Title: A new integration method

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Abstract: TBA

Implicit integration method for black hole perturbations and back-action computation

Sofiane Aoudia and Alessandro D.A.M. Spallicci

[1] Max Planck Institut für Gravitationphysik, A. Einstein
Mühlenberg 1, 14476 Golm, Deutschland
E-mail: aoudia@aei.mpg.de

[2] Observatoire des Sciences de l'Univers, Université d'Orléans
LPC2E - CNRS, 3A Av. de la Recherche Scientifique, 45071 Orléans, France
E-mail: spallicci@cnrs-orleans.fr

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Background and Plan

Schwarzschild-Droste black hole perturbations: computation milestones

- Odd perturbations vacuum equation : 1957 Regge and Wheeler
- Odd and even perturbations equation with a source : 1970 Zerilli
  Simulations in frequency domain until...
- Finite difference scheme in time domain with the particle falling-in from finite distance : 1997 Lousto and Price
- Initial conditions parametrisation : 2002 Martel and Poisson

Drawbacks and Trade-off RW - LdD

- Analysis mostly developed in RW gauge
- Wave equation and wavefunction's gauge invariance - Moncrief 1974

...but

- Complexity in assessing the continuity of the perturbations at the particle position
- Self-force in the harmonic (Lorenz - de Donder) gauge *up today*

$\Rightarrow$ Alternative analysis in harmonic gauges (Barack and coworkers)

...but

- Unavailability of a wave equation

Gauge choice : *a priori* or *ad hoc* ? while waiting for the answer....

A New Integration Method: revert a weakness of the RW gauge into an underpinning feature

Plan of the talk

- Jump conditions for wavefunction and its derivatives: radial case
- New finite elements method of integration
- Waveforms at infinity and at the position of the particle
- Generalisation: Higher order; tackling generic orbits
- Conclusions

Jump conditions

- Two heuristic arguments (Lousto 2000 and Nakano 2009[1]) for even metric perturbations $\in C^0$ continuity class at the position of the particle, in RW gauge (radial case)
  Argument 1: integration over r of the Hamiltonian constraint, tt component of the Einstein equations
  Argument 2: structure of selected even perturbation equations
- LN derive the jump conditions on the wavefunction and its derivatives for $l = 2$ (from Zerilli equation)

Alternative based on formal solutions of Zerilli equation, for all modes
Determination of the jump conditions that the wavefunction and its derivatives have to satisfy, to guarantee the continuity of the perturbations at the position of the particle

Inverse relations for the perturbation K, H_2, H_1, as functions of the wavefunction and its derivatives:

$$K = \frac{6M^2 + 3M\lambda r + \lambda(\lambda+1)r^2}{r^2(\lambda r + 3M)}\Psi + \left(1 - \frac{2M}{r}\right)\Psi_{,r} - \frac{\kappa\, u^0 (r-2M)^2}{(\lambda+1)(\lambda r + 3M) r}\delta$$

$$H_2 = -\frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(\lambda+1)r^3}{r^2(\lambda r + 3M)^2}\Psi + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)}\Psi_{,r} + (r-2M)\Psi_{,rr}$$
$$\frac{\kappa u^0 (r-2M)(\lambda^2 r^2 + 2\lambda M r - 3Mr + 3M^2)}{r(\lambda+1)(\lambda r + 3M)^2}\delta - \frac{\kappa u^0 (r-2M)^2}{(\lambda+1)(\lambda r + 3M)}\delta'$$

$$H_1 = \frac{\lambda r^2 - 3M\lambda r - 3M^2}{(r-2M)(\lambda r + 3M)}\Psi_{,t} + r\Psi_{,tr} - \frac{\kappa\, u^0\, \dot{z}_u\, (\lambda r + M)}{(\lambda+1)(\lambda r + 3M)}\delta + \frac{\kappa\, u^0\, \dot{z}_u\, r(r-2M)}{(\lambda+1)(\lambda r + 3M)}\delta'$$

