Title: Mode-Coupling, Hydrodynamic Long-time tails from Anti de Sitter Space

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Mode-Coupling, Hydrodynamic Long-Time Tails from Anti de Sitter Space

Mode-Coupling, Hydrodynamic Long-Time Tails from Anti de Sitter Space

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Emergence and Entanglement
Perimeter Institute, May 25-29, 2010

Paper

- Simon Caron-Huot and Omid Saremi
- arXiv 09094525

Long-Time Tails: (A bit of) History

- Conserved currents relax exponentially in time on long time scales $t \sim k^{-2}$
- MD Simulations for hard-disk/hard-sphere fluids in two/three dimensions [Alder, Wainwright 1960s]
- Surprise: autocorrelations live much longer than exponential relaxation time ($t \sim k^{-2}$) implicated by diffusion

Long-time tails and Mode-coupling:

- The existence of a back flow for a labeled particle
- Low-densities: particle is pushed along the initial motion also scatterings are rare: building up long-time positive correlation in the velocity field
- Vortex like pattern develops. After a few mean collision time, it matches the prediction of the Navier-Stokes equations
- High-densities: initial motion rapidly randomizes. The long-time autocorrelation will be lost

Long-time tails and Mode-Coupling

- Positive long-time autocorrelation: coupling between the particle motion and the hydrodynamic response of the fluid
- Momentum $p^i(0)$ after time t gets re-distributed
- Evolution of $V(t)$ is due to diffusion of the transverse component of the momentum $R(t) = (\gamma_\eta t)^{\frac{1}{2}}$ $p^i(0)/nV(t)$
- If there is charge diffusion $R(t) = ((\mathcal{D} + \gamma_\eta)t)^{\frac{1}{2}}$

Long-time tails: Numerical experiments

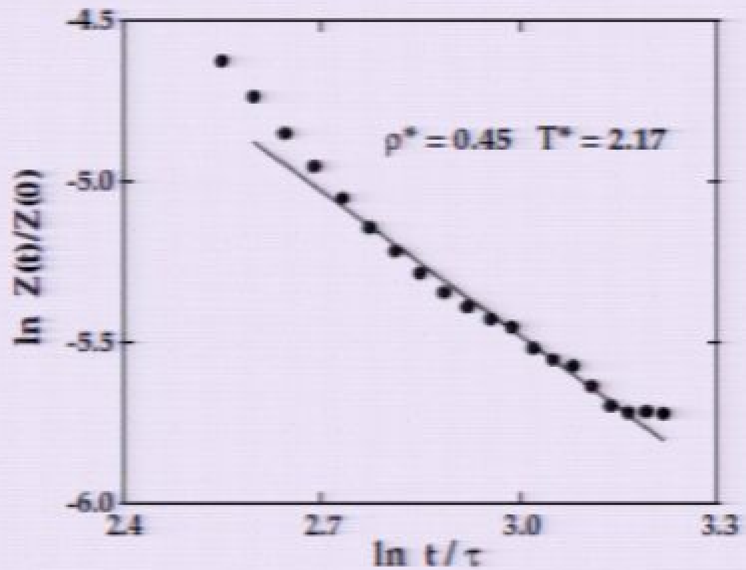

FIG. 8.5. Log–log plot of the velocity autocorrelation function versus time for a system of particles interacting through a truncated Lennard-Jones potential. The points are molecular-dynamics results and the line is drawn with a slope equal to $-\frac{3}{2}$. The unit of time is $\tau = (m\sigma^2/48\varepsilon)^{1/2}$. After Levesque and Ashurst.[22]

