{(\rm trans)}(\tilde{\tau}) + \epsilon J^{(1)}(\psi_\alpha, \tilde{\tau}) + O(\epsilon^{3/2})$$

for circular/small eccentricity orbits: larger shifts

$$T_{\rm trans} \sim O(\epsilon^{-1/5}),\ O(\epsilon^{-1/6})$$

$$\Delta J_\lambda^{(\rm trans)} \sim A_1\, \epsilon^{4/5} f_1,\ \ A_2\, \epsilon^{5/6} M^{5/6}\, \zeta(\tfrac{1}{6}, b_0)$$

Summary: Kerr inspirals via a two-timescale expansion

- ⋆ Developed a systematic 2-timescale approximation scheme that resolves the difficulties with standard perturbation theory for inspirals.

- ⋆ Passages near separatrices in the phase space (resonances, transition) make the orbit sensitive to changes in the initial data.

- ⋆ Transient resonances give rise to $O(\sqrt{M/\mu})$ phase corrections that depend on currently unknown pieces of the self force.

  Numerical indications: the net size of these corrections is ~ 10 cycles for $\epsilon \sim 10^{-6}$.

- ⋆ Analyzed the transition from inspiral to plunge via local expansions:

  for generic orbits, this gives rise to $O(\epsilon \log \epsilon)$ jumps in E, L_z, Q

  [for nearly circular orbits the shifts are $O(\epsilon^{5/6})$, $O(\epsilon^{4/5})$]

- ⋆ Regime of validity of the augmented expansion covers the entire inspiral

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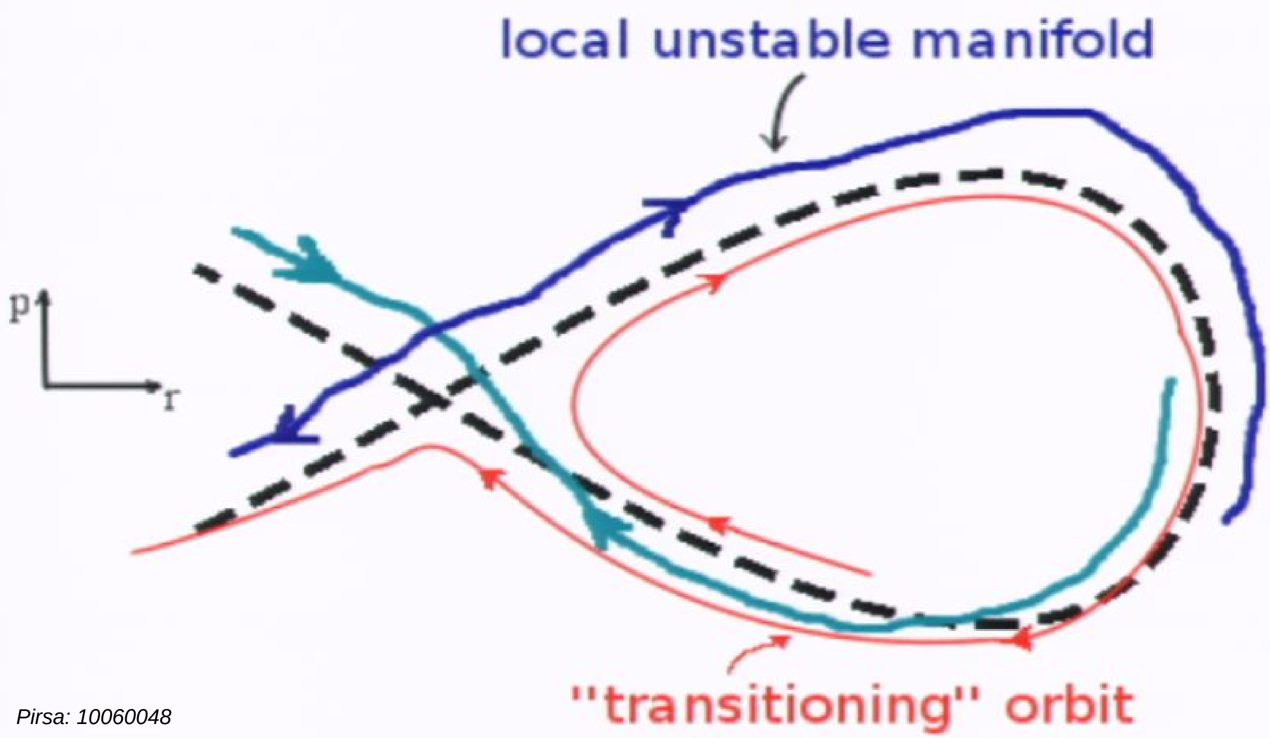

