

Title: Non-singular sources for self-force calculations in numerical relativity

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URL: <http://pirsa.org/10060044>

Abstract: Motivated by the goal of self-consistently calculating self-force-corrected orbits and waveforms with (3+1) evolution codes we derive a covariant expression for a non-singular representation of a scalar point charge moving along a geodesic of an arbitrary spacetime. This differs from previous representations that were anchored to the use a particular locally-inertial coordinate system.

Review of the (3+1) approach

- “Regularization before evolution”
- Solve the following equations simultaneously for ψ^R and u^α :

$$\begin{aligned}\square\psi^R &= S(x^\alpha, u^\alpha) \\ u^\beta\nabla_\beta u^\alpha &= g^{\alpha\beta}\nabla_\beta\psi^R\end{aligned}$$

- Advantages:
 - ▶ Non-post-processing approach to self-consistent evolution
 - ▶ Does not rely on the underlying symmetries of the spacetime
 - ▶ Uses existing numerical relativity infrastructure
- Disadvantages:
 - ▶ Lower accuracy – (could be a deal breaker)
 - ▶ Inherits the difficulties that confront any (3+1) calculation

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Effective Source

Away from the worldline, the effective source is defined by

$$S = -\square(W\tilde{\psi}^S) = -W\square\tilde{\psi}^S - 2\nabla^\alpha W\nabla_\alpha\tilde{\psi}^S - \tilde{\psi}^S\square W$$

where

- $\tilde{\psi}^S$ is an accurate-enough (explicit) approximation to the exact Detweiler-Whiting singular field, and
- W is an appropriately chosen window function, defined so that
 - ▶ $\lim_{x\rightarrow\gamma}\nabla_\alpha\psi^R = F_\alpha$,
 - ▶ $\psi^R \rightarrow \psi^{\text{ret}}$ away from the source,
 - ▶ S_{eff} is regular on the worldline.

Example of a singular field approximation:

$$\psi^S = q/\rho + O(\rho^3),$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$ in THZ coordinates (t, x, y, z)

'New' covariant starting point

Covariant singular field (Haas and Poisson, 2006)

$$\psi^S = \frac{q}{2r} + \frac{q}{2r_{\text{adv}}} + O(\epsilon^3)$$

$$\psi^S = \frac{q}{s} \left(1 + \frac{\bar{r}^2 - s^2}{6s^2} R_{u\sigma u\sigma} + \frac{\bar{r}(\bar{r}^2 - 3s^2)}{24s^2} R_{u\sigma u\sigma|u} - \frac{(\bar{r}^2 - s^2)}{24s^2} R_{u\sigma u\sigma|\sigma} \right) + O(\epsilon^3)$$

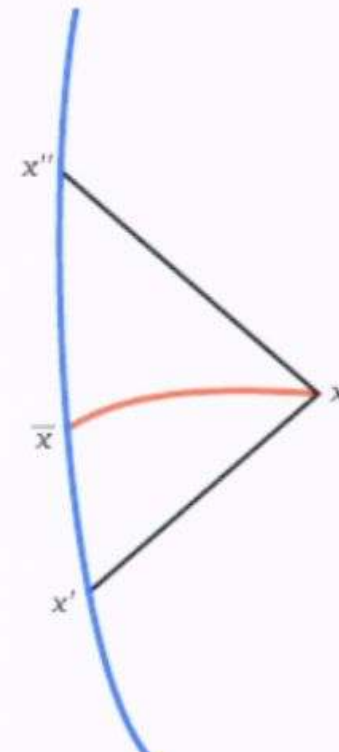
Five scalar functions

$$\{s, \bar{r}, R_{u\sigma u\sigma}, R_{u\sigma u\sigma|u}, R_{u\sigma u\sigma|\sigma}\}.$$

$$s^2 = (g^{\bar{\alpha}\bar{\beta}} + u^{\bar{\alpha}}u^{\bar{\beta}})\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}}$$