Long-time tails in an actual experiment

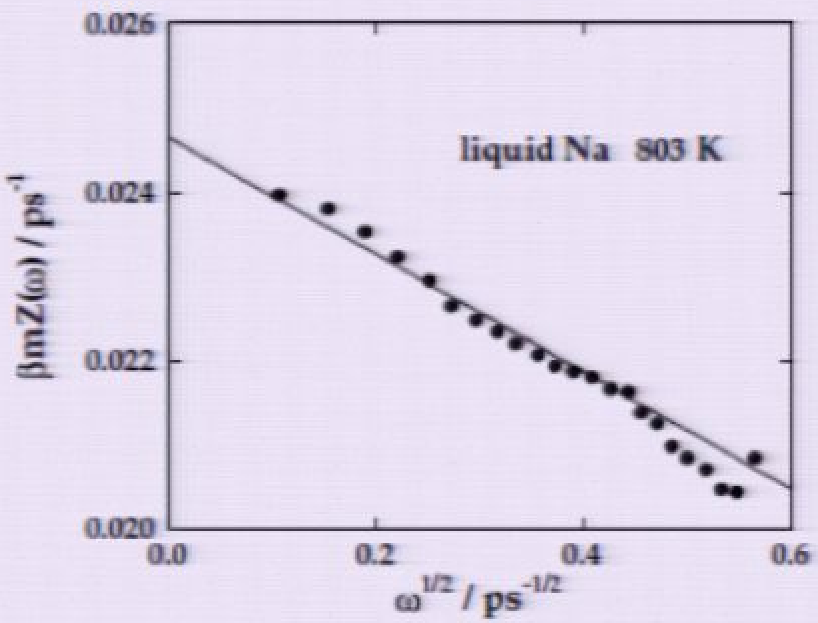

FIG. 8.6. Power spectrum of the velocity autocorrelation function of liquid sodium as a function of $\omega^{1/2}$. The points are derived from inelastic neutron-scattering measurements and the line is a least-squares fit to the data. After Morkel *et al.*[23]

Long-time tails and Mode-Coupling

- Mode-coupling formalism: decay of fluctuations into to a pair of hydrodynamic modes
- No Kubo formula in 2+1 dimensions for shear viscosity

Crash Course in Hydrodynamics

- Hydrodynamics: universal (interacting) theory of conserved currents
- Classical limit: states are highly populated $\omega/T,\ k/T \ll 1$
- Hydrodynamic variables : conserved currents
- Conservation laws: familiar Navier-Stokes equation and continuity in the NR-limit
- Constitutive relations to close the system of equations

$$j^a \simeq -D\nabla^a \delta\rho + \delta\rho\ \delta u^a + \dots$$

- Effective field theory philosophy: add any operator consistent with symmetries to the constitutive relations with unknown couplings

Long-time tails: convection term

- Near equilibrium there are two open channels for dissipation: diffusion and convection
- Diffusion acts on time scales $t \sim k^{-2}$. It detects inhomogeneities. Convection has no momentum dependence
- On largest scales diffusion shuts down. Although less efficient, convection dominates

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Long-time tail phenomenon

- Phenomenon associated with mode-coupling
- Long-time tails at $k = 0$ power law fall offs in correlation functions of conserved currents

$$\int d^{d-1}x \; \langle j^a(\tau, x) j^b(0,0) \rangle \propto \frac{\delta^{ab}}{\tau^{\frac{d-1}{2}}}$$

Hydrodynamic perturbation theory

- Hydrodynamic propagators and interactions

$$
\begin{aligned}
V^{-1}\langle \rho(\tau,k)\rho(0,-k)\rangle &= \Xi e^{-Dk^2|\tau|}, \\
V^{-1}\langle u^i(\tau,k)u^j(0,-k)\rangle &= \frac{T}{\epsilon+p}\left[(\delta^{ij}-\frac{k^i k^j}{k^2})e^{-\gamma_\eta k^2|\tau|}+\frac{k^i k^j}{k^2}e^{-\frac{1}{2}\gamma_s k^2|\tau|}\cos(kc_s\tau)\right]
\end{aligned}
$$

$$j^a \simeq -D\nabla^a\delta\rho + \delta\rho\,\delta u^a + \ldots$$

Long-Time Tails: Hydrodynamic Perturbation Theory

- Hydrodynamic loop diagram
- Using propagators in the long-time limit

$$\int d^{d-1}x \, \langle j^a(\tau, x) j^b(0) \rangle \simeq \int d^{d-1}x \, \langle \rho(\tau, x) u^a(t, x) \, \rho(0) u^b(0) \rangle$$

$$\langle \rho(\tau, x) u^a(\tau, x) \, \rho(0) u^b(0) \rangle = \langle \rho(\tau, x) \rho(0) \rangle \, \langle u^a(\tau, x) u^b(0) \rangle$$

$$\int d^{d-1}x \, \langle j^a(\tau, x) j^b(0) \rangle = \frac{T \Xi \delta^{ab}}{\epsilon + p} \frac{d-2}{d-1} \frac{1}{[4\pi(D+\gamma_\eta)|\tau|]^{\frac{d-1}{2}}}$$

