Title: Highly accurate EMRI self-force calculations

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Abstract: TBA

Highly accurate and efficient self-force computation using time-domain methods: Error estimates, validation, and optimization

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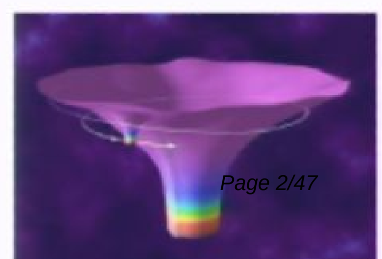

Why High Accuracy: LISA (1)

Matched filtering of the entire LISA data stream is impractically expensive, but once sources have been detected by hierarchical and/or time-frequency searches, matched filtering will be used to estimate parameters for source modelling, cosmology, GR tests, ...

Why High Accuracy: LISA (2)

High LISA S/N after matched filtering

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The **strongest** LISA EMRI sources will have quite large S/N after matched filtering:

a	$\mathcal{R}_{\rm MW}^{\rm BH}$ (Gyr^{-1})	LISA Performance 5yr,2chan	LISA Performance 2yr,1chan
0	4	140	16*
	40	460	180
	400	1300	490
0.5	4	160	20*
	40	560	210
	400	1500	590
0.9	4	240	46*
	40	780	330
	400	2100	860

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High LISA S/N after matched filtering

$\Rightarrow$ Need highly accurate calculation of GW phase over the full LISA timespan ($\sim 10^5$ EMRI orbits) to prevent template-phase errors from dominating the overall parameter-estimation error budget

$\Rightarrow$ Need highly accurate calculation of self-force: Huerta & Gair, PRD **79** (2009), 084021 find that a 10^{-6} fractional change in the EMRI self-force changes the cumulative GW phase by ~ 2 radians. Hence:

Self-Force Relative Error Tolerance

	$C = 1$	$C = 30$	$C = 1000$
$\rho_{\max} = 30$	2×10^{-8}	5×10^{-7}	2×10^{-5}
$\rho_{\max} = 300$	2×10^{-9}	5×10^{-8}	2×10^{-6}
$\rho_{\max} = 2000$	3×10^{-10}	8×10^{-9}	3×10^{-7}

[C measures the degeneracy (ill-conditioning) of the parameter estimation.]

Why High Accuracy: Orbit Correction

We can use the orbit-correction scheme of Gralla & Wald, CQG **25** (2008), 205009 to explore $\mathcal{O}(\mu^2)$ effects, where $\mu = m/M \sim 10^{-5}$ is the EMRI mass ratio.

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To reliably distinguish true $\mathcal{O}(\mu^2)$ effects due to the orbit correction from numerical errors in the $\mathcal{O}(\mu)$ self force, the self force needs to be calculated with a relative error $\ll \mu$.

Other uses of highly accurate self-force calculations:

- calibrate/constraint post-Newtonian expansions (e.g., Blanchet *et al.*, arXiv:0910.0207)
- test correctness of self-force calculations & theory by comparisons between calculations using different methods (e.g., Sago, Barack, & Detweiler, PRD **78** (2008), 124024)

Why High Efficiency: Orbit Correction

The orbit-correction scheme of Gralla & Wald, CQG **25** (2008), 205009 requires time-integrating a set of coupled ODEs for the orbital-parameter evolution on radiation-reaction and longer timescales, with the ODEs' right-hand-side functions being given by a self-force computation.

Even the most efficient ODE-integration schemes will require evaluating the right-hand-side functions (i.e., computing the self-force for some specified intermediate orbit) hundreds of times in the course of a single low-accuracy orbit-correction calculation, and many more for higher accuracies.

$\Rightarrow$ Self-force calculation should be as efficient as possible!

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[For comparison, Barack & Sago, PRD 81 (2010), 084021 report that their gravitational self-force calculation takes 4–8 cpu days at a 10^{-4} relative accuracy level.]

Barack-Ori mode-sum regularization [**conceptual**]

Lorenz gauge $\Rightarrow$ metric perturbation from point particle is relatively simple (locally isotropic).

Work in **time domain** $\Rightarrow$ allows arbitrary particle orbits.

Decompose metric perturbation into spherical harmonics (ℓ, m), compute each (ℓ, m) mode via time-domain **numerical integration** of the complex wave equation

$$\Box\phi_{\ell m} + V_\ell(r)\phi_{\ell m} = S_{\ell m}(t)\,\delta\big(r(t) - r_{\text{particle}}(t)\big)$$

Compute regularized self-force modes for each ℓ:

$$F^{(\pm)}_{\ell,\text{reg}} \sim \left(\sum_m (\nabla\phi_{\ell m})\Big|_{r=r_p^\pm} \right) \mp (\ell + \tfrac{1}{2})A - B$$

(known coefficients)

Compute self-force:

$$F_{\text{self}} = \sum^{\infty} F^{(\pm)}$$

Tail Force

The tail force $F_{\text{self,tail}}$ can be estimated via a "tail fit" of some subset of the $F_{\ell,\text{reg}}$ to the large-ℓ series

$$F_{\ell,\text{reg}} = \sum_{\substack{p \text{ even} \\ p \geq 2}} c_p f_p(\ell)$$

where the basis functions $f_p(\ell) = \mathcal{O}(\ell^{-p})$ are

$$\begin{aligned}
f_2(\ell) &= 1/((\ell - \tfrac{1}{2})(\ell + \tfrac{3}{2})) \\
f_4(\ell) &= 1/((\ell - \tfrac{3}{2})(\ell - \tfrac{1}{2})(\ell + \tfrac{3}{2})(\ell + \tfrac{5}{2})) \\
f_6(\ell) &= 1/((\ell - \tfrac{5}{2})(\ell - \tfrac{3}{2})(\ell - \tfrac{1}{2})(\ell + \tfrac{3}{2})(\ell + \tfrac{5}{2})(\ell + \tfrac{7}{2})) \\
&\cdots
\end{aligned}$$

Then $F_{\text{self,tail}} = \sum_p c_p \Gamma_p$ where $\Gamma_p = \sum_{\ell=K+1}^{\infty} f_p(\ell)$ can be computed analytically.

Adaptive Mesh Refinement (AMR)

I use **characteristic AMR** with ($\geq$) **4th order finite differencing** to numerically integrate the complex wave equation.

- Typically **speeds up computation by a factor of $\sim$ 30–40** over an equivalent-resolution (time-domain) unigrid computation.

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- CPU time now scales $\sim$ length of particle world-line (integration is cheap outside neighborhood of worldline).
  $\Rightarrow$ CPU time scales **linearly** with domain size for 1+1-D code
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  $\Rightarrow$ CPU time scales **linearly** with domain size for 1+1-D code
  $\Rightarrow$ relatively cheap to use very large domains for high accuracy (including waiting for low-ℓ initial-data transients to decay).
- Turn on AMR **gradually** $\Rightarrow$ don't waste high resolution on resolving the initial-data transient.