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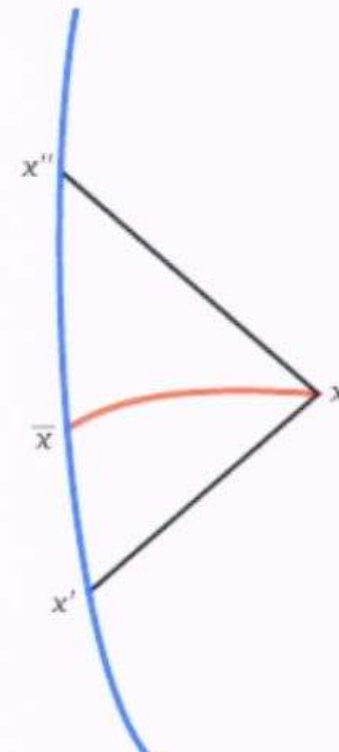
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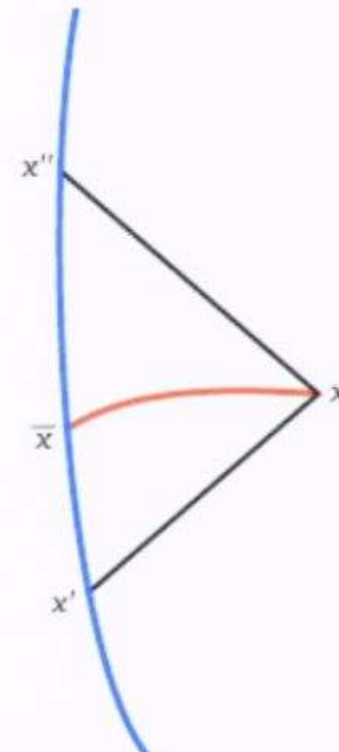
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Background coordinates

$\sigma_{\bar{\alpha}}$ can be expressed in terms of an expansion in the coordinate displacement $w^\alpha = (x^\alpha - \bar{x}^\alpha)$:

$$-\sigma_{\bar{\alpha}}(x, \bar{x}) = g_{\alpha\beta} w^\beta + A_{\alpha\beta\gamma} w^\beta w^\gamma + A_{\alpha\beta\gamma\delta} w^\beta w^\gamma w^\delta + A_{\alpha\beta\gamma\delta\epsilon} w^\beta w^\gamma w^\delta w^\epsilon + \dots$$

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$$A^\alpha{}_{\beta\gamma} = \frac{1}{2} \Gamma^\alpha{}_{\beta\gamma}$$

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are all quantities evaluated on γ .

Here then we have a perfectly legitimate (generic) singular field approximation in terms of the background coordinates.

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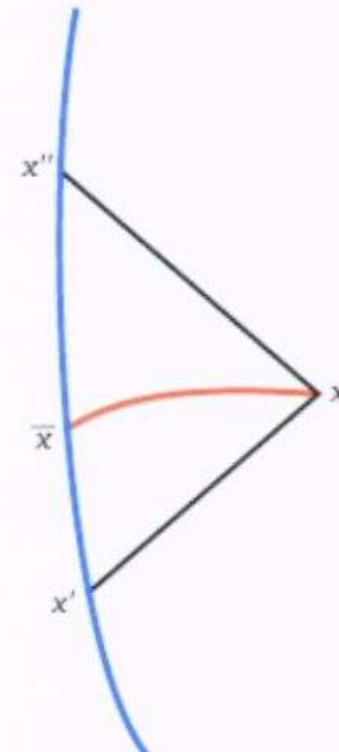
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A potential issue with the covariant approach?

- One approach that was originally pursued was to find a covariant expression of the effective source. (Barry's talk).
- Derive covariant expressions for $\square\tilde{\psi}^S$ and $\nabla_\alpha\tilde{\psi}^S$ from Eric's covariant approximation of the singular field.
- This requires truncations of higher-order terms.
- Truncation introduces inconsistencies in the effective source that **may** ruin the key relation $\psi^R = \psi^{\text{ret}} - W\tilde{\psi}^S$
- The benefit in pursuing a coordinate approach is that it affords one the ability to **commit** to the coordinate expression of $\tilde{\psi}^S$ and **not throw out anything** when computing $\square\tilde{\psi}^S$.

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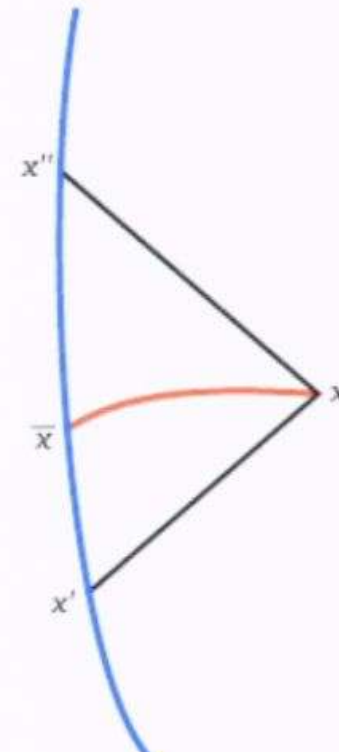
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Is truncation allowed in going from $\tilde{\psi}^S$ to $\square\tilde{\psi}^S$?