$\delta = \delta\,[r - z_u(t)]$ and $\delta' = \delta'\,[r - z_u(t)]$

Perturbations may be written as:

$$K = f_1(r)\Psi + f_2(r)\Psi_{,r} + f_3(r)\delta$$

$$H_2 = f_4(r)\Psi + f_5(r)\Psi_{,r} + f_6(r)\Psi_{,rr} + f_7(r)\delta + f_8(r)\delta'$$

From the visual inspection of the Zerilli wave equation:

$$\frac{d^2\Psi_l(t,r)}{dr^{*2}} - \frac{d^2\Psi_l(t,r)}{dt^2} - V_l(r)\Psi_l(t,r) = S_l(t,r)$$

$\Psi \in C^{-1}$ continuity class $\Rightarrow$

$$\Psi(t,r) = \Psi^+(t,r)\,\Theta_1 + \Psi^-(t,r)\,\Theta_2$$

$$\Psi_{,r} = \Psi^+_{,r}\Theta_1 + \Psi^-_{,r}\Theta_2 + \left(\Psi^+ - \Psi^-\right)\delta$$

$$\Psi_{,rr} = \Psi^+_{,rr}\Theta_1 + \Psi^-_{,rr}\Theta_2 + \left(\Psi^+_{,r} - \Psi^-_{,r}\right)\delta + \left(\Psi^+ - \Psi^-\right)_{z_u}\delta'$$

$$\Psi_{,t} = \Psi^+_{,t}\Theta_1 + \Psi^-_{,t}\Theta_2 - \left(\Psi^+ - \Psi^-\right)\dot{z}_u\delta$$

$$\Psi_{,tr} = \Psi^+_{,tr}\Theta_1 + \Psi^-_{,tr}\Theta_2 + \left(\Psi^+_{,t} - \Psi^-_{,t}\right)\delta - \left(\Psi^+ - \Psi^-\right)_{z_u}\dot{z}_u\delta'$$

$\Theta_1 = \Theta[r - z_u(t)]$, $\Theta_2 = \Theta[z_u(t) - r]$ Heaviside step distributions

One of the properties of the Dirac delta distribution, at the position of

Goal : conditions on Ψ and its derivatives cancel discontinuities in K, H_2 and H_1
$\Longrightarrow$ coefficients of Θ_1 must be = coefficients of Θ_2
coefficients of δ and δ' must vanish
Equivalent (thus consistent) conditions from K:

$$\left(\Psi^+ - \Psi^-\right)_{z_u} = -\frac{f_3}{f_2} \qquad \left(\Psi^+_{,r} - \Psi^-_{,r}\right)_{z_u} = \frac{f_1 f_3}{f_2^2}$$

from $H_{0,2}$:

$$\left(\Psi^+ - \Psi^-\right)_{z_u} = -\frac{f_8}{f_6} \qquad \left(\Psi^+_{,r} - \Psi^-_{,r}\right)_{z_u} = \frac{1}{f_6}\left(\frac{f_5 f_8}{f_6} - f_7 + f_{8,r} - \frac{f_{6,r} f_8}{f_6}\right)$$

from H_1:

$$\left(\Psi^+ - \Psi^-\right)_{z_u} = \frac{f_{12}}{\dot{z}_u f_{10}} \quad .$$

Other conditions coming from $H_{0,2}$ and H_1 are:

$$\left(\Psi^+_{,rr} - \Psi^-_{,rr}\right)_{z_u} = -\frac{f_4\left(\Psi^+ - \Psi^-\right) + f_5\left(\Psi^+_{,r} - \Psi^-_{,r}\right)}{f_6}$$

$$\left(\Psi^+_{,t} - \Psi^-_{,t}\right)_{z_u} = \left.\frac{d\left(\Psi^+ - \Psi^-\right)}{dt}\right|_{z_u} - \left(\Psi^+_{,r} - \Psi^-_{,r}\right)\dot{z}_u = \frac{\left(f_9 - f_{10,r}\right)\dot{z}_u\left(\Psi^+ - \Psi^-\right) - f_{11} + f_{12,r}}{f_{10}}$$

$$\left(\Psi^+_{,tr} - \Psi^-_{,tr}\right)_{z_u} = -\frac{f_9\left(\Psi^+_{,t} - \Psi^-_{,t}\right)}{f_{10}} \quad .$$