See Thornburg, arXiv:0909.0036 for details & **source code (C++).**

Floating-Point Arithmetic

In some cases, my highest-accuracy results are limited in accuracy by floating-point rounding errors in solving the complex wave equation.

To investigate these effects, I compare results computed using standard IEEE "double" (64-bit) floating-point arithmetic versus IEEE "double-extended" (80-bit) floating-point arithmetic. Double-extended has a factor of $2^{11} = 2048$ smaller rounding errors, but is a factor of ~ 2 slower.

Test Parameters

Scalar-field particle in circular geodesic orbit in Schwarzschild spacetime at $r_{\text{areal}} = 10M$ (orbital period $\approx 199M$).

Double precision:

AMR error tolerance 10^{-14}, 10^{-15}, 10^{-16},
FMR may be none or $\times 2$, $\times 3$, $\times 4$, $\times 6$, or $\times 8$

Double-extended precision:

AMR error tolerance 10^{-14} down to 10^{-19},
FMR may be none or $\times 2$

Complex wave equation solved for $0 \leq \ell \leq 30$, $\ell = 35$, and $\ell = 40$.

Numerical force sums modes for $0 \leq \ell \leq K{=}30$.

Tail fit coefficients $\{c_4, c_6\}$ fitted to modes $20 \leq \ell \leq 30$.
[For a circular particle orbit c_2 is given analytically by Detweiler, Messaritaki, & Whiting, PRD **67** (2003), 104016.]

Generic-orbit case ($\{c_2, c_4, c_6\}$ fitted) omitted here for brevity.

Error Estimates for Individual Regularized Modes

Difference between $F_{\ell,\mathrm{reg}}^{(+)}$ and $F_{\ell,\mathrm{reg}}^{(-)}$ $\Rightarrow$ "internal" error estimate:

$$F_{\ell,\mathrm{reg}} \pm \delta F_{\ell,\mathrm{reg}}^{(\mathrm{internal})} \equiv \frac{1}{2}\left(F_{\ell,\mathrm{reg}}^{(+)} + F_{\ell,\mathrm{reg}}^{(-)}\right) \pm \frac{1}{2}\left|F_{\ell,\mathrm{reg}}^{(+)} - F_{\ell,\mathrm{reg}}^{(-)}\right|$$

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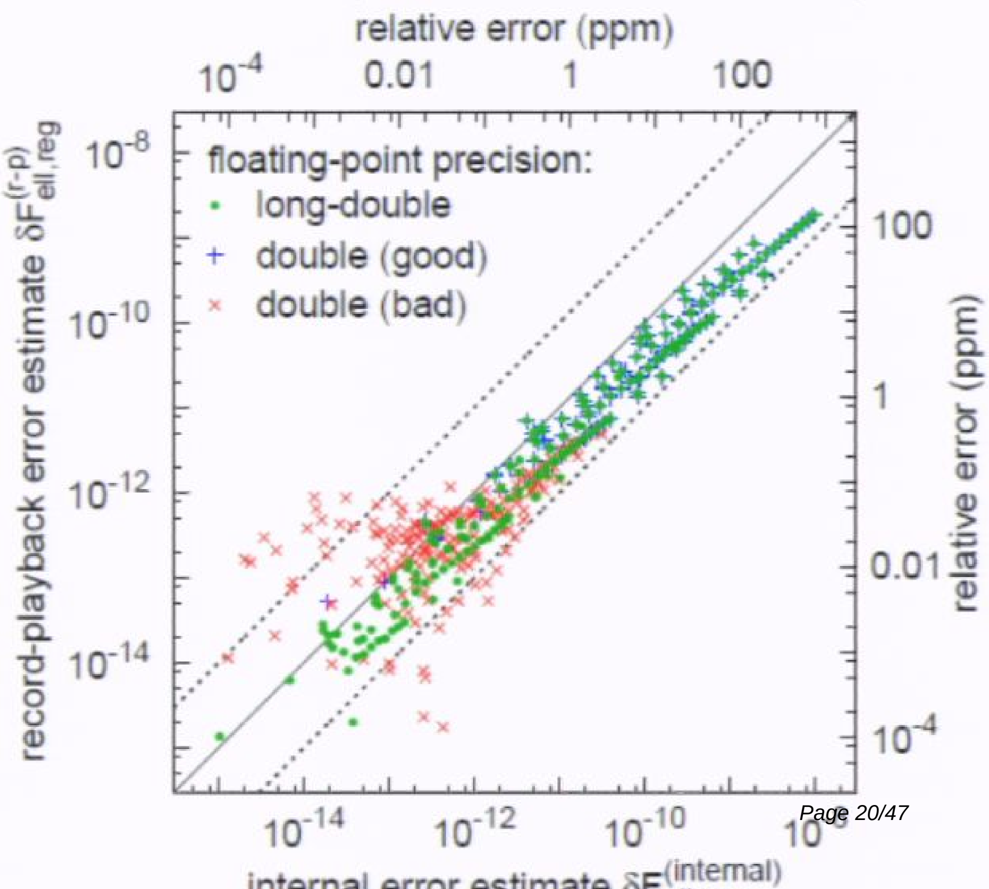
Validate by comparing with convergence-test ("record-playback") error estimates:



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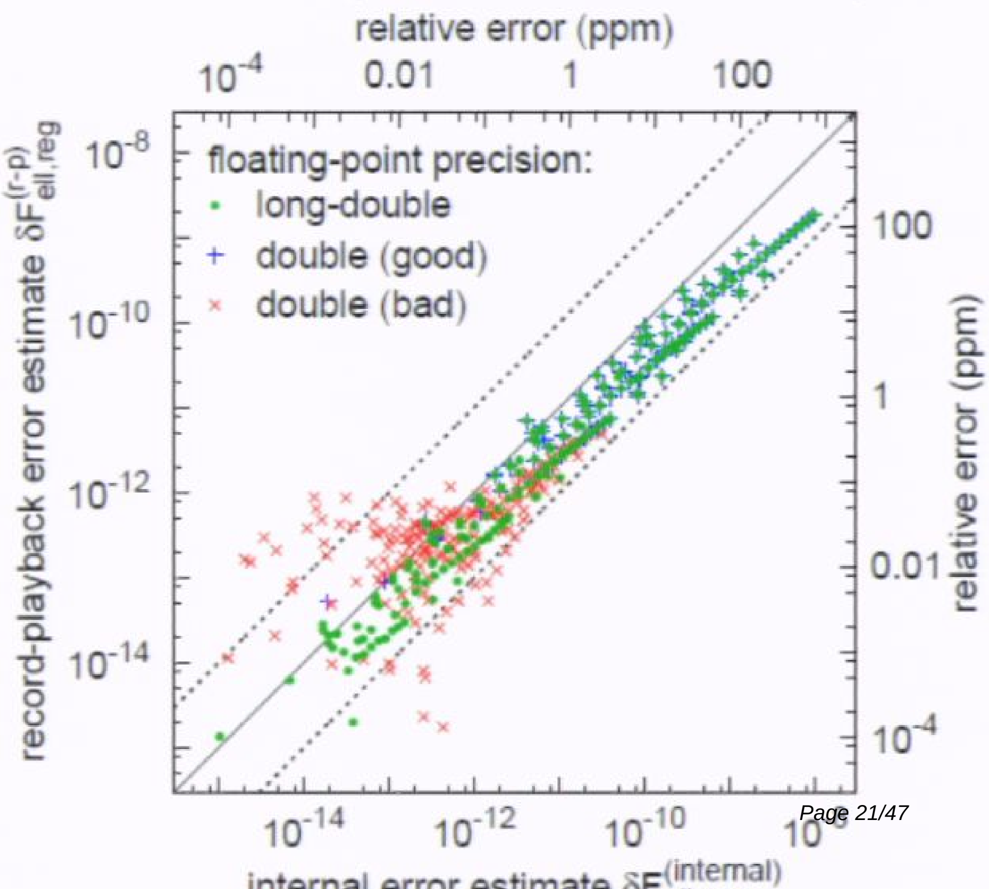
Using double-extended floating point arithmetic, the internal error estimate is reliable & somewhat conservative, (pessimistic), overestimating the convergence-test errors by ~ 1–$10\times$.



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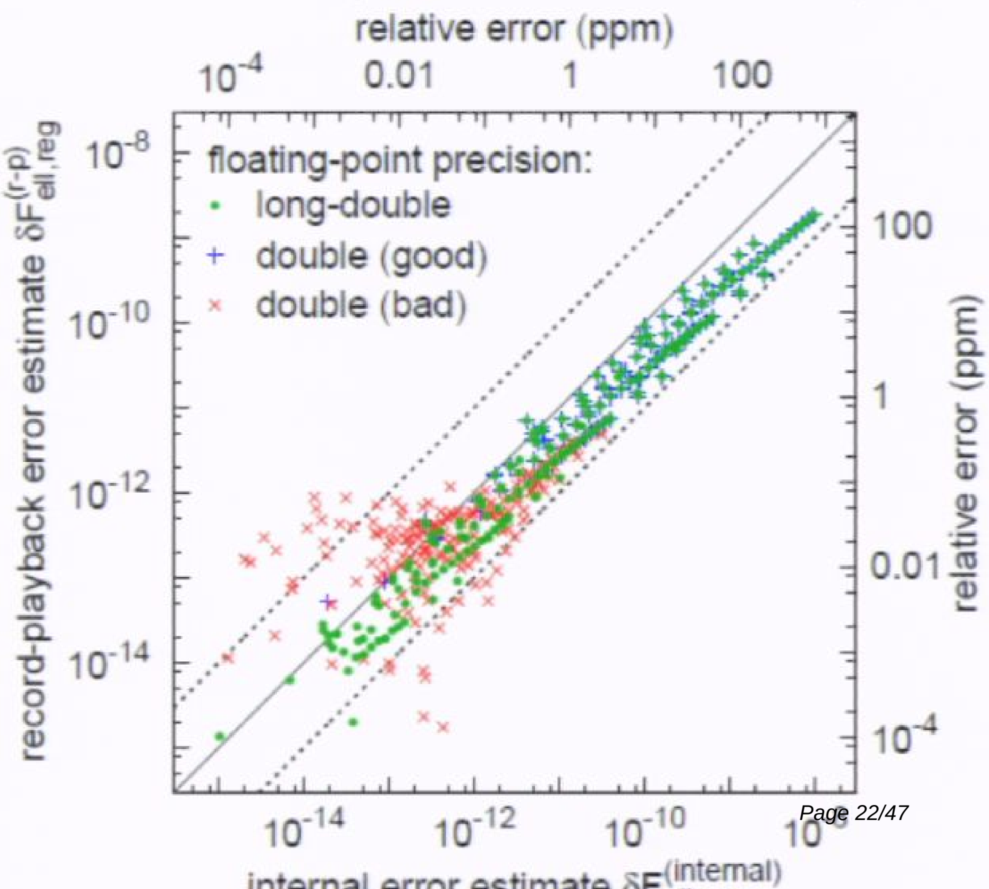
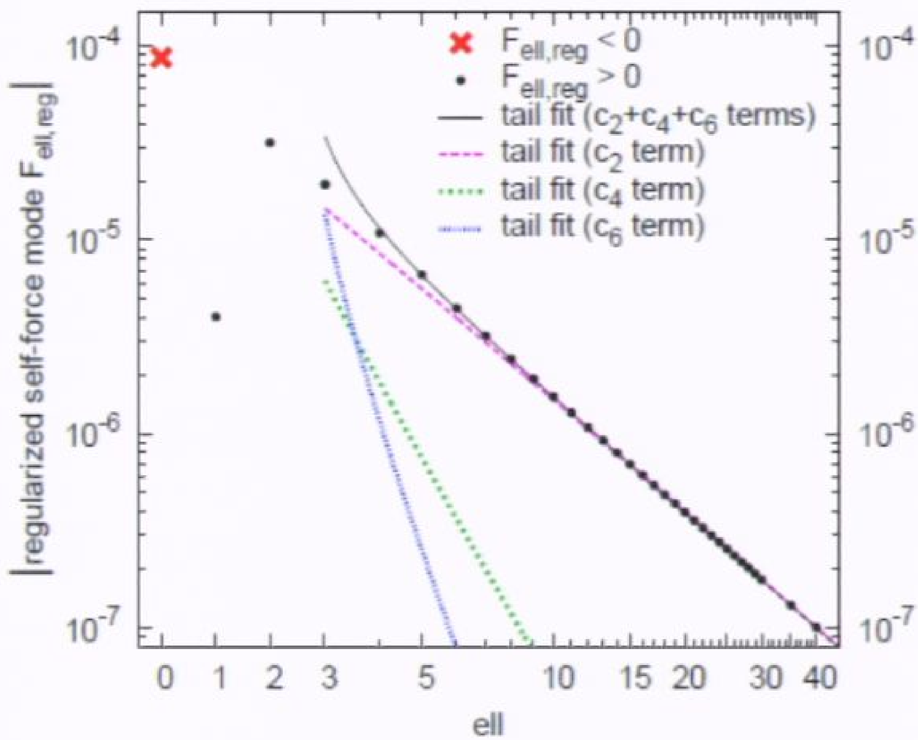
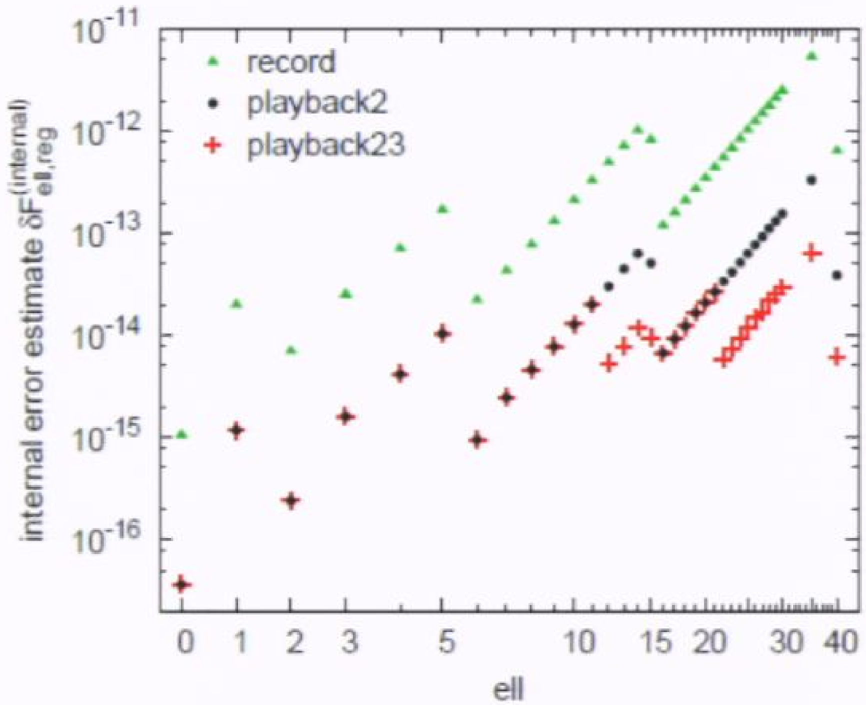
Using standard "double" arithmetic, the internal error estimates is ok for $\delta F^{(\text{internal})}_{\ell,\text{reg}} \gtrsim 0.1$ ppm, but unreliable for $\delta F^{(\text{internal})}_{\ell,\text{reg}} \lesssim 0.1$ ppm due to floating-point rounding errors in solving the complex wave equation.



