Title: The 4d Superconformal Index and 2d Topological QFT

Date: May 18, 2010 11:00 AM

URL: http://pirsa.org/10050102

Abstract:

A new paradigm for 4d $\mathcal{N} = 2$ susy gauge theories (Gaiotto, ...)

Sharpest statements for superconformal theories (16 supercharges). Perhaps they all arise from compactification of the $(2,0)$ 6d theory on a 2d surface Σ, with punctures and twist lines.

A new paradigm for 4d $\mathcal{N} = 2$ susy gauge theories (Gaiotto, ...)

Sharpest statements for superconformal theories (16 supercharges). Perhaps they all arise from compactification of the $(2,0)$ 6d theory on a 2d surface Σ, with punctures and twist lines.

- Complex moduli space of Σ = moduli space of the 4d theory.
- Moore-Seiberg groupoid of Σ = (generalized) 4d S-duality

Vast generalization of "$\mathcal{N} = 4$ S-duality as modular group of T^2".

A new paradigm for 4d $\mathcal{N} = 2$ susy gauge theories (Gaiotto, ...)

Sharpest statements for superconformal theories (16 supercharges). Perhaps they all arise from compactification of the $(2,0)$ 6d theory on a 2d surface Σ, with punctures and twist lines.

- Complex moduli space of Σ = moduli space of the 4d theory.
- Moore-Seiberg groupoid of Σ = (generalized) 4d S-duality

Vast generalization of "$\mathcal{N} = 4$ S-duality as modular group of T^2".

6=4+2: beautiful and unexpected 4d/2d connections. For ex.,

- Correlators of Liouville/Toda on Σ compute the 4d partition functions (on S^4)

In this talk we will uncover another surprising connection:

- A protected 4d quantity, the superconformal index, is computed by topological QFT on Σ.

A "microscopic" 2d definition of the TQFT still lacking. We will define it in terms of its abstract operator algebra.

In this talk we will uncover another surprising connection:

- A protected 4d quantity, the superconformal index, is computed by topological QFT on Σ.

A "microscopic" 2d definition of the TQFT still lacking. We will define it in terms of its abstract operator algebra.
Index = twisted partition function on $S^3 \times S^1$. Independent of the gauge theory moduli and invariant under S-duality.
It encodes the protected spectrum of the 4d theory. Useful tool.

- Computing the index in different duality frames gives very non-trivial checks of Gaiotto's dualities.
- Conversely, assuming S-duality we will explicitly compute the index of 4d theories lacking a Lagrangian description.
- Surprising connection with elliptic hypergeometric function, an active area of mathematical research.

Outline

- Review of the superconformal index.
- The index of the A_1 theories (generalized $SU(2)$ quivers) and its TQFT interpretation.
- Elliptic hypergeometric cookbook.
  Index of a chiral superfield = elliptic Gamma function
- $SU(3)$ generalized quivers. The index of the E_6 SCFT.
- 32 and 8 supersymmetries (briefly).

In this talk we will uncover another surprising connection:

- A protected 4d quantity, the superconformal index, is computed by topological QFT on Σ.

A "microscopic" 2d definition of the TQFT still lacking. We will define it in terms of its abstract operator algebra.
Index = twisted partition function on $S^3 \times S^1$. Independent of the gauge theory moduli and invariant under S-duality.
It encodes the protected spectrum of the 4d theory. Useful tool.

- Computing the index in different duality frames gives very non-trivial checks of Gaiotto's dualities.
- Conversely, assuming S-duality we will explicitly compute the index of 4d theories lacking a Lagrangian description.
- Surprising connection with elliptic hypergeometric function, an active area of mathematical research.

Outline

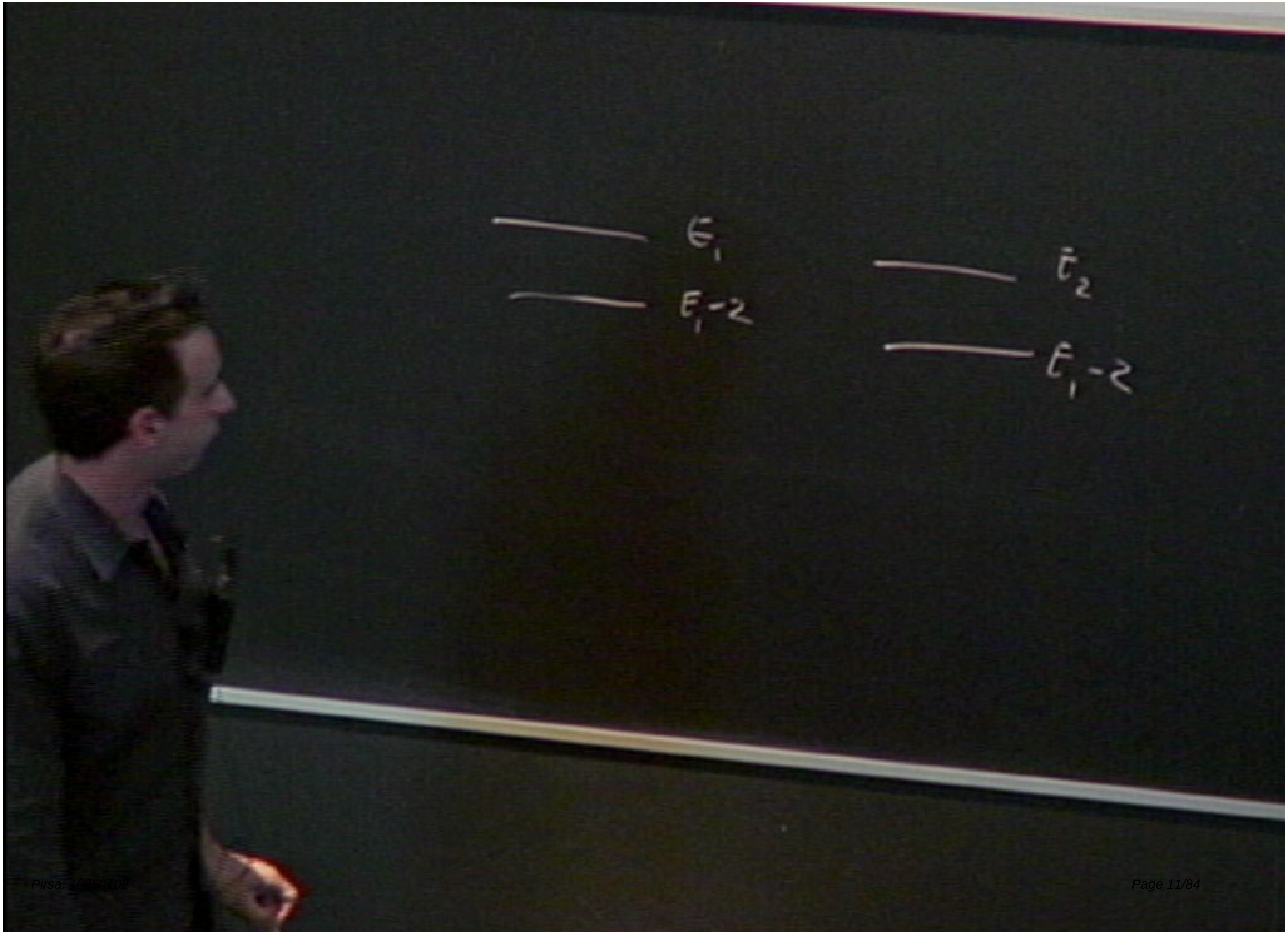
- Review of the superconformal index.
- The index of the A_1 theories (generalized $SU(2)$ quivers) and its TQFT interpretation.
- Elliptic hypergeometric cookbook.
  Index of a chiral superfield = elliptic Gamma function
- $SU(3)$ generalized quivers. The index of the E_6 SCFT.
- 32 and 8 supersymmetries (briefly).

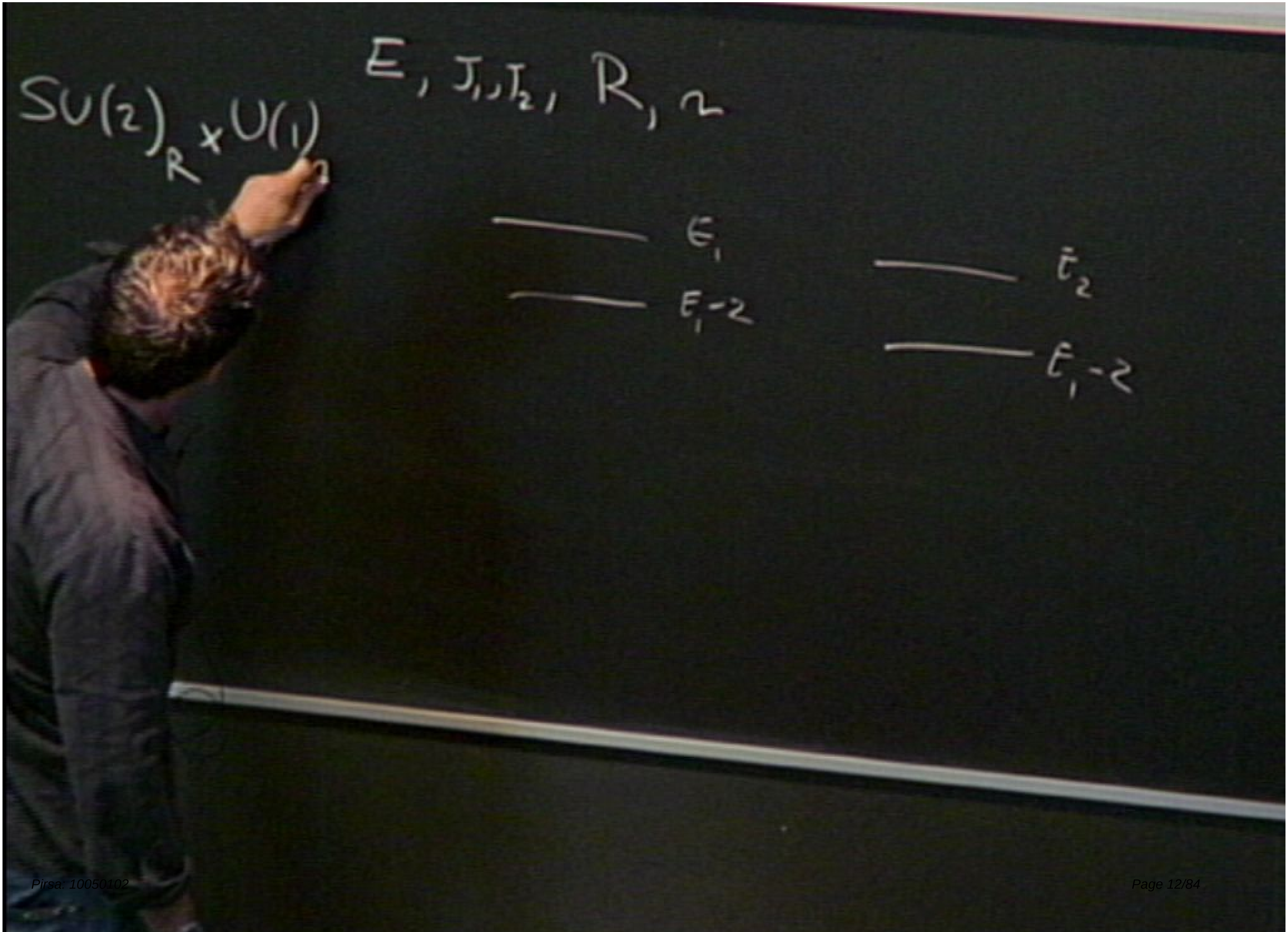
The Superconformal Index

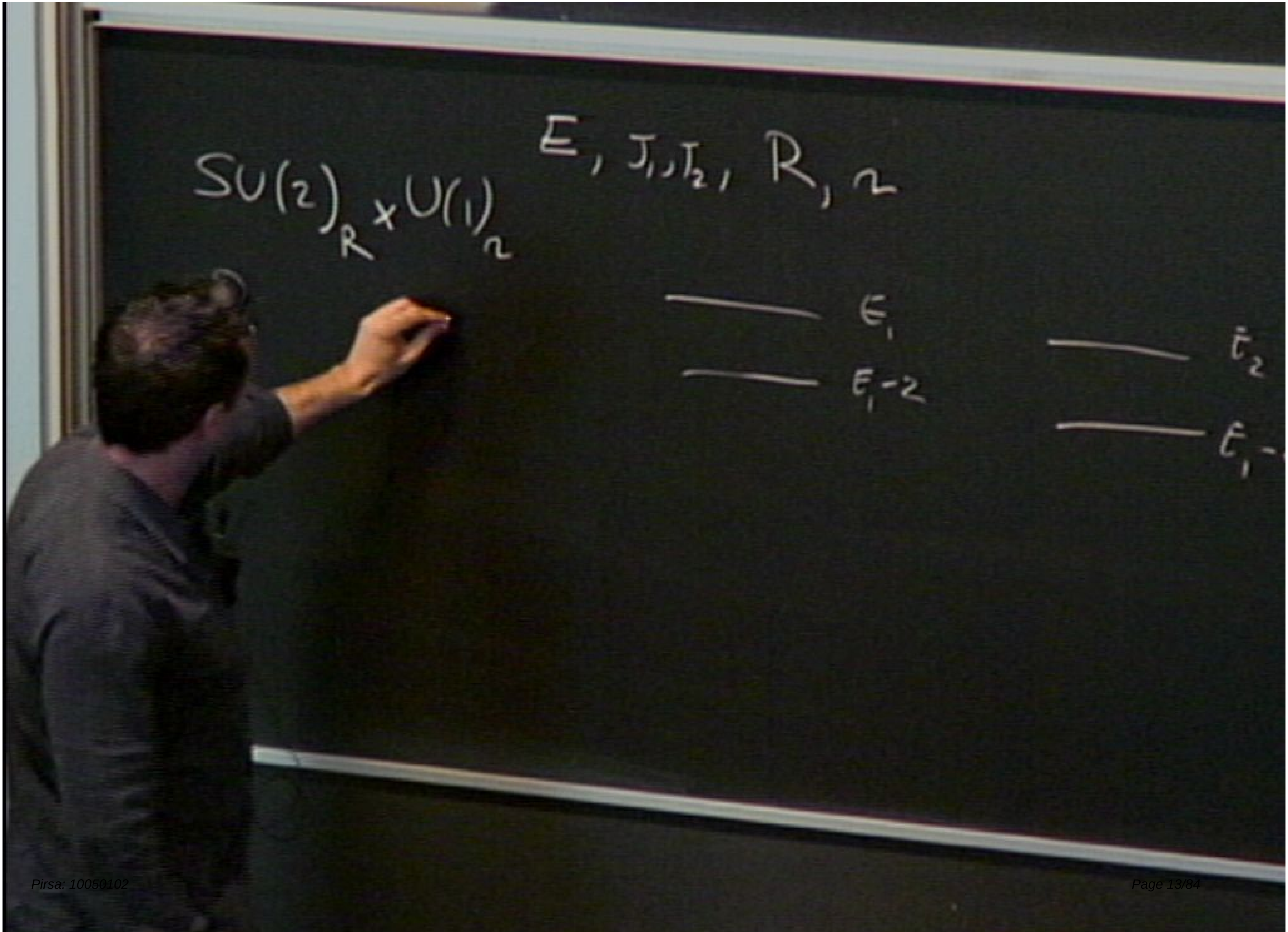
Romelsberger; Kinney, Maldacena, Minwalla, Raju 2005

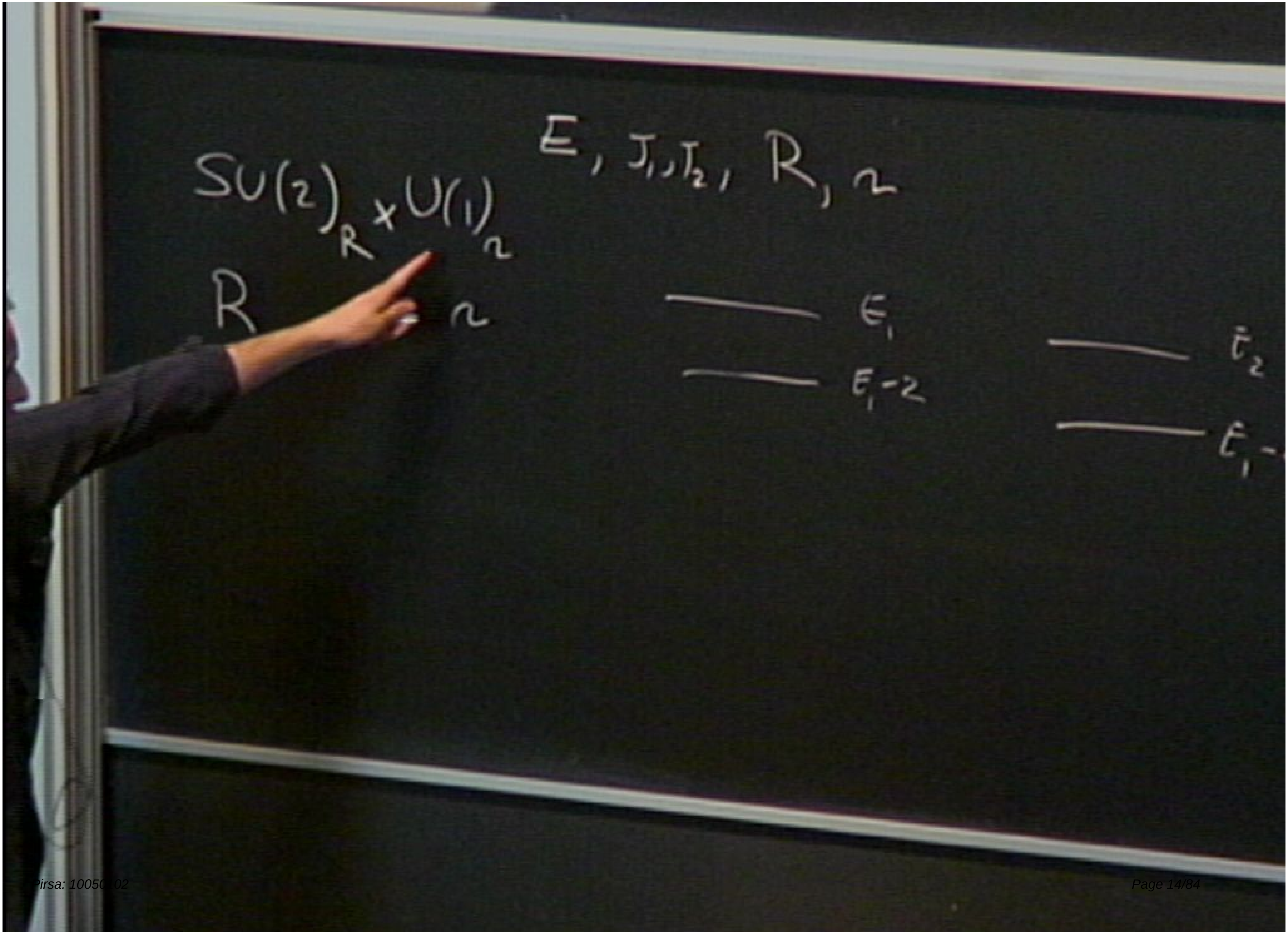
The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

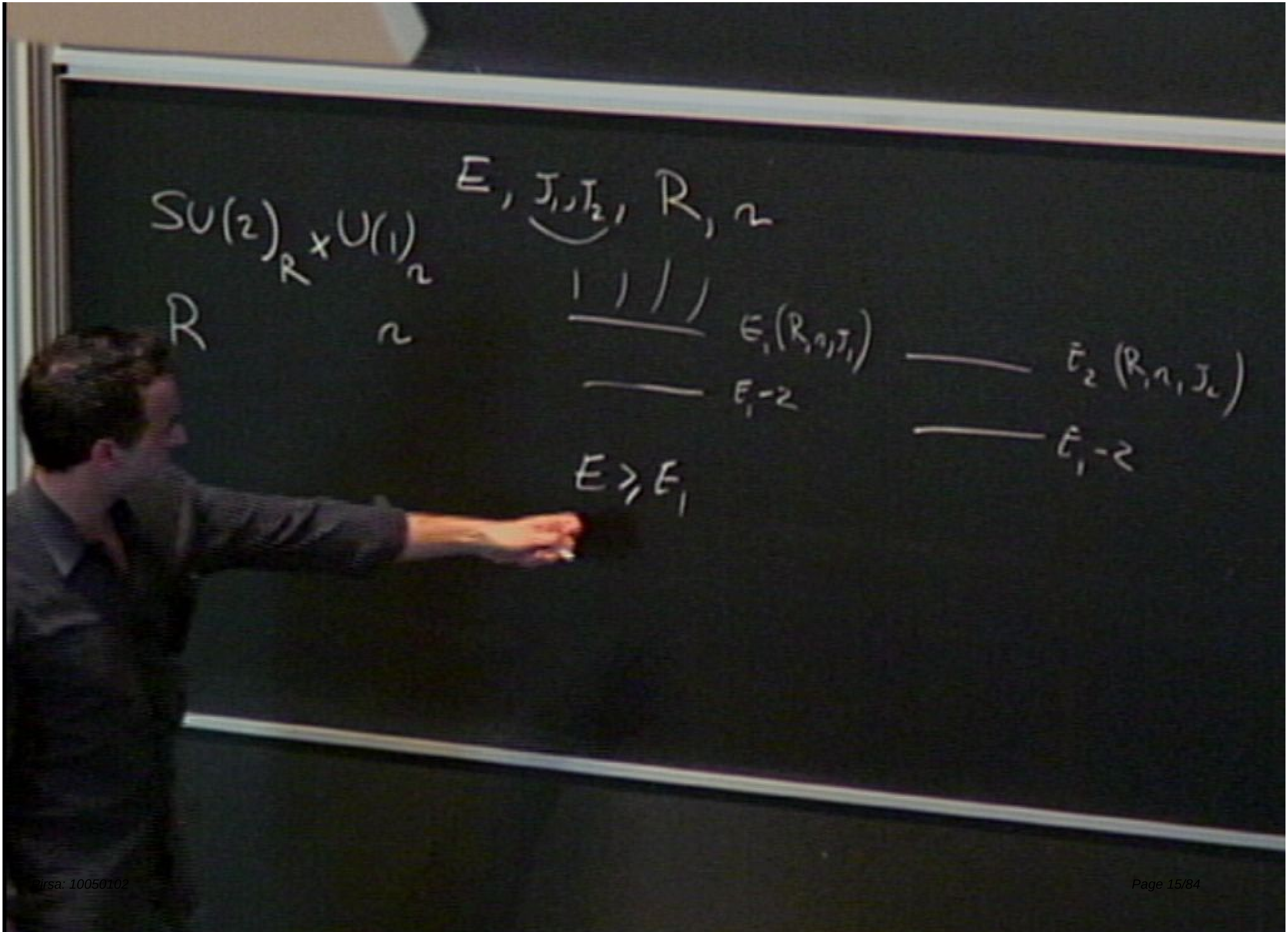
$$\mathcal{I}(t, v, y, \dots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \dots .$$


