Title: A New Derivation of the Effective Source for Self-Force Calculations

Date: Jun 21, 2010 09:50 AM

URL: http://pirsa.org/10060045

Abstract: TBA

A New Derivation of The Effective Source for Self-Force Calculations

Barry Wardell
Albert Einstein Institute

Puncture Method

- Barack & Golbourn '07, Detweiler & Vega '07
- Subtract 'puncture' field at the level of the wave equation.

$$\Box\Phi = \Box\Phi_{\mathrm{S}} + \Box\Phi_{\mathrm{R}}$$

- Solve for residual field with an effective source which is regular everywhere

$$\Box\Phi_{\mathrm{R}} = S_{\mathrm{eff}} \qquad S_{\mathrm{eff}} \equiv \int_\gamma \delta(x, z(\tau'))d\tau' - \Box\Phi_{\mathrm{S}}$$

Effective Source

- VD derived an effective source from a representation of the singular field in terms of THZ coordinates.
- Valid for circular geodesic motion in Schwarzschild
- $\mathcal{C}_0$ - leads to second order convergence
- Smoother source would give better convergence - $\mathcal{C}^\infty$ source would be exponentially convergent
- Ideally want a smooth effective source for generic orbits in Kerr

Effective Source

- BG used an expansion in m-modes.
- Lower order approximation to the effective source corresponding to the $\mathcal{O}(s^{-1})$ contribution to the singular field
- Sufficient to give field, but not self-force
- Higher order approximations required for self-force and to improve convergence

Singular Field

- Need

$$S_{\text{eff}} \equiv \int_\gamma \delta(x, z(\tau'))d\tau' - \Box\Phi_{\text{S}}$$

- Given the Detweiler-Whiting Green function,

$$G_{\text{DW}}(x,x') = \frac{1}{2}\left\{U(x,x')\delta\left(\sigma(x,x')\right) + V(x,x')\theta\left(\sigma(x,x')\right)\right\}$$

Φ_S is given by

$$\Phi_{\text{S}}(x) = q\int_\gamma G_{\text{DW}}(x,x')d\tau'$$

Singular Field

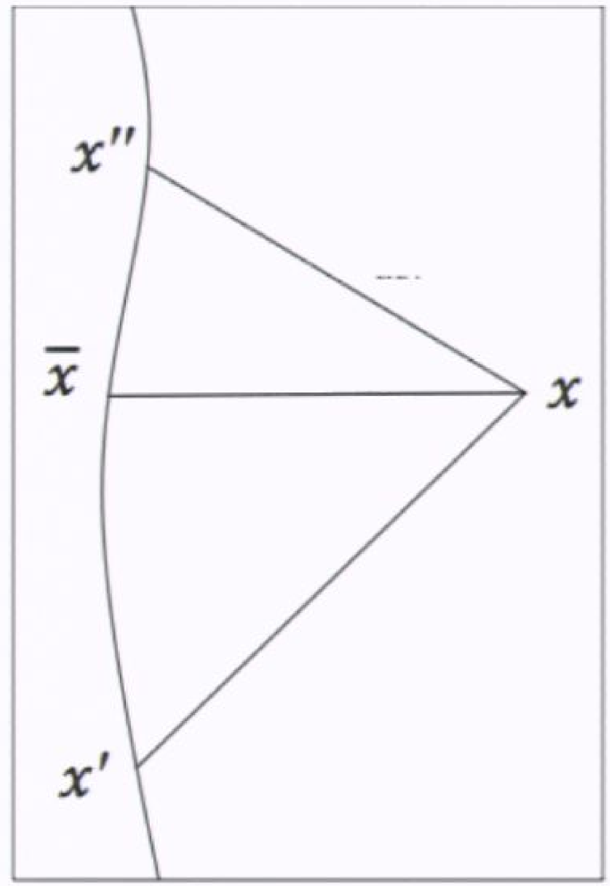
- So

$$\Phi_S(x) = \frac{q}{2}\left[\frac{U(x,x')}{r_{\rm ret}} + \frac{U(x,x'')}{r_{\rm adv}} - \int_{x'}^{x''} V(x,z(\tau'))d\tau'\right]$$

