Title: A fast frequency-domain calculator for the gravitational self force

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Abstract: TBA

A Fast Frequency-Domain Calculator for Gravitational Self-Force in Schwarzschild Spacetime

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Ultimate Goal: Gravitational Wave Detection

- To see GWs (from EMRIs) in the LISA output, we need very accurate GW templates.
- To get these templates, we need a very good modeling of orbits in strong gravity.
- In short, we need very precise, fast Self-Force computations.
- F-domain SF computations: fast(er) $\Rightarrow$ cover parameter space (p, e) in Schwarzschild.
- **Schwarzschild**: Solved (T-domain: Barack, Sago (full GSF); F-domain: Poisson, Detweiler). **Next**: Cover parameter space (F- vs. T-domains). **Kerr:** Eccentric scalar SF (Warburton); GSF: to come. Parameter space is huge for generic orbits.

Background

- A point particle of mass μ in Schwarzschild (Sch.) background with mass M.
- Particle induces a perturbation on the background spacetime. Geodesics are perturbed.
- We solve perturbed Einstein's equation in non-flat background.

$$G_{\mu\nu}(g_{\mu\nu} + h_{\mu\nu}) = 8\pi T_{\mu\nu}$$

- Retain only linear order $\mathcal{O}(h)$ perturbation. Recall $G(g) = 0$ (vacuum).

  Further simplifications:

  - Use $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h$
  - Lorenz gaugewhere $\nabla_\mu \bar{h}^{\mu\nu} = 0$ and with $\nabla_\mu T^{\mu\nu} = 0$

$$\Box \bar{h}_{\mu\nu} + 2R^{\alpha\ \beta}_{\ \mu\ \nu} \bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}.$$

  10 2nd order coupled wave-like equations sourced by $T_{\mu\nu} \propto \delta(r - r_p(t))$.

- The GSF is constructed from the gradient of $\bar{h}_{\mu\nu}$ projected in the direction orthogonal to 4-velocity u^μ.

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Separation of Variables: Tensor Spherical Harmonics

- Separate the angular part. Use tensor spherical harmonics
  10-dim. basis for any rank-2, symmetric 4-dim. tensor.
  In Kerr, spheroidal harmonics.

$$\bar{h}^{\ell m}_{\mu\nu} = \frac{\mu}{r} \sum_{\ell,m} \sum_{i=1}^{10} a^{(i)\ell} \bar{h}^{(i)\ell m}(t,r) Y^{\ell m}_{\mu\nu}(\theta,\phi;r).$$

- Angular variables decouple. Einstein equation becomes (at each ℓ, m)

$$\Box_{sc} \bar{h}^{(i)} + \mathcal{M}^{(i)}_{(j)} h^{(j)} = \mathcal{S}^{(i)}. \tag{1}$$

  $\Box_{sc}$ is the usual scalar field wave operator:

$$\Box_{sc} = \frac{1}{4}\left[\partial_t^2 - \partial_{r_*}^2 + f(r)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)\right]$$

  where $f(r) = 1 - \frac{2M}{r}$.

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Field Equations in Frequency Domain

- Bound orbits, up to $e = e_{max}(?)$, frequency domain faster. Determine e_{max}.
- For now: Circular orbits i.e. $r_p(t) = r_0$

$$\bar{h}^{(i)}_{\ell m}(t,r) = \sum_m R^{(i)}_{\ell m}(r) \exp^{-i\omega_m t} \qquad \omega_m = m\Omega_\phi$$

$$\tilde{\Box}_{sc} R^{(i)}(r) + \mathcal{M}^{(i)}_{(j)} R^{(j)}(r) = \mathcal{S}^{(i)}(r).$$

- 10 2nd order coupled ODEs. $\mathcal{M}^{(i)}_{(j)} R^{(j)}$ contain the couplings.

$$\mathcal{M}^{(1)}_{(j)} R^{(j)} = \frac{1}{2} f^2 f' R^{(3)}_{,r} + \frac{f}{2r^2}(1 - 4M/r)\left(R^{(1)} - R^{(5)} - fR^{(3)}\right) - \frac{f^2}{2r^2}(1 - 6M/r)R^{(6)}$$

- 4 gauge equations from $\nabla_\mu \bar{h}^{\mu\nu} = 0 \Rightarrow$ 4 1st order coupled ODEs.
- Parity: Tensor spherical harmonics decouple:

  10 ODEs $\Rightarrow$ 7 even $\oplus$ 3 odd. 4 gauge $\Rightarrow$ 3 even $\oplus$ 1 odd.

  Even modes: $\ell + m =$ even

  $(i) = 1 \cdots 7$ eqtns. for $R^{(1)\cdots(7)}$ w/ $R^{(8)} = R^{(9)} = R^{(10)} = 0$

  Odd modes: $\ell + m =$ odd

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  Odd modes: $\ell + m = \text{odd}$

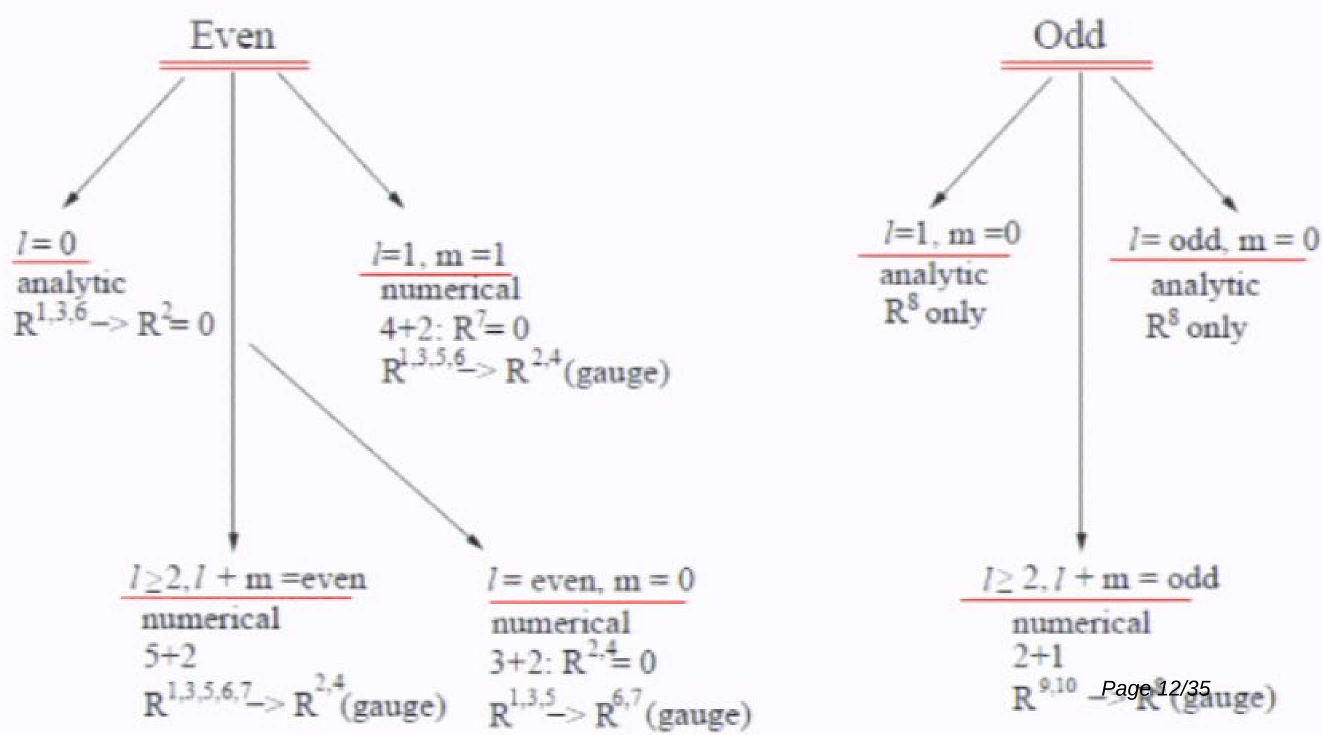
Hierarchical Solving Scheme

- Not everything couples together at each ℓ, m.
  Solve certain sets of ODEs first $\Rightarrow$ Use solutions to obtain other fields.
  Can also use the gauge equations.

	Even ($\ell + m = 2N$)	Odd ($\ell + m = 2N + 1$)
$\ell = 0$	$(i) = 1, 3, 6 \to 2$	0
$\ell = 1$	$m = 1 : (i) = 1, 3, 5, 6 \to 2, 4$	$m = 0 : (i) = 8$ only
$\ell \geq 2$	$(i) = 1, 3, 5, 6, 7 \to 2, 4$	$(i) = 9, 10 \to 8$

ODE Solving Tree:

Analytic solutions: monopole, odd dipole.