E1
E1-Z
E2
E1-Z

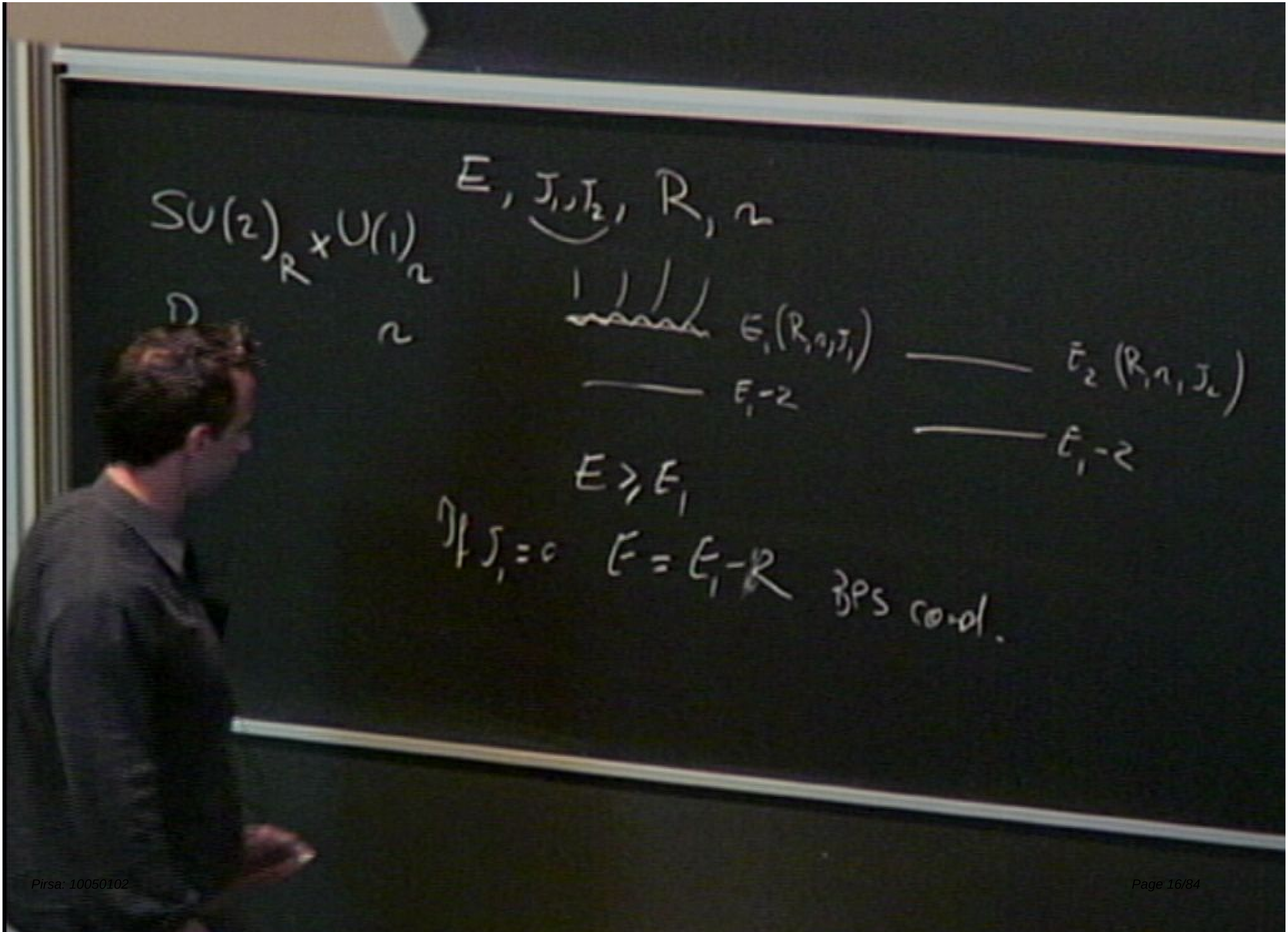

SU(2)_R × U(1)
E, J_1, J_2, R, r
E_1
E_1 - 2
E_2
E_1 - 2

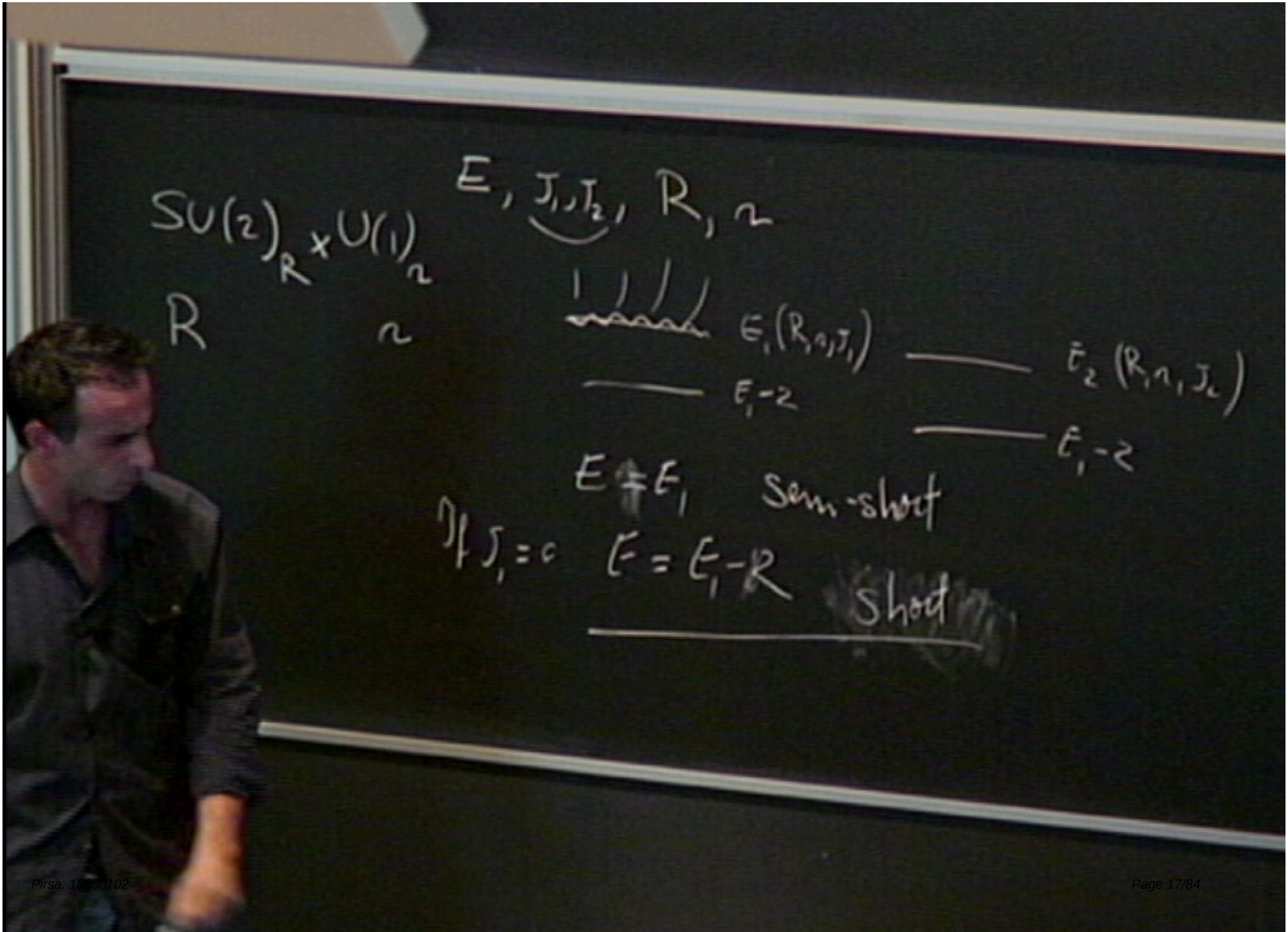

$SU(2)_R \times U(1)_z$
E, J_1, J_2, R, z
E_1
$E_1 - 2$
E_2
$E_1 -$

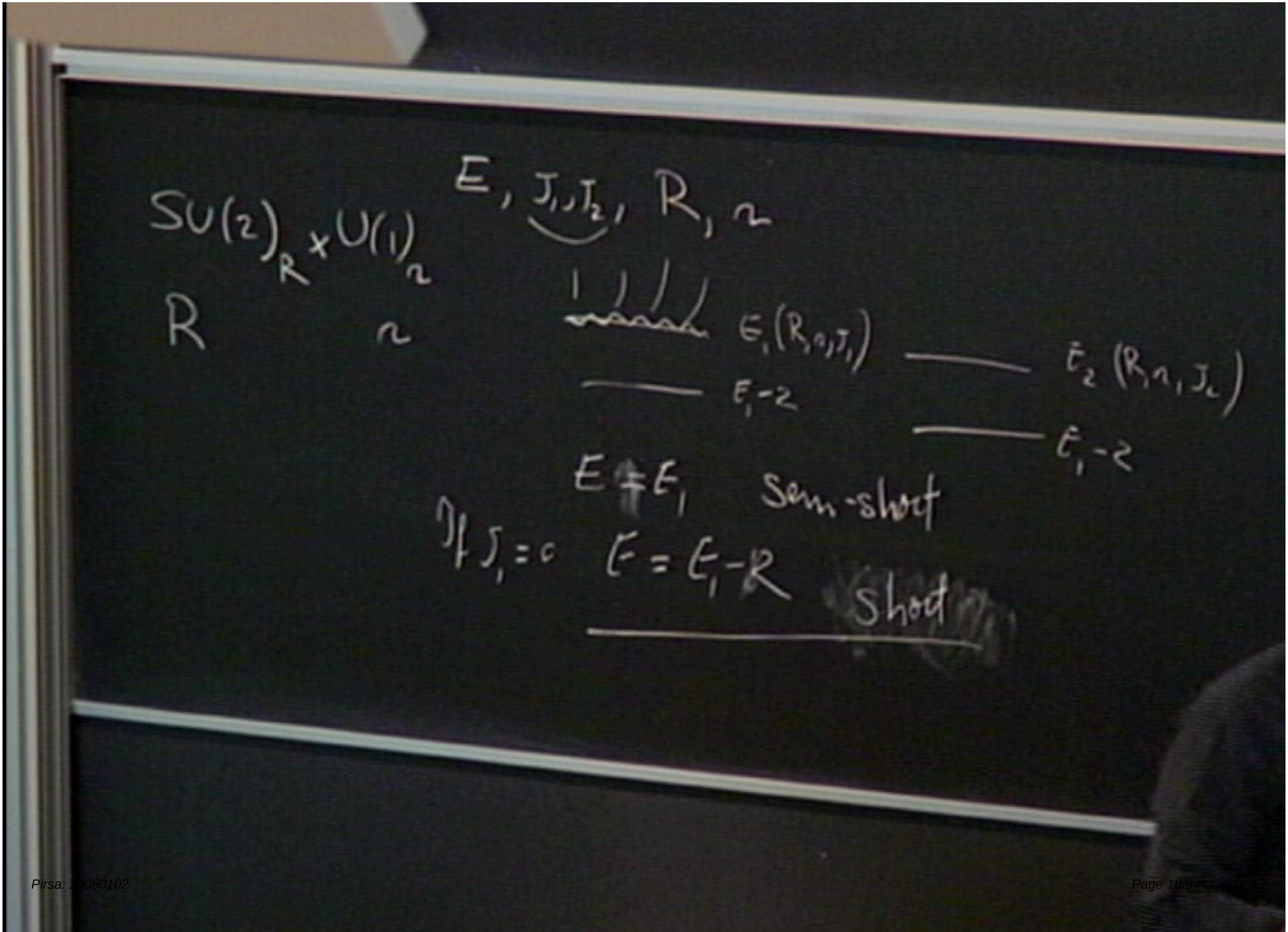

E, J1, J2, R, r
SU(2)R x U(1)r
R
r
E1
E1-2
E2



E, J1, J2, R, r
SU(2)R x U(1)r
R
r
E1(R,r,J1)
E1-2
E2(R,r,J2)
E1-2
E ≥ E1


SU(2)R × U(1)r
E, J1, J2, R, r
E1(R,r,J1)
E1-2
E2(R,r,J2)
E1-2
E ≥ E1
If J1=0 E=E1-R BPS cond.


SU(2)_R x U(1)_r
E, J_1, J_2, R, r
R
r
E_1(R, r, J_1)
E_1 - 2
E_2 (R, r, J_2)
E_1 - 2
E = E_1 semi-short
If J_1 = 0 E = E_1 - R
short


SU(2)_R × U(1)_r
E, J_1, J_2, R, r
R
r
E_1(R, r, J_1)
E_1 - 2
E_2(R, r, J_2)
E_1 - 2
E = E_1 Semi-short
If J_1 = 0 E = E_1 - R Short

The Superconformal Index

Romelsberger; Kinney, Maldacena, Minwalla, Raju 2005

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}(t, v, y, \dots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \dots .$$

The Superconformal Index

Romelsberger; Kinney, Maldacena,Minwalla, Raju 2005

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}(t, v, y, \dots) = \text{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \dots .$$

Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

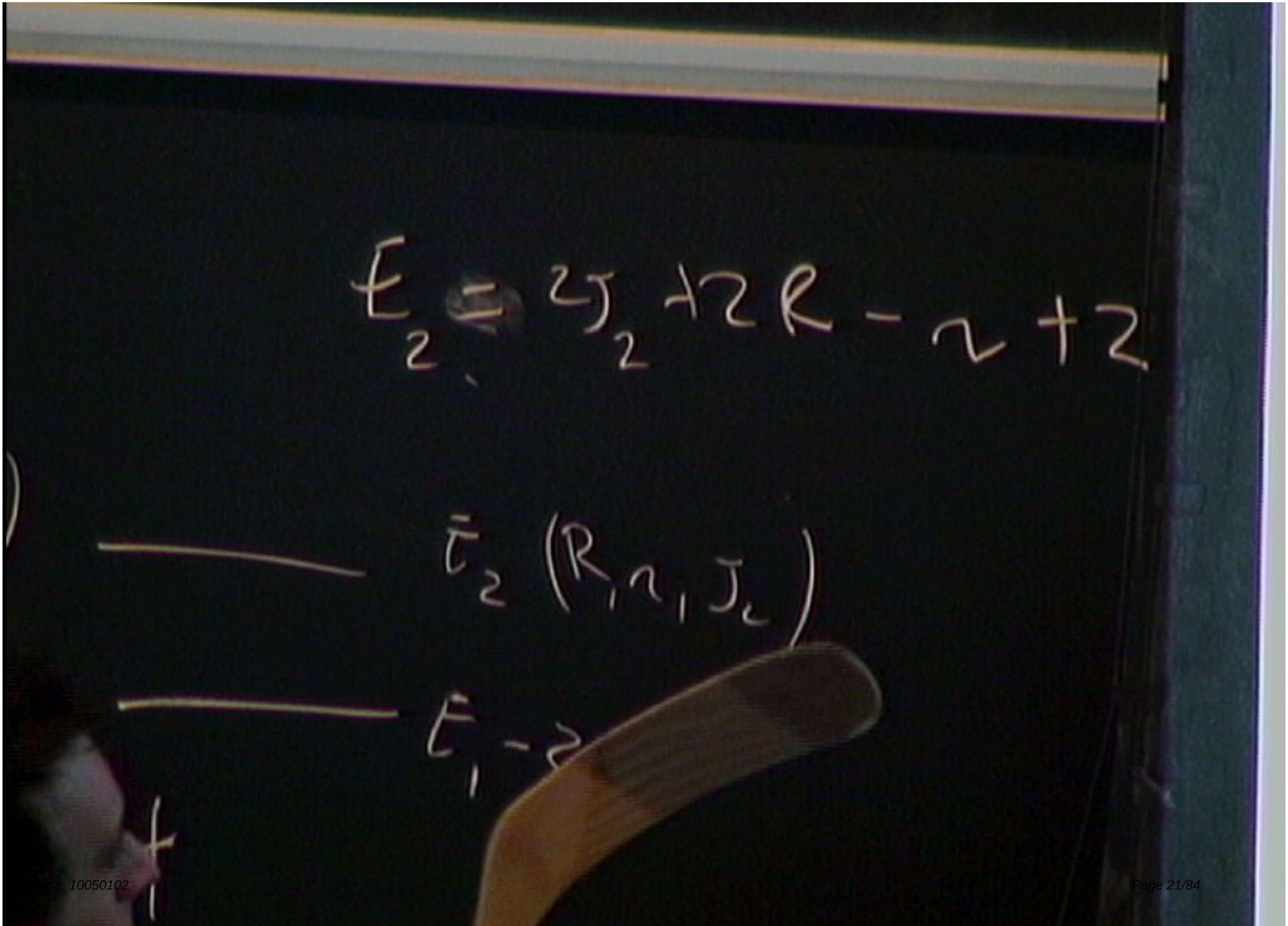
- The superconformal algebra implies (taking $Q = \bar{Q}_{2+}$)

$$2\{S, Q\} = E - 2j_2 - 2R + r \equiv \delta \geq 0 .$$

  where E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and (R, r) the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.
- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F e^{-\beta\delta + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.
- States with $\delta > 0$ come in pairs, boson + fermion, and cancel out,so $\mathcal{I}$ is β-independent.


$E_2 = 2J_2 + 2R - n + 2$
$E_2 (R, n, J_2)$
$E_1 - 2$

The Superconformal Index

Romelsberger; Kinney, Maldacena, Minwalla, Raju 2005

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}(t, v, y, \dots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \dots .$$

Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

- The superconformal algebra implies (taking $Q = \bar{Q}_{2+}$)

$$2\{S, Q\} = E - 2j_2 - 2R + r \equiv \delta \geq 0 .$$

  where E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and (R, r) the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.

- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F e^{-\beta\delta + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.

- States with $\delta > 0$ come in pairs, boson + fermion, and cancel out, so $\mathcal{I}$ is β-independent.

No Signal
VGA-1

The Superconformal Index

Romelsberger; Kinney, Maldacena, Minwalla, Raju 2005

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}(t, v, y, \dots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \dots .$$

Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

- The superconformal algebra implies (taking $Q = \bar{Q}_{2+}$)

$$2\{S, Q\} = E - 2j_2 - 2R + r \equiv \delta \geq 0 .$$

  where E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and (R, r) the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.
- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F e^{-\beta\delta + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.
- States with $\delta > 0$ come in pairs, boson + fermion, and cancel out, so $\mathcal{I}$ is β-independent.

The Index as a Matrix Integral

If the theory has Lagrangian description there is a simple recipe to compute the index.

- One defines a single-letter partition function as the index evaluated on the set of the basic objects (letters) in the theory with $\delta = 0$ and in a definite representation of the gauge and flavor groups:

$$f^{\mathcal{R}_j}(t, y, v),$$

where $\mathcal{R}_j$ labels the representation.

- Then the index is computed by enumerating the gauge-invariant words,

$$\boxed{\mathcal{I}(t, y, v, \mathbf{V}) = \int [d\mathbf{U}] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(\mathbf{U}^n, \mathbf{V}^n)\right)},$$

Here $\mathbf{U}$ is the matrix of the gauge group, $\mathbf{V}$ the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups.

- $\chi_{\mathcal{R}_j}(\mathbf{U})$ is the character of the group element in representation $\mathcal{R}_j$.
- The measure of integration $[d\,\mathbf{U}]$ is the invariant Haar measure.

$$\int [d\mathbf{U}] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(\mathbf{U}) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n\,.$$

The Superconformal Index

Romelsberger; Kinney, Maldacena, Minwalla, Raju 2005

The SC Index counts (with signs) the (semi)short multiplets, up to equivalence relations that sets to zero $\oplus_i$ Short$_i$ =Long.

$$\mathcal{I}(t, v, y, \ldots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \ldots .$$

Consider a 4d SCFT. On $S^3 \times \mathbb{R}$ (radial quantization), $Q^\dagger = S$.