- For x close to the world-line,

$$\begin{aligned} U(x,x') = U(x,x'') &= 1 + \mathcal{O}(\delta x^4) \\ V(x,z(\tau')) &= \mathcal{O}(\delta x^4) \end{aligned}$$

and we expand $r_{\rm ret}$ and $r_{\rm adv}$ in the geodesic distance between x and the world-line (Haas and Poisson) to get



$$\Phi_S = \frac{1}{s} + \frac{\bar{r}^2 - s^2}{6s^3} R_{\bar{a}\bar{b}\bar{c}\bar{d}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} + \frac{(\bar{r}^2 - 3s^2)\bar{r} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} u^{\bar{e}} \sigma^{\bar{b}} \sigma^{\bar{d}} - (\bar{r}^2 - s^2)\bar{r} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} \sigma^{\bar{e}}}{24 s^3} + \cdots$$

Singular Field

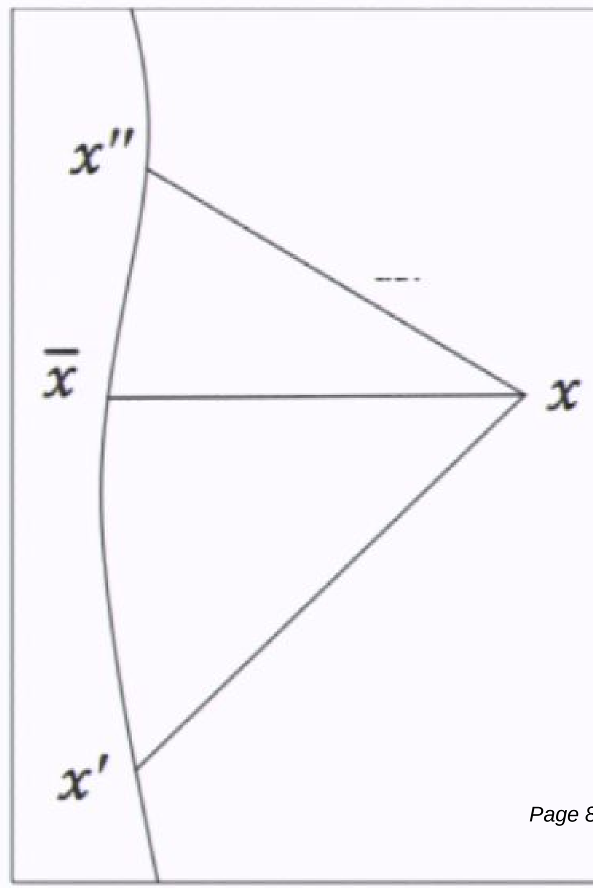
$$\Phi_S = \frac{1}{s} + \frac{\bar{r}^2 - s^2}{6s^3} R_{\bar{a}\bar{b}\bar{c}\bar{d}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} + \frac{(\bar{r}^2 - 3s^2)\bar{r} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} u^{\bar{e}} \sigma^{\bar{b}} \sigma^{\bar{d}} - (\bar{r}^2 - s^2)\bar{r} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} \sigma^{\bar{e}}}{24s^3} + \cdots$$

where

$$s^2 = (g^{\bar{a}\bar{b}} + u^{\bar{a}} u^{\bar{b}}) \sigma_{\bar{a}} \sigma_{\bar{b}}$$

$$\bar{r} = \sigma_{\bar{a}} u^{\bar{a}}$$

and $\sigma(x, \bar{x})$ is the Synge world-function.



Effective Source

- When it comes to evaluating these expressions in a particular spacetime, we only need to calculate $\sigma^{\bar{a}}(x,\bar{x})$, and components of the Riemann tensor at $\bar{x}$.
- Proceed by expanding about $\bar{x}$ in the coordinate separation of the points.
- Want to delay this expansion as long as possible because it leads to large expressions.