The Crux: Boundary Conditions (BC)

- In Time Domain: No BC. Inaccurate IC cause spurious radiation.
- In Freq. Domain, we solve 1-dim. ODE so must specify correct BCs.
- **Ansatz** $m \neq 0$ dynamical modes: in/outgoing waves at EH/infinity.

$$R_{in}^{(i)} = e^{-i\omega r_{in}^*} \sum_k b_k^{(i)} (r_{in} - 2M)^k \,, \quad R_{out}^{(i)} = e^{i\omega r_{out}^*} \sum_k \frac{a_k^{(i)}}{r_{out}^k}$$

- BC for $m = 0$ static modes are determined by regularity.

  Series is infinite but truncate at some k_{max}. $k = k_{start}$ are free.
- Substitute ansatz into each ODE. Obtain recursion relations for $a_k^{(i)}, b_k^{(i)}$. E.g.

$$\begin{aligned} 4M^2 k(k - 4Mi\omega) b_k^3 &= G_{k-1}^3 b_{k-1}^3 + 4M(b_{k-1}^1 - b_{k-1}^5 + b_{k-1}^6) \\ &\quad + H_{k-2}^3 + 2(b_{k-2}^1 - b_{k-2}^5 - b_{k-2}^6) - F_{k-3}^1 b_{k-3}^3 \end{aligned}$$

where $G_k^3 = 2M\,[k(2k-1) - \ell(\ell+1) + 1 - 12Mi\omega k]$.

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- 28 BCs in total. $k_{max} = 5$ to 10 usually.

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Matching the Inner and Outer Solutions

- 2 Junction Conditions at $r = r_0$:

  (1) $R^{(i)}_{in,out}$ are continuous, **(2)** $\frac{dR^{(i)}_{in,out}}{dr}$ have a jump discontinuity $\propto S^{(i)}$.
- Single field: Undergrad. QM, like solving Schrodinger eqtn. across $\delta-$potential.
- Many coupled fields: Follow Detweiler & Poisson 2003.
  - a^i_k, b^i_k that start the recursion relations for BC form a vector space.

    Example: Even sector, $\ell \geq 2, m > 1$.

    7 field and 3 gauge equations $\Rightarrow$ 4 d.o.f manifest in $a^{1,5,6,7}_{k=0}, b^{1,5,6,7}_{k=0}$

    4-dim. vector space for both inner (-) and outer (+) solutions.

    Basis vectors: (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)
  - Solve coupled ODEs with each basis vector as BC $\rightarrow$

    4 times each for inner/outer solutions $\rightarrow$ Solve ODEs 8 times.

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Matching continued

- Output at each ℓ, m: fields $A^{1,2,3,4}_{in/out}, B^{1,2,3,4}_{in/out}, C^{1,2,3,4}_{in/out}, D^{1,2,3,4}_{in/out}$
- Similarly, for the derivatives: $A'^{1,2,3,4}_{in/out}, B'^{1,2,3,4}_{in/out}, C'^{1,2,3,4}_{in/out}, D'^{1,2,3,4}_{in/out}$

  e.g. A^1_{out} is $R^{(1)}_{+}$ obtained by using the vector $(1, 0, 0, 0)$
  i.e. $a^1_0 = 1$, $a^{5,6,7}_0 = 0$.
- The output fields and their derivatives form a 8×8 complex matrix.
- To obtain the normalized, inhomogeneous solutions, we must solve for the complex amplitudes x_i in the following equation.

$$\begin{pmatrix} -A^1_{in} & -A^2_{in} & -A^3_{in} & -A^4_{in} & A^1_{out} & A^2_{out} & A^3_{out} & A^4_{out} \\ -B^1_{in} & \cdots & & -B^4_{in} & B^1_{out} & \cdots & & B^4_{out} \\ -C^1_{in} & \cdots & & -C^4_{in} & C^1_{out} & \cdots & & C^4_{out} \\ -D^1_{in} & \cdots & & -D^4_{in} & D^1_{out} & \cdots & & D^4_{out} \\ -A'^1_{in} & -A'^2_{in} & -A'^3_{in} & -A'^4_{in} & A'^1_{out} & A'^2_{out} & A'^3_{out} & A'^4_{out} \\ -B'^1_{in} & \cdots & & -B'^4_{in} & B'^1_{out} & \cdots & & B'^4_{out} \\ -C'^1_{in} & \cdots & & -C'^4_{in} & C'^1_{out} & \cdots & & C'^4_{out} \\ -D'^1_{in} & \cdots & & -D'^4_{in} & D'^1_{out} & \cdots & & D'^4_{out} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ J_1 \\ 0 \\ J_6 \\ J_7 \end{pmatrix}$$

- Inhomogeneous sol.: e.g. $R^{(1)}_{in} = x_1 A^1_{in} + x_2 A^2_{in} - x_3 A^3_{in} + x_4 A^4_{in}$ etc.