In the explicit form, the jump conditions become:

$$\left(\Psi^{+} - \Psi^{-}\right)_{z_u} = \frac{\kappa E z_u}{(\lambda+1)(3M+\lambda z_u)}$$

$$\left(\Psi^{+}_{,r} - \Psi^{-}_{,r}\right)_{z_u} = \frac{\kappa E \left[6M^2 + 3M\lambda z_u + \lambda(\lambda+1)z_u^2\right]}{(\lambda+1)(2M-z_u)(3M+\lambda z_u)^2}$$

$$\left(\Psi^{+}_{,rr} - \Psi^{-}_{,rr}\right)_{z_u} = -\frac{\kappa E \left[3M^3(5\lambda-3) + 6M^2\lambda(\lambda-3)z_u + 3M\lambda^2(\lambda-1)z_u^2 - 2\lambda^2(\lambda+1)z_u^3\right]}{(\lambda+1)(2M-z_u)^2(3M+\lambda z_u)^3}$$

$$\left(\Psi^{+}_{,t} - \Psi^{-}_{,t}\right)_{z_u} = -\frac{\kappa E z_u \dot{z}_u}{(2M-z_u)(3M+\lambda z_u)}$$

$$\left(\Psi^{+}_{,tr} - \Psi^{-}_{,tr}\right)_{z_u} = \frac{\kappa E \left(3M^2 + 3M\lambda z_u - \lambda z_u^2\right)\dot{z}_u}{(2M-z_u)^2(3M+\lambda z_u)^2} \quad .$$

These conditions ensure that all perturbations are C° at the particle position

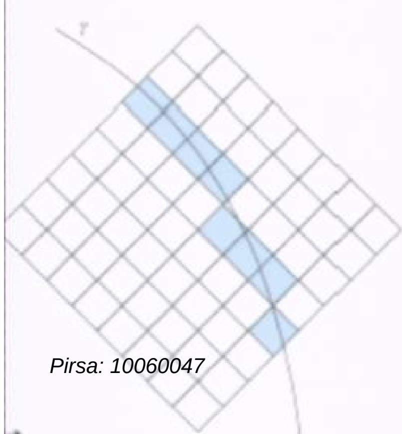
New method

What doesn't change:
Integration domain discretised by 2-dimensional uniform mesh (t, r_*)
e.g. $-450 < r_*/2M < 1500$ $0 < t < t_{end}$, t_{end} larger than falling time
Cell area $2h^2$, $\sqrt{2}h$ = dimension of an edge

Initial data (Lousto and Price 1997, Martel and Poisson 2002)

The integration method considers the cells of two groups:

cells never crossed by the world-line (Lousto and Price 1997)

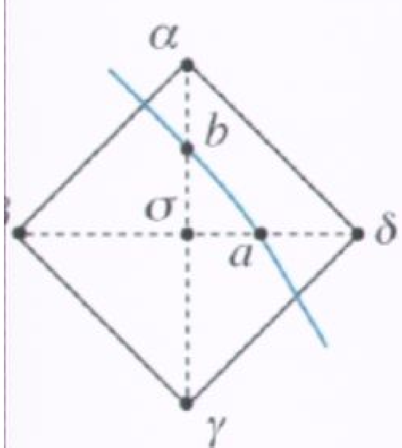


What does change:
For the cells crossed by a particle, 4 sub-cases/trajectories
Jump conditions: from t, r to t, r_* variables
1^{st} order Taylor expansion but 2^{nd} has been developed (SA AS)

