Title: A tale of two spectral problems

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Abstract: This talk will discuss some surprising links which have emerged in the last few years between two at first sight distinct areas of mathematical physics: the spectral properties of certain simple schroedinger-like equations, and the Bethe ansatz techniques which are used to compute the energies of states in integrable quantum field theories. No knowledge of either area will be assumed.

P.I. 2/6/10

A tale of two spectral problems

...work with

Roberto Tateo
Clare Dunning
& also Junji Suzuki, Ferdinando Gliozzi

PLAN:

1: The first spectral problem

- Integrable models
- Bethe Ansatz
- TQ & fusion hierarchies
- CFT & minimal models

Baxter; Klümper-Batchelor-Pearce; Destri-de Vega; Bazhanov-Lukyanov-Zamolodchikov.

2: The second spectral problem

- ODEs and eigenvalues
- WKB, Stokes phenomenon, spectral determinants...

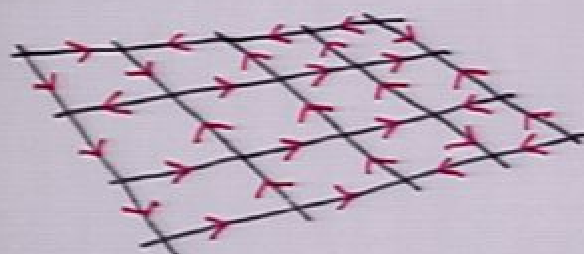
Sibuya; Voros; Bessis-Zinn-Justin; Bender-Boettcher.

3: Minimal models and McNuggets (if time)

4: Conclusions

1: Integrable models and the Bethe Ansatz:

Example: the 6-vertex model



– a generalisation of "square ice"

• : oxygen

• : hydrogen

"ice rule": near each oxygen, exactly two hydrogens

i.e, flux of arrows is conserved

A key part of Lieb's calculation was the Bethe Ansatz.

First generalise a little and ask for a weighted sum, a.k.a. a partition function:

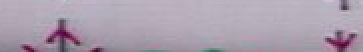
$$Z = \sum_{\substack{\text{allowed} \\ \text{arrow configs} \\ \{\sigma\}}} P(\{\sigma\})$$

the relative probability of $\{\sigma\}$, equal to a product over all vertices of the local Boltzmann weights.

The local Boltzmann weights depend only on the arrows incident on that vertex:

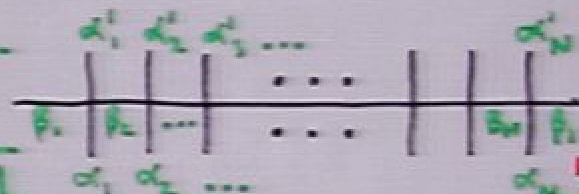
eg:  $= a =$ 

 $= b =$  (6 vertices, 3 parameters)



One approach to finding $\mathbb{Z}$ starts by computing the transfer matrix $\mathbb{T}$:

$$\mathbb{T}^{\alpha'_1 \dots \alpha'_N}_{\alpha_1 \dots \alpha_N} = \sum_{\{\beta\}}$$



[NB: $\beta_{N+1} = \beta_1$
ie periodic b.c. in space]

Product of the local Boltzman weights along a single row.

Then $\mathbb{Z} = \mathrm{Tr}(\mathbb{T}^M)$ and the leading (large M) behaviour of $\mathbb{Z}$ is given by the largest eigenvalues of $\mathbb{Z}$.

(Recall for the square ice $\mathbb{Z} \sim W^{M.N}$ & we wanted to

· Our original problem has reduced to diagonalising a -big- ($2^N \times 2^N$) matrix.
This is still hard to do!

the first spectral problem

· The Bethe Ansatz serves as an intelligent guess for an eigenvector of $\mathbb{T}$ which depends on a relatively small number of parameters called Bethe roots: $e_0, e_1 \dots e_k$ (say, where $k \sim O(N)$).

· The guess works iff the roots satisfy a set of coupled equations - the Bethe Ansatz Equations (BAE).

· 6-vertex example:

$$\boxed{\prod_{i=0}^{k}\left(\frac{e_i - q^2 e_j}{e_i - q^{-2} e_j}\right) = -1 \;, j=0..k}$$

(k+1 equations in k+1 unknowns)
(q depends on a, b & c)

· NB: [illegible]

TQ:

- The 'raw' Bethe ansatz is a significant improvement ($O(2^N) \to O(N)$ unknowns) but we still want $N \to \infty$ so the number of unknowns, k_i, is still diverging.

