Title: Review of self-force computations in a radiation gauge

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Abstract: TBA

Self-force in a gauge appropriate to separable wave equations

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The self-force per unit mass is the acceleration with respect to the background metric:

$$u^\beta \nabla_\beta u^\alpha = -(g^{\alpha\beta} + u^\alpha u^\beta) u^\gamma u^\delta (\nabla_\gamma h^{\text{ren}}{}_{\beta\delta} - \frac{1}{2} \nabla_\beta h^{\text{ren}}{}_{\gamma\delta}) = f$$

Point-particle renormalization (MiSaTaQuWa):

Particle follows a geodesic of $h_{\text{renormalized}}$

$$h_{\text{renormalized}} = h_{\text{retarded}} - h_{\text{singular}}$$

Compute h_{ret} and h_{singular} with a regulator – e.g. a cutoff in spherical harmonic l

$$h_{\text{renormalized}} = \lim_{l\to\infty}\left[h(l) - h_{\text{singular}}(l) \right]$$

Teukolsky equation: separable operator $\mathcal{T}$

$$\mathcal{T}\psi_0 = S,$$

Source function S = $\mathcal{O}$ T,

T = energy-momentum tensor

$\mathcal{O}$ a 2nd-order differential operator

$$\mathcal{T} := +\partial_r \Delta \partial_r + \partial_{\cos\theta} \sin^2\theta \, \partial_{\cos\theta} - \frac{1}{\Delta}\left[(r^2+a^2)\partial_t + a\partial_\phi - 2(r-M)\right]^2$$

$$-8(r+ia\cos\theta)\partial_t + \frac{1}{\sin^2\theta}\left[a\sin^2\theta\partial_t + \partial_\phi + 2i\cos\theta\right]^2$$

The decoupled, separable Teukolsky equation gives ψ_0 and ψ_4 but not *h*.

In vacuum, there is a prescription for finding *h*

(Chrzanowski, Cohen-Kegeles, Wald, Lousto-Whiting, Ori)

•The lRG gauge

$$h_{\alpha\beta} l^{\beta} = 0 \qquad\qquad h = 0$$

constraints, agreeing for outgoing waves in flat
pace with a transverse-tracefree gauge.

Find h via hertz potential,

$$\psi_0 = \frac{1}{8}[L^4\overline{\Psi} + 12\Psi_2\varrho^{-3}\partial_t\Psi]$$

$= (4 \text{ angular derivatives} + \text{time derivative})\Psi$

Algebraic solution
for each frequency and angular harmonic

The solution:

$$\bar{\Psi}_{\ell m\omega} = 8\frac{(-1)^m D\bar{\psi}_{0\ell -m-\omega} + 12iM\omega\psi_{0\ell m\omega}}{D^2 + 144M^2\omega^2},$$

D the Teukolsky-Starobinsky constant.

There is an alternative equation for Y, involving four radial derivatives, and for each angular harmonic it gives the same $\Psi_{lm\omega}$. (Ori '03)

Reconstructed metric:

$$_{11} = -\frac{r^2}{2}(\bar{\eth}^2\Psi + \eth^2\bar{\Psi}) = O(r^{-3}),$$

$$_{13} = \frac{r^3}{2\sqrt{2}}\left(\partial_t - f\partial_r - \frac{2}{r}\right)\bar{\eth}\Psi = O(r^{-2}),$$

$$_{33} = \ldots = O(r^{-1}) \qquad (f = 1 - 2M / r$$

This is the same behavior as a Lorenz gauge for outgoing radiation:

and there is a corresponding 1RG metric for outgoing radiation related to a Lorenz (transverse tracefree) gauge by an asymptotically vanishing gauge vector,

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Outline of method

Compute ψ_0^{ret} from the Teukolsky equation as a mode sum over ℓ, m, ω

Find the potential Ψ^{ret} from ψ_0^{ret} with algebraic solution for each mode for $r \neq r_0$.

$\Psi^{\text{ret}}_{\ell m \omega}$ unique.

Find, in a radiation gauge, the components of $h^{\text{ret}}_{\alpha\beta}$ nd its derivatives that occur in the expression for f^{α}.

(h does not yet have contributions from mass, spin)

Compute $a^{\text{ret}\,\alpha}$ from the perturbed geodesic equation s a mode sum truncated at $\ell_{\max}$

Compute the renormalization vectors A^{α} and B^{α}

Determine the contribution to $h_{\alpha\beta}^{\rm ret}$ and then $f^{\,\rm ret\,\alpha}$ of the perturbations in mass, angular momentum and change in CM.

6 real parameters:

$\delta m, \delta a$ in, e.g. perturbed Boyer-Lindquist gauge

two angles for gauge transformation $r > r_0$ to rotate spin

one parameter for gauge transformation $r > r_0$

to remove asymptotic dipole moment.

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Mode sum renormalization (a view of Barack-Ori)

To find a decomposition of h_0^{ret}, can extend h_0^{ret} smoothly to the sphere $r = r_0$

Two smooth extensions differ by a smooth function. The coefficients in their angular harmonic expansion agree – difference between the harmonic expansions disagreement falls off faster than any power of L.

That is, the expansion

$$h^{ret}_{Lorenz} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^{R}_{Lorenz}$$

is a property of h^{ret}_{Lorenz} itself.

The first three terms renormalize h^{ret}_0 to the order needed to compute the self-force.