- Long-time tail is suppressed by (entropy) density

Long-time tail: Gravity dual

- What gravitational phenomenon is dual to this effect?
- In $N=4$ SYM, this will be an $\mathcal{O}(1/N_c^2)$ effect
- In the bulk, it appears at the order g_s^2
- The bulk AdS (quantum) gravity captures this effect at one-loop
- Real-time signature perturbation theory is more transparent

Things (not) to worry about!

- The region $k << 1/\sqrt{t}$ dominates the loop momentum integral
- Non-renormalizability of perturbative quantum gravity in AdS is not an issue. The UV region's contribution is negligible so cutoff can be removed safely and still remain finite

What to Compute

- Hydrodynamic correlators are real so the symmetric function is what to compute

$$\frac{1}{2}\int d^{d-1}x \; {}^{(1)}\langle\{j^x(\omega,x), j^x(0)\}\rangle = (1+2n_B(\omega))\,\mathrm{Im}\,{}^{(1)}G^{xx}_R(\omega)$$

- Related to the retarded correlator in equilibrium

$$\begin{aligned} {}^{(1)}G^{xx}_R(\tau) &\equiv \int d^{d-1}x \; {}^{(1)}\langle j^x(\tau,x,r\to\infty)\; j^x(0,r\to\infty)\rangle_R\,, \\ {}^{(1)}G^{xy,xy}_R(\tau) &\equiv \int d^{d-1}x \; {}^{(1)}\langle t^x{}_y(\tau,x,r\to\infty)\; t^x{}_y(0,r\to\infty)\rangle_R \end{aligned}$$

- We use Schwinger-Keldysh formalism in ra-basis

Setup: Bulk Spacetime

- Bulk theory: Einstein or Einstein/Maxwell (Yang-Mills)+ cosmological constant: we do not need the full supergravity on AdS5
- Bulk geometry is an AdS_{d+1}-Schwarzschild black hole

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = \left(\frac{4\pi T \ell}{d}\right)^2 dr^2 + p(r)(-q(r)dt^2 + \delta_{ij} dx^i dx^j)$$

$$p(r) = \cosh^{\frac{4}{d}}(2\pi T r), \quad q(r) = \tanh^2(2\pi T r)$$

- In perturbation theory, we gauge-fix to Gaussian normal coordinate

$$h_{r\mu} = 0, \quad h_{rr} = 0$$

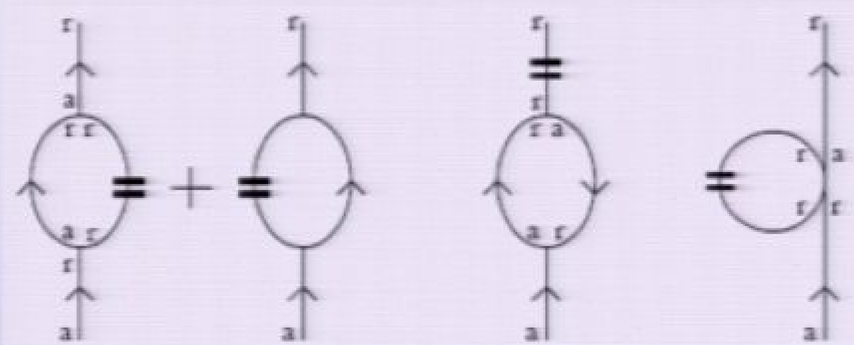
- Space-like axial gauge for the gauge field $A_r = 0.$