- Baxter, motivated by the 8-vertex model, found a neat reformulation:

  the $\boxed{\text{TQ relation}}$

- This needs an extra ingredient: recall the 6-vertex Boltz. wts. depended on 3 parameters a, b, c and so far we only saw 1 in the BAE, namely q.

- Wrap the remaining nontrivial parameter dependence in a second quantity ξ – the spectral

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- Wrap the remaining nontrivial parameter dependence in a second quantity ξ – the $\boxed{\text{spectral parameter}}$

Hence $\mathbb{T} \to \mathbb{T}(\xi)$

FACT: $[\mathbb{T}(\xi), \mathbb{T}(\xi')] = 0$

(the $\mathbb{T}$s are 'commuting transfer matrices')
($\rightsquigarrow$ conservation laws)

So diagonalise the $\mathbb{T}(\xi)$ simultaneously, and concentrate on a single eigenvalue $T(\xi)$. (eg the largest - ground state - one)

KEY FEATURE: $T(\xi)$ is an entire function of ξ.

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<u>KEY FEATURE</u>: $T(\xi)$ is an <u>entire</u> function of ξ.

<u>CLAIM</u>: The requirement that $T(\xi)$ follows from the 6-vertex BAE is equivalent to the demand that there exist a second entire function $Q(\xi)$ such that $\boxed{T(\xi)Q(\xi) = Q(q^{-2}\xi) + Q(q^{2}\xi)}$

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$$\boxed{T(\zeta)Q(\zeta) = Q(q_-^{-2}\zeta) + Q(q_+^{2}\zeta)}$$

(The T-Q relation)

TQ ⇒ BAE:

$$T(\zeta)Q(\zeta)=Q(q^{-2}\zeta)+Q(q^{2}\zeta)$$

at this stage, nothing to do with the Bethe roots

- Let the zeros of Q be $\{e_0, e_1, \dots e_K\}$

Then $Q(\zeta)=Q(0)\prod_{i=0}^{K}(1-\zeta/e_i)$

- Put $\zeta=e_j$ in the TQ relation. LHS vanishes (since $Q=0$ & T can't have poles) so, from the RHS,

$$Q(q^{-2}e_j)=-Q(q^{2}e_j)$$ must be true.

$$\Rightarrow -1=\frac{Q(q^2e_j)}{Q(q^{-2}e_j)}=\prod_{i=0}^{K}\left(\frac{1-q^2e_j/e_i}{1-q^{-2}e_j/e_i}\right)$$

$$=\prod_{i=0}^{K}\left(\frac{e_i-q^2e_j}{e_i-q^{-2}e_j}\right)$$ ← BAE

TQ $\Rightarrow$ BAE:

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Message: if T & Q satisfy TQ, the zeros of Q satisfy the BAE (& are highly constrained).

Note: if $K \to \infty$, you should worry about convergence of the product; but in fact TQ always holds...

TQ ⇒ BAE:

$$T(\zeta)Q(\zeta)=Q(q^{-2}\zeta)+Q(q^{2}\zeta)$$

- Let the zeros of Q be $\{e_0, e_1, \dots e_k\}$ at this stage, nothing to do with the Bethe roots

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Another strand – The Fusion hierarchy

- It is natural to define a set of "fused" transfer matrices $\mathbb{T}_j(\xi)$, $j=0,\frac{1}{2},1,\ldots$ leading to "fused" eigenvalues $T_j(\xi)$ after simultaneous diagonalisation, as follows:

$$\boxed{T_j(q^{-1}\xi)\,T_j(q\,\xi) = 1 + T_{j+\frac{1}{2}}(\xi)\,T_{j-\frac{1}{2}}(\xi)}$$

the fusion hierarchy

with
$$\boxed{\begin{aligned} T_0(\xi) &= 1 \\ T_{\frac{1}{2}}(\xi) &= T(\xi) \end{aligned}}$$
← 'initial conditions'

- This in itself doesn't constrain $T(\xi)$ (it just defines an infinite tower of further unknown functions)

BUT when q is a root of unity the hierarchy truncates:

Eg: if $q = e^{2\pi i/(2n+5)}$ then $T_{n+\frac{1}{2}}(\xi) = 1$

The hierarchy becomes a closed set of

Another point of view: ("field theory")

In fact Bazhanov, Lukyanov & Zamolodchikov (BLZ) defined their T & Q operators directly in Hilbert spaces of $c \le 1$ (2d) CFTs (conformal field theories) (a continuous family - if $q = e^{i\pi\beta^2}$

then $c = 1 - 6(\beta - \beta^{-1})^2$)

(NB: "c" is the central charge, a characteristic of scale-invariant QFTs in 2D)

In general these CFTs are non-unitary and non-rational.