Effective Source

- Easiest (but messiest) approach is to expand the singular field and compute d'Alembertian by taking partial derivatives. Leads to large expressions and delicate cancellations near the world-line.
- More work (but tidier) to compute the covariant d'Alembertian of the singular field. Cleanly see the cancellation of divergent terms up to the order determined by the approximation of the singular field.

COVARIANT

$$\Phi_{\rm S} = \frac{1}{s}$$

$$\begin{aligned} \Box\Phi_{\rm S} &= \frac{3(\bar{r}^2+s^2)}{s^5} + \frac{3\bar{r}^2-s^2}{s^5}\nabla_a\bar{r}\nabla^a\bar{r} - \frac{\sigma_a{}^a}{s^3} - \frac{\bar{r}\Box\bar{r}}{s^3} \\ &\approx \frac{3(\bar{r}^2+s^2)}{s^5} + \frac{3\bar{r}^2-s^2}{s^5}\left(-1+\frac{1}{3}R_{\bar{a}\bar{b}\bar{c}\bar{d}}u^{\bar{a}}u^{\bar{c}}\sigma^{\bar{b}}\sigma^{\bar{d}}\right) - \frac{4}{s^3} \\ &= \frac{3\bar{r}^2-s^2}{3s^5}R_{\bar{a}\bar{b}\bar{c}\bar{d}}u^{\bar{a}}u^{\bar{c}}\sigma^{\bar{b}}\sigma^{\bar{d}} \\ &= \mathcal{O}(s^{-1}) \end{aligned}$$

Cancellation Structure

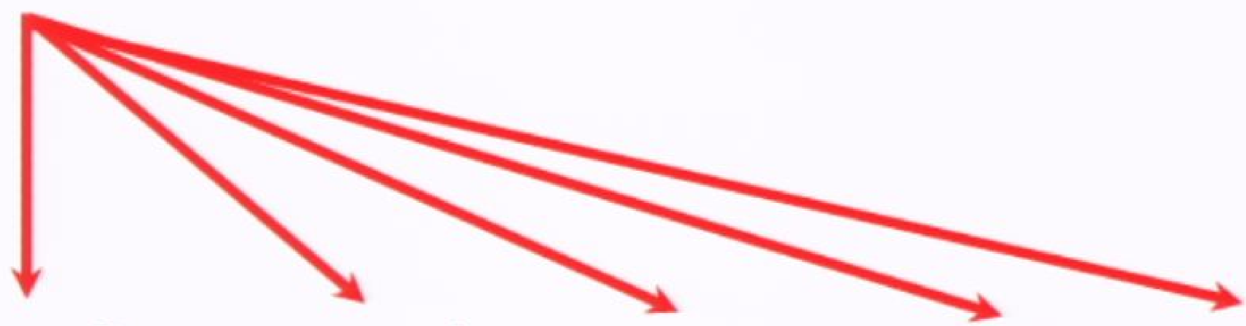
$$\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-1}) + \mathcal{O}(s) + \mathcal{O}(s^2)$$

$$\Box\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-3}) + \mathcal{O}(s^{-1}) + \mathcal{O}(1) + \mathcal{O}(s)$$

Cancellation Structure

$$A_\mu, B_\mu$$

$$\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-1}) + \mathcal{O}(s) + \mathcal{O}(s^2)$$