- The superconformal algebra implies (taking $Q = \bar{Q}_{2+}$)

$$2\{S, Q\} = E - 2j_2 - 2R + r \equiv \delta \geq 0 .$$

  where E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \otimes SU(2)_2$ Lorentz spins, and (R, r) the quantum numbers under the $SU(2)_R \otimes U(1)_r$ R-symmetry.
- The SC index is the Witten index

$$\mathcal{I} = Tr(-1)^F e^{-\beta\delta + M}$$

  Here M is a generic combination of charges (weighted by chemical potentials) which commutes with S and Q.
- States with $\delta > 0$ come in pairs, boson + fermion, and cancel out, so $\mathcal{I}$ is β-independent.

The Index as a Matrix Integral

If the theory has Lagrangian description there is a simple recipe to compute the index.

- One defines a single-letter partition function as the index evaluated on the set of the basic objects (letters) in the theory with $\delta = 0$ and in a definite representation of the gauge and flavor groups:

$$f^{\mathcal{R}_j}(t, y, v),$$

where $\mathcal{R}_j$ labels the representation.

- Then the index is computed by enumerating the gauge-invariant words,

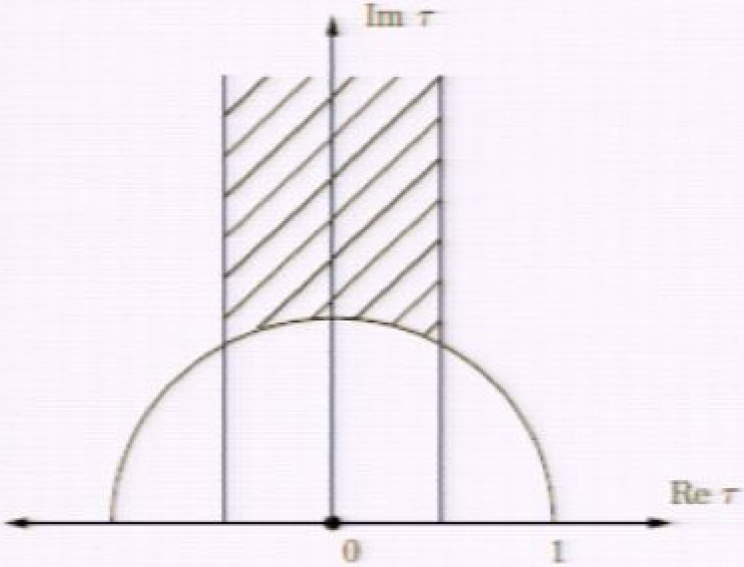
$$\boxed{\mathcal{I}(t, y, v, \mathbf{V}) = \int [d\mathbf{U}] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(\mathbf{U}^n, \mathbf{V}^n)\right)},$$

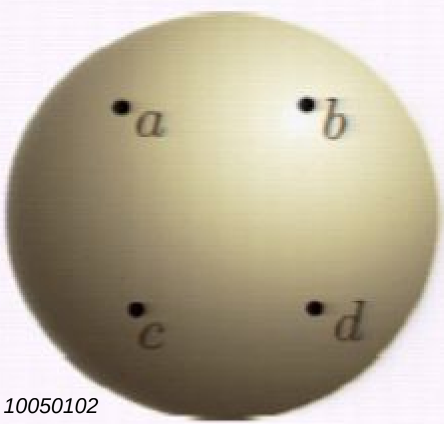
Here $\mathbf{U}$ is the matrix of the gauge group, $\mathbf{V}$ the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups.

- $\chi_{\mathcal{R}_j}(\mathbf{U})$ is the character of the group element in representation $\mathcal{R}_j$.
- The measure of integration $[d\,\mathbf{U}]$ is the invariant Haar measure.

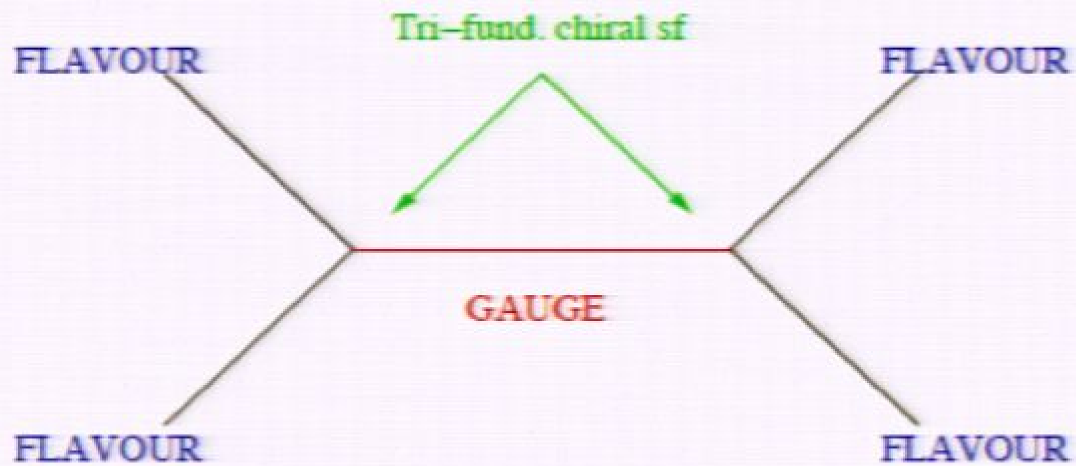
$$\int [d\mathbf{U}] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(\mathbf{U}) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n .$$

S-duality for $\mathcal{N} = 2$ $SU(2)$ SYM with $N_f = 4$





- $\mathbf{2} \sim \bar{\mathbf{2}}$ and thus we have eight $\mathcal{N} = 1$ χsf in fundamental of $SU(2)$.
- Generalized quivers: internal edges = gauge groups; external edges = flavour groups; vertices = Tri-Fundamental χsf.



- S-duality $\tau \to -\frac{1}{\tau}$ is accompanied by an $SO(8)$ triality transformation, which permutes the four $SU(2)$ flavor factors. On the diagrams this is implemented as channel crossing.

The Index as a Matrix Integral

If the theory has Lagrangian description there is a simple recipe to compute the index.

- One defines a single-letter partition function as the index evaluated on the set of the basic objects (letters) in the theory with $\delta = 0$ and in a definite representation of the gauge and flavor groups:

$$f^{\mathcal{R}_j}(t, y, v),$$

where $\mathcal{R}_j$ labels the representation.

- Then the index is computed by enumerating the gauge-invariant words,

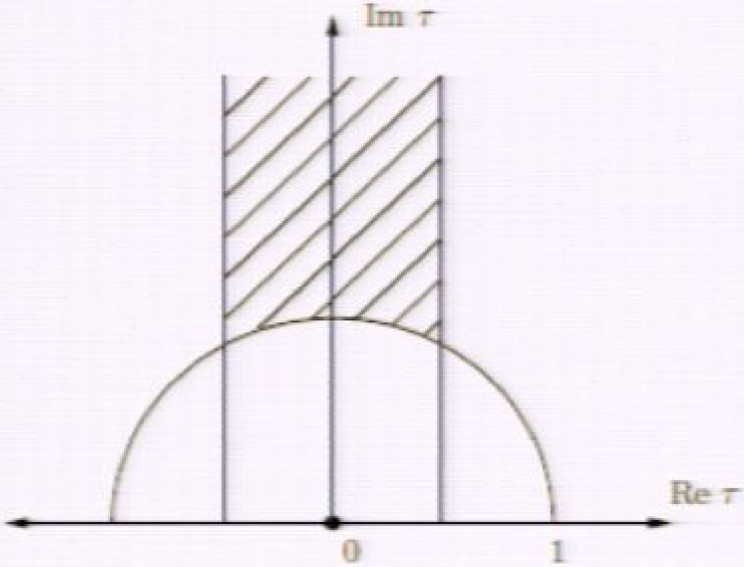
$$\boxed{\mathcal{I}(t, y, v, \mathbf{V}) = \int [d\mathbf{U}] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(\mathbf{U}^n, \mathbf{V}^n)\right)},$$

Here $\mathbf{U}$ is the matrix of the gauge group, $\mathbf{V}$ the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups.

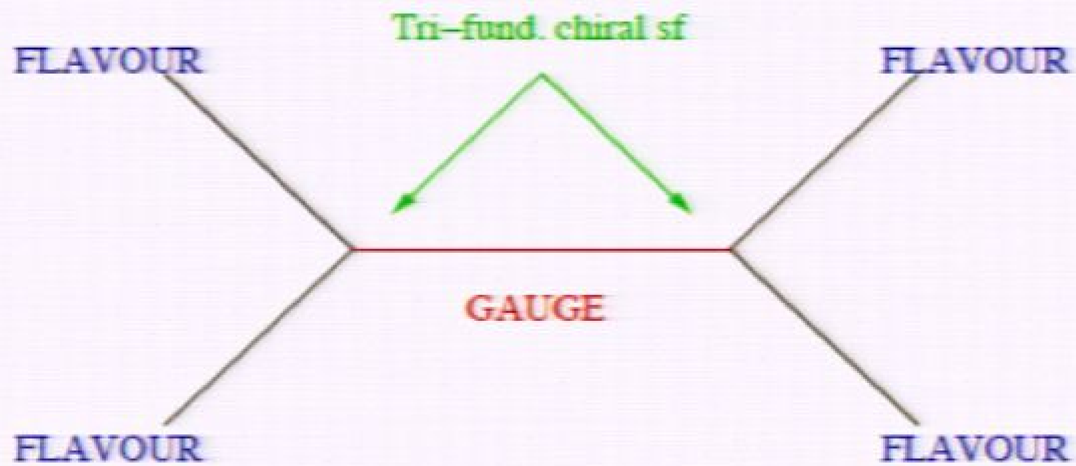
- $\chi_{\mathcal{R}_j}(\mathbf{U})$ is the character of the group element in representation $\mathcal{R}_j$.
- The measure of integration $[d\,\mathbf{U}]$ is the invariant Haar measure.

$$\int [d\mathbf{U}] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(\mathbf{U}) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n \,.$$

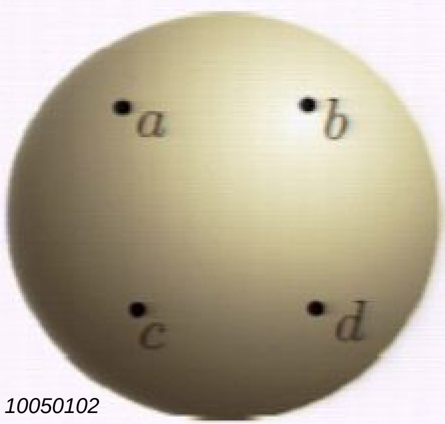
S-duality for $\mathcal{N} = 2$ $SU(2)$ SYM with $N_f = 4$

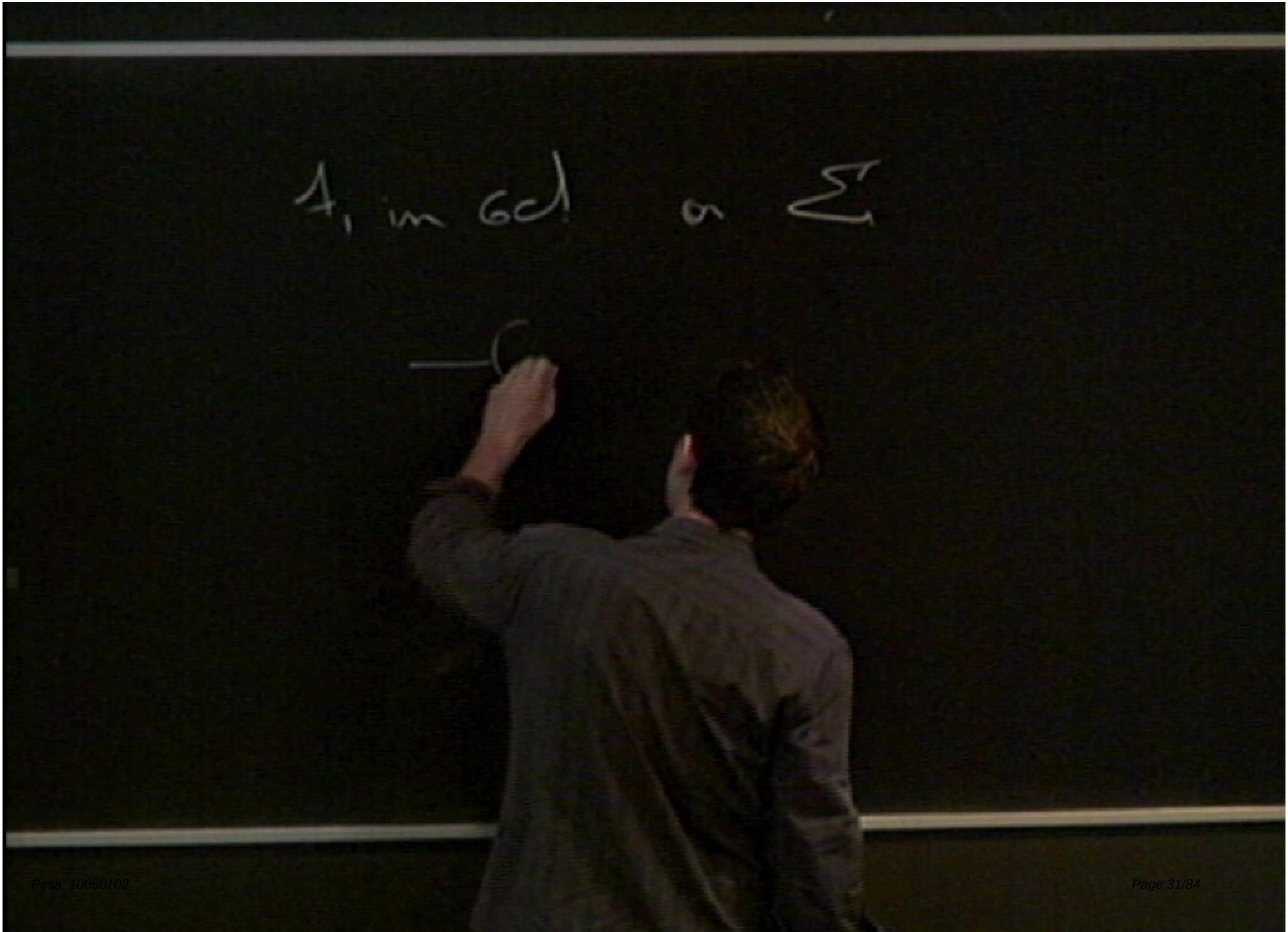


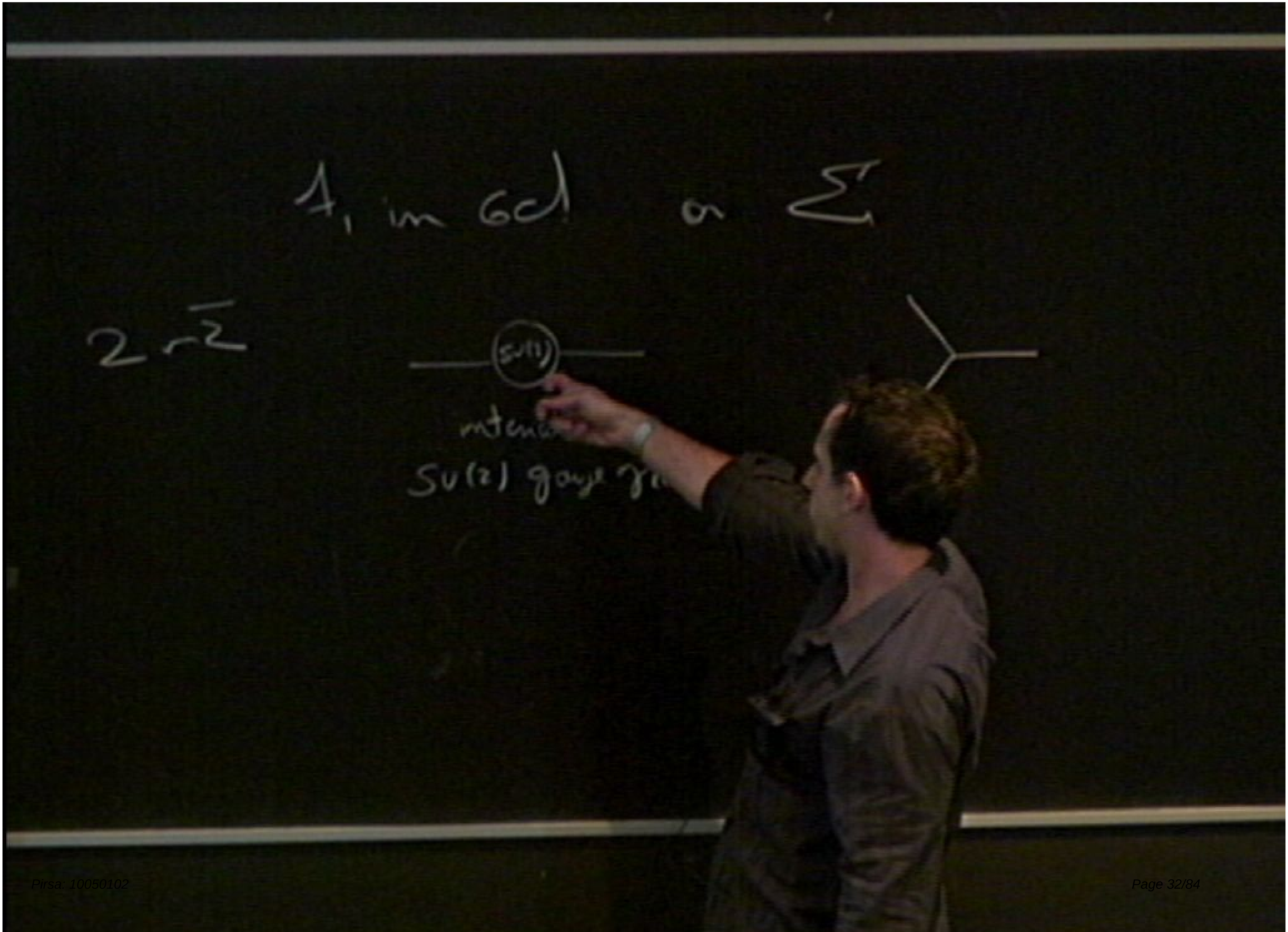
- $2 \sim \bar{2}$ and thus we have eight $\mathcal{N} = 1$ χsf in fundamental of $SU(2)$.
- Generalized quivers: internal edges = gauge groups; external edges = flavour groups; vertices = Tri-Fundamental χsf.