Case 1: particle crosses the β - δ line at a and γ - α line at b

Definitions: $\epsilon_a = \min\{a\delta, a\beta\}$, $\epsilon_b = \min\{b\alpha, b\gamma\}$; $\Psi^{\pm}_{b} = \Psi^{\pm}(t_b, r^*_b)$, $\Psi^{\pm}_{a} = \Psi^{\pm}(t_a, r^*_a)$

6 numerical equations



$$\Psi^+_\alpha = \Psi^+(t_b + \epsilon_b, r^*_b) = \Psi^+_b + \epsilon_b \, \Psi^+_{,t}\big|_b \quad (1)$$

$$\Psi^-_\sigma = \Psi^-(t_b - (h - \epsilon_b), r^*_b) = \Psi^-_b - (h - \epsilon_b) \, \Psi^-_{,t}\big|_b \quad (2)$$

$$\Psi^-_\gamma = \Psi^-(t_b - 2h + \epsilon_b, r^*_b) = \Psi^-(t_\sigma - h, r^*_b) = \Psi^-_\sigma - h \, \Psi^-_{,t}\big|_\sigma \quad (3)$$

$$\Psi^+_\delta = \Psi^+(t_a, r^*_a + \epsilon_a) = \Psi^+_a + \epsilon_a \, \Psi^+_{,r^*}\big|_a \quad (4)$$

$$\Psi^-_\sigma = \Psi^-(t_a, r^*_a - (h - \epsilon_a)) = \Psi^-_a - (h - \epsilon_a) \, \Psi^-_{,r^*}\big|_a \quad (5)$$

$$\Psi^-_\beta = \Psi^-(t_a, r^*_a - 2h + \epsilon_a) = \Psi^-(t_\sigma, r^*_\sigma - h) = \Psi^-_\sigma - h \, \Psi^-_{,r^*}\big|_\sigma \quad (6)$$

6 analytic expressions:

$$\left(\Psi^+ - \Psi^-\right)_a = [\Psi]_a \qquad \left(\Psi^+_{,r^*} - \Psi^-_{,r^*}\right)_a = [\Psi_{,r^*}]_a \qquad \left(\Psi^+_{,t} - \Psi^-_{,t}\right)_a = [\Psi_{,r^*}]_a$$

Our aim: determination of the value of Ψ^+_α, knowing those of Ψ^-_β, Ψ^-_γ, Ψ^+_δ, ϵ_a, ϵ_b, $[\Psi]_{a,b}$, $[\Psi_{,r}]_{a,b}$ and $[\Psi_{,t}]_{a,b}$

Algebraic manipulation. Subtracting (1) and (2):

$$\Psi^+_\alpha = \Psi^-_\sigma + [\Psi]_b + [\Psi_{,t}]_b + h\,\Psi^-_{,t}\Big|_b \tag{7}$$

Subtracting (4) and (5):

$$\Psi^+_\delta = \Psi^-_\sigma + [\Psi]_a + [\Psi_{,r^*}]_a + h\,\Psi^-_{,r^*}\Big|_a \tag{8}$$

Summing (3) and (7), (6) and (8), and combining the results, it provides:

$$\Psi^+_\alpha = \Psi^-_\beta - \Psi^-_\gamma + \Psi^+_\delta - [\Psi]_a + [\Psi]_b - \epsilon_a\,[\Psi_{,r^*}]_a + \epsilon_b\,[\Psi_{,t}]_b + \mathcal{O}(h^2)$$

- No need of direct integration of the singular source
- Top cell value depending upon analytic expressions (and other cell's corners)