Diagrams

- In Schwinger-Keldysh formalism in ra-basis

$$\phi_r = \frac{1}{2}(\phi_1 + \phi_2), \quad \phi_a = \phi_1 - \phi_2.$$

$$G = \begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & G_{aa} \end{pmatrix} = \begin{pmatrix} -i(G_R - G_A)\left(\frac{1}{2} + n_B(\omega)\right) & -iG_R \\ -iG_A & 0 \end{pmatrix}$$



Bulk counter-terms

- The Symanzik procedure [Symanzik '85]

$$\langle \mathcal{O}^{i_1}(x_1)\cdots\mathcal{O}^{i_n}(x_n)\rangle_{\rm bdy} = \lim_{r_1,\dots,r_n\to r_c}\langle \Pi^{i_1}(r_1,x_1)\cdots\Pi^{i_n}(r_n,x_n)\rangle_{\rm bulk}.$$

- We do not keep track of the contact terms. They are irrelevant at long times

- Play no-role at long times : stuck to the boundary (surface interactions)

Diagrams

- In Schwinger-Keldysh formalism in ra-basis

$$\phi_r = \frac{1}{2}(\phi_1 + \phi_2), \quad \phi_a = \phi_1 - \phi_2.$$

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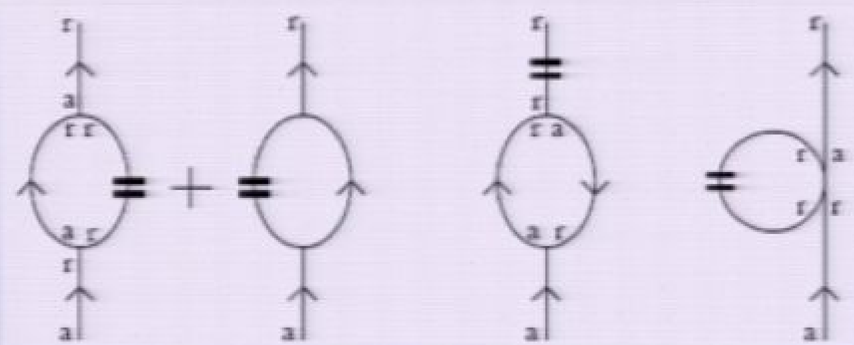


The origin of the boundary KMS conditions

- In QFT in curved s pace limit of QG: BH is in Thermal equilibrium with the environment to all orders in perturbation theory
- [Gibbons-Perry '76]

$$G_{12}(r,r';p) = -in_{\mathrm{B}}(p)(G_{\mathrm{R}}(r,r';p) - G_{A}(r,r';p)), \quad \text{etc.}$$

- Fluctuations around H-H state are thermal to all order in perturbation theory
- The KMS conditions are obeyed in the bulk
- So in an AdS/CFT setting, the boundary KMS conditions are inherited from the bulk

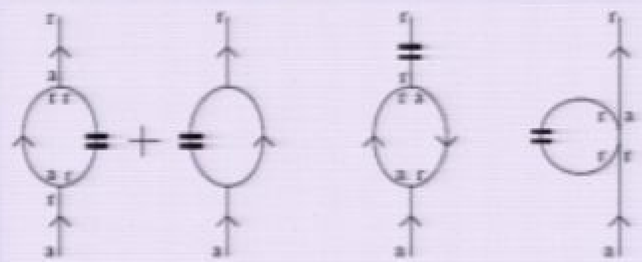
Dominant fields in the hydrodynamic regime

- Both perturbations in the loop: long-lived and diffusive
- Charge fluctuations dominate over current perturbations

  where $A_t/A_z \sim k/\omega$
- h_{tx} dominates over h_{zx} : momentum fluctuation dominates over stress
- Spin-2 fluctuations are negligible : the boundary data for non-trace part of $\delta t^{ij}_{\rm bdy}$

  i.e., $\delta t^{ij}_{\rm bdy} \simeq \delta p \delta^{ij} + \eta \nabla^{(i} u^{j)} + \mathcal{O}(\nabla^2)$

is suppressed by viscous contributions .