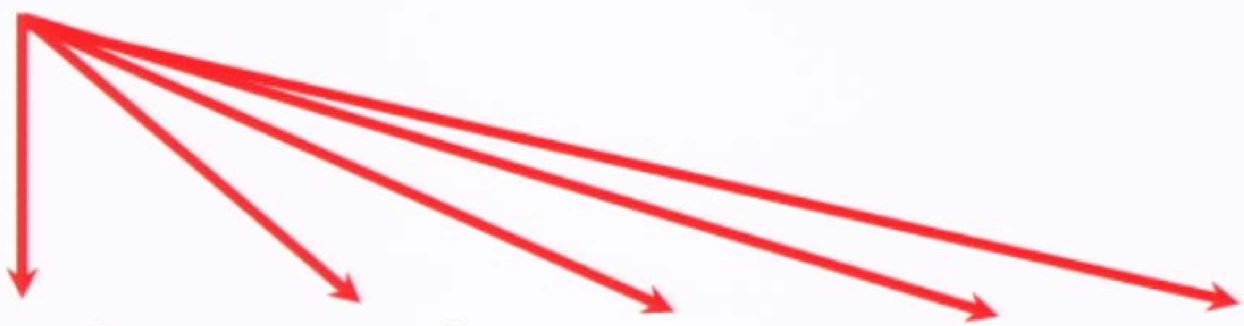
$$\Box\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-3}) + \mathcal{O}(s^{-1}) + \mathcal{O}(1) + \mathcal{O}(s)$$

$$= \frac{3\bar{r}^2 - s^2}{3s^5} R_{\bar{a}\bar{b}\bar{c}\bar{d}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}}$$

Cancellation Structure

$$\overset{\color{red}{A_\mu, B_\mu}}{} \qquad \overset{\color{blue}{C_\mu}}{}$$

$$\Phi_{\mathrm{S}} \approx \; \mathcal{O}(s^{-1}) \; + \; \mathcal{O}(s) \; + \; \mathcal{O}(s^2)$$



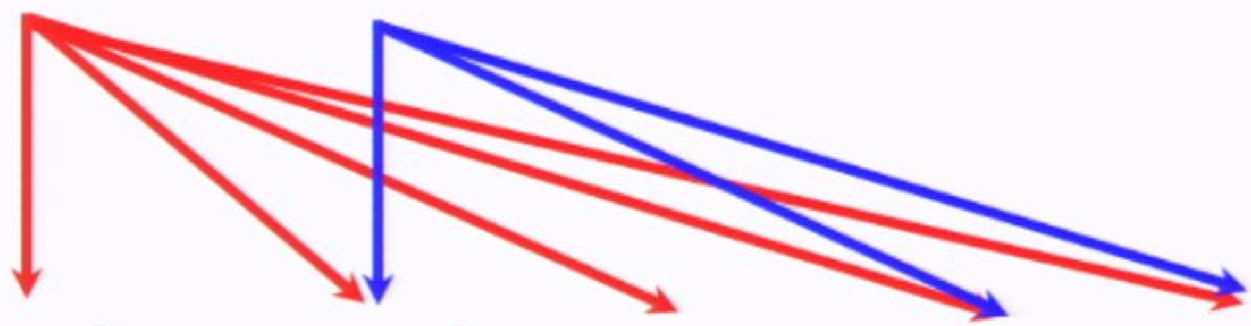
$$\Box\Phi_{\mathrm{S}} \approx \; \mathcal{O}(s^{-3}) + \mathcal{O}(s^{-1}) + \; \mathcal{O}(1) \; + \; \mathcal{O}(s)$$

$$= \quad -\frac{3\bar{r}^2 - s^2}{6s^5} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} \sigma^{\bar{e}}$$

Cancellation Structure

$$\overset{\color{red}{A_\mu, B_\mu}}{} \qquad \overset{\color{blue}{C_\mu}}{}$$

$$\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-1}) + \mathcal{O}(s) + \mathcal{O}(s^2)$$