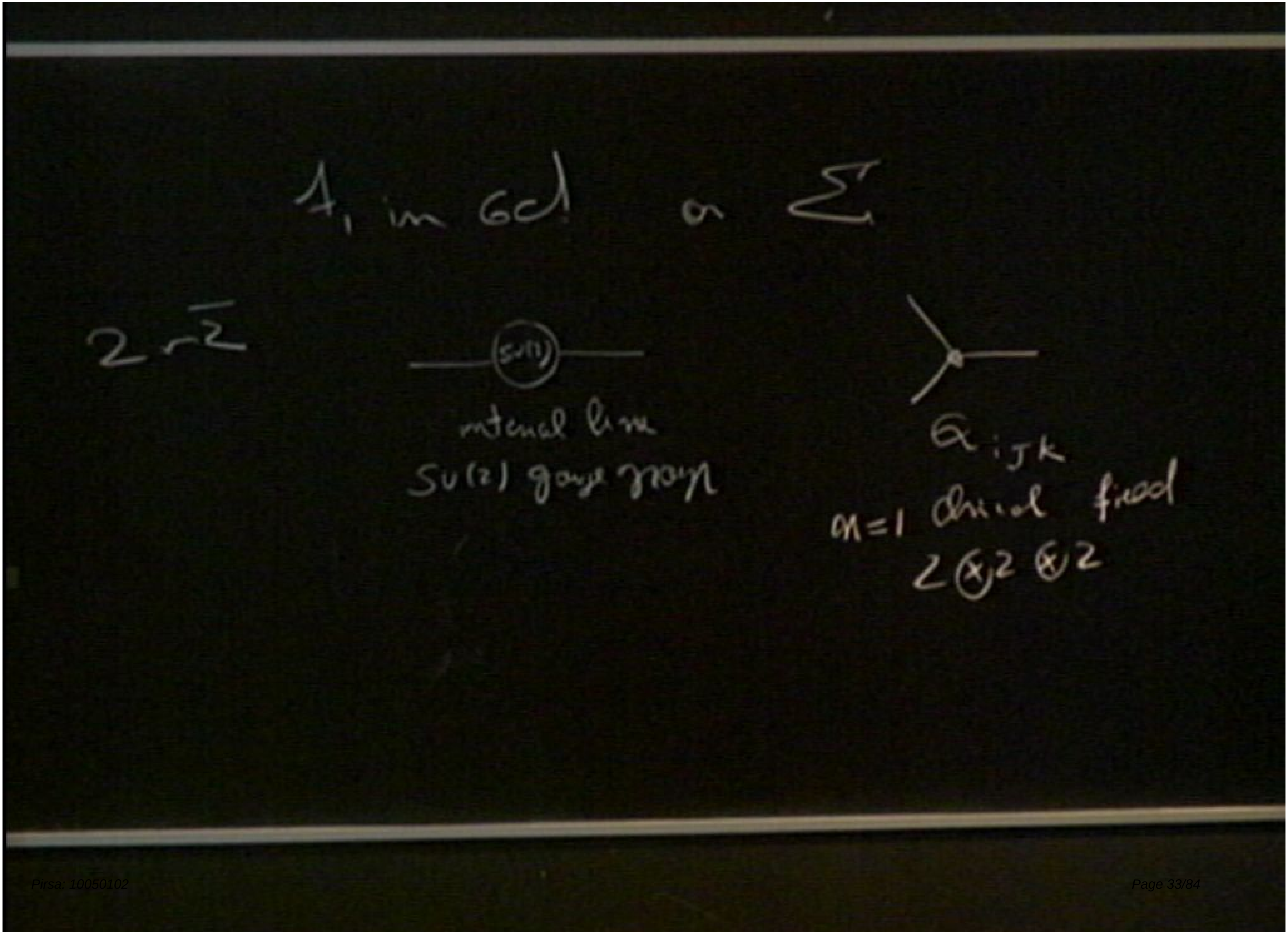


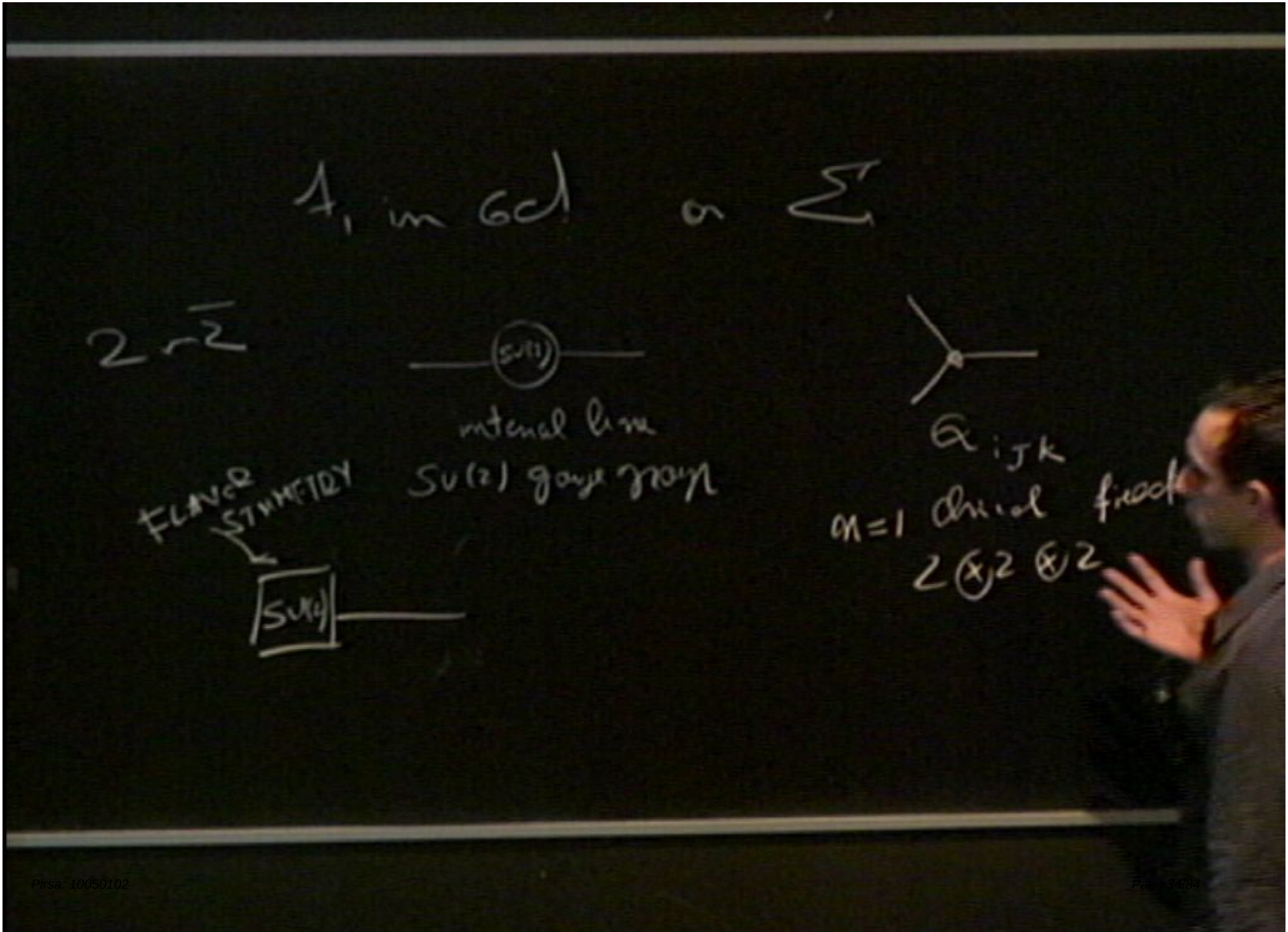
- S-duality $\tau \to -\frac{1}{\tau}$ is accompanied by an $SO(8)$ triality transformation, which permutes the four $SU(2)$ flavor factors. On the diagrams this is implemented as channel crossing.

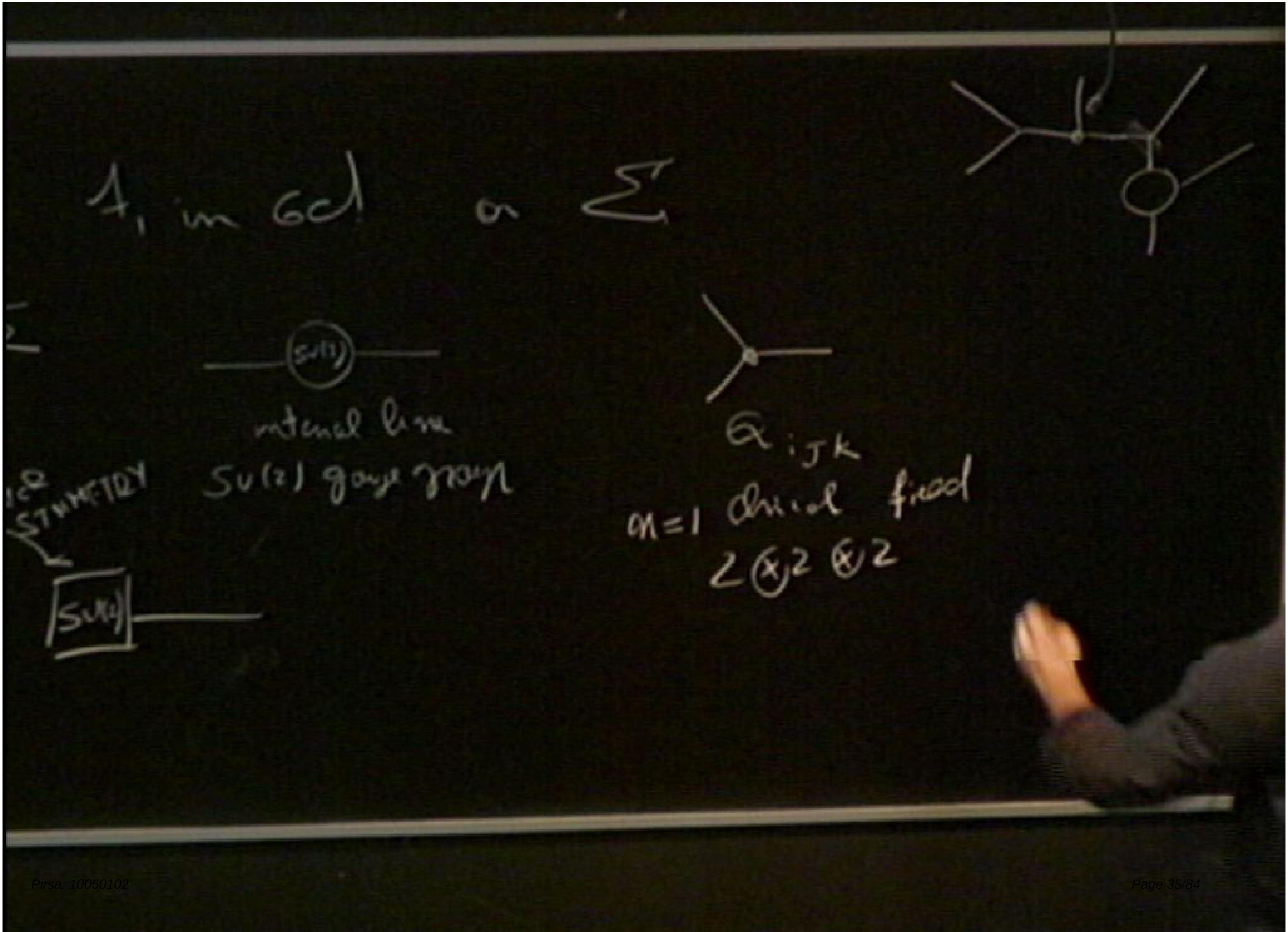


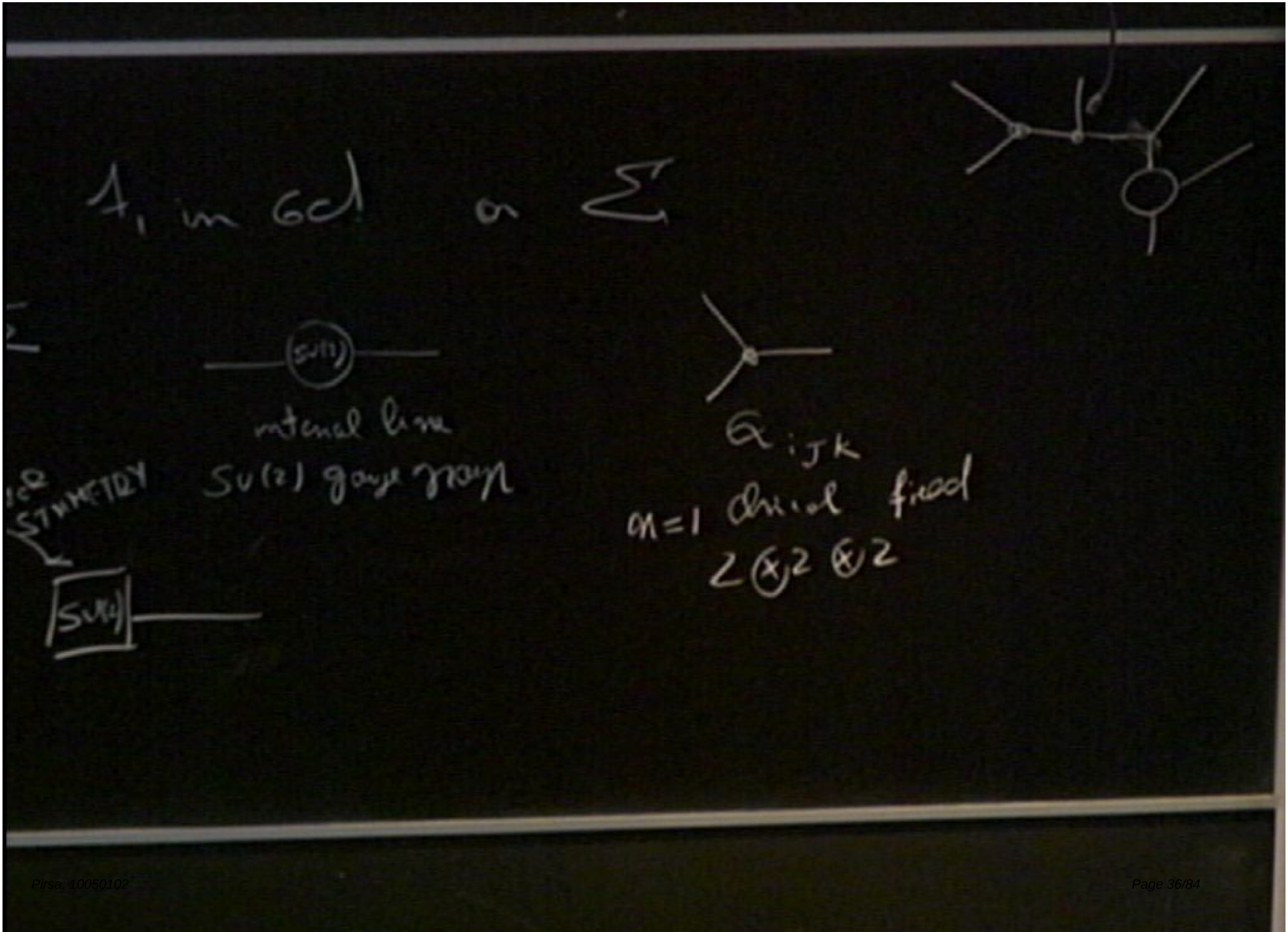

4, in 6d on Σ

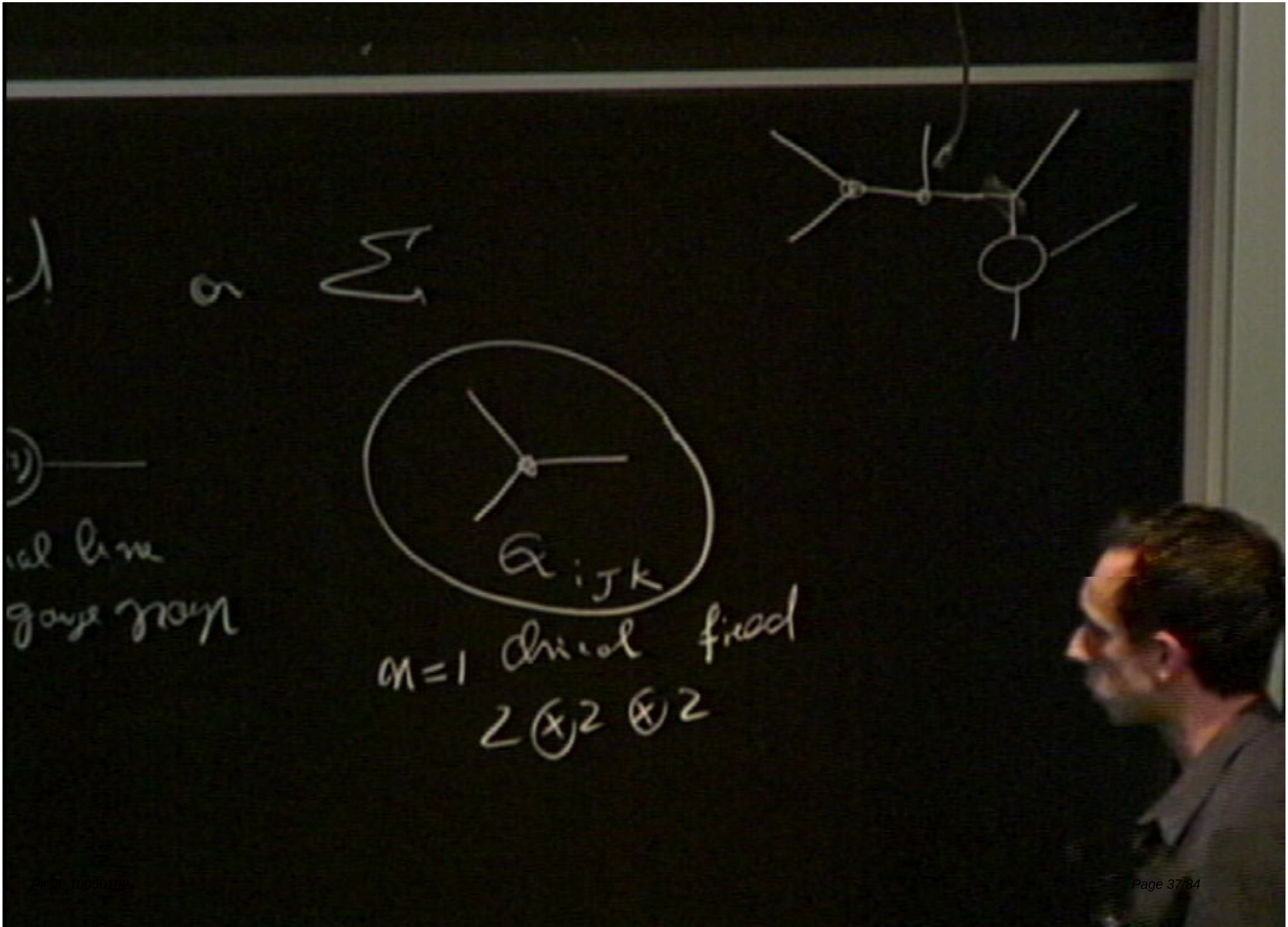

1, in 6d on Σ
SU(2)


A1 in 6d on Σ
SU(2)
internal line
SU(2) gauge group
Q ijk
n=1 chiral field
2 ⊗ 2 ⊗ 2

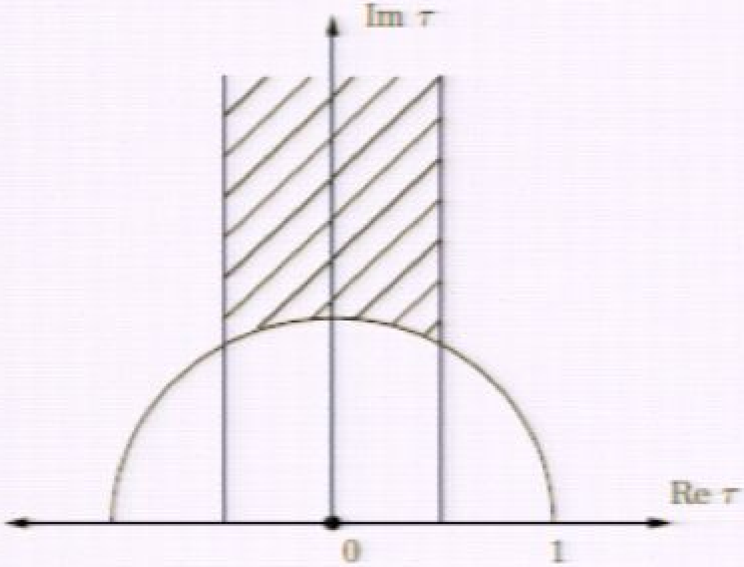

T1 in 6d on Σ
internal line
SU(2) gauge group
FLAVOR SYMMETRY
SU(2)
N=1 chiral field
2 ⊗ 2 ⊗ 2

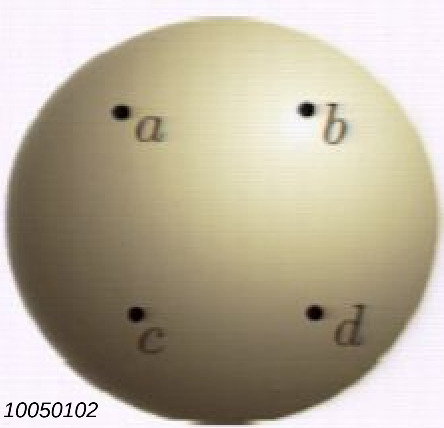

internal line
SU(2) gauge group
SYMMETRY
SU(2)
Q IJK
N=1 chiral field
2 ⊗ 2 ⊗ 2


A1 in 6d on Σ
SU(2)
internal line
SU(2) gauge group
SYMMETRY
SU(2)
Q iJK
N=1 chiral field
2 ⊗ 2 ⊗ 2

