Title: Quantum Entanglement and Quantum Criticality

Date: May 28, 2010 04:00 PM

URL: http://pirsa.org/10050086

Abstract: The entanglement entropy of a pure quantum state of a bipartite system is defined as the von Neumann entropy of the reduced density matrix obtained by tracing over one of the two parts. Critical ground states of local Hamiltonians in one dimension have entanglement that diverges logarithmically in the subsystem size, with a universal coefficient that is is related to the central charge of the associated conformal field theory. In this talk I will discuss the extension of these ideas to two dimensional systems, either at a special quantum critical point or in a topological phase. We find the entanglement entropy for a standard class of z=2 quantum critical points in two spatial dimensions with scale invariant ground state wave functions: in addition to a nonuniversal ‘‘area law’’ contribution proportional to the size of the boundary of the region under observation, there is generically a universal logarithmically divergent correction, and in its absence a universal finite piece is found. This logarithmic term is completely determined by the geometry of the partition into subsystems and the central charge of the field theory that describes the equal-time correlations of the critical wavefunction. On the other hand, in a topological phase there is no such logarithmic term but instead a universal constant term. We will discuss the connection between this universal entanglement entropy and the nature of the topological phase.

Quantum Entanglement and Quantum Criticality

Talk at the Workshop "Emergence and Entanglement"
Perimeter Institute (Waterloo, Ontario, Canada), May 24–29, 2010

Eduardo Fradkin

Department of Physics
University of Illinois at Urbana Champaign

May 27, 2010

Collaborators and References

- Benjamin Hsu, Eytan Grosfeld, Stefanos Papanikolaou (Cornell), Kumar Raman (Riverside), Robert G. Leigh, Shiying Dong, Sean Nowling (Helsinki)
- Joel E. Moore (Berkeley) , Michael Mulligan (MIT), Eun-Ah Kim (Cornell), Paul Fendley (Virginia)

- Benjamin Hsu, Eytan Grosfeld, and Eduardo Fradkin, *Quantum noise and entanglement generated by a local quantum quench*, Phys. Rev. B **80**, 235412 (2009), arXiv:0908.2622.
- Eduardo Fradkin, *Scaling of Entanglement Entropy at 2D quantum Lifshitz fixed points and topological fluids*, J. Phys. A: Math. Theor. **42**, 504011 (2009), (special issue on Entanglement Entropy ,P. Calabrese, J. Cardy and B. Doyon, editors); arXiv:0906.1569v1.
- Benjamin Hsu, Michael Mulligan, Eduardo Fradkin and Eun-Ah Kim, *Universal behavior of the entanglement entropy in 2D conformal quantum critical points*, Phys. Rev. B **79**, 115421 (2009).
- S. Papanikolaou, K. S. Raman, and E. Fradkin, *Topological phases and topological entropy of two-dimensional systems with finite correlation length*, Phys. Rev. B **76**, 224421 (2007).
- E. Fradkin and J. E. Moore, *Entanglement entropy of 2D conformal quantum critical points: hearing the shape of a quantum drum*, Phys. Rev. Lett. **97**, 050404 (2006).
- S. Dong, E. Fradkin, R. G. Leigh and S. Nowling, *Topological Entanglement Entropy in Chern-Simons Theories and Quantum Hall Fluids*, JHEP **05**, 016 (2008).

Outline

- Motivation

Outline

- Motivation
- Entanglement Entropy and Quantum Phase Transitions

Outline

- Motivation
- Entanglement Entropy and Quantum Phase Transitions
- Entanglement entropy and quantum criticality in $1+1$ Dimensions

Outline

- Motivation
- Entanglement Entropy and Quantum Phase Transitions
- Entanglement entropy and quantum criticality in $1+1$ Dimensions
- Conformal Quantum Critical Points in $2+1$ dimensions

Outline

- Motivation
- Entanglement Entropy and Quantum Phase Transitions
- Entanglement entropy and quantum criticality in $1+1$ Dimensions
- Conformal Quantum Critical Points in $2+1$ dimensions
- Entanglement Entropy in Topological Phases
- Applications to abelian and non-abelian fractional quantum Hall states and $\mathbb{Z}_2$ topological phases

Motivation

- Entanglement entropy measures quantum mechanical correlations in many body systems and field theories

Motivation

- Entanglement entropy measures quantum mechanical correlations in many body systems and field theories
- It is a very non-local quantity which very difficult to measure (see, however, Klich and Levitov (2009); Hsu, Grosfeld, Fradkin (2009)