Bulk Variable

- The bulk computation is best organized in momentum variables rather than in field space.

$$t^{\mu}{}_{\nu} = 2\frac{\delta S_{\text{bulk}}}{\delta \gamma_{\mu\nu}} = \frac{\sqrt{-\gamma}}{8\pi G_N}(\delta^{\mu}{}_{\nu} K - K^{\mu}{}_{\nu}),$$

$$j^{\mu} = \frac{\delta S_{\text{bulk}}}{\delta A_{\mu}} = \frac{\sqrt{-\gamma}}{g^2_{d+1}} F^{\mu r},$$

- Consider them as extending in the bulk: local density of the currents

$$t^{\mu}_{\nu} = t^{\mu}_{\nu}(r), \quad j^{\mu} = j^{\mu}(r)$$

Gravity Vertices: j^x

- Rewriting in terms of the bulk local currents

$$
\begin{aligned}
V_{A-A-g} &= \int d^{d-1}x\, \delta j^t(\tau)\, \delta t^t{}_x(\tau) \int \frac{dr\, 16\pi G g^{xx}}{\sqrt{-g}|g^{tt}|} + \mathcal{O}(1/\tau) \\
&= \int d^{d-1}x \frac{\delta j^t(\tau)\, \delta t^t{}_x(\tau)}{\epsilon + p},
\end{aligned}
$$

- Gauge-gauge-gauge vertices are also sub-leading : $\delta A^x \partial_r \delta A^t$
- and $\delta A^t \partial_r \delta A^x$ involve current fluctuation

Gravity Vertices : t^{xy}

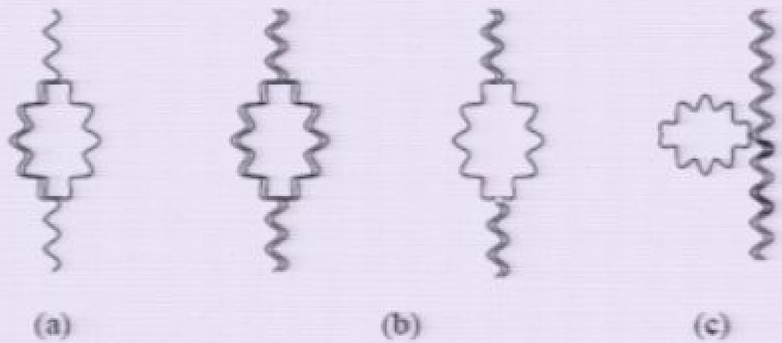
- Recasting it in the bulk stress tensor variables

$$
\begin{aligned}
V_{g-g-g} &= \int d^{d-1}x\, \delta t^t{}_x\, \delta t^t{}_y \int dr \frac{16\pi G g^{xx}}{\sqrt{-g}|g^{tt}|} + \mathcal{O}(1/\tau) \\
&= \frac{1}{\epsilon+p} \int d^{d-1}x\, \delta t^t{}_x\, \delta t^t{}_y,
\end{aligned}
$$

- Graviton-gauge-gauge is sub-leading: couples to $F^{xr}F^{yr} \sim j^x j^y$ which is sub-dominant

Diagrams

- At one loop



Feynman diagrams contributing to (a) current correlator (b) stress tensor correlator. Four-point vertices (c) will be found to have negligible effects. Wavy lines are bulk Yang-Mills fields and double lines are gravitons.

External Wavefunctions

- Let us focus on $< j_x j_x >$
- External wave function for A_x :boundary conditions

$$\frac{1}{\mathcal{G}}\partial_r[\mathcal{G}\partial_r\psi_\omega(r)] + |g^{tt}|\omega^2\psi_\omega(r) = 0. \qquad \mathcal{G} = \sqrt{-g}g^{xx}$$

$$\psi_\omega(r) \propto r^{-i\frac{\omega}{2\pi T}} \text{ as } r \to 0 \qquad \psi_\omega(r) = 1 \text{ when } r \to \infty$$

$$\psi_\omega^{\rm ret}(r) = \frac{{}_2F_1(-\frac{(1+i)}{2}\mathfrak{w}, -\frac{(-1+i)}{2}\mathfrak{w}; 1 - i\mathfrak{w}; \tanh^2(\pi T r))}{{}_2F_1(-\frac{(1+i)}{2}\mathfrak{w}, -\frac{(-1+i)}{2}\mathfrak{w}, 1 - i\mathfrak{w}, 1)} \tanh^{-i\mathfrak{w}}(\pi T r)$$