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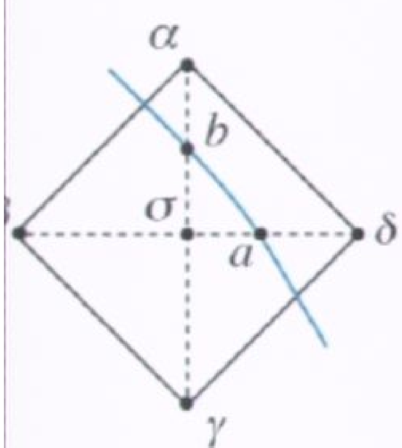

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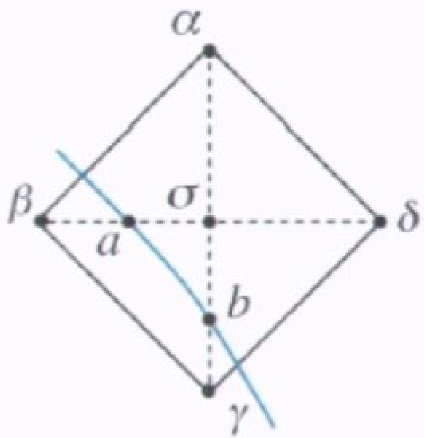
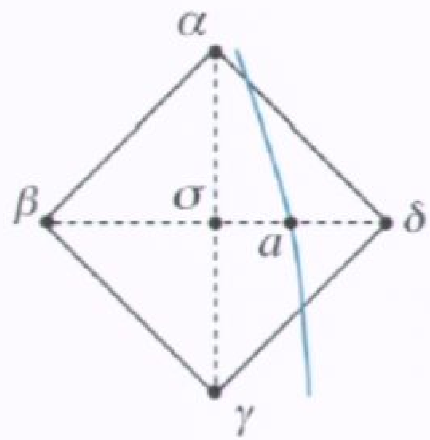
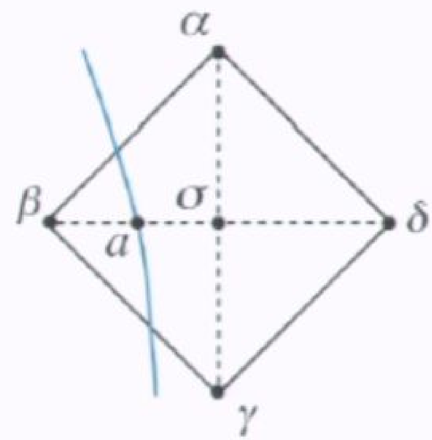
Similar relations for the other three cases