Motivation

- Entanglement entropy measures quantum mechanical correlations in many body systems and field theories
- It is a very non-local quantity which very difficult to measure (see, however, Klich and Levitov (2009); Hsu, Grosfeld, Fradkin (2009)
- Its behavior at generic quantum critical points is not understood, except in 1D (from CFT)

Motivation

- Entanglement entropy measures quantum mechanical correlations in many body systems and field theories
- It is a very non-local quantity which very difficult to measure (see, however, Klich and Levitov (2009); Hsu, Grosfeld, Fradkin (2009)
- Its behavior at generic quantum critical points is not understood, except in 1D (from CFT)
- We will discuss the scaling of the entanglement entropy at a class of special 2D QCP with scale invariant wave functions: universal finite terms.

Motivation

- Entanglement entropy measures quantum mechanical correlations in many body systems and field theories
- It is a very non-local quantity which very difficult to measure (see, however, Klich and Levitov (2009); Hsu, Grosfeld, Fradkin (2009)
- Its behavior at generic quantum critical points is not understood, except in 1D (from CFT)
- We will discuss the scaling of the entanglement entropy at a class of special 2D QCP with scale invariant wave functions: universal finite terms.
- Topological phase $\Rightarrow$ universal topological entropy
- Can the structure of the topological field theory of a topological phase be determined by a suitable set of entanglement measurements?

Physics Illinois

Time Reversal Invariant Systems: Quantum Dimer Models

Time Reversal Invariant Systems: Quantum Dimer Models

- Quantum dimer models at the Rokhsar-Kivelson (RK) point have an exact ground state wave function has the short range RVB form

Time Reversal Invariant Systems: Quantum Dimer Models

- Quantum dimer models at the Rokhsar-Kivelson (RK) point have an exact ground state wave function has the short range RVB form

$$|\Psi_{\mathrm{RVB}}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

Time Reversal Invariant Systems: Quantum Dimer Models

- Quantum dimer models at the Rokhsar-Kivelson (RK) point have an exact ground state wave function has the short range RVB form

$$|\Psi_{\mathrm{RVB}}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

-
  - Bipartite lattices: quantum (multi) critical points
    Effective field theory with $z = 2$ and massless deconfined spinons

Time Reversal Invariant Systems: Quantum Dimer Models

- Quantum dimer models at the Rokhsar-Kivelson (RK) point have an exact ground state wave function has the short range RVB form

$$|\Psi_{\mathrm{RVB}}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

-
  - Bipartite lattices: quantum (multi) critical points
    Effective field theory with $z = 2$ and massless deconfined spinons
  - Non-bipartite lattices: Topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus (Moessner and Sondhi, 1998)

Physics Illinois

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin; Henley; Ardonne, Fendley and Fradkin

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin; Henley; Ardonne, Fendley and Fradkin

- QDM on a square lattice ⇔ 2D height model
- Physical Operators are invariant under $\varphi(x) \to \varphi(x) + 2\pi r$.

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin; Henley; Ardonne, Fendley and Fradkin

- QDM on a square lattice $\Leftrightarrow$ 2D height model
- Physical Operators are invariant under $\varphi(x) \to \varphi(x) + 2\pi r$.
- Quantum Lifshitz Model Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 \left(\nabla^2\varphi\right)^2 \right]$$

- The Ground State Wave Function $\Psi_0[\varphi]$ is Scale Invariant

$$\Psi_0[\varphi] \propto e^{-\frac{k}{8\pi}\int d^2x \, \left(\nabla\varphi(\mathbf{x})\right)^2}$$

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin; Henley; Ardonne, Fendley and Fradkin

- QDM on a square lattice $\Leftrightarrow$ 2D height model
- Physical Operators are invariant under $\varphi(x) \to \varphi(x) + 2\pi r$.
- Quantum Lifshitz Model Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 \left(\nabla^2\varphi\right)^2 \right]$$

- The Ground State Wave Function $\Psi_0[\varphi]$ is Scale Invariant

$$\Psi_0[\varphi] \propto e^{-\frac{k}{8\pi}\int d^2x \, \left(\nabla\varphi(\mathbf{x})\right)^2}$$

- The norm of the 2D wave function is the partition function of a classical critical conformally invariant system!