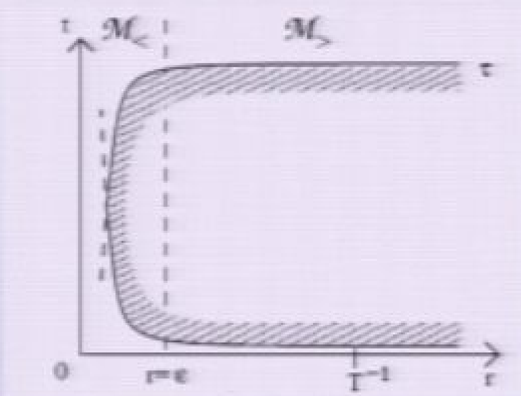
- Spectrum of the quasi-normal modes: gapped and off the imaginary axis

$$\mathfrak{w}_n = n(\pm 1 - i), \quad n = 1, 2, \ldots.$$

External Wavefunction

- A_x relaxes microscopically. A_x dissipates on time scale T^{-1}
- The field profile falls into the horizon along a null ray

$$A_x^{\text{ret}}(r,t) \to A_x^{\text{ret}}(t - r_*(r))$$



Bulk-bulk correlators

- The upshot: bulk-bulk correlators are equal to the boundary-boundary correlators up to corrections which become large near the horizon.
- The $G_{\mathrm{R}}^{tt}(r, r'; p)$ propagator

$$[\partial_r(C\partial_r)+(|g^{tt}|\omega^2-g^{zz}k^2)C]\left[G_{\mathrm{R}}^{tt}(r,r';p)+\frac{\delta(r-r')}{Cg^{zz}}\right]=-k^2\delta(r-r')$$

- Solve in the hydro approximation
- The combination with Neumann boundary condition

$$G(r)=(1-i\frac{\sigma k^2}{\omega\Xi})F(r)+(1+i\frac{\sigma k^2}{\omega\Xi})F^*(r)$$

Emergence of non-linear Hydro from Anti de Sitter Space

- Tip of the half-almond $r_0(\tau) \approx \frac{\sqrt{2}}{2\pi T} e^{-\pi T \tau}$.
- The shaded region $T^{-1} \ll t$
- Not all of the half-almond contributes: the external disturbances relax on microscopic scale T^{-1}
- Vertices near the tip would correspond to a scale of non-locality in the boundary of the order $\sim \tau$
- In contrast, in hydro the vertices are instantaneous on the microscopic time scale T^{-1}
- A minimal choice for ϵ is $\epsilon(\tau) \sim \frac{1}{T} e^{-\pi T \tau/2}$
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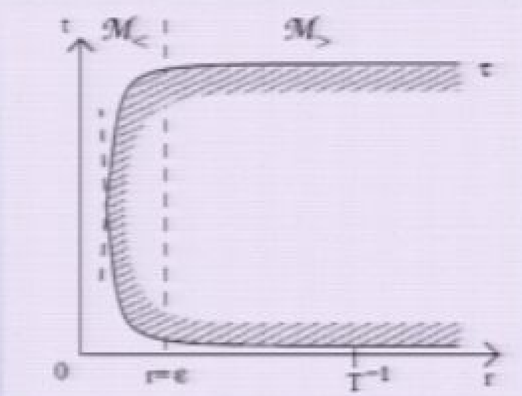