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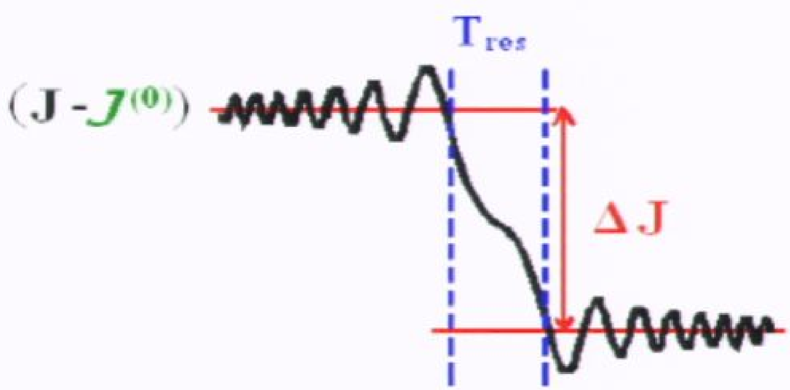

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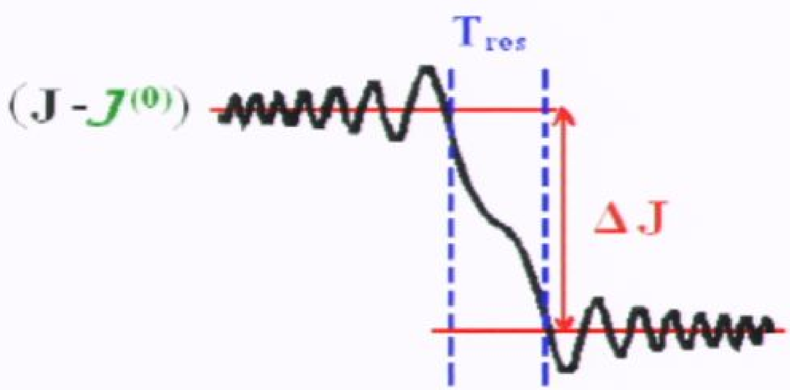

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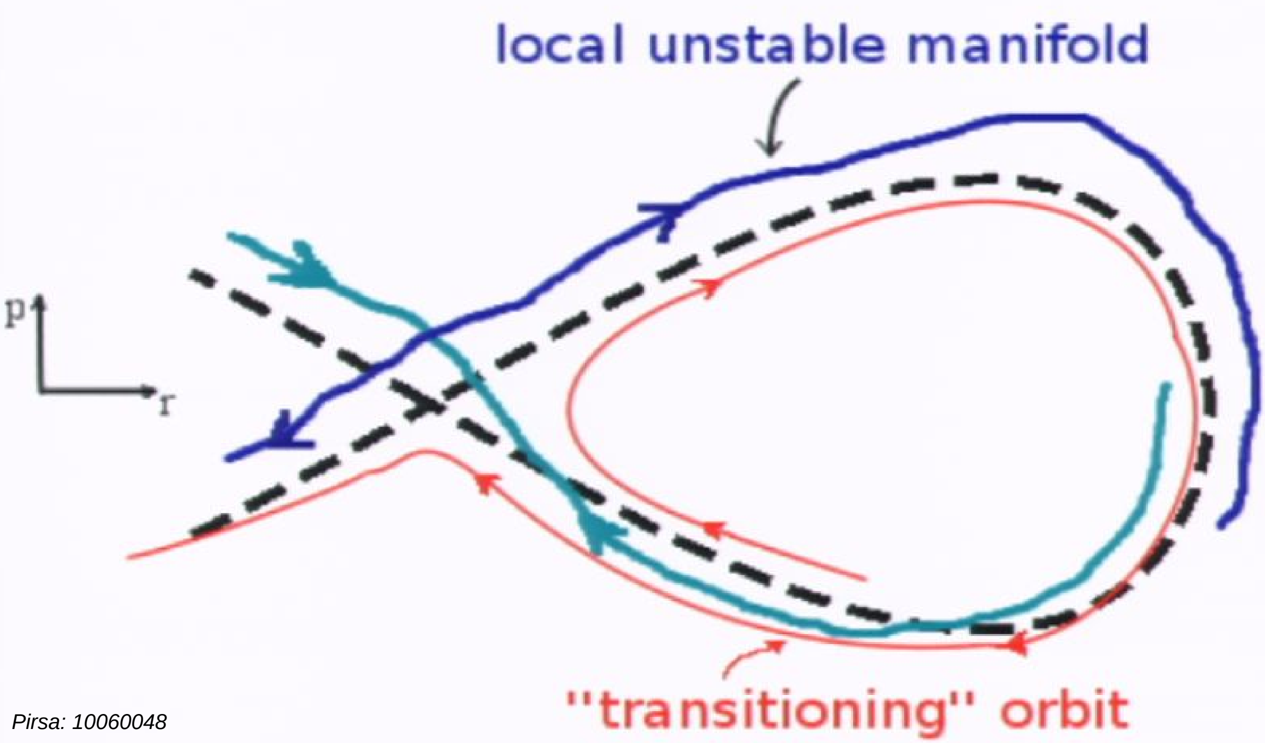

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