But when truncation occurs they can be restricted and so-called minimal models (© BPZ) recovered.

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Eg: $q = e^{2\pi i/(2n+3)} \rightarrow c = 1 - 3\frac{(2n+1)^2}{(2n+3)}$

The minimal model $\mathcal{M}_{2,2n+3}$

NB: Minimal models have many special properties - eg finite # of "primary" fields with certain scaling dimensions, etc etc...

Summary:

- To solve integrable lattice models you need to diagonalise big matrices;
- The Bethe ansatz ≡ inspired eigenvector guess;
- ∃ reformulations in terms of sets of functional relations, either:

(A) {TQ + entirety of T & Q}

or

(B) {Fusion hierarchy + truncation for q a root of unity}

Either way the resulting 'closed' set of equations can be converted into nonlinear <u>integral</u> equations for T etc... • "DdV"/"KBP" in case (A);

• "TBA" in case (B).

No indices! & # Bethe roots is a parameter, making $N \to \infty$ (thermodynamic) limit easier.

Method used with great success by Baxter; Klümper-Batchelor-Pearce (KBP); Bazhanov-Lukyanov-Zamolodchikov (BLZ)... (90's)

2: The second spectral problem:

Consider the following non-standard 'cubic oscillator':

$$\left[-\frac{d^2}{dx^2} - (ix)^3\right]\psi(x) = E\,\psi(x)$$

with boundary conditions

$\psi(x) \to 0$ as $x \to \pm\infty$ on the real axis.

... a well-defined, but non-Hermitian, spectral problem.

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at $c = -22/5$ $(\mathcal{M}_{2,5})$

Initial question: $\left(-\frac{d^2}{dx^2} - (ix)^3\right)\psi = E\psi$

$$\psi(x \to \pm\infty) \to 0$$

¿What does the spectrum look like?

¿Is it real?

Bessis & Zinn-Justin, early 90's, conjectured that the answer is yes.

(motivation from non-unitary QFTs arising in stat. mech.)

... but it is not at all clear how to prove this...

(eg Mezincescu, 2000, showed any complex eigenvalues E are constrained by $\frac{\mathrm{Re}(E)}{|E|^2} < 10^{-5}$; this later improved all the way to 10^{-15} )

ODEs 101...

- Since the b.c.s are imposed at $x \to \pm\infty$ let's use WKB to see what solutions look like in this region —

The ODE is...

$$\left(-\frac{d^2}{dx^2} \underbrace{-(ix)^3 - E}_{P(x) \equiv -(ix)^3 - E}\right)\psi(x) = 0$$

... and WKB says

$$\psi(x) \sim \frac{1}{P(x)^{1/4}} \exp\left(\pm \int^x \sqrt{P(t)}\, dt\right)$$

$$\sim \frac{1}{P(x)^{1/4}} \exp\left(\pm\, i \int^x (it)^{3/2} dt\right)$$

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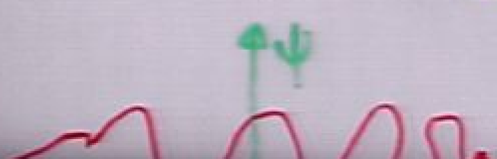
$$\psi(x) \sim \frac{\text{const}}{x^{3/4}} \exp\left(\pm \frac{2}{5}(ix)^{5/2}\right)$$

As $x \to +\infty$ one solution grows in modulus (is dominant) and one shrinks (is subdominant).

Ditto as $x \to -\infty$.

An eigenvalue occurs when a subdominant solution at $x \to +\infty$ is also subdominant at $x \to -\infty$.