Overview of Modes and Error Estimates





Error Estimates for the Numerical Force (1)

We don't *a priori* know whether or to what extent the actual errors in the individual modes $F_{\ell,\mathrm{reg}}$ for different ℓ might be correlated, so it's not obvious how to sum these to obtain an internal error estimate for the numerical force:

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Worst case: Actual numerical errors in different modes are perfectly correlated $\Rightarrow$ should add individual modes' internal error estimates arithmetically [Barack & Sago, PRD 81 (2010), 084021 use this method].

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Best case: Actual numerical errors in different modes are (close enough to) statistically independent $\Rightarrow$ should add individual modes' internal error estimates in quadrature.

Error Estimates for the Numerical Force (2)

Compute numerical-force internal error estimates $\delta F^{(\text{internal})}_{\text{self,num}}$ both ways (arithmetic & quadrature sums) and compare with convergence-test error estimates to see which works better:

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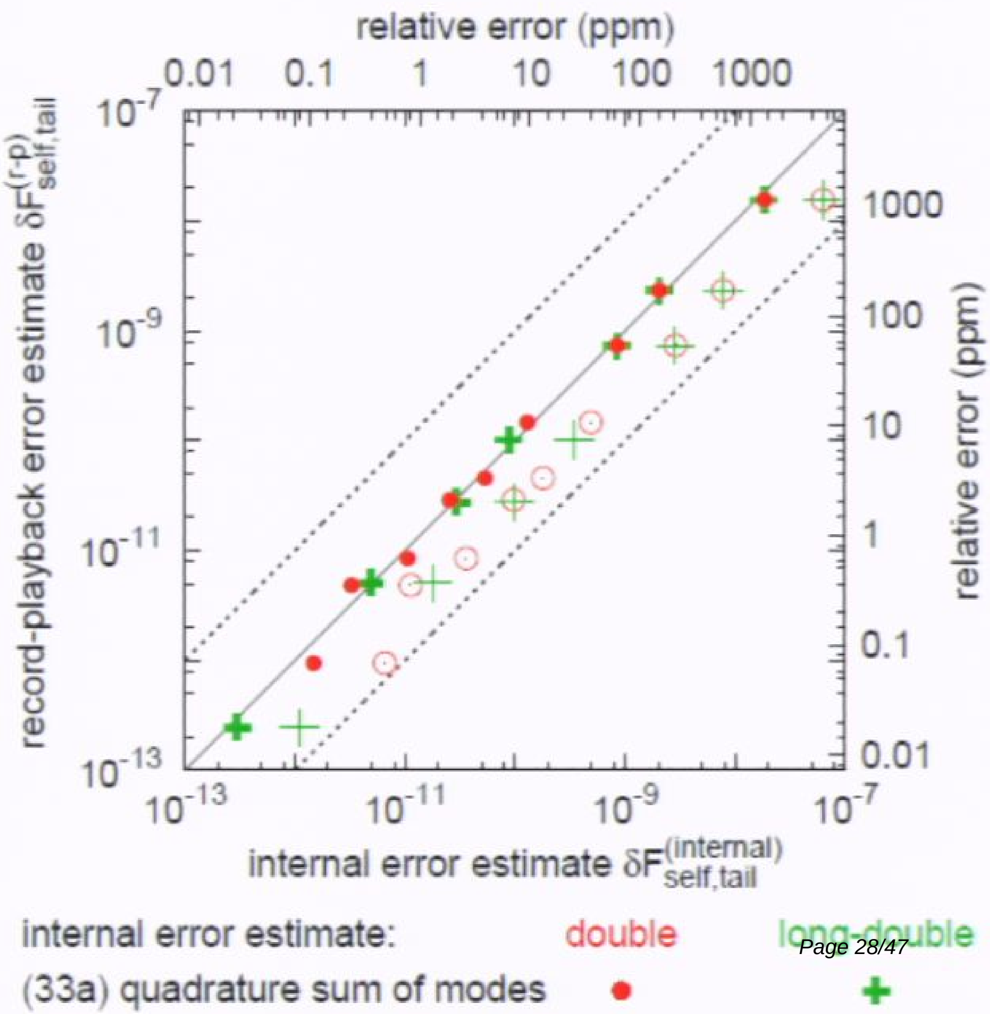

Error Estimates for the Numerical Force (2)