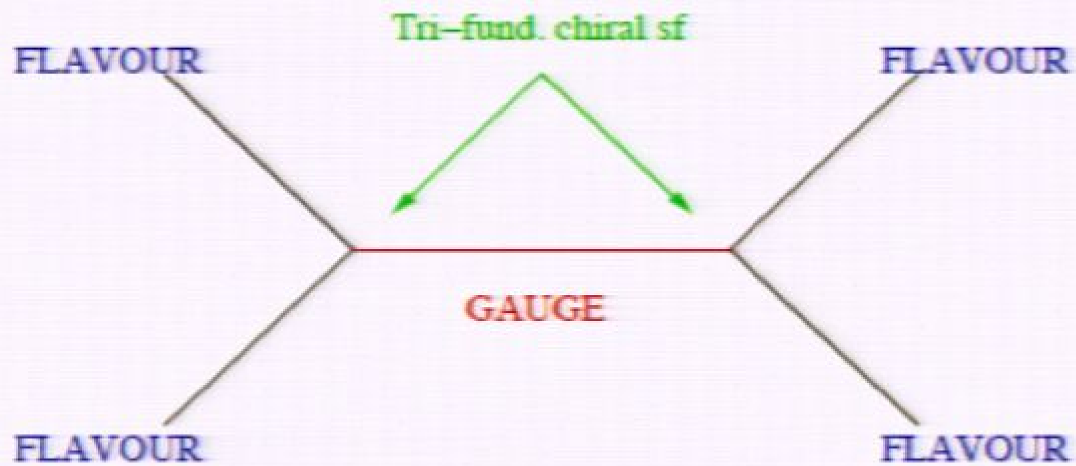

or Σ
gauge group
Q iJK
N=1 chiral field
Z ⊗ Z ⊗ Z

S-duality for $\mathcal{N} = 2$ $SU(2)$ SYM with $N_f = 4$

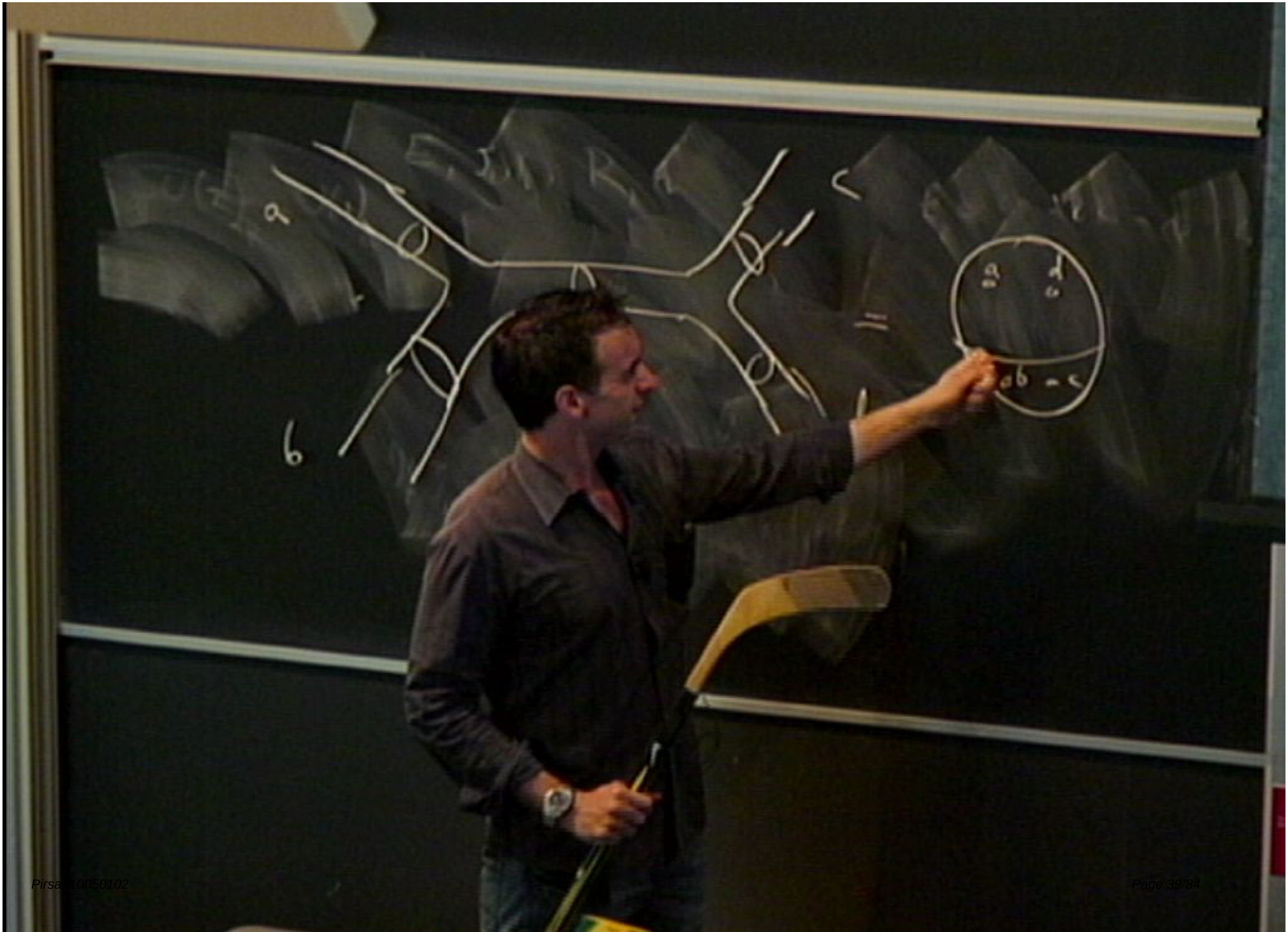


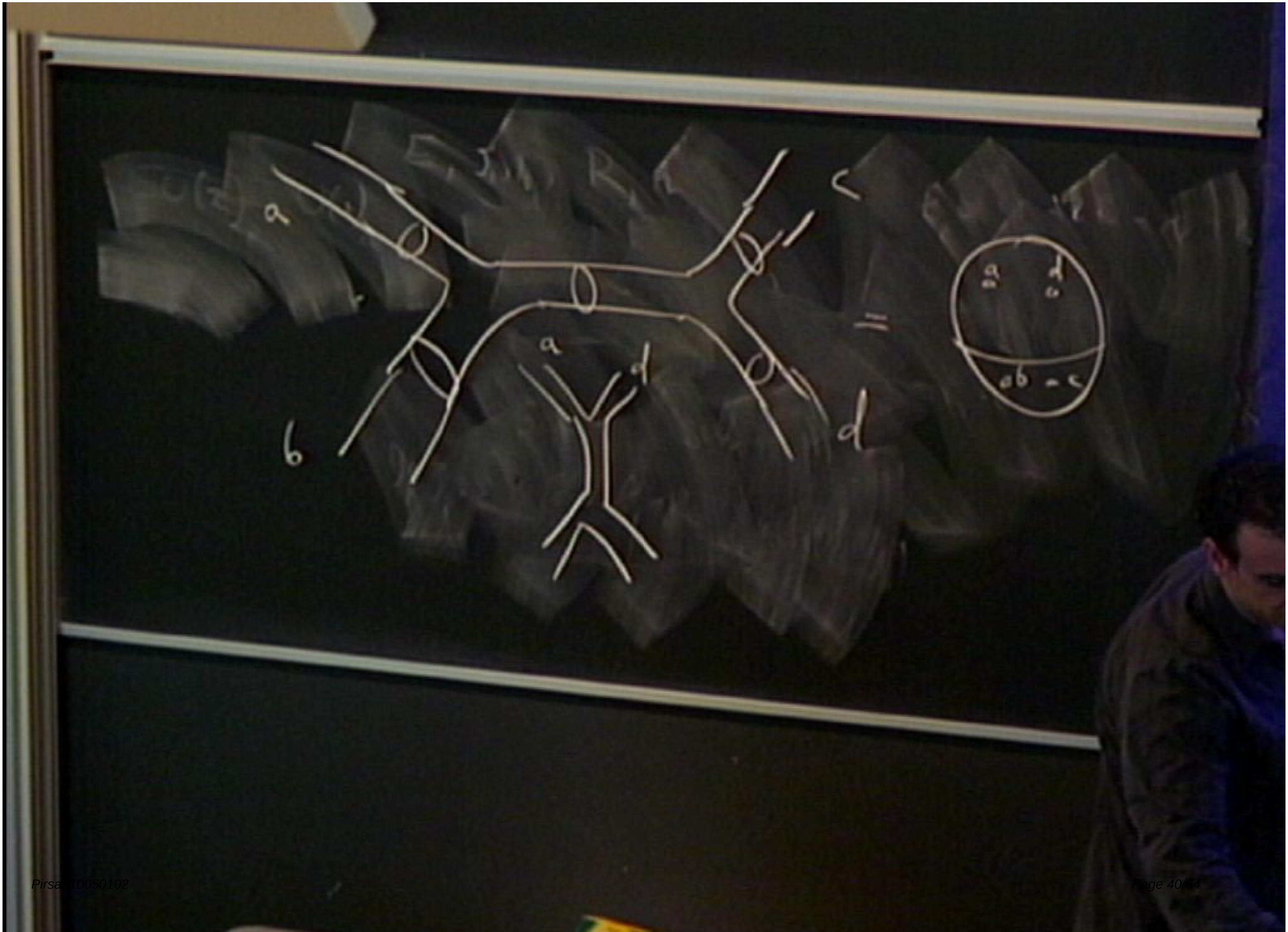


- $\mathbf{2} \sim \bar{\mathbf{2}}$ and thus we have eight $\mathcal{N} = 1$ χsf in fundamental of $SU(2)$.
- Generalized quivers: internal edges = gauge groups; external edges = flavour groups; vertices = Tri-Fundamental χsf.



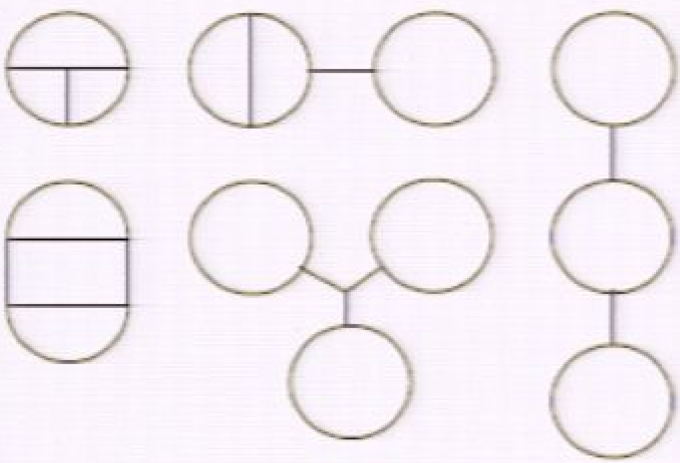
- S-duality $\tau \to -\frac{1}{\tau}$ is accompanied by an $SO(8)$ triality transformation, which permutes the four $SU(2)$ flavor factors. On the diagrams this is implemented as channel crossing.


a
b
c

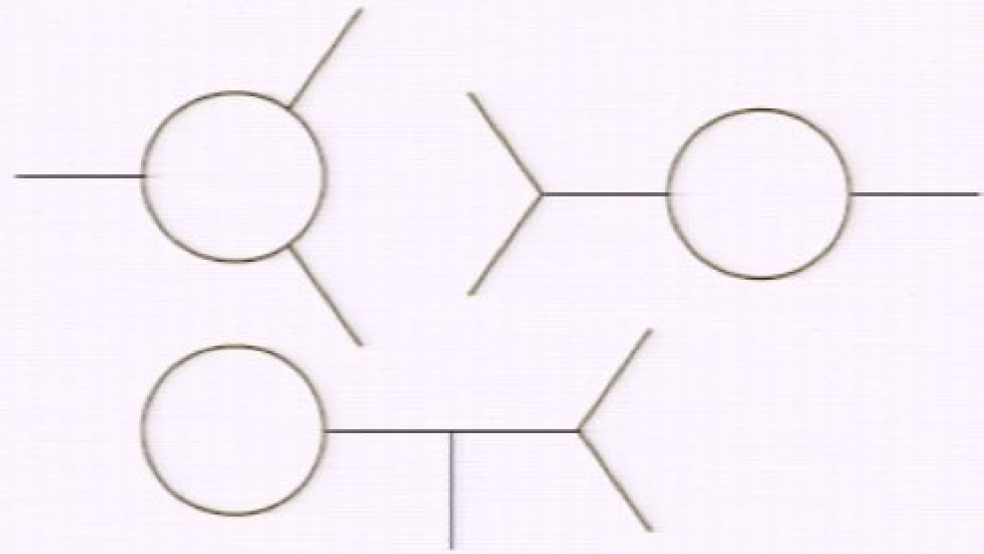

a
c
b
a
d
d

Generalized $SU(2)$ quivers

Some examples:

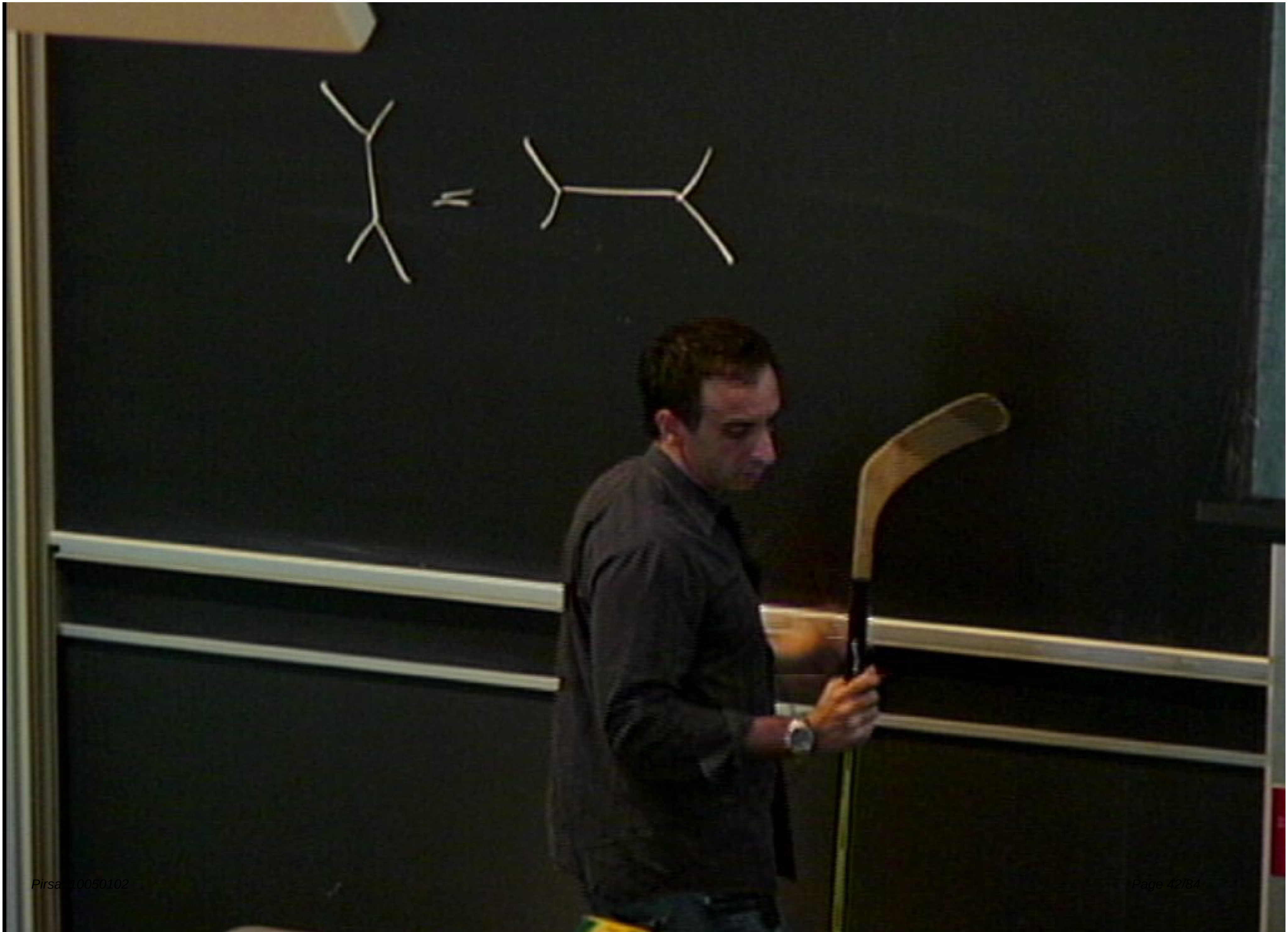


(a)



(b)

The generalized quivers in (a) arise from different pairs-of-paint decomposition of the same Riemann surface. The corresponding 4d theories are related by S-dualities. They must have the same superconformal index. The same applies to (b).



The index for the A_1 theories

The index is read off from the quiver

$$\mathcal{I} = \int \left[\prod_{I=1}^{N_G} d\,U_I\right] \quad e^{\sum_{I\in \text{Edges}} \sum_{n=1}^{\infty} \frac{1}{n} f_{adj}(t^n, y^n, v^n)\, \chi_{adj}(U_I^n)}$$

$$e^{\sum_{\{I,J,K\}\in \text{Vertices}} \sum_{n=1}^{\infty} \frac{1}{n} f_{3-fund}(t^n, y^n, v^n)\, \chi_{3-fund}(U_I^n, U_J^n, U_K^n)} .$$

Index of a chiral superfield = elliptic Gamma function

- Mathematicians have a name for the index of the chiral superfield: elliptic Gamma function

$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1 - z^{-1}\, p^{j+1} q^{k+1}}{1 - z\, p^j q^k}.$$

- The index of a χsf is (Dolan and Osborn - 2008)

$$\exp\left[\sum_{k=1}^{\infty} \frac{1}{k} f^{chi}\left(t^k, v^k, y^k\right)\right] = \Gamma\left(\frac{t^2}{\sqrt{v}}; p, q\right), \quad p = t^3 y,\ q = t^3 y^{-1}.$$

Index of a chiral superfield = elliptic Gamma function

- Mathematicians have a name for the index of the chiral superfield: elliptic Gamma function

$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1 - z^{-1}\, p^{j+1} q^{k+1}}{1 - z\, p^j q^k}.$$

- The index of a χsf is (*Dolan and Osborn - 2008*)

$$\exp\left[\sum_{k=1}^{\infty} \frac{1}{k} f^{chi}\left(t^k, v^k, y^k\right)\right] = \Gamma\left(\frac{t^2}{\sqrt{v}}; p, q\right), \quad p = t^3 y,\ q = t^3 y^{-1}.$$

Index of a chiral superfield = elliptic Gamma function

- Mathematicians have a name for the index of the chiral superfield: elliptic Gamma function

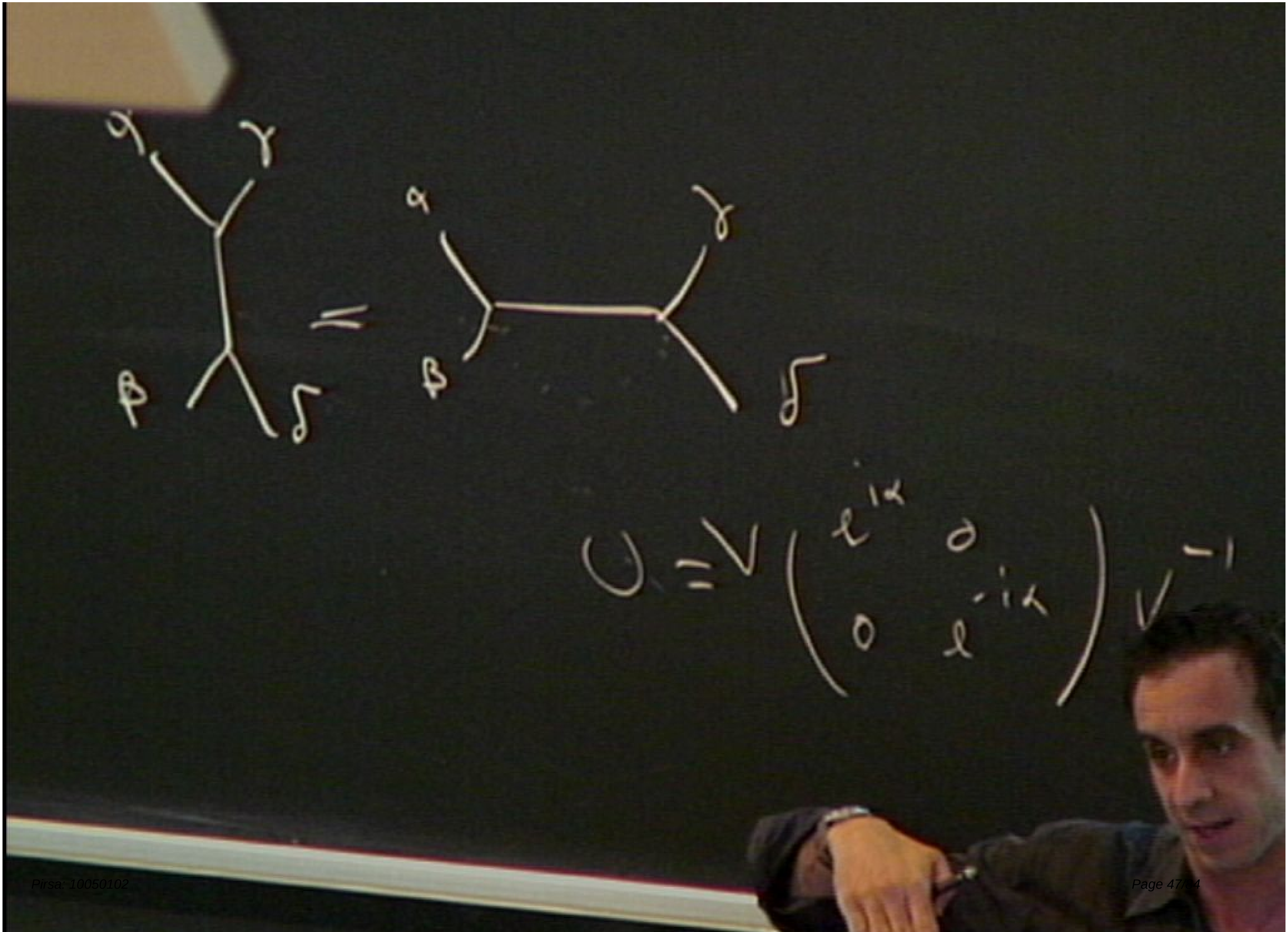
$$\Gamma(z;p,q) \equiv \prod_{j,k\geq 0} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-z\,p^j q^k}.$$

- The index of a χsf is (*Dolan and Osborn - 2008*)

$$\exp\left[\sum_{k=1}^{\infty}\frac{1}{k}f^{chi}\left(t^k,v^k,y^k\right)\right]=\Gamma\left(\frac{t^2}{\sqrt{v}};p,q\right),\quad p=t^3y,\ q=t^3y^{-1}.$$

- The Elliptic Beta integral is a generalization of the celebrated Euler Beta integral (*Spiridonov - 2001*)

$$\kappa\oint\frac{dz}{2\pi iz}\frac{\prod_{i=1}^{6}\Gamma(t_i z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)}=\prod_{1\leq i<j\leq 6}\Gamma(t_it_k;p,q)\quad\rightarrow\quad\int_0^1 dt\,t^{\alpha-1}(1-t)^{\beta-1}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$


α γ β δ = α γ β δ
$U = V \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} V^{-1}$

Elliptic Cookbook

Recall the character of the (anti)fundamental representation of $SU(n)$

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar{f}} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1\,.$$

- The index of a hypermultiplet in fundamental of $SU(n)$

$$\prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1}; p, q\right)$$

Elliptic Cookbook

Recall the character of the (anti)fundamental representation of $SU(n)$

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar{f}} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1\,.$$

- The index of a hypermultiplet in fundamental of $SU(n)$

$$\prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}}\, a_i^{\pm 1}; p, q\right)$$

Elliptic Cookbook

Recall the character of the (anti)fundamental representation of $SU(n)$

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar{f}} = \sum_{i=1}^{n} \frac{1}{a_i}, \qquad \prod_{i=1}^{n} a_i = 1.$$

- The index of a hypermultiplet in fundamental of $SU(n)$

$$\prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1}; p, q\right)$$

- When an $SU(n)$ symmetry is gauged we add a vector multiplet and integrate over the gauge group

$$\frac{\left[2\Gamma(t^2 v; p, q)\,\kappa\right]^{n-1}}{n!} \oint_{\mathbb{T}_{n-1}} \prod_{i=1}^{n-1} d\mu(a_i) \prod_{i\neq j} \frac{\Gamma(t^2 v\, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)} \cdots \Bigg|_{\prod_{i=1}^{n} a_i = 1}$$

* For brevity we will often omit the parameters p and q from the expression of the Gamma function.