Problem: WKB is not valid in the intermediate region between $x = +\infty$ and $x = -\infty$:



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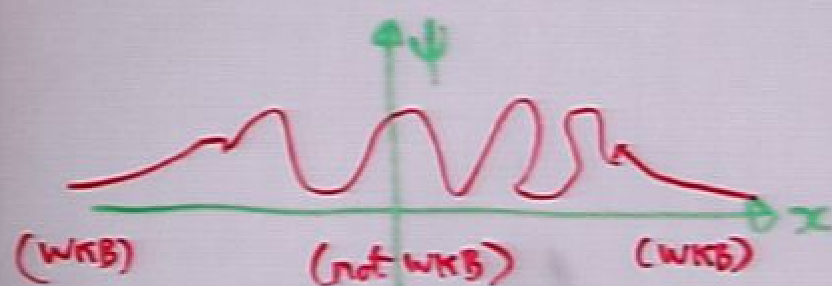


Options:

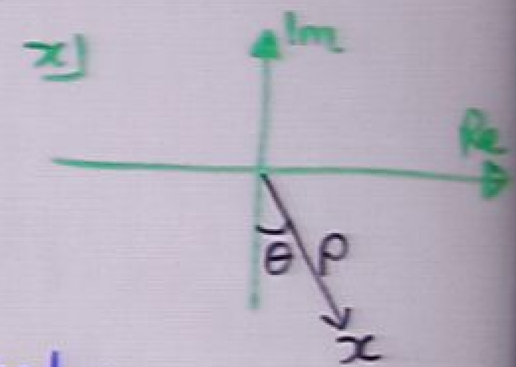
(a) use some sort of asymptotic matching to more complicated solutions (eg Airy functions for linear turning points)

or

(b) enlarge the perspective & consider the solution in the whole complex plane.

Set $x = \rho/_{i}\, e^{i\theta}$

$(\rho, \theta \in \mathbb{R})$



For $\rho \to \infty$ WKB still applies and so

$\psi(x) \sim \frac{\text{const}}{\cdots} \exp\left(\pm 2\, e^{i\frac{5\theta}{2}} \ldots 5/2 \right)$

for linear turning points)

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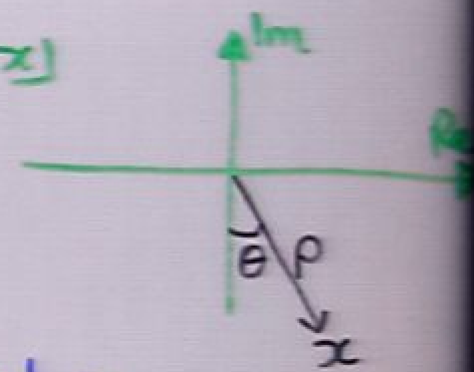

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$$\psi(x) \sim \frac{\text{const}}{\rho^{3/4}} \exp\left(\pm \frac{2}{5} e^{i\frac{5\theta}{2}} \rho^{5/2}\right)$$

ODE: $\left(-\frac{d^2}{dx^2} - (ix)^3 - E\right)\psi(x) = 0$ (*) $x = \rho_{r_i} e^{i\theta}$

WKB: $\psi(x) \sim \frac{\text{const}}{\rho^{3/4}} \exp\left(\pm 2_{/5} e^{i\frac{5\theta}{2}} \rho^{5/2}\right)$ (**)

Mini-paradox:

General nonsense states that solutions to (*) are entire and in particular single-valued.

But this is not true of the WKB solutions (**), even if we go round the origin at large ρ where WKB should work.

Resolution:

"The continuation of an asymptotic is not always the same as the asymptotic of a continuation"

a.k.a. the STOKES phenomenon.

WKB: $\psi(x) \sim \frac{\text{const}}{\rho^{3/4}} \exp\left(\pm \tfrac{2}{3}\, e^{[illegible]} \rho^{3/2} \right)$ $(**)$

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Stokes 101:

... take a closer look at the pattern of dominance and subdominance:

$$\psi_{\pm}(x) \sim \frac{\text{const}}{\rho^{3/4}} \exp\left(\pm \frac{2}{5} \boxed{e^{i\frac{5\theta}{2}}} \rho^{5/2}\right)$$

→ complex phase

- So long as the complex phase has a nonzero real part, then one solution grows (is dominant) and one decays (is subdominant) as $\rho \to \infty$.