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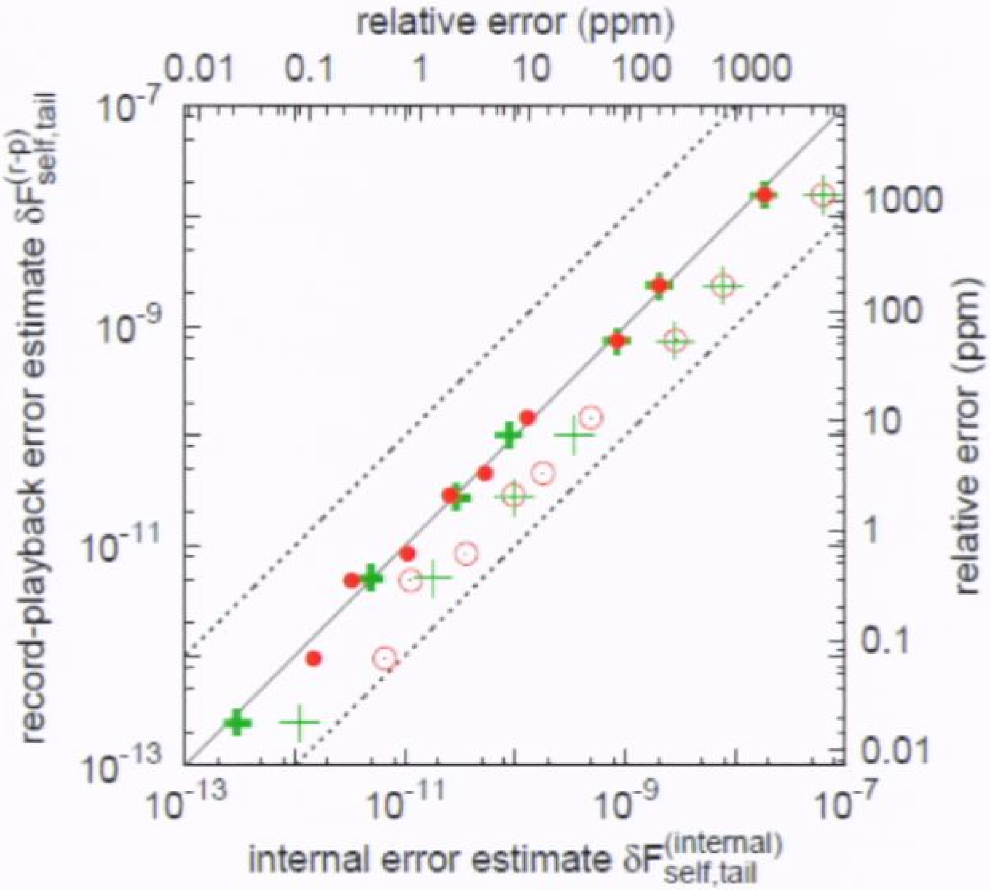
The arithmetic sum systematically overestimates the convergence-test error by $\sim 3\times$, while the quadrature sum gives an excellent estimate of the convergence-test error.

This is true for both double and double-extended arithmetic, *even for errors so small that the individual modes' internal error estimates are unreliable!* (This is apparently due to the averaging inherent in summing



internal error estimate: double long-double

(33a) quadrature sum of modes

Ill-Conditioning of the Tail Fit

The tail-fit basis functions $\{f_p\}$ are nearly linearly dependent, so the tail fit is numerically ill-conditioned. That is, there are linear combinations $\sum_p b_p f_p$ where the linear-combination coefficients $\{b_p\}$ have unit 2-norm, yet the linear combination $\sum_p b_p f_p$ is much smaller than the largest f_p.

Coefficients Being Fitted	Condition Number κ: Basis is $\{f_p\}$	Condition Number κ: Basis is $\{\overline{f}_p\}$
$\{c_4, c_6\}$	2.6×10^3	11
$\{c_4, c_6, c_8\}$	5.9×10^6	100
$\{c_2, c_4\}$	2.2×10^3	8.3
$\{c_2, c_4, c_6\}$	4.4×10^6	66
$\{c_2, c_4, c_6, c_8\}$	8.0×10^9	460

The ill-conditioning can be greatly reduced by renormalizing the basis functions to have similar magnitudes over the range of ℓ used

Error Estimates for the Tail Fit (1)

Again, we don't *a priori* know whether or to what extent the actual errors in the individual modes $F_{\ell,\text{reg}}$ for different ℓ might be correlated, so it's not obvious that we can apply the standard "statistics-101" formulae for error propagation in least-squares fitting:

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Best case: Actual numerical errors in different modes are (close enough to) statistically independent that we can use the standard statistical error estimate.

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Best case: Actual numerical errors in different modes are (close enough to) statistically independent that we can use the standard statistical error estimate.

Worst case: Estimate self-force errors by trying a trial fit for each possible combination of $\{-, 0, +\}$ signs of the different modes' internal error estimates. (For Q modes in the fit, there are 3^Q combinations, and hence 3^Q trial fits are needed.)

Error Estimates for the Tail Fit (2)

Compute tail-fit internal error estimates $\delta F^{(\text{internal})}_{\text{self,tail}}$ both ways (statistical and worst-case-of-3^Q-combinations-of-error-signs) and compare with convergence-test error estimates to see which works better:

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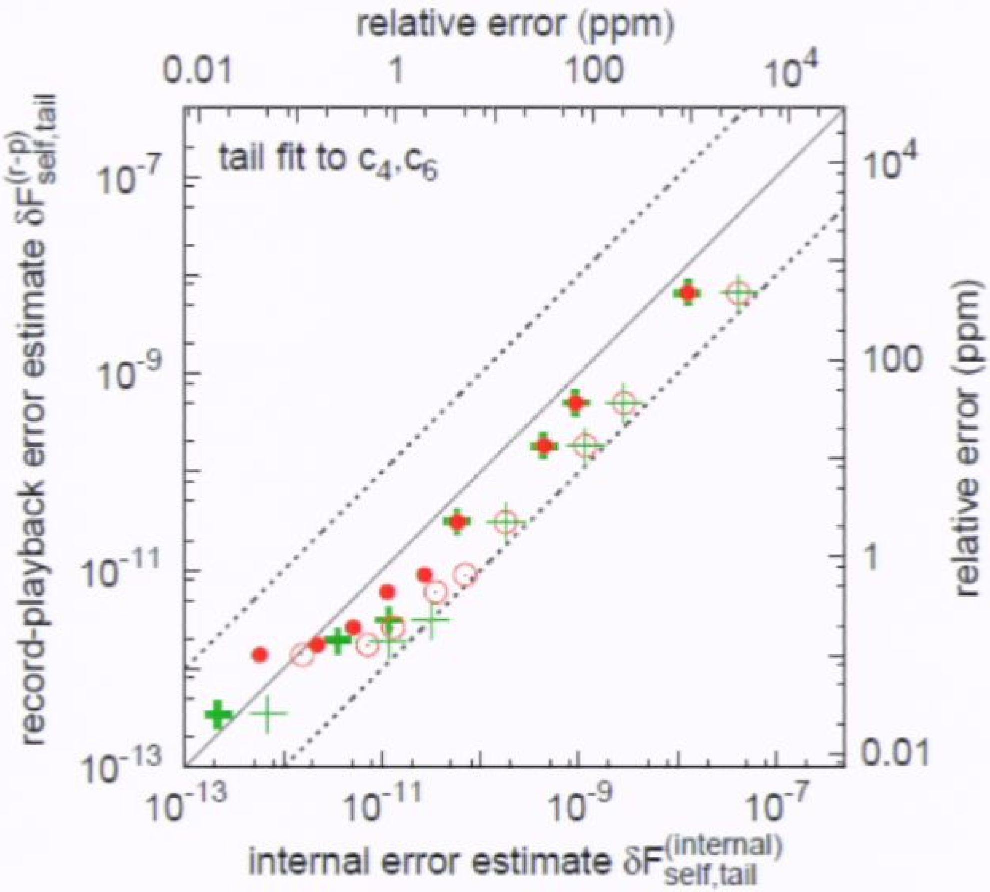
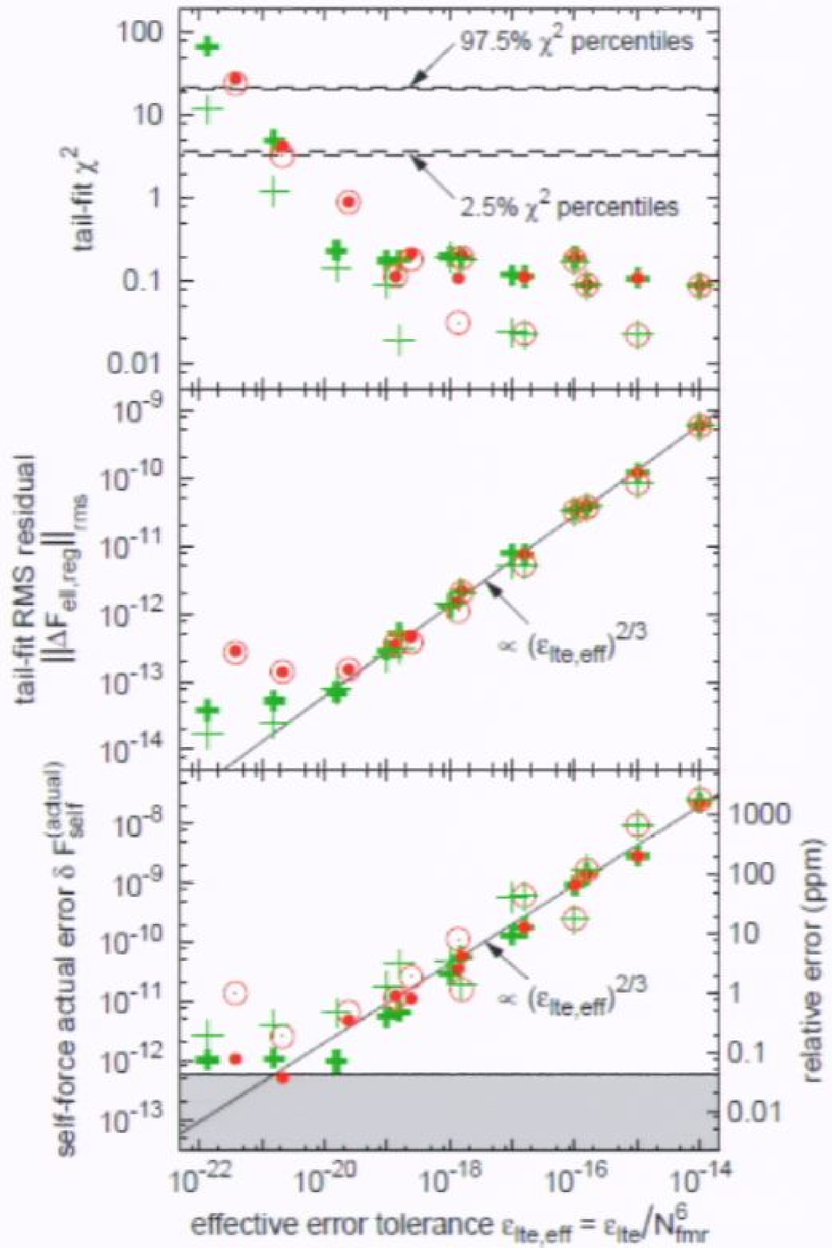


internal error estimate: double long-double

(34a) statistical error in tail fit

Tail-Fit Results



tail fit coefficients: double long-double

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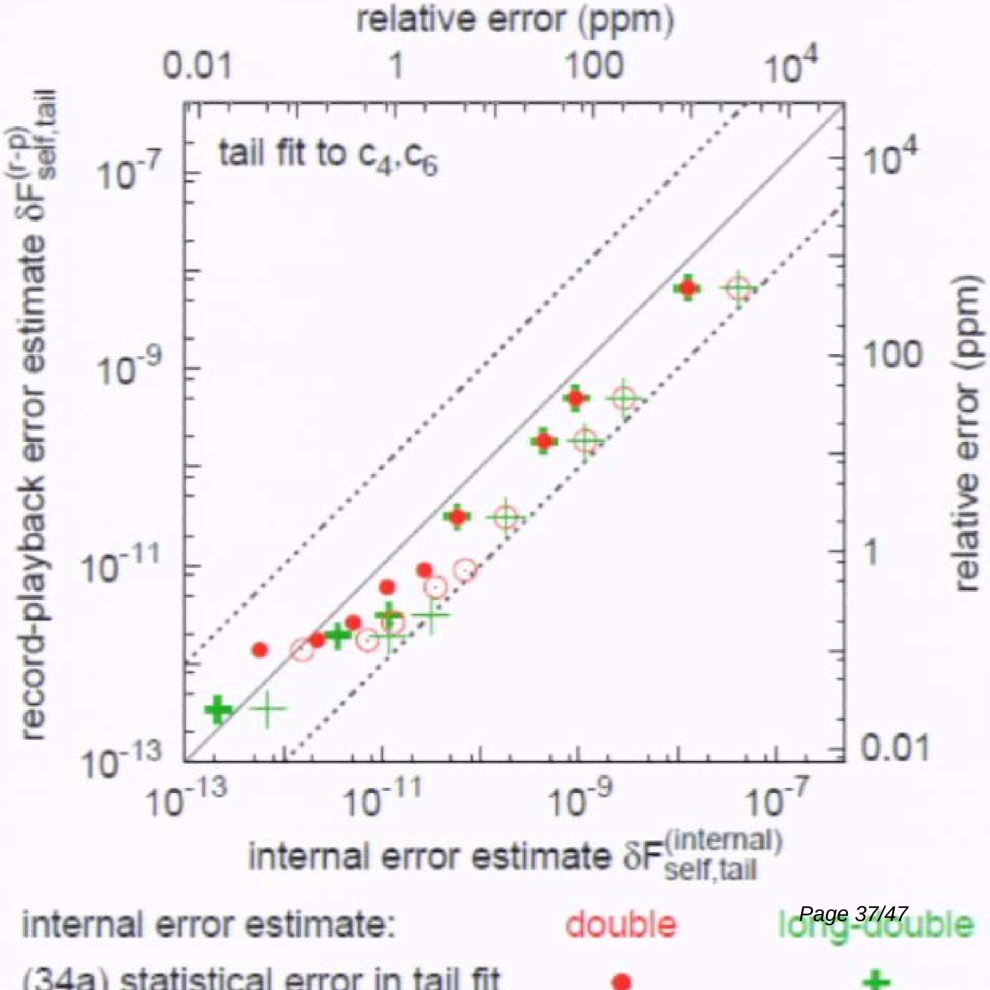
The worst-case error estimate systematically overestimates the convergence-test error by $\sim 6\times$, while the statistical error estimate is a fairly good (conservative by $\sim 2\times$) estimate of the convergence-test error.