Recall the wave equation with the effective source:

$$\square\psi^R = -\tilde{\psi}^S\square W - 2\nabla^\alpha W\nabla_\alpha\tilde{\psi}^S - W\square\tilde{\psi}^S$$

Determine a nice covariant expression for $\square\tilde{\psi}^S$; call it $F(\sigma)$.

$$\square\tilde{\psi}^S = F(\sigma) + R\sigma^n.$$

Use this truncated version in the effective source:

$$\square\bar{\psi}^R = -\tilde{\psi}^S\square W - 2\nabla^\alpha W\nabla_\alpha\tilde{\psi}^S - WF(\sigma)$$

Is there any crucial difference between ψ^R and $\bar{\psi}^R$?

$$\square(\psi^R - \bar{\psi}^R) = WR\sigma^n$$

Does this imply that $(\psi^R - \bar{\psi}^R)$ vanishes on γ ?

Choosing the approximate singular field

1) Re-package the singular field as

$$\tilde{\psi}^S = \frac{q}{s^3} K$$

where

$$K := s^2 + \frac{(\bar{r}^2 - s^2)}{6} R_{u\sigma u\sigma} + \frac{\bar{r}(\bar{r}^2 - 3s^2)}{24} R_{u\sigma u\sigma|u} - \frac{(\bar{r}^2 - s^2)}{24} R_{u\sigma u\sigma|\sigma}.$$

2) Truncate s^2 and F at the appropriate order.

$$s^2 = B_{\alpha\beta} w^\alpha w^\beta + B_{\alpha\beta\gamma} w^\alpha w^\beta w^\gamma + B_{\alpha\beta\gamma\delta} w^\alpha w^\beta w^\gamma w^\delta + B_{\alpha\beta\gamma\delta\epsilon} w^\alpha w^\beta w^\gamma w^\delta w^\epsilon$$

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Write the approximate singular field and effective source in terms of 8 polynomials in w^α and their (partial) derivatives

Coding up the effective source

- Following the original THZ code, I wanted the effective source to be computed entirely through evaluations of the helping polynomials and their explicit (partial) derivatives.
- 72 C/C++ functions
- These helping polynomials are generally VERY long and nasty.
- Many many thanks to Maple and its code generation facility!
- ... and to `grOptionDefaultSimp := 0` in grtensor. (Whoever laughs at this is a true pro!)

C/C++ code for effective scalar charge moving along a generic geodesic in Schwarzschild → DONE!

Since nothing said so far is specific to Schwarzschild, Kerr should now be trivial.

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Interpolation

A problem that arises in calculating the effective source is the subtraction of very large numbers occurring close to the particle.