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Numerical Details

- Code in C, using GNU Scientific Library (GSL) repository.

  Rewrote 2nd order ODEs as 1st order ODEs.
  Used Embedded Runge-Kutta Prince-Dormand (rk8pd) integrator
  Empirically shown to be the fastest integrator (the coarsest).
  Typical ODE solver error threshold: $(10^{-10}, 10^{-10})$.
- At each ℓ, m, same set of ODEs is solved many times:
  - 4 times for inner/outer, real/imag solutions
  - Another N times to cover the N-dim. vector space

    e.g. Even sector $\ell \geq 2, m > 1$: 4-d vector space $\Rightarrow$ Solve 16 times!
    Similarly in Odd sector $\ell \geq 2$: 2-d vector space $\Rightarrow$ Solve 8 times
- For $\ell = 0$ *to* 15, ODEs are solved
  $(12 + 7 \times 12 + 56 \times 8 + 63 \times 16) = 1552$ times!

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- Code in C, using GNU Scientific Library (GSL) repository.

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Numerics contd.

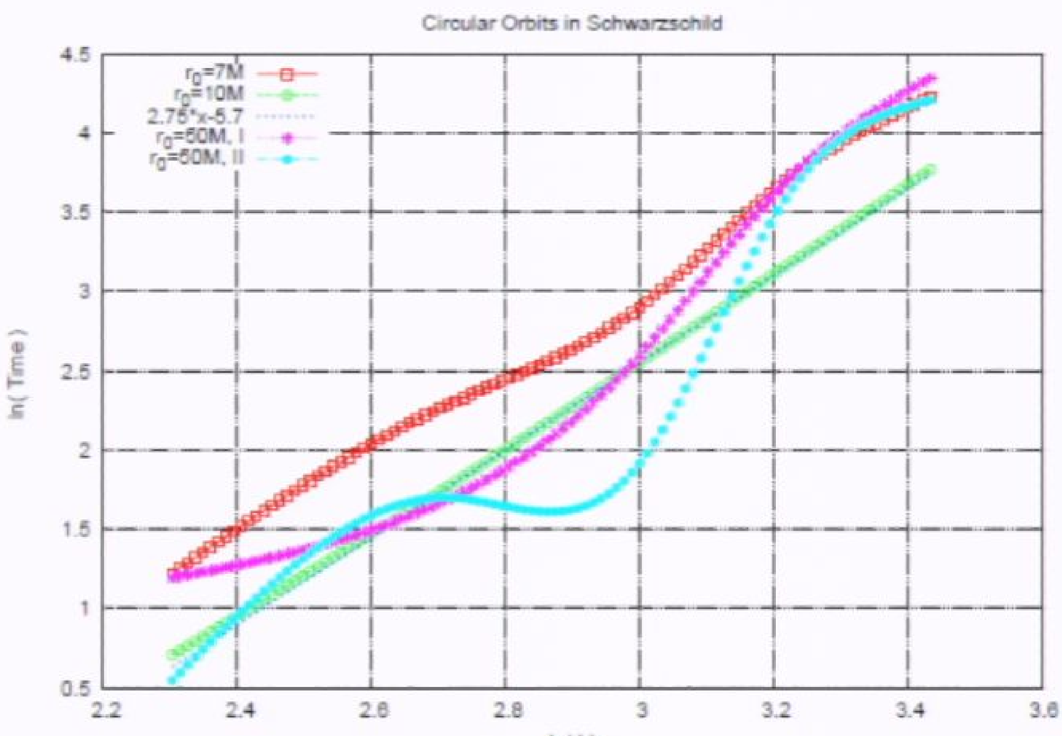
Speed tests:

	$r_0 = 50M^{(I)}$	$r_0 = 50M^{(II)}$	$r_0 = 10M$	$r_0 = 7M$
ℓ	time (m:s)	time (m:s)	time (m:s)	time (m:s)
10	3:18	1:44	2:02	3:32
15	5:22	5:29	5:48	9:45
20	13:10	6:39	12:43	17:58
25	40:20	36:31	23:43	40:40
31	1:17:31	1:07:21	43:30	1:08:43