The index of the $SU(2)$ generalized quivers in terms of elliptic Gamma functions

The index of $N_f = 4$ $SU(2)$ gauge theory can be written as

$$\kappa \Gamma\,(t^2 v) \oint \frac{dz}{2\pi i\, z} \frac{\Gamma(t^2\, v\, z^{\pm 2})}{\Gamma(z^{\pm 2})}\, \Gamma(\frac{t^2}{\sqrt{v}} a^{\pm 1} b^{\pm 1} z^{\pm 1})\, \Gamma(\frac{t^2}{\sqrt{v}} c^{\pm 1} d^{\pm 1} z^{\pm 1}).$$

The index of the $SU(2)$ generalized quivers in terms of elliptic Gamma functions

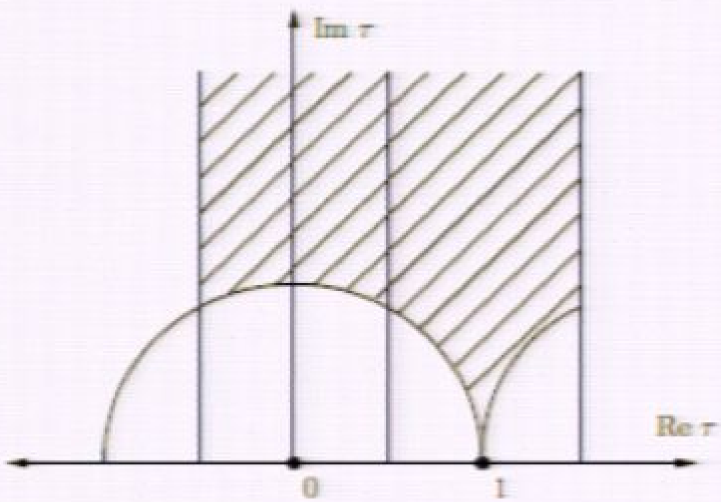
The index of $N_f = 4$ $SU(2)$ gauge theory can be written as

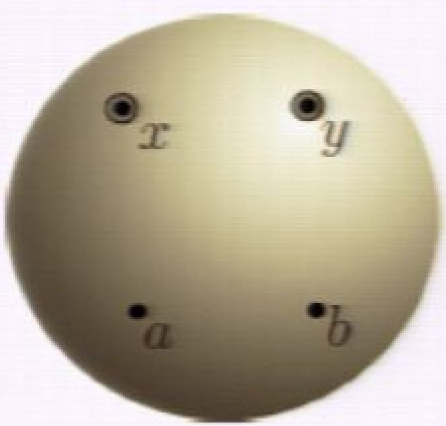
$$\kappa \Gamma(t^2 v) \oint \frac{dz}{2\pi i z} \frac{\Gamma(t^2 v z^{\pm 2})}{\Gamma(z^{\pm 2})} \Gamma(\frac{t^2}{\sqrt{v}} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \Gamma(\frac{t^2}{\sqrt{v}} c^{\pm 1} d^{\pm 1} z^{\pm 1}).$$

This integral was recently shown to be invariant under exchanging a and c (more generally, under the Weyl group of F_4) (van de Bult 2009)

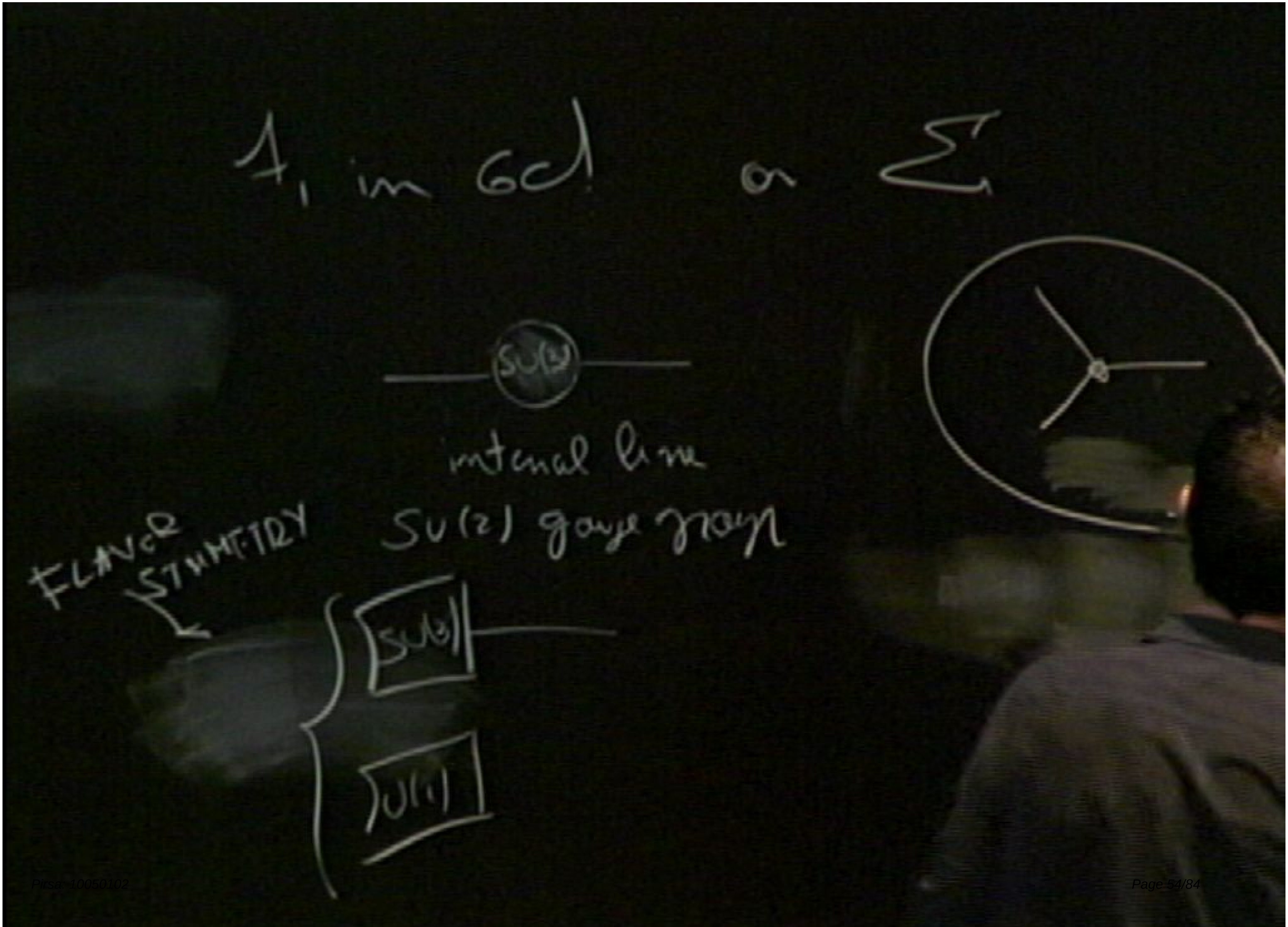
This checks associativity of the A_1 TQFT, or equivalently, S-duality for the index of the A_1 theories

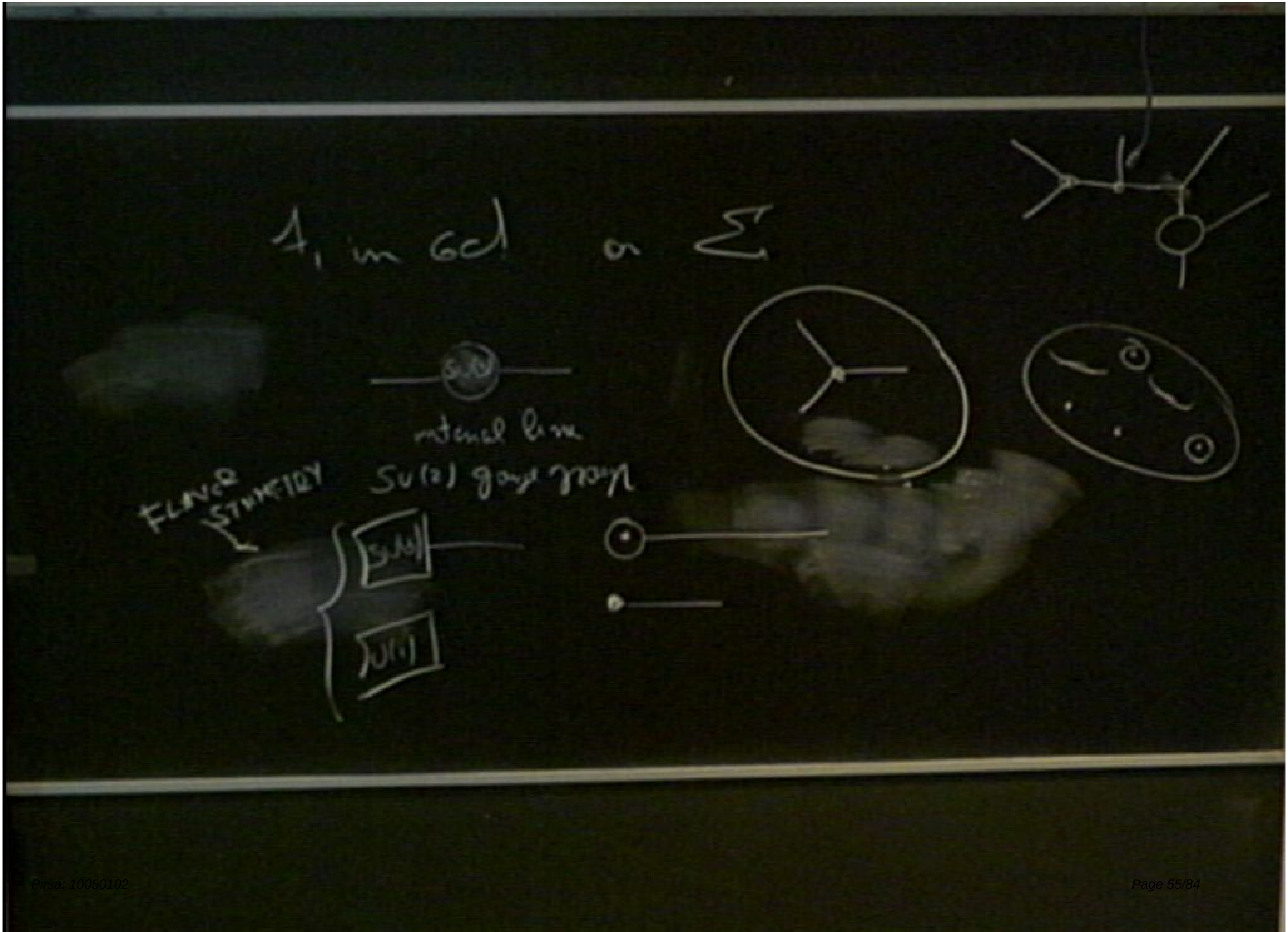
A_2 generalized quivers



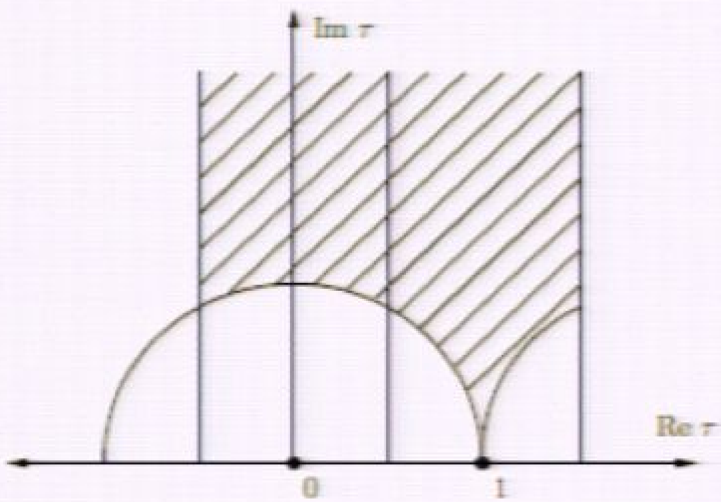


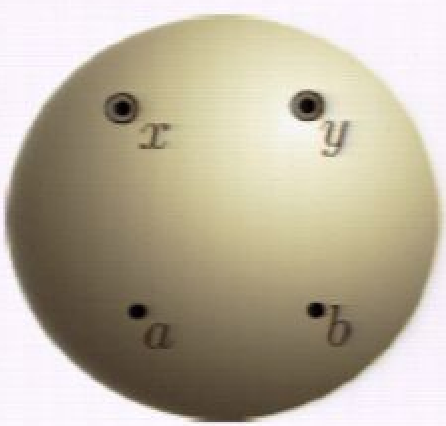
- Generalized quivers: internal edges = $SU(3)$ gauge groups; external edges = flavour groups, either $U(1)$ or $SU(3)$; vertices = hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres–Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.


T1 in 6d on Σ
SU(2)
internal line
SU(2) gauge group
FLAVOR SYMMETRY
SU(3)
U(1)


FLAVOR SYMMETRY
internal line
SU(2) gauge group
SU(N)
U(1)

A_2 generalized quivers

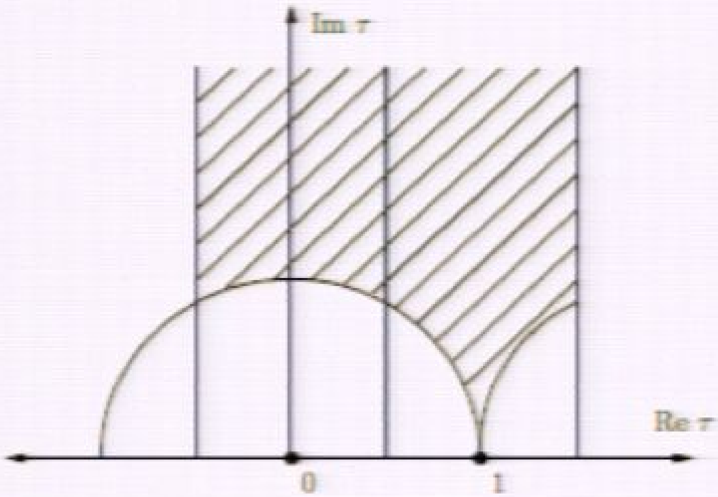


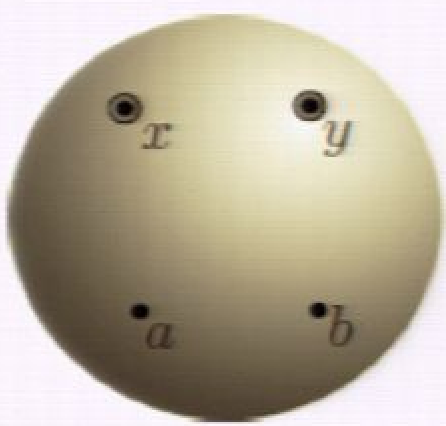


- Generalized quivers: internal edges $= SU(3)$ gauge groups; external edges $=$ flavour groups, either $U(1)$ or $SU(3)$; vertices $=$ hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres-Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.

No Signal
VGA-1

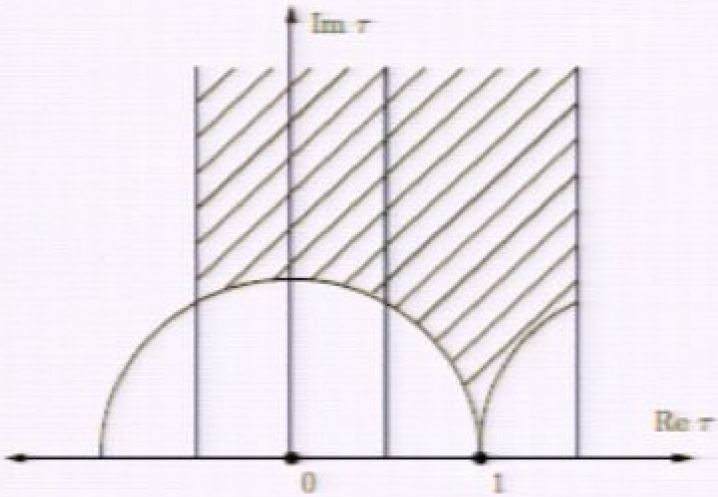
A_2 generalized quivers

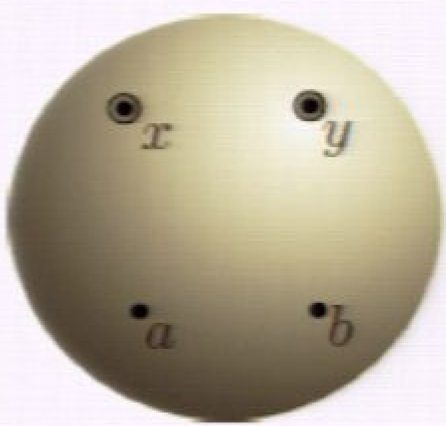




- Generalized quivers: internal edges = $SU(3)$ gauge groups; external edges = flavour groups, either $U(1)$ or $SU(3)$; vertices = hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres-Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.

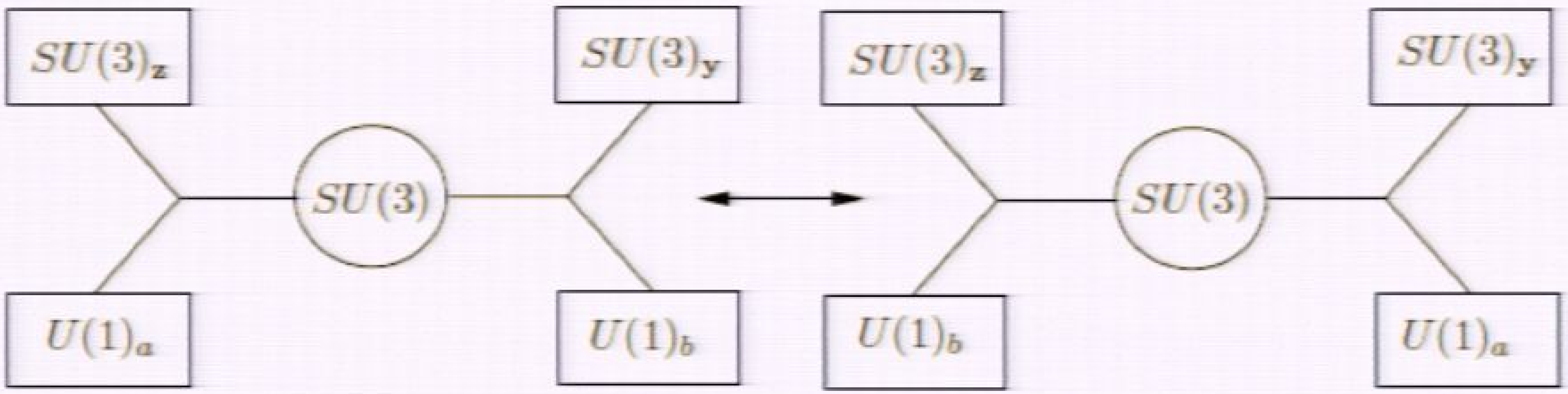
A_2 generalized quivers





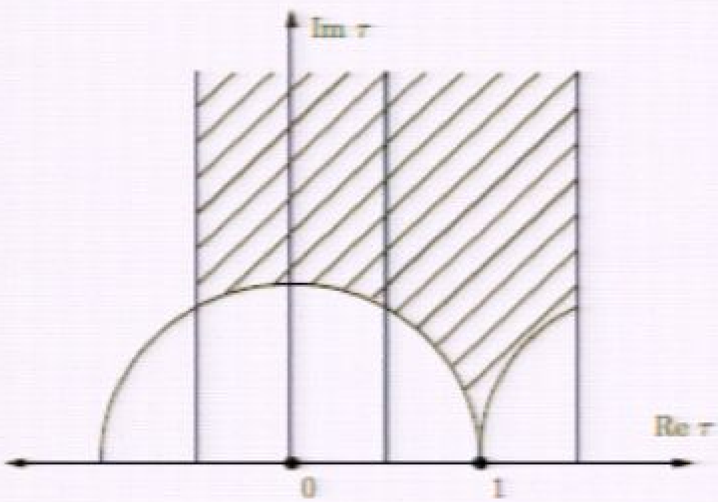
- Generalized quivers: internal edges = $SU(3)$ gauge groups; external edges = flavour groups, either $U(1)$ or $SU(3)$; vertices = hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres-Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.