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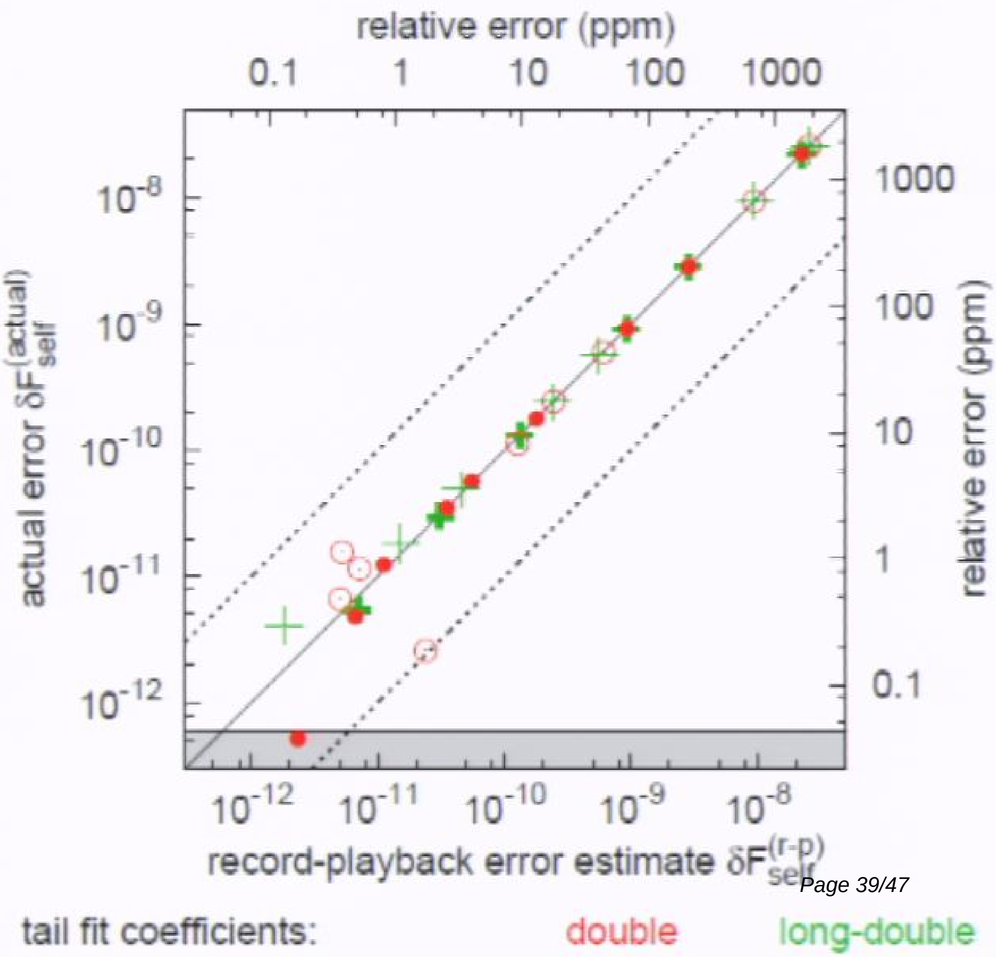
/alidate Overall Self-Force Error Estimates

Validate the convergence-test error estimates for the overall self-force [quadrature sum of numerical-force and tail-force error; details omitted for brevity] by comparing with highly-accurate frequency-domain results from Detweiler, Messaritaki, & Whiting, PRD **67** (2003), 104016:

Very good agreement except for high-accuracy results with double floating-point precision.

[Shaded region shows error $\le 3\times$ Detweiler, Messaritaki, & Whiting error estimate]



Error Estimates for the Self-Force

Compute this error estimate as quadrature sum of error estimates for numerical force & tail force [details of validation of quadrature vs. arithmetic sum omitted for brevity].

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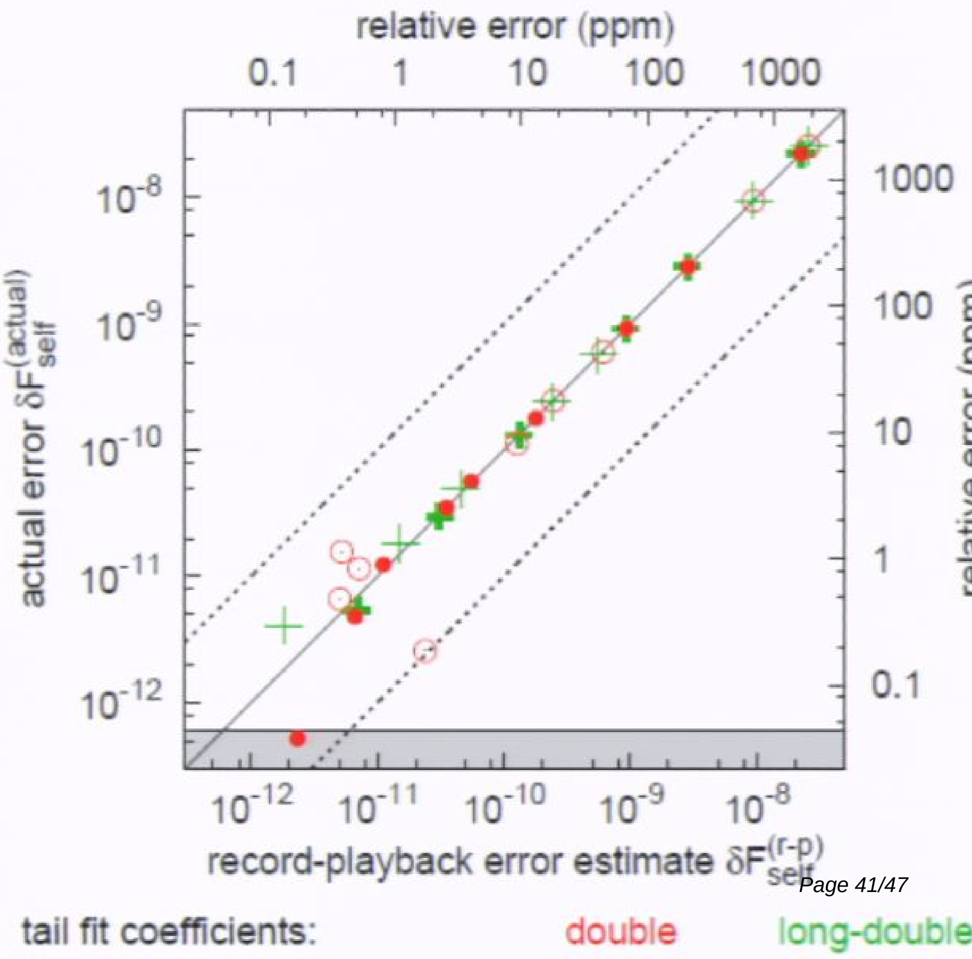