Physics Illinois

Mapping to a 2D Euclidean CFT

Effective field theory: the Quantum Lifshitz Model

Moessner, Sondhi and Fradkin; Henley; Ardonne, Fendley and Fradkin

- QDM on a square lattice ⇔ 2D height model
- Physical Operators are invariant under $\varphi(x) \to \varphi(x) + 2\pi r$.
- Quantum Lifshitz Model Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{k}{4\pi}\right)^2 \left(\nabla^2\varphi\right)^2\right]$$

- The Ground State Wave Function $\Psi_0[\varphi]$ is Scale Invariant

$$\Psi_0[\varphi] \propto e^{-\frac{k}{8\pi}\int d^2x \ (\nabla\varphi(\mathbf{x}))^2}$$

- The norm of the 2D wave function is the partition function of a classical critical conformally invariant system!

$$\|\Psi_0\|^2 = \int \mathcal{D}\varphi \ e^{-\frac{k}{4\pi}\int d^2x \, (\nabla\varphi(\mathbf{x}))^2} = \text{“}Z\text{''}$$

Physics Illinois

Mapping to a 2D Euclidean CFT

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- The equal-time expectation value of operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$.

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- The equal-time expectation value of operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$.
- Time-dependent correlators: dynamical exponent $z = 2$.
- Matching the correlation functions of the RK and Lifshitz models, one finds $k = r = 1$.

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- The equal-time expectation value of operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$.
- Time-dependent correlators: dynamical exponent $z = 2$.
- Matching the correlation functions of the RK and Lifshitz models, one finds $k = r = 1$.
- For the 2D quantum Baxter wave function, $r = 1$ but k varies continuously

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- The equal-time expectation value of operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$.
- Time-dependent correlators: dynamical exponent $z = 2$.
- Matching the correlation functions of the RK and Lifshitz models, one finds $k = r = 1$.
- For the 2D quantum Baxter wave function, $r = 1$ but k varies continuously
- Multicritical point with many relevant perturbations: *e.g.* diagonal dimers drive the system into a $\mathbb{Z}_2$ topological phase.

Physics Illinois

Mapping to a 2D Euclidean CFT

- The amplitude of $|\varphi\rangle$ is the Gibbs weight of a Euclidean 2D free massless scalar field: scale invariant wave functions
- The equal-time expectation value of operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$.
- Time-dependent correlators: dynamical exponent $z = 2$.
- Matching the correlation functions of the RK and Lifshitz models, one finds $k = r = 1$.
- For the 2D quantum Baxter wave function, $r = 1$ but k varies continuously
- Multicritical point with many relevant perturbations: *e.g.* diagonal dimers drive the system into a $\mathbb{Z}_2$ topological phase.
- This construction generalizes to states with non-Abelian braid statistics (Fendley and Fradkin, 2005)

Entanglememt

Entanglement Entropy

- Given the pure state $\Psi[\varphi_A, \varphi_B]$ and the (trivial) density matrix of the combined system $A \cup B$

Entanglement Entropy

- Given the pure state $\Psi[\varphi_A, \varphi_B]$ and the (trivial) density matrix of the combined system $A \cup B$

$$\langle \varphi_A, \varphi_B | \rho_{A\cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^*[\varphi'_A, \varphi'_B]$$

- The reduced density matrix for A, which acts only on the states $\{\varphi_A\}$, is constructed by tracing over the degrees of freedom in B:

Entanglement Entropy

- Given the pure state $\Psi[\varphi_A, \varphi_B]$ and the (trivial) density matrix of the combined system $A \cup B$

$$\langle \varphi_A, \varphi_B | \rho_{A \cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^*[\varphi'_A, \varphi'_B]$$

- The reduced density matrix for A, which acts only on the states $\{\varphi_A\}$, is constructed by tracing over the degrees of freedom in B:

$$\rho_A = \mathrm{tr}\,_B\, \rho_{A \cup B}$$

- The von Neumann entanglement entropy is

Physics Illinois

Scaling Behavior of Quantum Entanglement

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\ L^{D-1} + \ldots$ (Srednicki, 1993; Bombelli et al, 1986).

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\ L^{D-1} + \ldots$ (Srednicki, 1993; Bombelli et al, 1986). This the non-universal area law is the leading generic behavior in all dimensions.