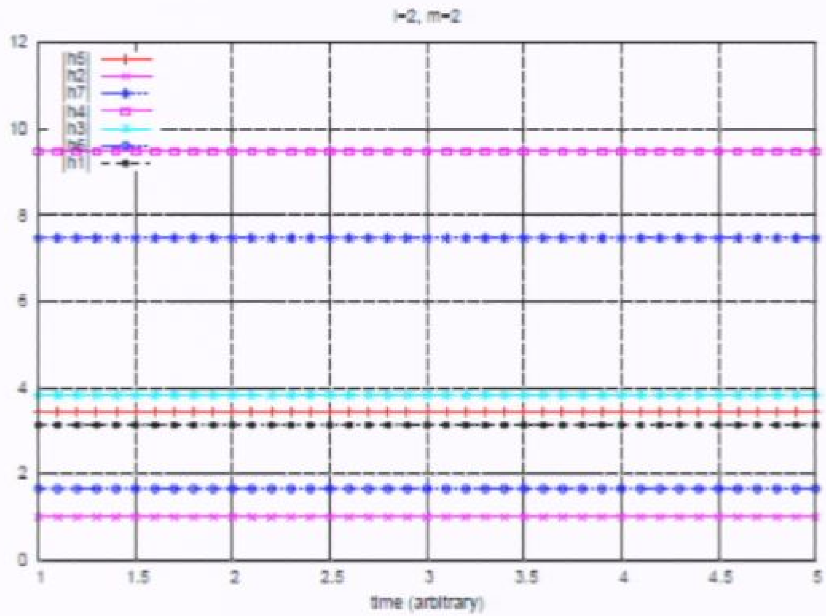
I: 10^{-11} for rel. & abs. accuracies.

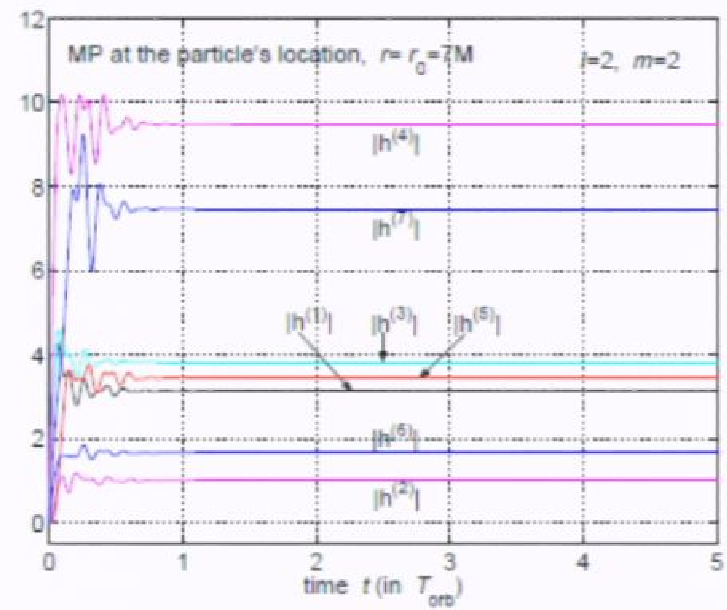
II: 10^{-10} for rel. accuracy.

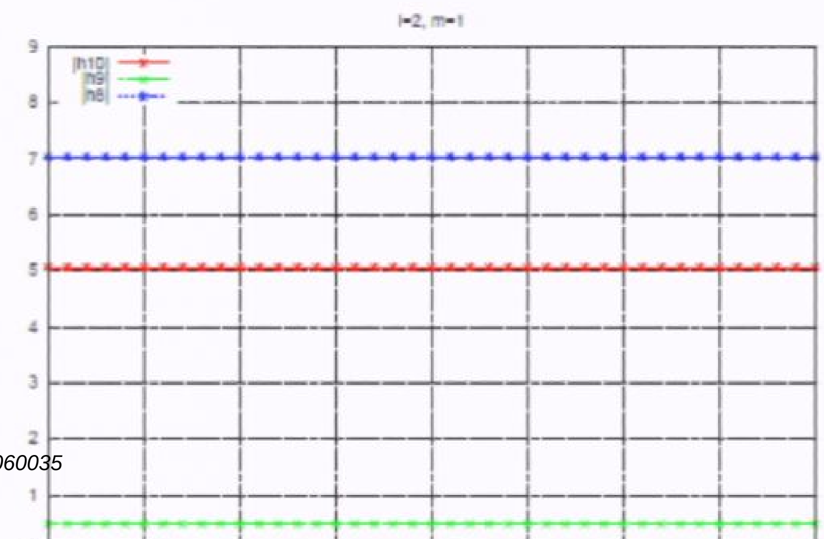
Some Power law scaling for run time as i.e. $t \sim \ell^{\alpha}$ with $\alpha \simeq 2.75$.