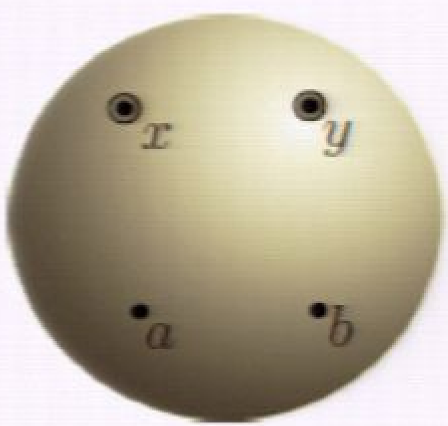
Weakly-coupled frame



$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\,(b\, y_i\, x_j)^{\pm 1}\right)\ \prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

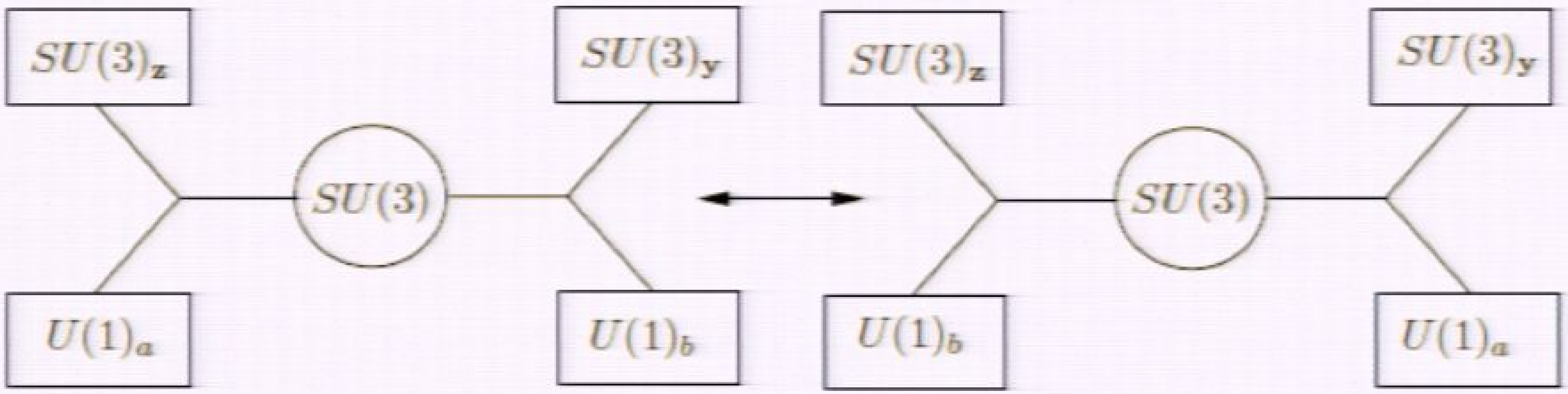
A_2 generalized quivers





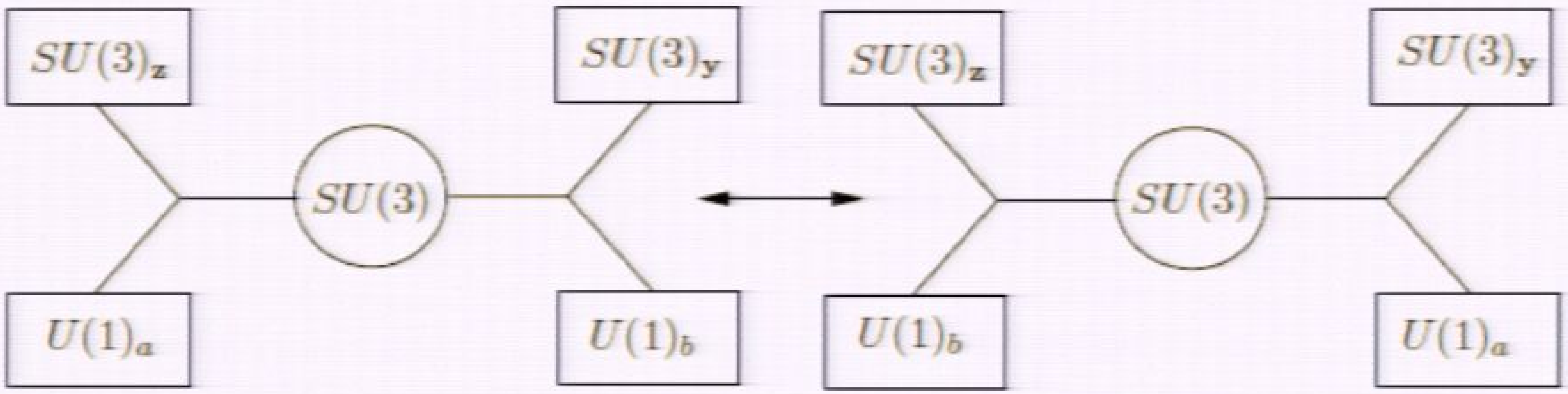
- Generalized quivers: internal edges = $SU(3)$ gauge groups; external edges = flavour groups, either $U(1)$ or $SU(3)$; vertices = hypermultiplets.
- Basic example $N_f = 6$ $SU(3)$ SYM
- S-duality group generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$.
- Three possible degenerations of the four-punctured sphere: different types of punctures collide (2 possibilities), or two like punctures collide.
- "Usual" S-duality is the equivalence of the two degenerations when different types of punctures collide (interchange of the two flavor $U(1)$ or $SU(3)$ factors)
- Argyres-Seiberg duality brings us to the frame when two like punctures collide. The theory consists of an $SU(2)$ vector multiplet coupled to a fundamental hyper and to a strongly coupled rank-one SCFT with E_6 flavor symmetry.

Weakly-coupled frame



$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\,(b\, y_i\, x_j)^{\pm 1}\right)\ \prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

Weakly-coupled frame

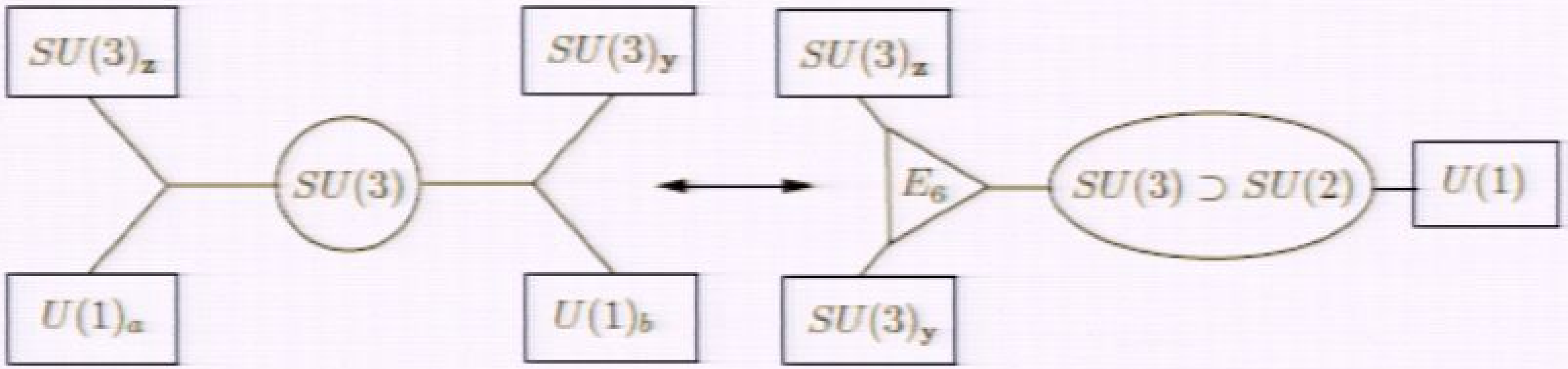


$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}(b\, y_i\, x_j)^{\pm 1}\right)\ \prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

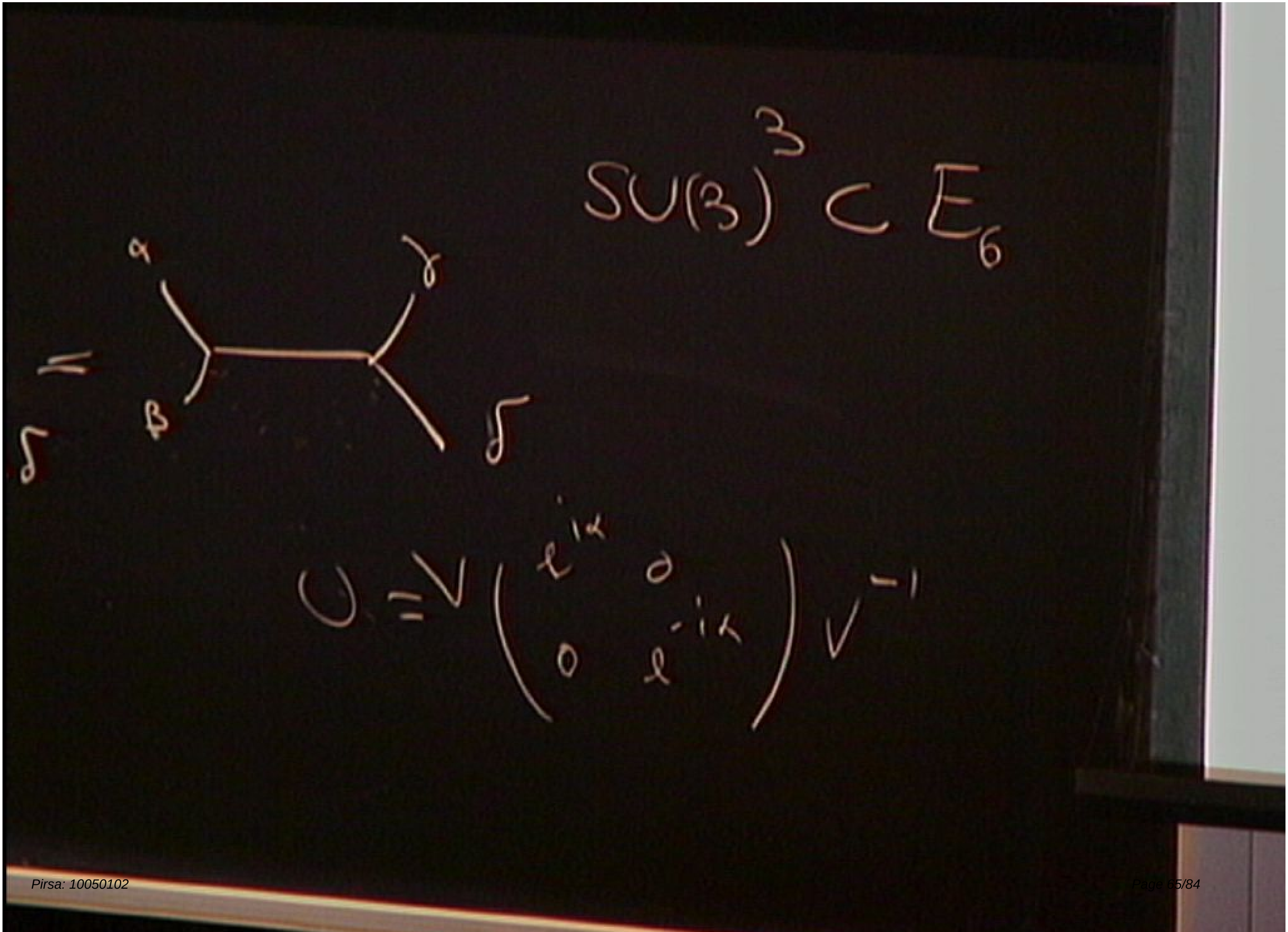
S-duality implies symmetry under $a \leftrightarrow b$

Checked perturbatively in t and analytically proved for $t = v$.

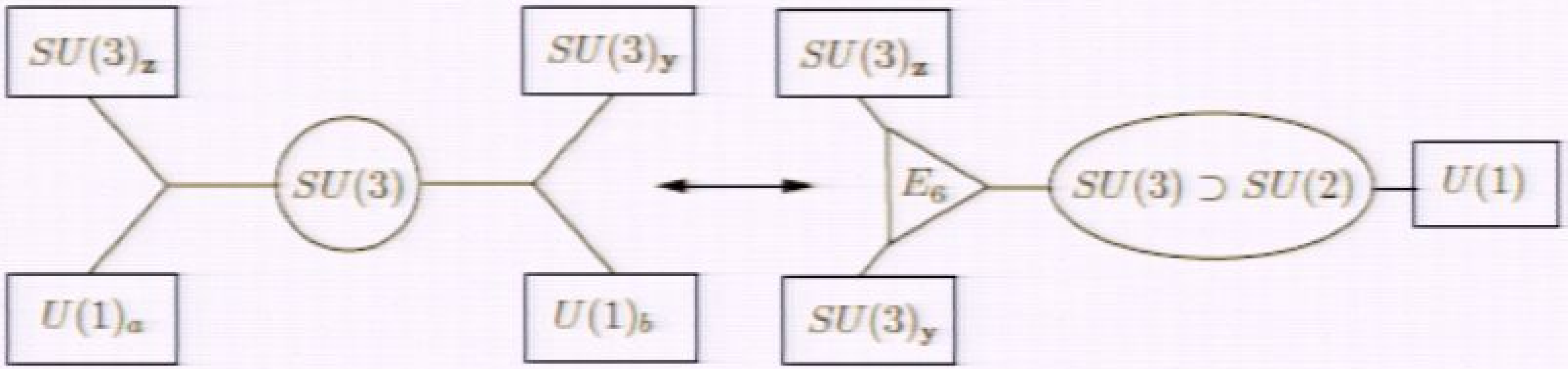
Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

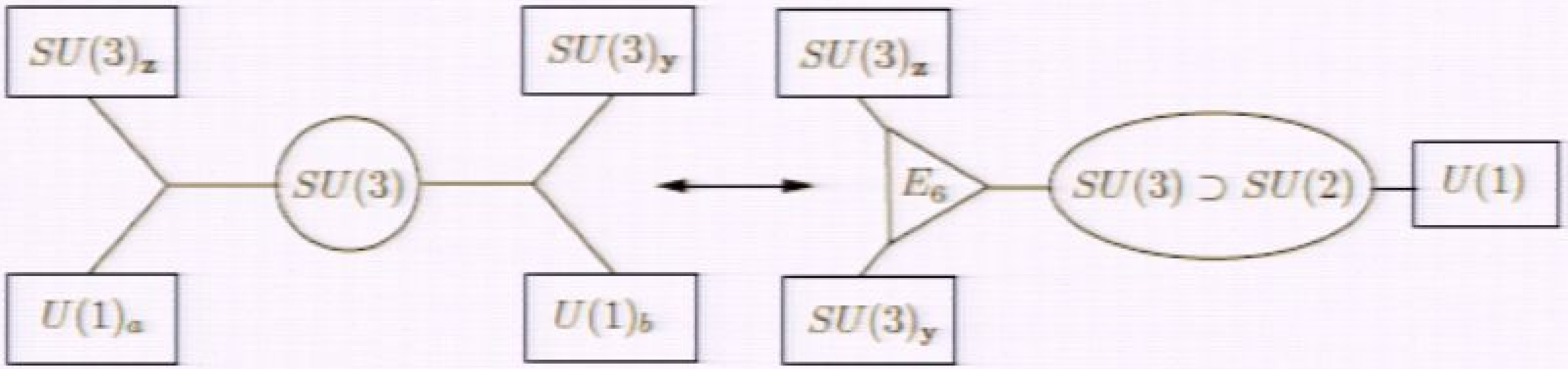

$SU(3)^3 \subset E_6$
α
γ
β
δ
$U = V \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} V^{-1}$

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Let $C^{(E_6)}(\mathbf{x},\mathbf{y},\mathbf{z})$ denote the index of rank one E_6 SCFT.

- In the strongly-coupled frame, the index reads

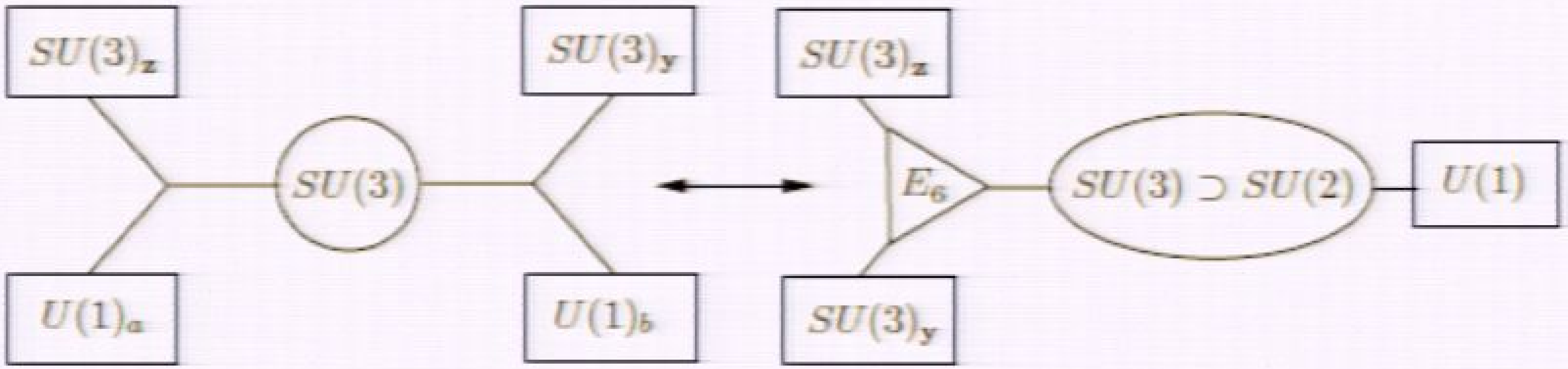
$$\hat{\mathcal{I}}(s,r;\mathbf{y},\mathbf{z}) = \kappa\,\Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma\left(\frac{t^2}{\sqrt{v}} e^{\pm 1} s^{\pm 1}\right) C^{(E_6)}((e,r),\mathbf{y},\mathbf{z}) \,.$$

- Argyres-Seiberg duality implies

$$\boxed{\hat{\mathcal{I}}(s,r;\mathbf{y},\mathbf{z}) = \mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} \qquad s = (a/b)^{3/2}, \qquad r = (a\,b)^{-1/2}}\,.$$

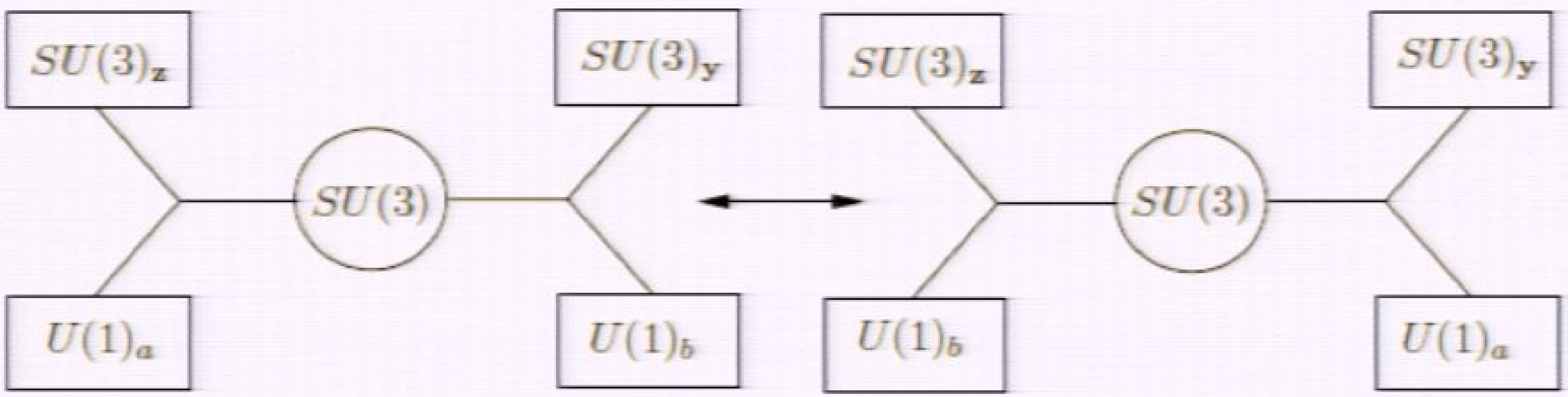
where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is the index in the weakly-coupled frame.