Physics Illinois

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\, L^{D-1} + \ldots$ (Srednicki, 1993; Bombelli et al, 1986). This the non-universal area law is the leading generic behavior in all dimensions.
- Universal behavior in $d = 1$ critical systems (CFT):
  - Generic scaling behavior in $d = 1$ (CFT)

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\ L^{D-1} + \ldots$ (Srednicki, 1993; Bombelli et al, 1986). This the non-universal area law is the leading generic behavior in all dimensions.
- Universal behavior in $d = 1$ critical systems (CFT):
  - Generic scaling behavior in $d = 1$ (CFT)

$$S = \frac{c}{3} \log\left(\frac{L}{a}\right) + \text{finite terms} \qquad \text{Wilczek et al, Calabrese and Cardy}$$

  - Also at 1D quantum random fixed points (Refael and Moore, 2004).

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\ L^{D-1} + \dots$ (Srednicki, 1993; Bombelli et al, 1986). This the non-universal area law is the leading generic behavior in all dimensions.
- Universal behavior in $d = 1$ critical systems (CFT):
  - Generic scaling behavior in $d = 1$ (CFT)

    $$S = \frac{c}{3} \log\left(\frac{L}{a}\right) + \text{finite terms} \qquad \text{Wilczek et al, Calabrese and Cardy}$$

  - Also at 1D quantum random fixed points (Refael and Moore, 2004).
  - Away from criticality, the correlation length ξ is finite and $S = \frac{c}{6} \log\left(\frac{\xi}{a}\right) + \text{finite terms}$
- *Universal* subleading terms at QCP for $d > 1$?

Physics Illinois

Scaling Behavior of Quantum Entanglement

- Massive relativistic free field theories obey an "area law" $S = \text{const.}\, L^{D-1} + \ldots$ (Srednicki, 1993; Bombelli et al, 1986). This the non-universal area law is the leading generic behavior in all dimensions.
- Universal behavior in $d = 1$ critical systems (CFT):
  - Generic scaling behavior in $d = 1$ (CFT)

$$S = \frac{c}{3} \log\left(\frac{L}{a}\right) + \text{finite terms} \qquad \text{Wilczek et al, Calabrese and Cardy}$$

  - Also at 1D quantum random fixed points (Refael and Moore, 2004).
  - Away from criticality, the correlation length ξ is finite and $S = \frac{c}{6} \log\left(\frac{\xi}{a}\right) + \text{finite terms}$
- *Universal* subleading terms at QCP for $d > 1$?
- Universal $O(1)$ term in *topological phases* in 2D

$$S = \alpha\, L - \gamma + O(L^{-1}), \qquad \text{Kitaev and Preskill, Levin and Wen}$$

α is non-universal; γ *depends only on topological invariants*

Physics Illinois

Entanglement Entropy of Conformal Wave Functions

with Joel Moore

Entanglement Entropy of Conformal Wave Functions

with Joel Moore

- We split a large region into two disjoint regions A and B, sharing a common boundary Γ.

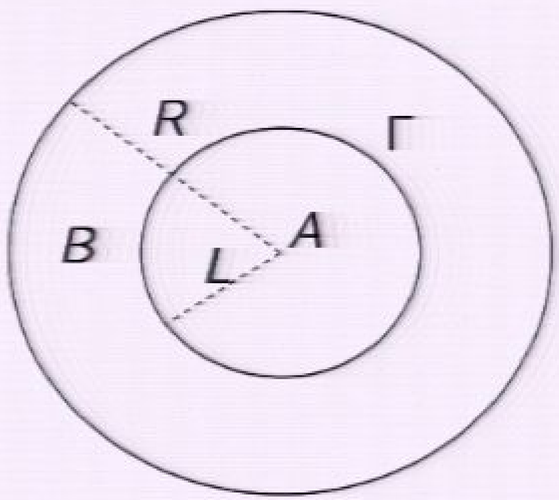
Entanglement Entropy of Conformal Wave Functions

with Joel Moore

- We split a large region into two disjoint regions A and B, sharing a common boundary Γ.
- Configurations are glued at the boundary

$$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$

- Dirichlet boundary conditions on Γ for $n-1$ fields $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_i$ not restricted on Γ.