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Validate by comparing with highly-accurate frequency-domain results from Detweiler, Messaritaki, & Whiting, PRD **67** (2003), 104016:

Very good agreement except for errors $\lesssim 0.1$ ppm.

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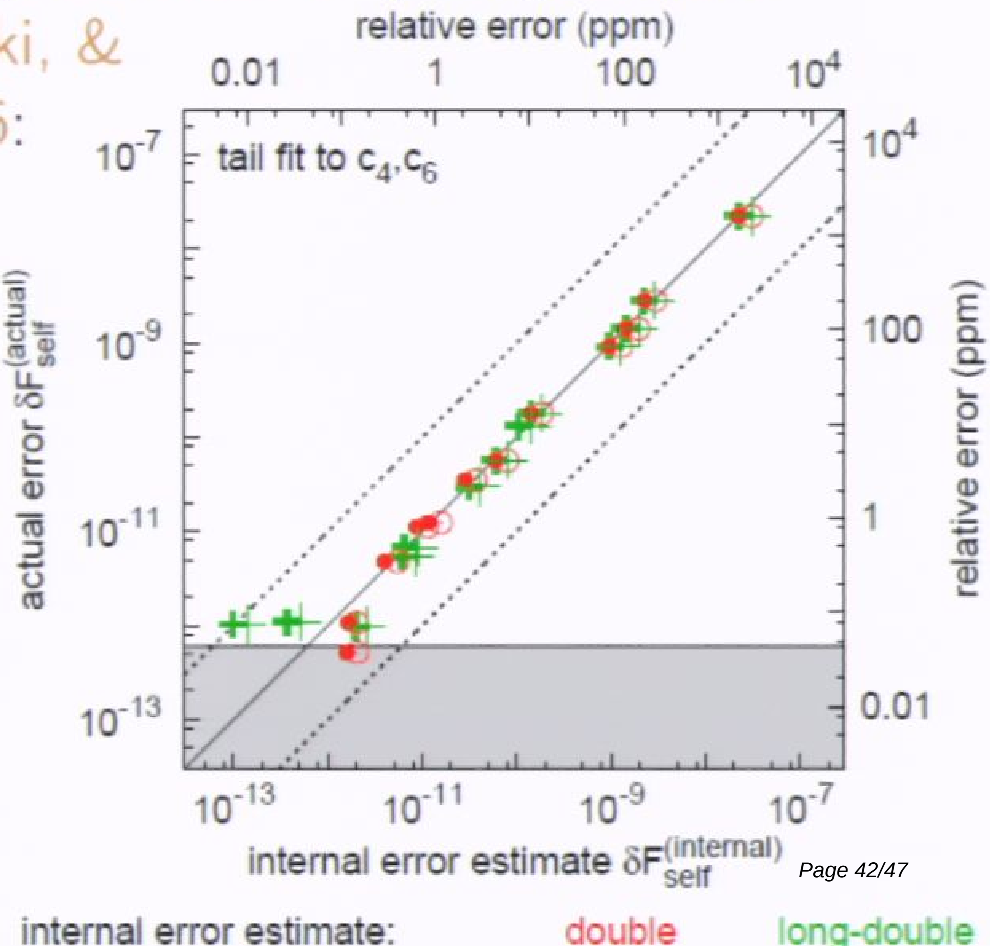

Sample Results

AMR error tolerance 10^{-19} (double-extended floating-point)

method	F_{self} (q^2/M)	$\delta F_{\text{self}}^{\text{(internal)}}$ (ppm)	$\delta F_{\text{self}}^{\text{(actual)}}$ (ppm)
record	$1.378\,448\,82 \times 10^{-5}$	0.44	0.39
playback2	$1.378\,448\,169\,8 \times 10^{-5}$	0.027	0.080
D., M., & W.	$1.378\,448\,28 \times 10^{-5}$	0.015	

CPU time is currently quite slow at more modest accuracy, because numerical parameters are still tuned for the very highest-accuracy test runs:

- Mostly same AMR error tolerance for each ℓ (more efficient would be less-accurate tolerance for modes with small $F_{\ell,\text{reg}}$)
- $\ell = 0$ integrations use a domain size of $30\,000M$ or $100\,000M$.
- Complex wave equation solved for $0 \leq \ell \leq 30$, $\ell = 35$, and

Conclusions

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- Internal error estimate from $|\text{outside} - \text{inside}|$.
- It's safe to treat different modes' internal error estimates as statistically independent for purposes of computing error estimates for the numerical force and tail force.
- Tail fit is highly ill-conditionined when fitting many orders simultaneously; renormalizing basis functions mostly solves this problem.

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- Floating-point roundoff limits my AMR code to ~ 0.1 ppm.
- My results agree beautifully with highly-accurate frequency-domain results of Detweiler, Messaritaki, & Whiting, PRD **67** (2003), 104016.
  $\Rightarrow$ confirms correctness of both codes & theories

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- Tail fit is highly ill-conditionined when fitting many orders simultaneously; renormalizing basis functions mostly solves this problem.
- Floating-point roundoff limits my AMR code to ~ 0.1 ppm.
- My results agree beautifully with highly-accurate frequency-domain results of Detweiler, Messaritaki, & Whiting, PRD **67** (2003), 104016.

  $\Rightarrow$ confirms correctness of both codes & theories

See Thornburg, arXiv:1006.xxxx for details (submitted to gr-qc