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Weakly-coupled frame

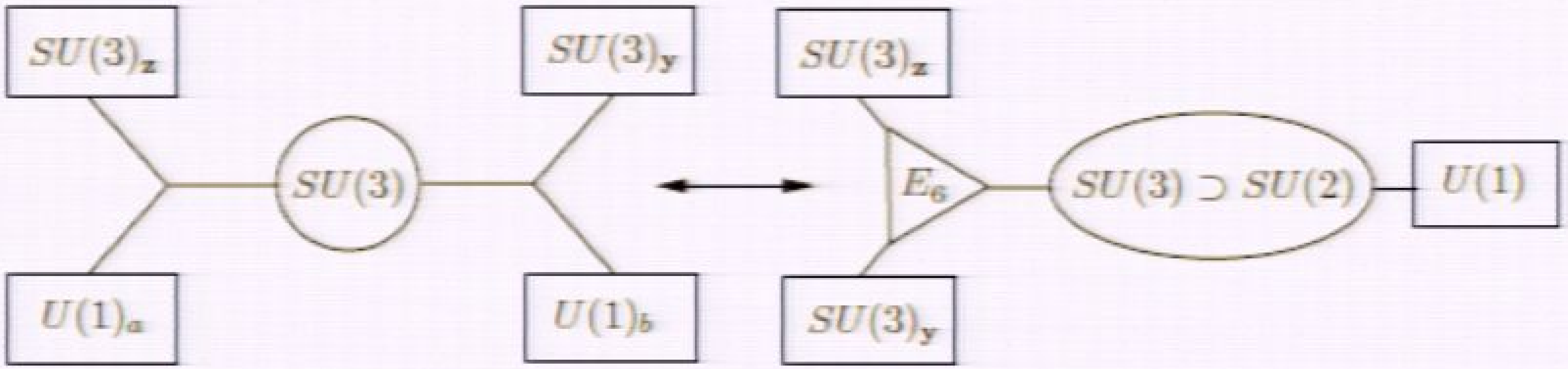


$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \frac{2}{3}\kappa^2\Gamma(t^2v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right) \Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\, y_i\, x_j\right)^{\pm 1}\right) \prod_{i\neq j}\Gamma\left(t^2 v\, \frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

S-duality implies symmetry under $a \quad \leftrightarrow \quad b$

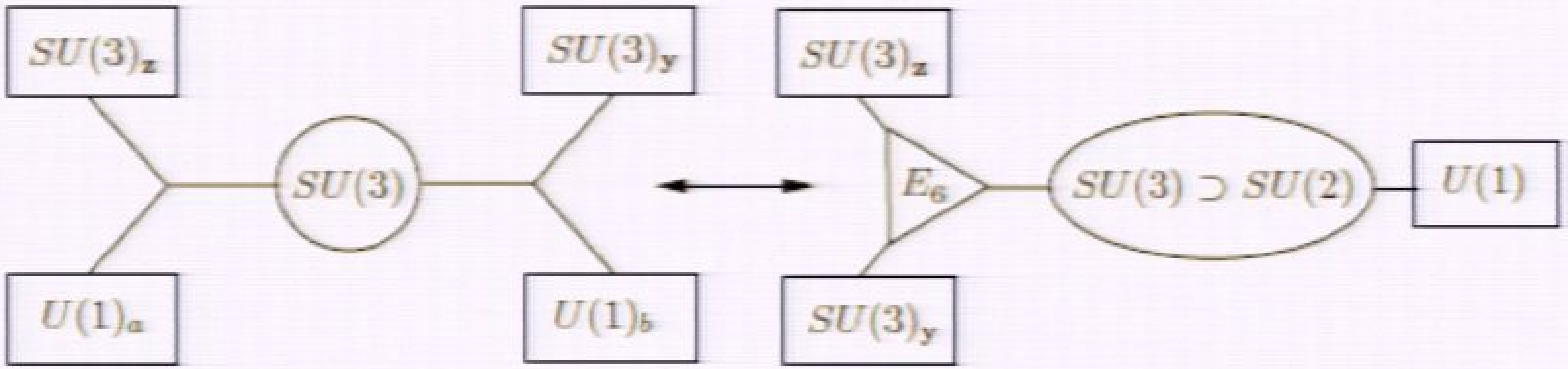
Checked perturbatively in t and analytically proved for $t = v$.

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Strongly-coupled frame



The E_6 SCFT has no Lagrangian description

Let $C^{(E_6)}(\mathbf{x},\mathbf{y},\mathbf{z})$ denote the index of rank one E_6 SCFT.

- In the strongly-coupled frame, the index reads

$$\hat{\mathcal{I}}(s,r;\mathbf{y},\mathbf{z}) = \kappa\,\Gamma(t^2 v)\oint_{\mathbb{T}} \frac{de}{2\pi i\,e}\frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})}\Gamma\!\left(\frac{t^2}{\sqrt{v}}e^{\pm 1}\,s^{\pm 1}\right) C^{(E_6)}((e,r),\mathbf{y},\mathbf{z})\,.$$

- Argyres-Seiberg duality implies

$$\boxed{\hat{\mathcal{I}}(s,r;\mathbf{y},\mathbf{z}) = \mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} \qquad s=(a/b)^{3/2}, \qquad r=(a\,b)^{-1/2}}\,.$$

where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is the index in the weakly-coupled frame.

Inverting the $SU(2)$ integral

$$\hat{\mathcal{I}}(s, r; \mathbf{y}, \mathbf{z}) \;=\; \kappa \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(\frac{t^2}{\sqrt{v}} e^{\pm 1} s^{\pm 1})}{\Gamma(\frac{t^4}{v})\Gamma(e^{\pm 2})} \Gamma(t^2 v e^{\pm 2})\, C^{(E_6)}((e, r), \mathbf{y}, \mathbf{z}) \,.$$

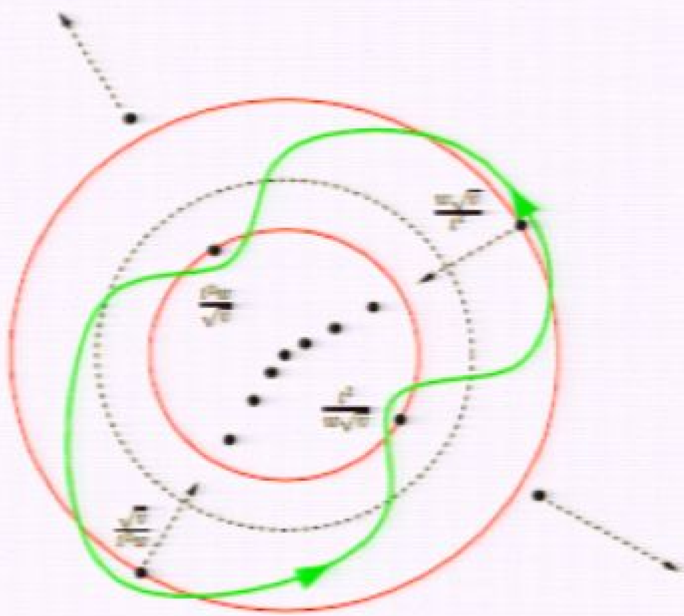
Inverting the $SU(2)$ integral

$$\hat{\mathcal{I}}(s,r;\mathbf{y},\mathbf{z}) \quad = \quad \kappa \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma(\frac{t^2}{\sqrt{v}} e^{\pm1}\, s^{\pm1})}{\Gamma(\frac{t^4}{v})\Gamma(e^{\pm2})} \Gamma(t^2 v e^{\pm2})\, C^{(E_6)}((e,\, r),\mathbf{y},\mathbf{z})\ .$$

Inversion formula: Under certain assumptions the following holds:
(Spiridonov-Warnaar 2004)

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{ds}{2\pi i\, s}\, \delta\left(s, w; \left(\frac{t^2}{\sqrt{v}}\right)^{-1}\right) f(s) \quad \longrightarrow \quad f(s) = \kappa \oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\, \delta\left(e, s; \frac{t^2}{\sqrt{v}}\right) \hat{f}(e)\,.$$

The integration contour C_w is a deformation of the unit circle



The index of the E_6 SCFT

Using the inversion formula we obtain the index of the E_6 SCFT

$$C^{(E_6)}((w,r),\mathbf{y},\mathbf{z}) = \frac{2\kappa^3\Gamma(t^2v)^2}{3\Gamma(t^2 v\, w^{\pm 2})}\oint_{C_w}\frac{ds}{2\pi i\, s}\frac{\Gamma(\frac{\sqrt{v}}{t^2}w^{\pm 1}\, s^{\pm 1})}{\Gamma(\frac{v}{t^4},\, s^{\pm 2})}\times$$

$$\times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\, x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac{1}{3}}\, z_i}{x_j\, r}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac{1}{3}}\, y_i\, x_j}{r}\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

Spectrum of protected operators from the index

$$\mathcal{C}^{(E_6)} \equiv \sum_{k=0}^{\infty} a_k\, t^k\,.$$

$$a_0 = 1,\quad a_1 t = a_2 t^2 = a_3 t^3 = 0,\quad a_4 t^4 = \frac{t^4}{v}\chi^{E_6}_{\mathbf{78}},\quad a_5 t^5 = 0$$

$$a_6 t^6 = -\,t^6 \chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3,\quad a_7 t^7 = \frac{t^7}{v}\left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y + \frac{1}{y}\right) - t^7 v^2 \left(y + \frac{1}{y}\right)$$

$$a_8 t^8 = \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v,\quad a_9 t^9 = -t^9 \left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9 \left(y + \frac{1}{y}\right) + t^9 v^3 \left(y + \frac{1}{y}\right)$$

$$a_{10} t^{10} = -\,\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\,\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2 + 1 + \frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^{10}}{v}\left(y + \frac{1}{y}\right)^2 - t^{10} v^2 \left(y + \frac{1}{y}\right)^2 .$$

The index is E_6 covariant

Spectrum of protected operators from the index

$$\mathcal{I}(t, v, y, \ldots) = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)} \ldots .$$

$$a_0 = 1, \quad a_1 t = a_2 t^2 = a_3 t^3 = 0, \quad a_4 t^4 = \frac{t^4}{v}\chi^{E_6}_{\mathbf{78}}, \quad a_5 t^5 = 0$$

$$a_6 t^6 = -t^6 \chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3, \quad a_7 t^7 = \frac{t^7}{v}\left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y + \frac{1}{y}\right) - t^7 v^2 \left(y + \frac{1}{y}\right)$$

$$a_8 t^8 = \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v, \quad a_9 t^9 = -t^9\left(y + \frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9\left(y + \frac{1}{y}\right) + t^9 v^3 \left(y + \frac{1}{y}\right)$$

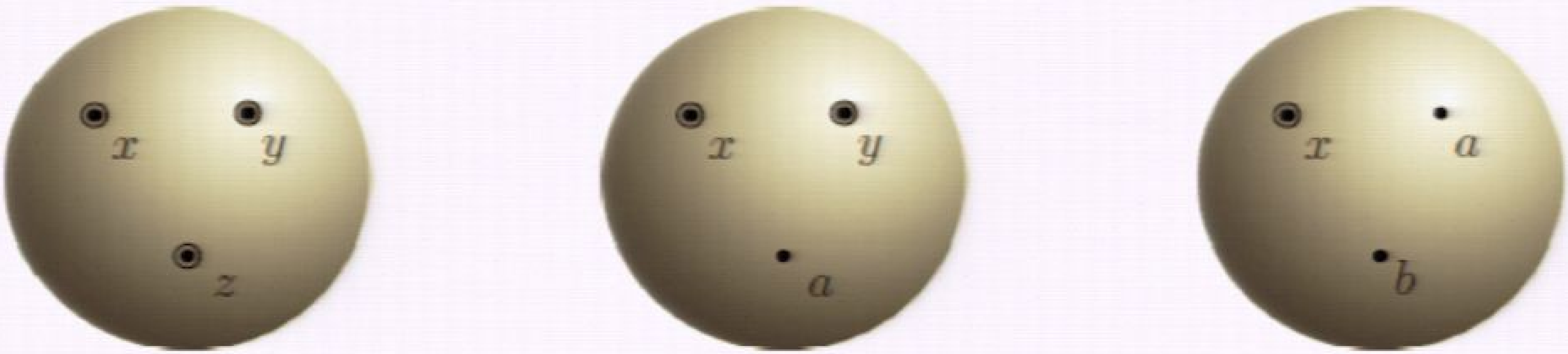
$$a_{10} t^{10} = -\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2 + 1 + \frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^{10}}{v}\left(y + \frac{1}{y}\right)^2 - t^{10} v^2 \left(y + \frac{1}{y}\right)^2 .$$

$$\mathbb{X} \to \frac{t^4/v - t^6}{(1-t^3 y)(1-t^3/y)}, \quad u \to \frac{t^6 v^3 - t^7 v^2 (y + \frac{1}{y}) + t^8 v}{(1-t^3 y)(1-t^3/y)}, \quad T \to \frac{-t^6 + \frac{t^7}{v}(y + \frac{1}{y}) + t^8 v - t^9 (y + \frac{1}{y})}{(1-t^3 y)(1-t^3/y)} .$$

	E	r	R	j_1	j_2
$\mathbb{X}$	2	0	1	0	0
u	3	-3	0	0	0
T	2	0	0	0	0

Constraints : $\quad (\mathbb{X} \otimes \mathbb{X})|_{\mathbf{650} \oplus \mathbf{1}} = 0\,, \quad \mathbb{X} \otimes u = 0\,, \quad \mathbb{X} \otimes T = 0\,.$

$SU(3)$ TQFT



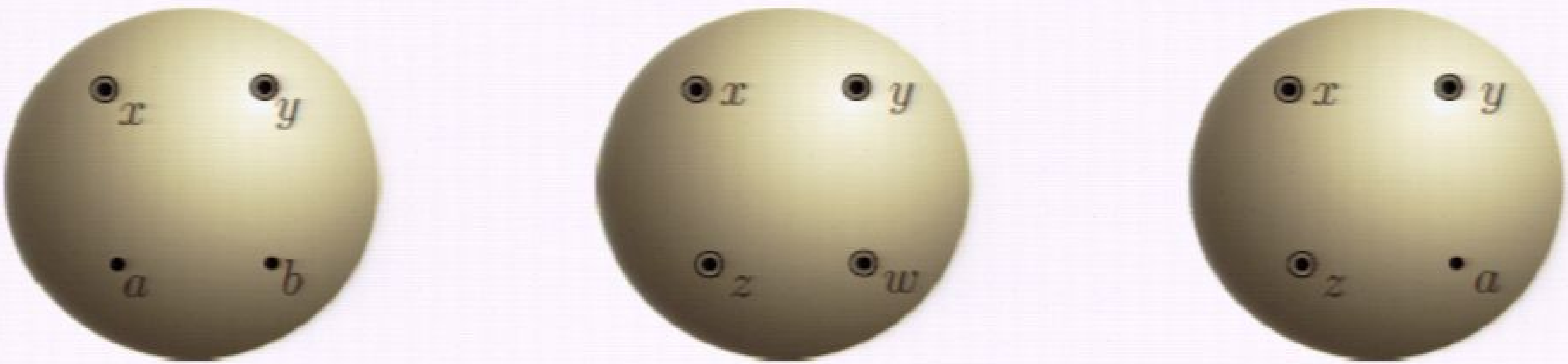
$C^{(333)}_{x,y,z}$ $C^{(133)}_{a,x,y}$ $C^{(113)}_{a,b,x}$

rank **1** : $C^{(333)}_{x,y,z}$: Index of E_6 SCFT.

rank **0** : $C^{(133)}_{a,x,y}$: Index of a hypermultiplet.

"rank -1" : $C^{(113)}_{a,b,x}$: An auxiliary construct to write the Argyres-Seiberg theory .

S-duality checks of the E_6 index



(a) (b) (c)

(a) $N_f = 6$ $SU(3)$ theory (in either of two S-dual frames), or Argyres-Seiberg theory.

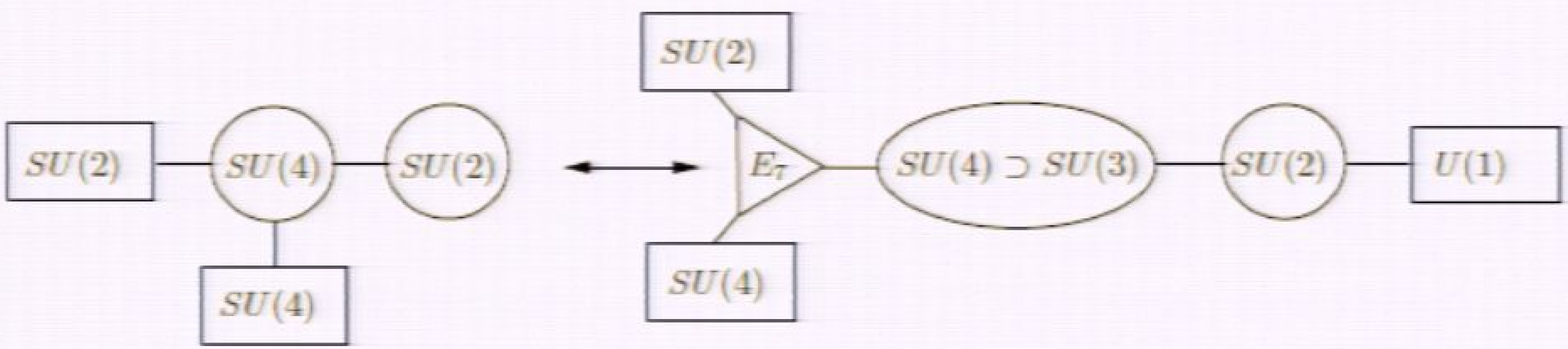
(b) Two E_6 theories "joined" by gauging an $SU(3)$ subgroup of the flavor symmetry.

(c) E_6 SCFT joined to hypers by an $SU(3)$ gauging.

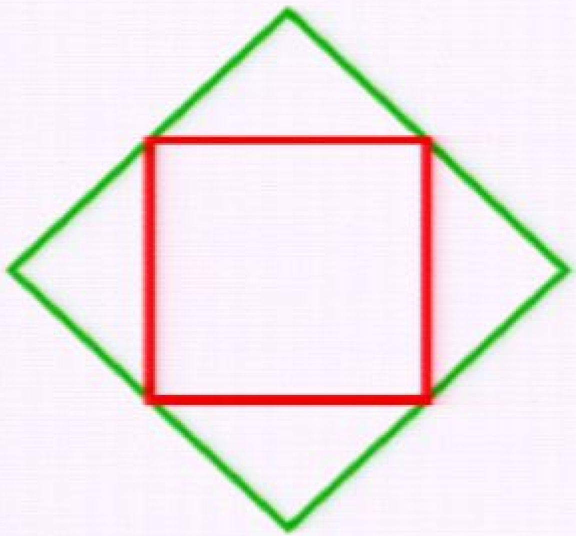
We checked associativity perturbatively in t.

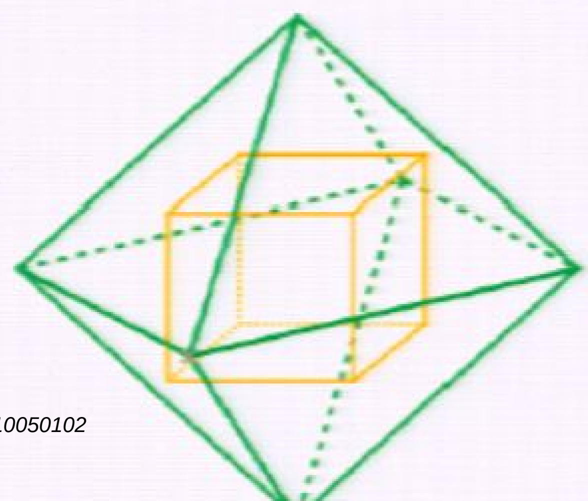
Higher rank

- Can in principle generalize the discussion to quivers with higher rank gauge groups.
- Get many intrinsically strongly coupled theories: E_7 SCFT, T_N theories ...
- To obtain the index of these higher rank theories have to learn to invert the superconformal tails (technically involved but doable).



32 supersymmetries: S-duality $SO(2n+1)/Sp(n)$





- The index on *root* system **X**

$$\mathcal{I}_{\mathcal{N}=4} \sim \oint \prod_j \frac{dz_j}{2\pi i z_j} \prod_{\alpha \in \mathbf{X}} \frac{\Gamma(t^2\, e^{\alpha}; p, q)^3}{\Gamma(e^{\alpha}; p, q)},$$

where we formally identify $z_i = e^{e_i}$.

- The root systems of $SO(2n+1)$ and $Sp(n)$ are

$$\begin{aligned} SO(2n+1) &: \quad \mathbf{X} = \{\pm e_i,\ \pm e_i \pm e_j, i<j\} \\ Sp(n) &: \quad \mathbf{X} = \{\pm 2\, e_i,\ \pm e_i \pm e_j, i<j\}, \end{aligned}$$

and they define dual polyhedra.

- $n=2$ $SO(5)$ and $Sp(2)$ are both squares.
- $n=3$ $SO(7)$ gives a cube and $Sp(3)$ is an octahedron.

8 Supersymmetries

Curious recipe (not yet understood): define the index by counting the states in the UV, but with the IR charge assignments (Romelsberger)

- Several Seiberg-dual pairs turn out to have the same index.
  (Romelsberger, Dolan Osborn, Spiridonov Vartanov)
- Remarkably, setting $v = t$ in the $\mathcal{N} = 2$ index gives the $\mathcal{N} = 1$ index of the SCFT obtained (in the IR) integrating out the chiral adjoints.
- We are working on the index of $\mathcal{N} = 1$ SCFTs that have an AdS_5 dual.
  For example there are closed formulas for the index of the SCFTs dual to $AdS_5 \times Y_{pq}$.
  Highly non-trivial matches with supergravity.

Outlook

- Many possible extensions to theories with 16 supercharges (higher rank, ADE)
- Possible to add line and surface operators

Outlook

- Many possible extensions to theories with 16 supercharges (higher rank, ADE)
- Possible to add line and surface operators
- Important to understand why the Romelsberger recipe for the $\mathcal{N} = 1$ index works.
  Make contact with counting of chiral operators (Hanany and many others) .

Outlook

- Many possible extensions to theories with 16 supercharges (higher rank, ADE)
- Possible to add line and surface operators
- Important to understand why the Romelsberger recipe for the $\mathcal{N} = 1$ index works.
  Make contact with counting of chiral operators (Hanany and many others) .
- It must be possible to obtain a "microscopic" Lagrangian description of the 2d TQFT by reduction of the twisted 6d (2,0) theory on $S^3 \times S^1$. This would give a uniform description of the index for all A_n theories. Highly non-trivial to reproduce the index by a non-perturbative 2d calculation (vortex dynamics?).
- Relation to Liouville/Toda?
- More systematic understanding of the connection with elliptic hypergeometric mathematics?