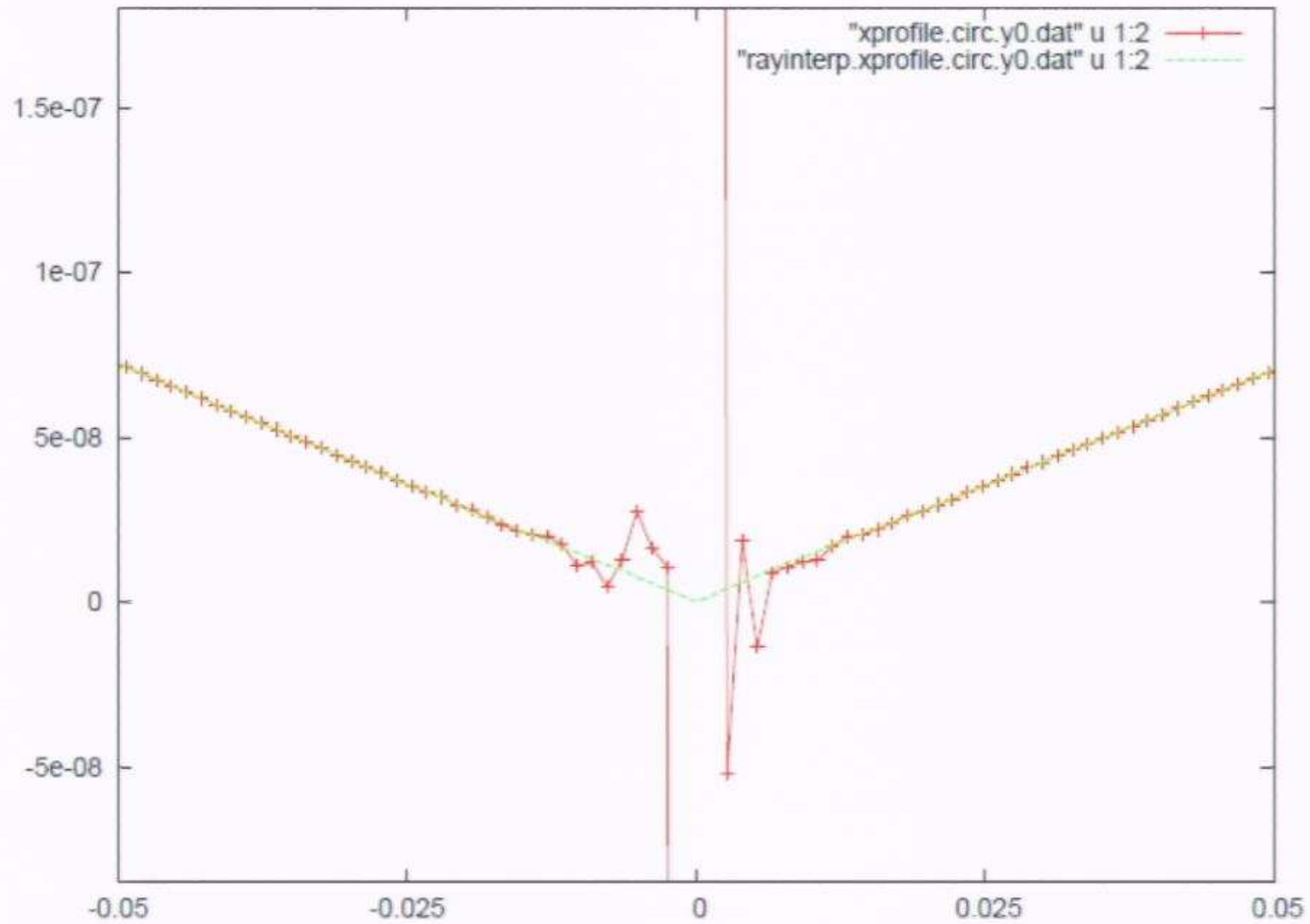
Two important facts:

- 1) $S = 0$ on γ (formally).
- 2) S is smooth everywhere except on γ , where it is just C^0 .

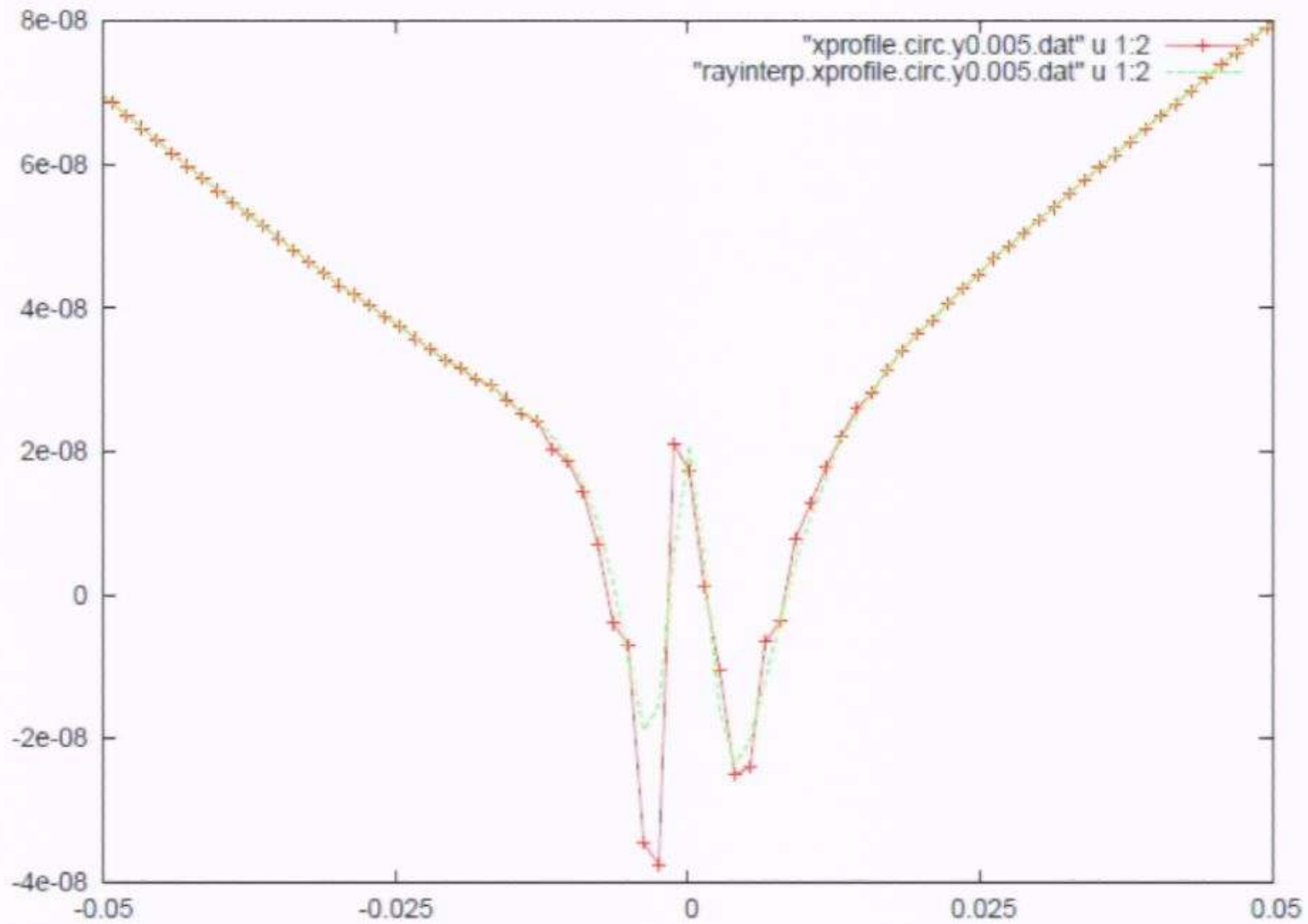
Consider $S(\lambda) := S(\vec{x}(\lambda))$ as a function of λ , along the coordinate ray, $\vec{x} = \vec{x}_0 + \lambda \hat{r}$, where $\hat{r} := \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|}$.