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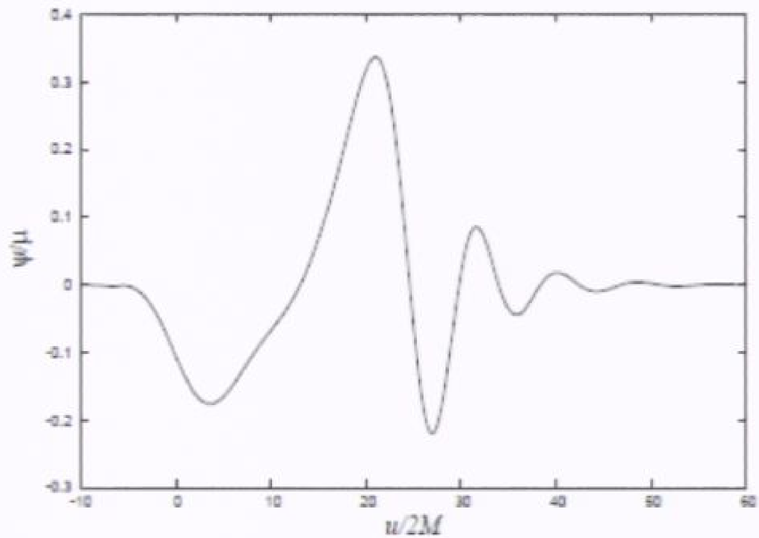
ψ/μ
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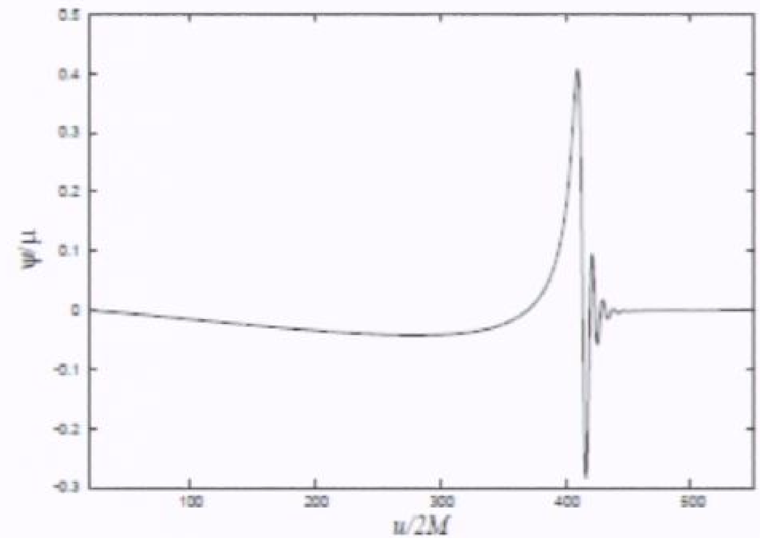
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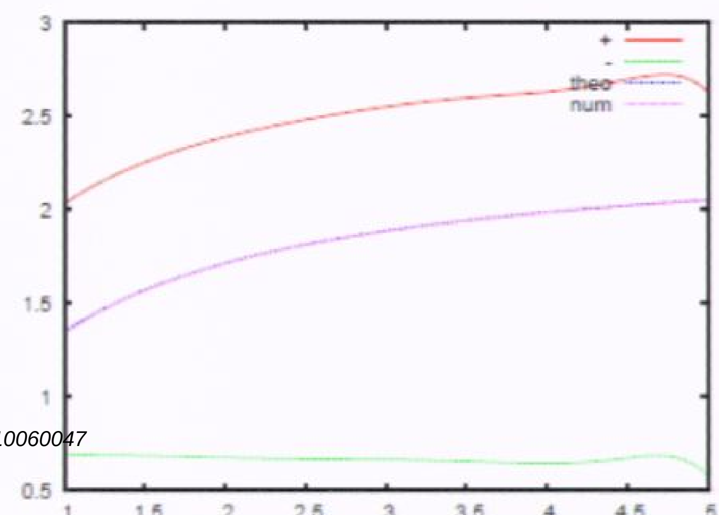
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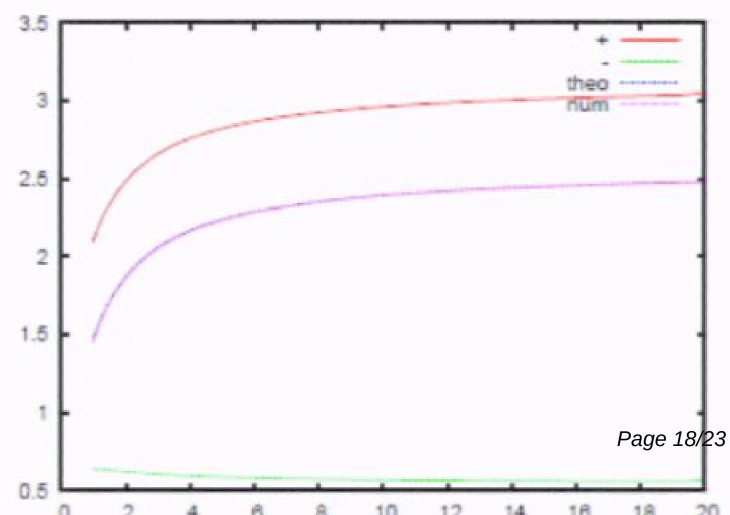
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Generalisation

Higher order

A second order scheme has been developed. It requires second order derivatives. Conceptually alike.