  [In particular this is true for $\theta = \pm \pi/2$, the +ve & -ve real axes, making the original eigenproblem well-defined]

- But if $\mathrm{Re}\left(e^{i\frac{5\theta}{2}}\right) = 0$ then both solutions oscillate, marking the direction where dominant & subdominant solutions swap over. (→ "anti Stokes lines")

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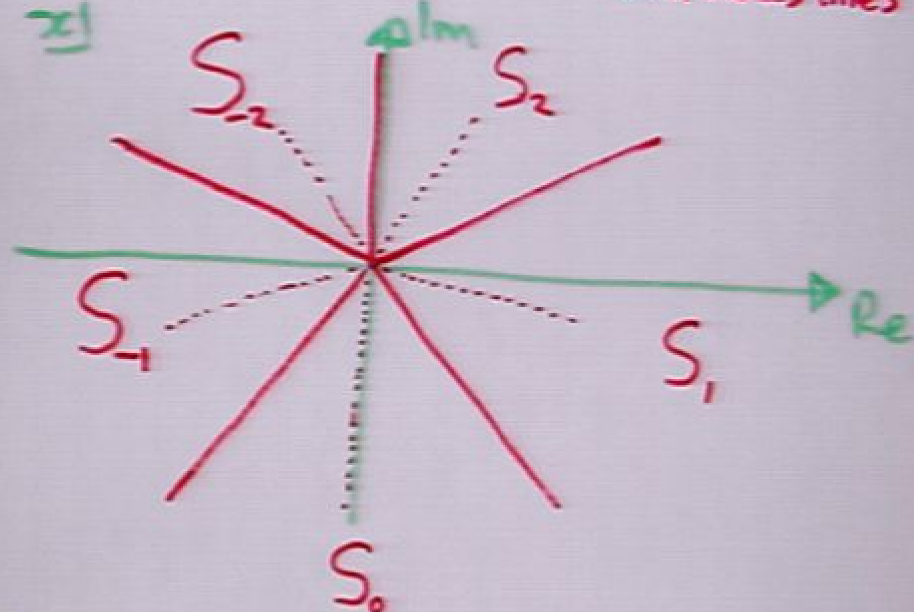
  [For the "real" cubic oscillator this happens on the real axis causing all sorts of grief]

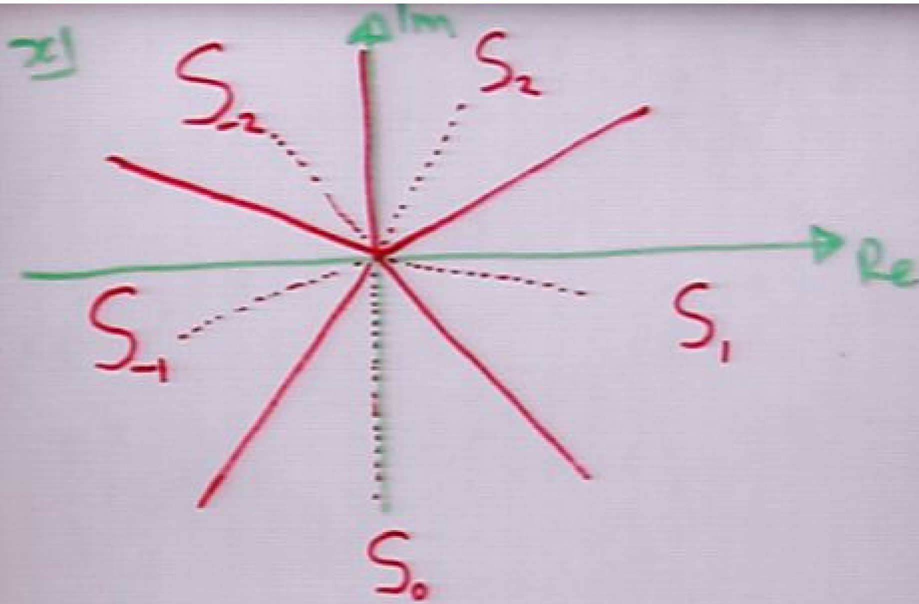
$$\mathrm{Re}\left[e^{i\frac{5}{2}\theta}\right]=0 \;\Rightarrow\; \theta=\frac{\pi}{5}+\frac{2n\pi}{5}$$

"anti-Stokes lines"



- The <u>antiStokes lines</u> split the complex plane into <u>Stokes sectors</u> S_0, $S_{\pm1}$, $S_{\pm2}$

- Within each Stokes sector one solution $\psi_\pm$ is dominant, the other $\psi_\mp$ is subdominant.
- In the middle of the stokes sector, on the (dotted) <u>Stokes lines</u>, dominance is maximal &



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- In the middle of the stokes sector, on the (dotted) <u>Stokes lines</u>, dominance is maximal & the coeff of the subdominant solution can jump – Stokes phenomenon – thus restoring single-valuedness.