Orders in powers of L same as for Lorenz gauge
(A, B, C different constants in each line below)

$$h^{ret}_{Lorenz} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^{R}_{Lorenz}$$

∂^2

$$\psi^{ret}_0 = \Sigma[A\,L^2 + B + \frac{C}{L} + \cdots] + \psi^{R}_0$$

∂^{-4}

$$\Psi = \Sigma[\frac{A}{L^2} + \frac{B}{L^3} + \frac{C}{L^4} + \cdots] + \Psi^{R}$$

∂^2

$$h^{ret}_{rad} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^{R}_{rad}$$

∂

Because the first three terms in the harmonic expansion of ψ_0^{ret} determine the first three terms in the harmonic expansion of h^{ret} one can in principle find ψ_0^R from ψ_0^{ret}

- Although the argument required the order

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ & & \downarrow \\ & & h^{R}_{rad} \end{array}$$

The diagram commutes

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ \downarrow & & \downarrow \end{array}$$

because of the linear

- Abhay Shah finds the singular parts of analytically, but a match to the asymptotic harmonic expansion of h^{ret} gives the expected agreement. To obtain more rapid convergence of the expansion for the renormalized field, he finds by numerical matching the next set of terms in the asymptotic harmonic expansion of ψ_0^{ret} .

TABLE I: The fractional error in the renormalization coefficient A and the error in B for $\langle\psi_0\rangle$ is given here for a particle in circular orbit in a Schwarzschild background at radius r_0. ΔA and ΔB are the differences between the coefficients obtained numerically and by using the analytic expression. The analytic value of B is zero.

r_0/M	A_{analytic}	$\lvert\Delta A/A\rvert$	$\lvert\Delta B\rvert$
8	-0.002507548110466834	6.717×10^{-10}	8.496×10^{-10}
10	-0.001214016915072354	1.666×10^{-11}	1.245×10^{-11}
15	-0.0003356104323965837	2.419×10^{-12}	6.003×10^{-13}
20	-0.0001370231924969076	1.365×10^{-14}	3.853×10^{-14}
25	-0.00006882828667366571	5.706×10^{-13}	7.462×10^{-15}
30	-0.00003933557520091981	7.843×10^{-13}	1.067×10^{-14}
35	-0.00002455304484596332	8.549×10^{-13}	8.341×10^{-15}
40	-0.00001634080095822354	7.960×10^{-13}	5.325×10^{-15}
45	-0.00001141847437793787	7.605×10^{-13}	3.673×10^{-15}
50	-0.000008290448479296679	7.722×10^{-13}	2.807×10^{-15}
55	-0.000006208226467936966	6.961×10^{-13}	1.912×10^{-15}
60	-0.000004768831841073202	6.739×10^{-13}	1.433×10^{-15}
70	-0.000002990259098529844	6.209×10^{-13}	8.488×10^{-16}
80	-0.000001996831417921701	6.316×10^{-13}	5.893×10^{-16}

ingular field in generic gauge (Gralla):

et $$f^\alpha[h] := -\mathrm{m}(g^{\alpha\delta} + u^\alpha u^\delta)\left(\nabla_\beta h_{\gamma\delta} - \frac{1}{2}\nabla_\delta h_{\beta\gamma}\right)u^\beta u^\gamma.$$

et h^s be tensor in a neighborhood of the particle's trajectory for which h^{ret} - h^s is differentiable and h^s has even parity to $O(\rho^{-1})$, here ρ is the geodesic distance to the particle's trajectory;

at each point P of the trajectory,

$$\lim_{\rho\to 0}\int f[h^s]\ d\Omega = 0,$$

here the integral is over a sphere of constant ρ,

hen the renormalized self-force is given in this gauge by

$$f^{\mathrm{ren}} = \lim_{\rho\to 0}\int (f[h^{\mathrm{ret}}] - f[h^s])d\Omega.$$

hat is, the first-order perturbative correction to the geodesic of the ckground spacetime is given by a connecting vector Z^α that satisfies hat is, the first-order perturbative correction to the eodesic of the background spacetime is given by a onnecting vector Z^{α} that satisfies

$$mu\cdot\nabla(u\cdot\nabla Z^{\alpha}) = f^{\text{ren}\,\alpha}$$

ow Abhay Shah's computation of f^{s} finds

$$f^{s} \propto \nabla\left(\frac{1}{\rho}\right)$$

or a particle in circular orbit in a Schwarzschild ackground, as is the case for a Lorentz gauge.

Numerical demonstration that $a^s = -(1-2M/r_0)\nabla(\frac{1}{\rho})$

r_0/M	A	B	$A/\tilde{A}$	$B/\tilde{B}$	$-f_0$
6	$1.9641855032960099 \times 10^{-2}$	$9.7199207699180032 \times 10^{-3}$	-0.6666666666667120	-0.6666666666336628	-0.6666666666666667
30	$1.0540925533899464 \times 10^{-3}$	$5.2303191867714355 \times 10^{-4}$	-0.9333333333333372	-0.9333333333323244	-0.9333333333333333
85	$1.3594386461871310 \times 10^{-4}$	$6.7777662345830980 \times 10^{-5}$	-0.9764705882352940	-0.9764705882355747	-0.9764705882352941

TABLE I: The first column shows the radius of the orbiting particle in terms of background Schwarzschild coordinate r. The second and the third columns show the leading and the sub-leading regularization parameters. The fourth and the fifth columns show the ratio of the leading and subleading regularization parameters that we find numerically and those from leading and subleading parts of $\partial_r(1/\rho)$. The last column shows the value of $f = -(1-2M/r_0)$

ecause

ψ_0 has even parity to leading order in ρ,

the tetrad vectors and spin coefficients are constant to leading order in ρ,

Ψ and $h_{\alpha\beta}$ are constructed from ψ_0 by derivatives along the tetrad vectors and multiplication by spin coefficients

$h_{\alpha\beta}$ has even parity to leading order in ρ.