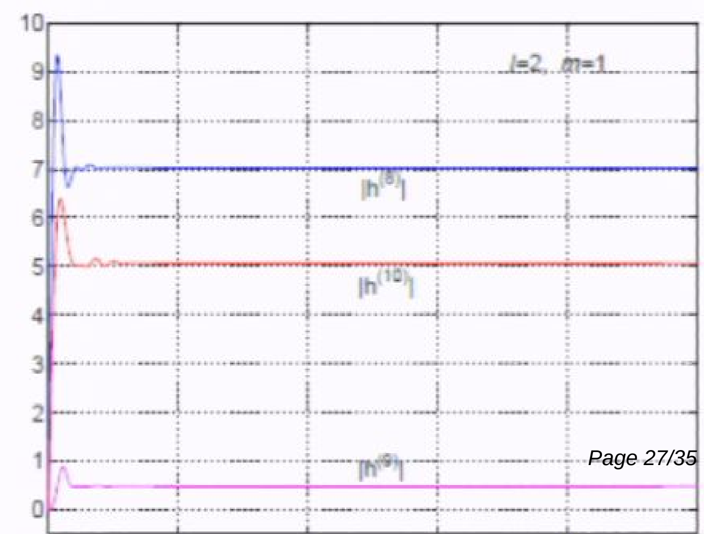


Results: Comparing $|h^{(i)}|$ with Barack & Sago 2007









What about the Self-Force?

- Physical SF F^{μ}_{SF} via "mode-sum" scheme: $F^{\mu}_{SF} = \sum_{\ell=0}^{\infty} F^{\mu reg\,\pm}_{\ell}$
- Regularization: Removes divergent parts of the field so ℓ-modes scale as L^{-2} in where $L = \ell + 1/2$ and

$$F^{\mu reg\,\pm}_{\ell} = F^{\mu\,\pm}_{\ell} - LA^{\mu}_{\pm} - B^{\mu}$$

  No t-regularization in circular orbits, only r.

  $\Rightarrow$ Only keeps the timelike support of the Green's function: tail field $\bar{h}^{tail}_{\mu\nu}$.

- Complication: Regularization is done using $Y^{\ell' m}$ but our decomposition is in $Y^{\ell m}_{\mu\nu}$.

  $\Rightarrow$ coupling between each tensor mode ℓ

  and up to 7 scalar modes $\ell - 3 \leq \ell' \leq \ell + 3$.

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$$\begin{aligned} F^{\mu\,\pm}_{\ell} &= \frac{\mu^2}{r_0^2} \sum_{m=-\ell}^{\ell} Y^{\ell m}(\pi/2, \phi_0) \times \\ & \left[\mathcal{F}^{\mu\ell-3,m}_{(-3)} + \mathcal{F}^{\mu\ell-2,m}_{(-2)} + \mathcal{F}^{\mu\ell-1,m}_{(-1)} + \mathcal{F}^{\mu\ell,m}_{(0)} + \mathcal{F}^{\mu\ell+1,m}_{(+1)} + \mathcal{F}^{\mu\ell+2,m}_{(+2)} + \mathcal{F}^{\mu\ell+3,m}_{(+3)}\right] \end{aligned}$$

Progress, Current Status and Outlook

- **Currently**: Code computes $F_\ell^{\mu reg\,\pm}, F_{SF}^{\mu}$ for $\ell = 0 \cdots 28$.
  For $r_0 = 7M$, time $\gtrsim$ 1 hour; $r_0 = 10M$, time $\sim$ 40 minutes.
  Not there yet: (1) $F_\ell^{\mu reg\,+} \neq F_\ell^{\mu reg\,-}$. (2) Fluxes.
  Soon: Large ℓ-tail.
- Speed of progress so far:

Scalar		Gravitational		
Prep.	Coding	Prep	Coding	
			odd (2+1)	even (5+2)
1 month	2 months	5 weeks	5 weeks	3 weeks

- **Future:** Eccentric orbits.
  First: Revisit scalar field then gravitational SF.
  Expect code to slowdown a lot depending on e.
  Circular $\ell = 15$ takes $\lesssim$ 10 minutes at $r_0 = 7M$ so still not bad.
  With an 8-core machine, we can do ~ 10-20 points in parameter space/day.

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