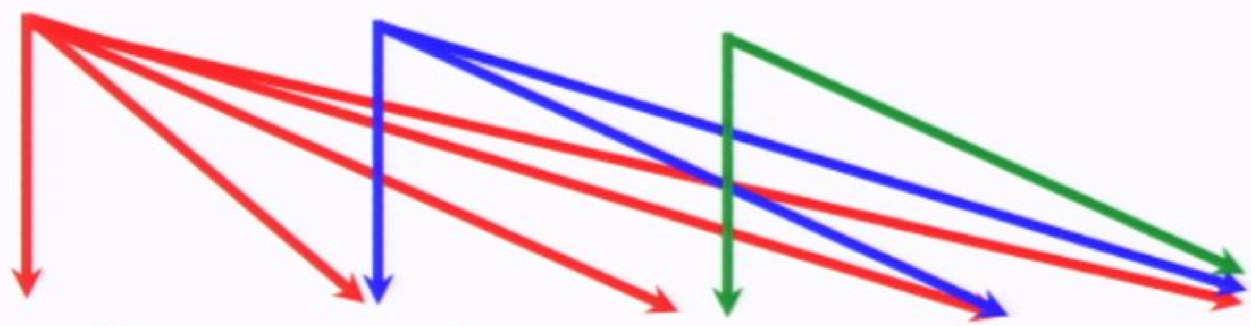
$$\Box\Phi_{\mathrm{S}} \approx \mathcal{O}(s^{-3}) + \mathcal{O}(s^{-1}) + \mathcal{O}(1) + \mathcal{O}(s)$$

$$= -\frac{3\bar{r}^2 - s^2}{6s^5} R_{\bar{a}\bar{b}\bar{c}\bar{d};\bar{e}} u^{\bar{a}} u^{\bar{c}} \sigma^{\bar{b}} \sigma^{\bar{d}} \sigma^{\bar{e}}$$

Cancellation Structure

$$\qquad\quad A_\mu, B_\mu \qquad\quad C_\mu \qquad\quad D_\mu$$

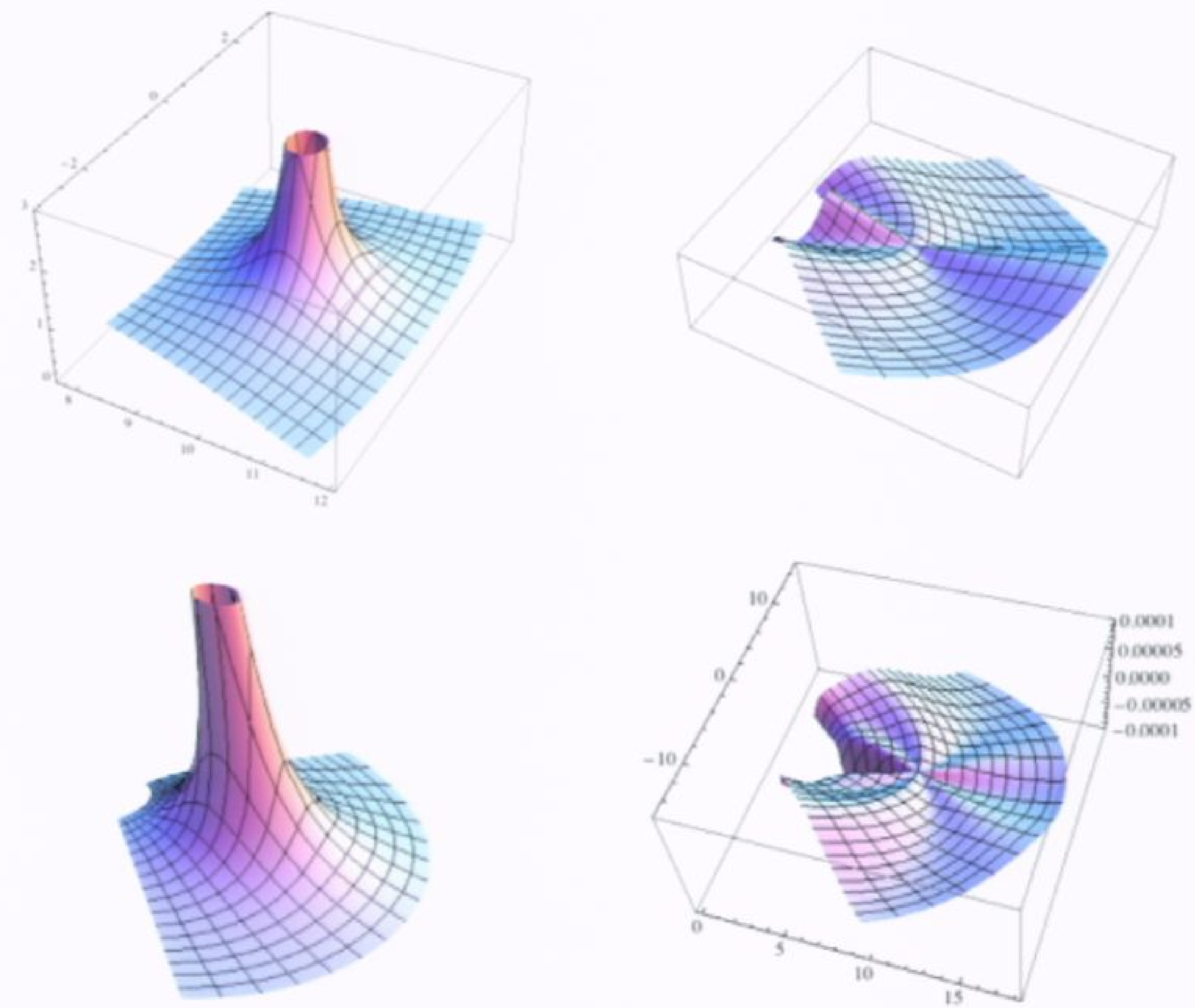
$$\Phi_{\rm S} \approx \mathcal{O}(s^{-1}) + \mathcal{O}(s) + \mathcal{O}(s^2)$$