Generic orbits

Computation of the jump conditions from the Regge-Wheeler or Zerilli (modified by $r^* \to r$) equations, just by inserting:

$$\Psi_{RW,Z}(t,r) = \Psi^+(t,r)\Theta_1 + \Psi^-(t,r)\Theta_2 = \Psi^+\Theta_1 + \Psi^-(1-\Theta_1) =$$

$$= (\Psi^+ - \Psi^-)\Theta_1 + \Psi^-$$

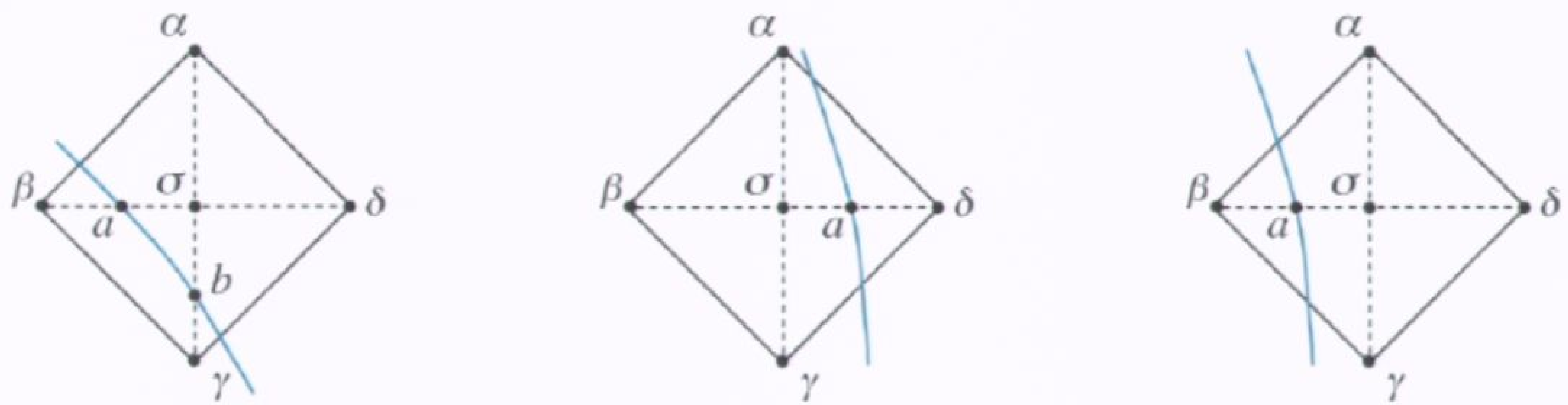
Conditions on Ψ and derivatives (up to second) are obtained

There are 8 (not just 4 as in the radial) cases for the particle trajectory inside the cell (also towards larger r)

Conclusions

- The method avoids the direct and explicit integration both of the wave equation and of the source term with the associated singularities. The information on the wave equation is implicitly given by the jump conditions.
- Conversely, for the cells not crossed by the particle world-line, the integrating method retains the 'traditional' approach.
- For the computation of the back-action, this method ensures a well behaved wavefunction at the particle position, since the approach is governed by the theoretical values of the jump conditions at the particle position.

Similar relations for the other three cases



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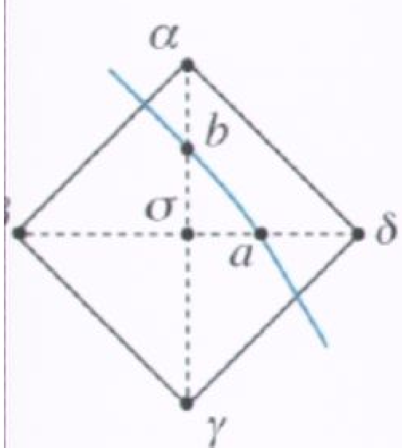

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