- Note y_k and y_{k+1} must be independent (different dominance in $S_k \cup S_{k+1}$) and hence $\{y_k, y_{k+1}\}$ is a basis of solutions to the ODE.

- So $y_{-1}(x,E)$ must be a linear combination of $y_0(x,E)$ and $y_1(x,E)$:

$$\boxed{y_{-1} = C y_0 + \tilde{C} y_1}$$

A Sto relati

Stokes multipliers

and 'taking Wronskians'

$$\boxed{\begin{aligned} C &= \frac{W[y_{-1}, y_1]}{W[y_0, y_1]} \\ \tilde{C} &= \frac{W[y_{-1}, y_0]}{W[y_1, y_0]} \end{aligned}}$$

the Wronskian

$(W[f,g] = fg' - f'g)$

- Overlapping WKB asymptotics show that $W[y_k, y_{k+1}]$ is independent of E so we can normalise the y_k such that $W[y_k, y_{k+1}] = 1$, and then

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A Stokes relation

Stokes multipliers

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- Overlapping WKB asymptotics show that $W[y_k, y_{k+1}]$ is independent of E so we can normalise the y_k such that $W[y_k, y_{k+1}] = 1$, and then

$$\boxed{C = W[y_{-1}, y_1] \text{ and } \tilde{C} = -1}$$

and the Stokes relation becomes

$$(\alpha)\quad \boxed{C(E)\,y_0(x,E) = y_{-1}(x,E) + y_1(x,E)}$$

... this looks like a TQ relation!

Two problems: must eliminate "k" and "x" from $y_k(x,E)$.
① ②

① For k, use Sibuya-Symanzik rescaling and set

$$\boxed{y_k(x,E) := \omega^{k/2}\,y_0(\omega^{-k}x,\omega^{2k}E)}$$

with $\omega = e^{2\pi i/5}$

- If y_0 is subdominant in S_0, this y_k ticks all the boxes (easy to see)

Then (α) becomes

$$(\beta)\quad \boxed{C(E)\,y_0(x,E) = \omega^{-1/2}\,y_0(\omega x,\omega^{-2}E) + \omega^{1/2}\,y_0(\omega^{-1}x,\omega^{2}E)}$$

$$\boxed{C(E)\,D(E) = \omega^{-1/2} D(\omega^{-2}E) + \omega^{1/2} D(\omega^{2}E)}$$

where $\boxed{D(E) := y_0(0,E)}$

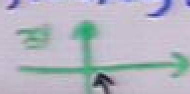
- This is exactly a 6-vertex TQ relation of the sort we saw earlier, at the special point $q = e^{2\pi i/5}$.

  [the factors $\omega^{\pm 1/2}$ emerge in the 6-vertex world if we impose suitable twists in the boundary conditions]

- Furthermore both C and D [which we can now think of as T and Q] are **Spectral Determinants**:

- $C(E) \equiv W[y_{-1}, y_1]$ vanishes iff the solutions y_{-1} and y_1 are proportional to each other:

- $D(E) \equiv y_0(0,E)$ vanishes iff y_0, vanishing as $x \to \infty$ on $-$ve imag axis, also vanishes at $x=0$



$$\boxed{C(E)D(E) = \omega^{-1/2} D(\omega^{-2}E) + \omega^{1/2} D(\omega^{2}E)}$$

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- $D(E) \equiv y_0(0,E)$ vanishes iff y_0, vanishing as $x \to \infty$ on -ve imag axis, also vanishes at $x=0$ (a *radial* spectral problem)

Conclusion (to this bit):

$$\boxed{\text{At } q=e^{2\pi i/5} \text{ the } T \;\&\; Q \text{ operators of BLZ encode spectral data for the complex cubic anharmonic oscillator } \left[-\frac{d^2}{dx^2}-(ix)^3\right]\psi=E\psi.}$$

This is $\beta^2 = 2/5$ (remember, $q = e^{i\pi\beta^2}$)

and hence $c = 1 - 6(\beta - \beta^{-1})^2$

$= -22/5$

... as was claimed a while ago.