To compute S at a point very close to γ , instead use (Lagrange) interpolation with $S(\lambda = 0) = 0$ and a few other evaluations of S along the coordinate ray.

Results after interpolation

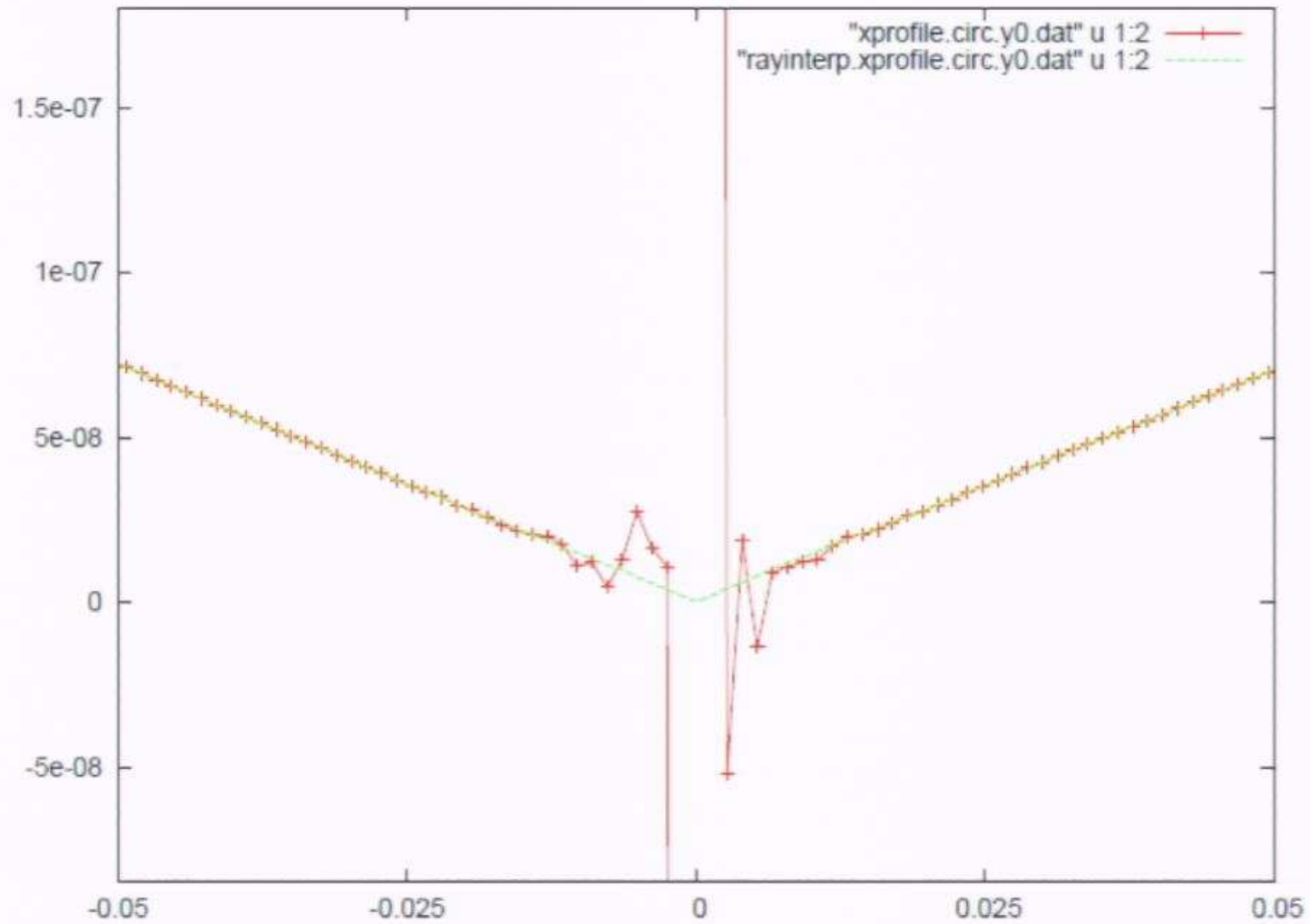


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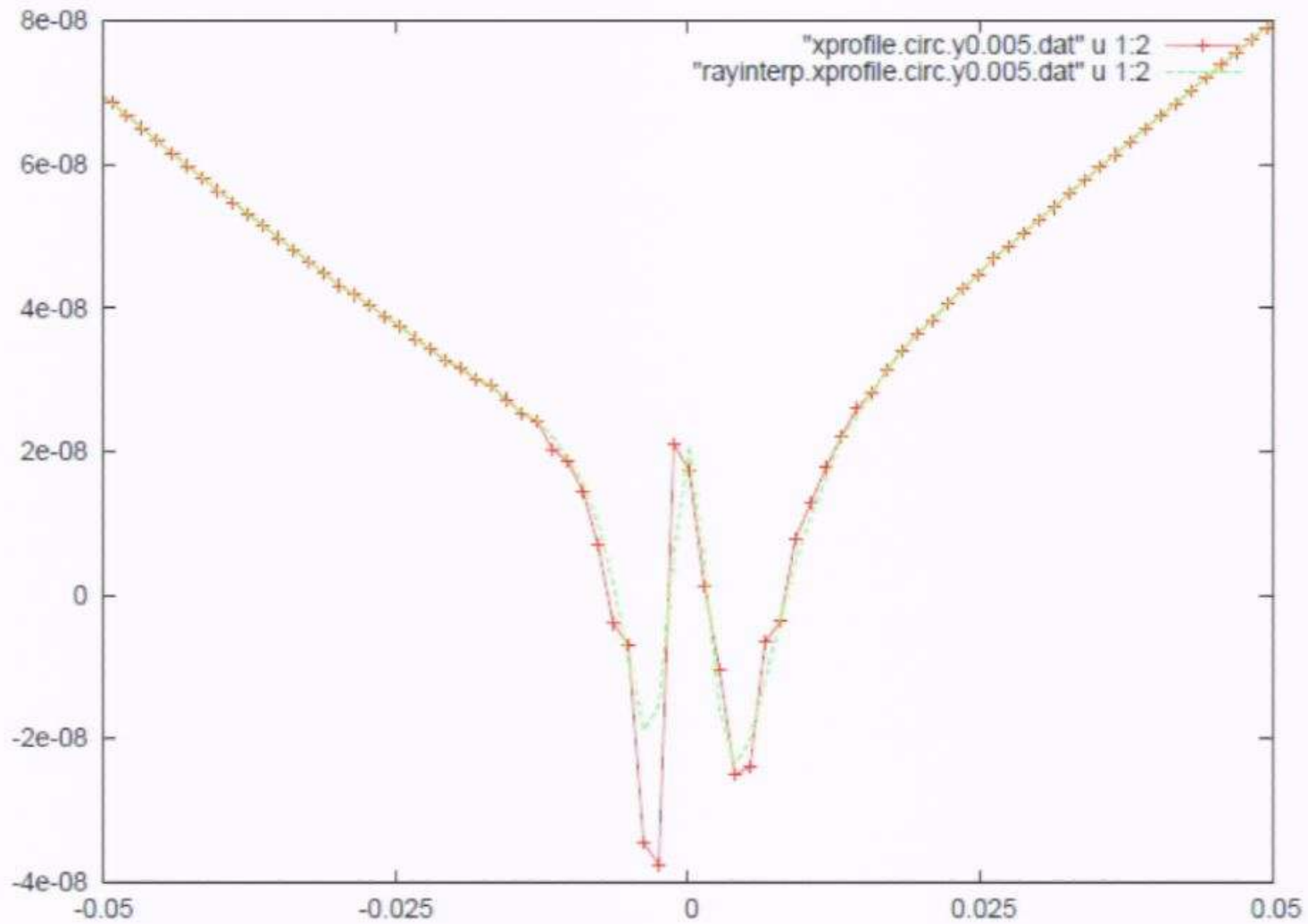