$$\Box\Phi_{\rm S} \approx \mathcal{O}(s^{-3}) + \mathcal{O}(s^{-1}) + \mathcal{O}(1) + \mathcal{O}(s)$$

$$= \frac{1}{45 s^3} R^{\bar\alpha}{}_{\bar a \bar\beta \bar b} R^{\bar\beta}{}_{\bar c \bar\alpha \bar d} \sigma^{\bar a}\sigma^{\bar b}\sigma^{\bar c}\sigma^{\bar d} + \cdots$$

Results



Gravitational Case

- Analogous to scalar case, the gravitational singular Green function is

$$G_{\mathrm{DW}}{}^{ab}{}_{a'b'}(x,x') = \frac{1}{2}\left\{U(x,x')^{ab}{}_{a'b'}\delta\left(\sigma(x,x')\right) + V(x,x')^{ab}{}_{a'b'}\theta\left(\sigma(x,x')\right)\right\}$$

In Lorentz gauge, the trace-reversed singular field is

$$\begin{aligned}\gamma_{\mathrm{S}}^{ab} &= \int_\gamma G_{\mathrm{DW}}{}^{ab}{}_{a'b'}(x,z(\tau'))u^{a'}u^{b'}d\tau' \\ &\approx 2Mu^{(a}u^{b)}\Phi_{\mathrm{S}}\end{aligned}$$