Reality proof:

[illegible]

$$C(E)D(E) = \omega^{-1/2} D(\omega^{-2}E) + \omega^{1/2} D(\omega^{2}E) \quad \leftarrow \text{(TQ)}$$

- The zeros E_i of $C(E)$ are the eigenvalues of the non-Hermitian spectral problem. Write $C(E) = \prod_{i=0}^{\infty}\left(1 - \frac{E}{E_i}\right)$.
- The (negated) zeros $-e_i$ of $D(E)$ are the eigenvalues of a Hermitian spectral problem - so they are all real.

Write $D(E) = \prod_{i=0}^{\infty}\left(1 + E/e_i\right)$.

Now consider TQ at a zero of C, say at $E = E_k$:

LHS=0, so $$\boxed{\frac{D(\omega^{-2}E_k)}{D(\omega^{2}E_k)} = -\omega}$$

ie $$\boxed{\prod_{i=0}^{\infty}\left(\frac{e_i + \omega^{-2}E_k}{e_i + \omega^{2}E_k}\right) = -\omega}$$

... where the e_is are all real (and positive) but we don't (yet) know about E_k.

Reality proof:

[DT99, TQ, KT]

$$C(E)D(E) = \omega^{-1/2} D(\omega^{-2}E) + \omega^{1/2} D(\omega^{2}E) \quad \leftarrow (TQ)$$

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... where the e_is are all real (and positive) but we don't (yet) know about E_k.

Key trick: take $|\ |^2$ to find

$$\prod_{i=0}^{\infty} \left| \frac{e_i + \omega^{-2} E_k}{e_i + \omega^{2} E_k} \right|^2 = 1 \qquad (\omega = e^{2\pi i/5})$$

Clearly true if E_k is real! (since each term in product $= 1$)

Otherwise suppose $E_k = |E_k| e^{i\delta_k}$. Then eqn is

$$\prod_{i=0}^{\infty} \left(\frac{e_i^2 + |E_k|^2 + 2e_i |E_k| \cos(\frac{4\pi}{5} - \delta_k)}{e_i^2 + |E_k|^2 + 2e_i |E_k| \cos(\frac{4\pi}{5} + \delta_k)} \right) = 1$$

Since all e_i are positive, if $\cos(\frac{4\pi}{5} - \delta_k) > \cos(\frac{4\pi}{5} + \delta_k)$ then <u>every</u> term in the product must be > 1 and the equality fails. Likewise $\cos(\frac{4\pi}{5} - \delta_k) < \cos(\frac{4\pi}{5} + \delta_k)$ is impossible.

Hence $\cos(\frac{4\pi}{5} - \delta_k) = \cos(\frac{4\pi}{5} + \delta_k)$

ie $\sin(\frac{4\pi}{5}) \sin(\delta_k) = 0$

and so $\delta_k = 0$ or π and E_k <u>is</u> real,

as conjectured by Bessis & Zinn-Justin.

This **generalises** and then **specialises** in various ways:

- Replace the 'potential' $-(ix)^3$ by $-(ix)^{2M}$ to get all other values of q: $\boxed{q = e^{i\pi/(M+1)}}$

- Add an 'angular momentum' term $\frac{L(L+1)}{x^2}$ [BLZ] to get other twisted boundary conditions in the six-vertex model (or, other primary fields in the CFT).

- Set $M=1$ to relate the free-fermion point of the six-vertex model ($q = e^{i\pi/2}$) to the simple harmonic oscillator ($-(ix)^{2M} = x^2$).

- Set $M = 1/2$ to map the 'Airy' (linear) potential to the $\Delta = 1/2$ ($q = e^{2\pi i/3}$) point of the six-vertex model where the RS conjectures arise. (RS ≡ Razumov Stroganov)

- etc etc...

$$\boxed{q = e}$$

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4:

Conclusions:

- At least in some cases there's a concrete link – via the Bethe Ansatz and TQ relations – between the spectral problem of finding the ground state in an integrable quantum field theory, and a spectral problem in some sort of quantum mechanics.
- Immediate spinoffs include a reality proof for the quantum mechanics, and a novel way to compute T & Q functions (by solving ODEs) in the QFT.
- The story goes further – to excited states, to other Bethe ansatz systems & QFTs, even cases which can't (yet?) be treated by Bethe Ansatz. ("paperclip brane")
- Finally, this looks to be the tip of a bigger iceberg relating to computations of gge theory scattering amplitudes (Alday, Maldacena, Sever, Vieira), wall-crossing (Gaiotto, Moore, Neitzke) etc...

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