The effective source along the ray ($x = 0, y = 0.005M$), before and after

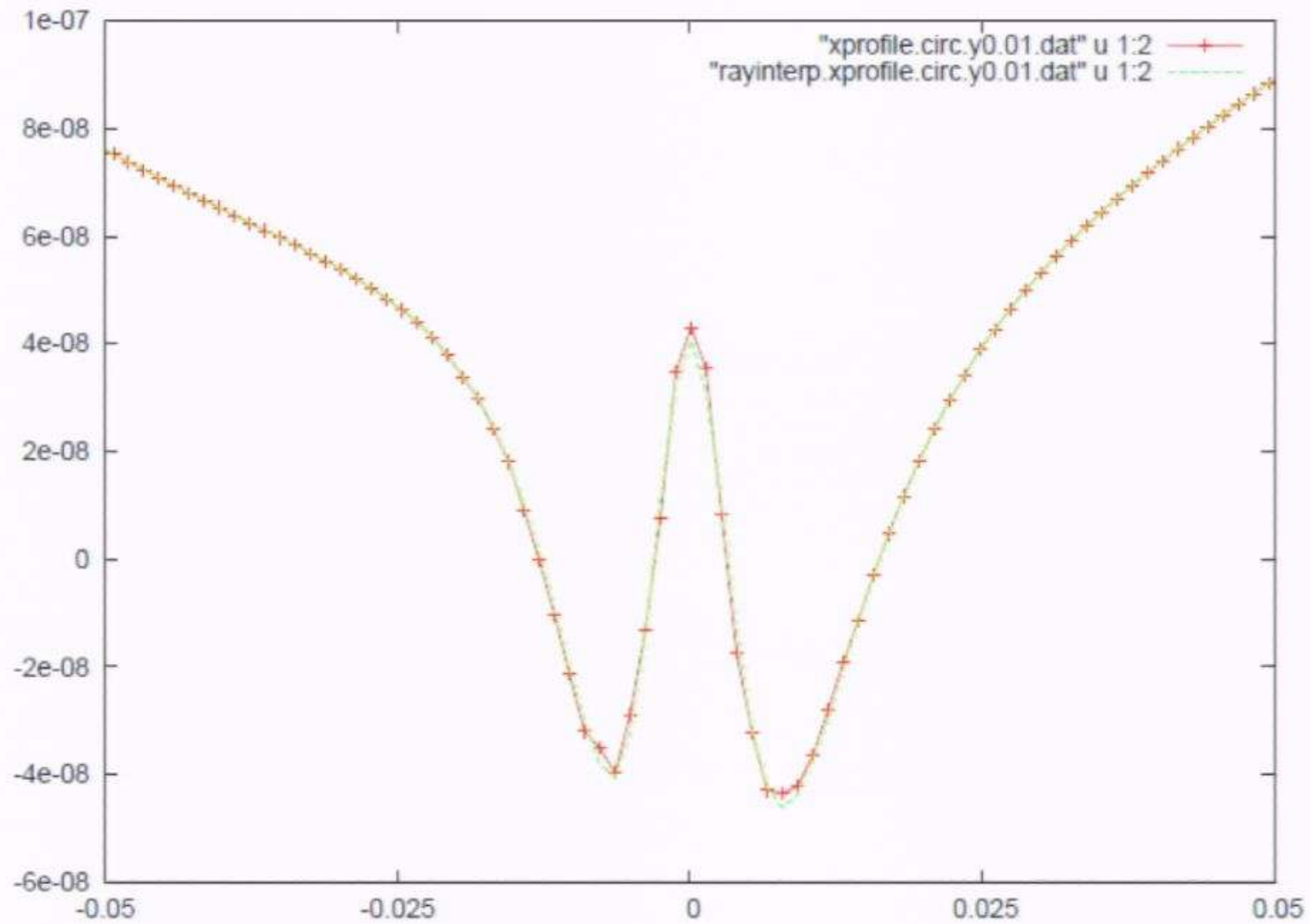
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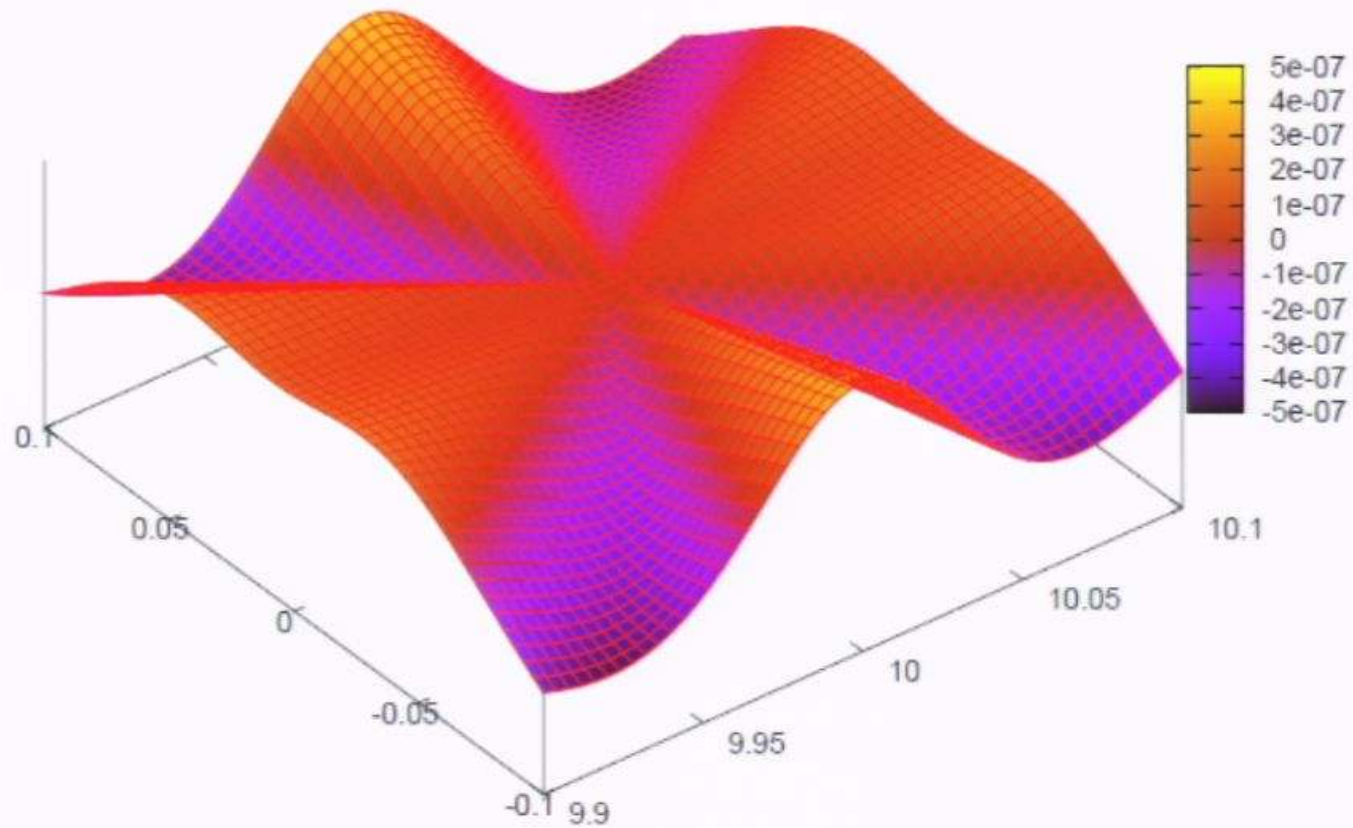


Results after interpolation



The effective source along the ray ($x = 0, y = 0.01M$), before and after

Effective source for a circular orbit in Schwarzschild



Take home message

- C/C++ code for an effective scalar charge moving along generic geodesics of Schwarzschild . (Kerr is to follow quickly).
- Interpolation effectively handles the errors arising from the delicate cancellations that occur close to the particle.
- With Peter Diener's (3+1) finite difference code, self-consistent evolution for a scalar point charge is now possible.