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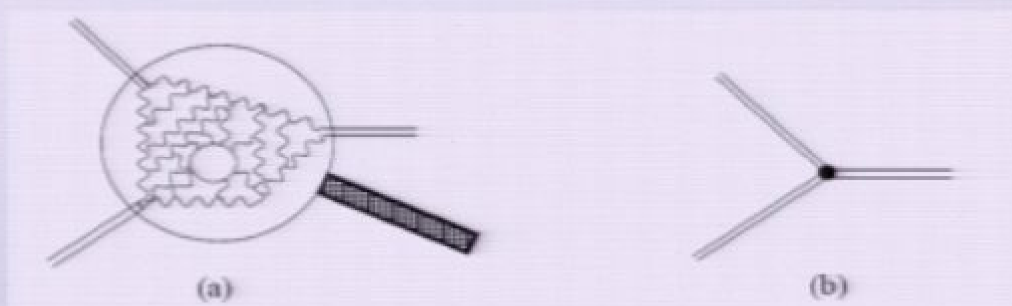
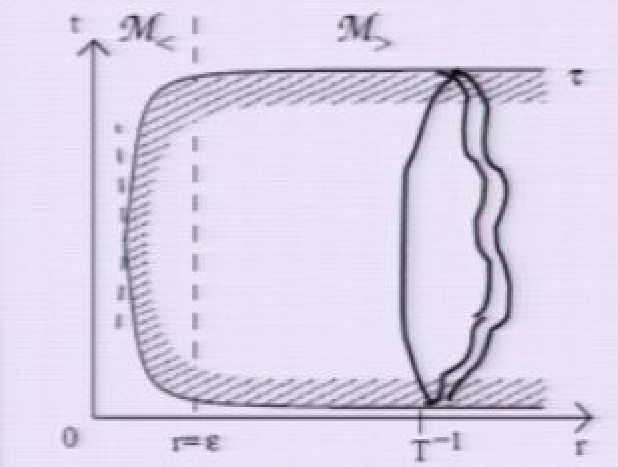


Emergence of Hydrodynamics from Anti de Sitter Space

- $r - t$ plane : the causal diamond: insertions at 0 and τ



Bulk-bulk correlators

- The full bulk-bulk propagator

$$
\begin{aligned}
G_{\mathrm{R}}^{tt}(r,r';p) + \frac{\delta(r-r')}{C(r)g^{zz}} &= i\sigma k^2 \frac{G(r)F(r')\theta(r-r') + F(r)G(r')\theta(r'-r)}{2(\omega + i\frac{\sigma}{\Xi}k^2)} \\
&= \frac{i\sigma k^2}{\omega + i\frac{\sigma}{\Xi}k^2} + \mathcal{O}(\omega \log\frac{1}{r}, k^2).
\end{aligned}
$$

- It loses its radial dependence up to corrections which are only large in $\mathcal{M}_<$

Radial Flow equation for the Hydrodynamic Variable

- Bulk Hamiltonian (Einstein and Yang-Mills) : the radial evolution of the momenta. It can be organized written as a derivative expansion

$$\partial_r(\delta t^\mu{}_\nu) = \frac{1}{8\pi G}\delta(\sqrt{-g}^{(d)} R^\mu{}_\nu) = 0 + \mathcal{O}(\omega^2, q^2) \quad \mu \neq \nu,$$

$$\partial_r(\delta j^\mu) = \frac{1}{g^2_{d+1}}\delta(\partial_\nu \sqrt{-g} F^{\nu\mu}) = 0 + \mathcal{O}(\omega^2, q^2),$$

$$\frac{1}{\sqrt{g}}\partial_r(\sqrt{g} K^\mu{}_\nu) = {}^{(d)}R^\mu{}_\nu + \frac{d}{\ell^2}\delta^\mu{}_\nu.$$

- Holds up to exponentially close to the horizon due to oscillations

$$r \sim e^{-2\pi T/\omega}$$

Relation to Long-time tails in Black hole Physics

- Evolution of probe fields on black hole spacetimes
- Late-time power-law tails in flat space $\Phi \sim t^{-n}$
- In AdS space they do not exist. Dirichlet boundary condition kill it
- We have argue that they do exist even in AdS but at quantum level!
- The argument for their existence comes from an unlikely (to the eyes of a relativist in the 70s) source: hydrodynamics!

Summary and outlook

- Emergence of hydrodynamic long-time tails from Anti de Sitter space
- Depends on very few entries of the AdS/CFT dictionary: Universality
- Counter-intuitive: must have come from deep IR. Hydro fluctuation are tide to the horizon but the scale of vertices are microscopic since they are local in time
- Mode coupling theory of the critical systems from gravity: diverging transport coefficients
- Log-running of the shear viscosity as the universal property of 2+1 dimensional horizons