So, compared to the scalar case, we additionally

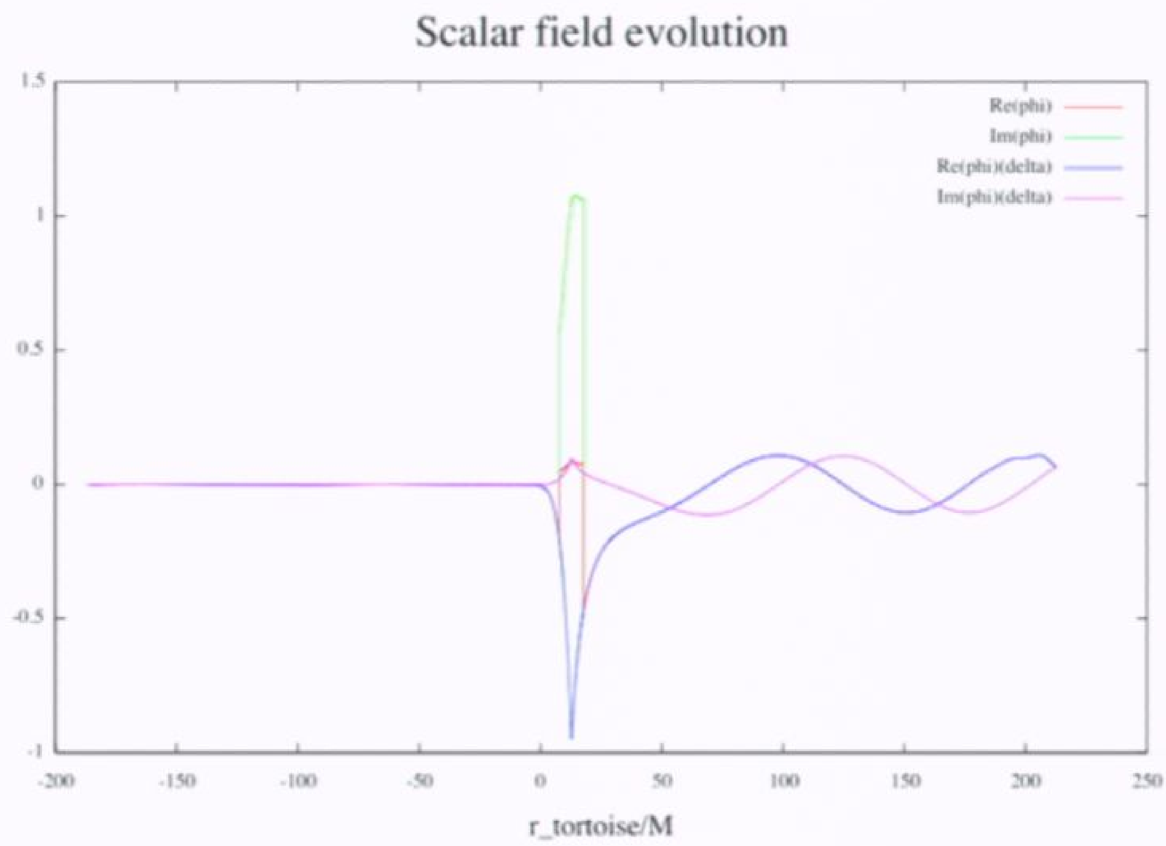
$$u^a = g^a{}_{\bar{a}}u^{\bar{a}}$$

Applications 1

- 2+1 m-mode (Barack and Dolan) - cf. Sam's talk yesterday
- World-tube approach
- Computed coordinate expansion of singular field for tube boundary condition, symbolically computed d'Alembertion of this for the effective source within the tube.

Applications 2

- 1+1 *l*,*m*-mode. Adaptation of the Sopuerta/Canizares/Jaramillo code to include effecti\



Applications 3

- 3+1 scalar wave equation
- Using Cactus Computational Toolkit and Llama multipatch finite difference code - Kerr-Schild
- Also without Llama for Schwarzschild
- Initially scalar field, circular orbits in Schwarzschild
- Code verified using Vega & Detweiler effsource code
- Testing with the covariant + cordinate source

Applications 3

ows Media Player

The file you are attempting to play has an extension (.mov) that does not match the file format. Playing the file may result in unexpected behavior.

Do you want the Player to try to play this content?

Don't ask me again for this extension (.mov).

Yes No

plications 3

Applications 3

CONCLUSIONS

- Computed singular field and effective source from coordinate expansion of world-function - for generic orbits in Kerr
- Implemented Detweiler-Vega 3+1 ‘puncture’ scheme using Cactus/Llama for circular orbits in Schwarzschild
- Higher order for a smoother source?
- Gravitational case?
- Singular field also useful for deriving regularization terms in mode-sum SF scheme? Poisson Haas '06