$R \gg L \gg a$

Entanglement Entropies and 2D Classical Free Energies

Entanglement Entropies and 2D Classical Free Energies

- In terms of the *partition functions* Z_D, for a field in the whole system $A \cup B$ that vanishes at the common boundary Γ, and $Z_{A \cup B}$, for a field that is free at the boundary

Entanglement Entropies and 2D Classical Free Energies

- In terms of the *partition functions* Z_D, for a field in the whole system $A \cup B$ that vanishes at the common boundary Γ, and $Z_{A\cup B}$, for a field that is free at the boundary

$$\text{tr}\, \rho_A^n = \frac{Z_D^{n-1} Z_F}{Z_F^n} = \left(\frac{Z_D}{Z_F}\right)^{n-1} \Rightarrow S = -\ln\left(\frac{Z_D}{Z_F}\right) = -\ln \frac{Z_D^A Z_D^B}{Z_{A\cup B}}$$

Entanglement Entropies and 2D Classical Free Energies

- In terms of the *partition functions* Z_D, for a field in the whole system $A \cup B$ that vanishes at the common boundary Γ, and $Z_{A\cup B}$, for a field that is free at the boundary

$$\mathrm{tr}\ \rho_A^n = \frac{Z_D^{n-1} Z_F}{Z_F^n} = \left(\frac{Z_D}{Z_F}\right)^{n-1} \Rightarrow S = -\ln\left(\frac{Z_D}{Z_F}\right) = -\ln \frac{Z_D^A Z_D^B}{Z_{A\cup B}}$$

- Entanglement entropy for a general conformal QCP:

$$S = F_A + F_B - F_{A\cup B}$$

Entanglement Entropies and 2D Classical Free Energies

- In terms of the *partition functions* Z_D, for a field in the whole system $A \cup B$ that vanishes at the common boundary Γ, and $Z_{A \cup B}$, for a field that is free at the boundary

$$\operatorname{tr} \rho_A^n = \frac{Z_D^{n-1} Z_F}{Z_F^n} = \left(\frac{Z_D}{Z_F}\right)^{n-1} \Rightarrow S = -\ln\left(\frac{Z_D}{Z_F}\right) = -\ln \frac{Z_D^A Z_D^B}{Z_{A \cup B}}$$

- Entanglement entropy for a general conformal QCP:

$$S = F_A + F_B - F_{A \cup B}$$

- For a more general 2D CQCP the boundary condition changes from Dirichlet to a conformal boundary condition in the conformal block of the identity.

Physics Illinois

Universal Contributions to the Entanglement Entropy

Universal Contributions to the Entanglement Entropy

- For a large bounded region of linear size L and smooth boundary, *F* obeys the 'Mark Kac Law' ('Can you hear the shape of a drum?')

Universal Contributions to the Entanglement Entropy

- For a large bounded region of linear size L and smooth boundary, F obeys the 'Mark Kac Law' ('Can you hear the shape of a drum?')

$$F = \alpha L^2 + \beta L - \frac{c}{6}\chi \ln L + O(1), \quad \text{(Cardy and Peschel)}$$

α and β are non-universal constants, c is the central charge of the CFT, and χ is the Euler characteristic of the region:

Universal Contributions to the Entanglement Entropy

- For a large bounded region of linear size L and smooth boundary, F obeys the 'Mark Kac Law' ('Can you hear the shape of a drum?')

$$F = \alpha L^2 + \beta L - \frac{c}{6}\chi \ln L + O(1), \quad \text{(Cardy and Peschel)}$$

α and β are non-universal constants, c is the central charge of the CFT, and χ is the Euler characteristic of the region:

$$\chi = 2 - 2h - b, \quad h = \#\ \text{handles},\ b = \#\ \text{boundaries}$$

$$\Delta S = -\frac{c}{6}(\chi_A + \chi_B - \chi_{A\cup B}) \log L$$

Universal Contributions to the Entanglement Entropy

- For a large bounded region of linear size L and smooth boundary, F obeys the 'Mark Kac Law' ('Can you hear the shape of a drum?')

$$F = \alpha L^2 + \beta L - \frac{c}{6}\chi \ln L + O(1), \quad \text{(Cardy and Peschel)}$$

α and β are non-universal constants, c is the central charge of the CFT, and χ is the Euler characteristic of the region:

$$\chi = 2 - 2h - b, \quad h = \#\,\text{handles},\ b = \#\,\text{boundaries}$$

$$\Delta S = -\frac{c}{6}(\chi_A + \chi_B - \chi_{A\cup B}) \log L$$

- The $O(1)$ correction has a *universal piece* related to the "boundary entropy" of Affleck and Ludwig

Universal Contributions to the Entanglement Entropy

- For regions $A \subseteq B$ the coefficient of the $\log L$ term vanishes since

Universal Contributions to the Entanglement Entropy

- For regions $A \subseteq B$ the coefficient of the $\log L$ term vanishes since

$$\chi_A + \chi_B = \chi_{A \cup B} \Rightarrow \Delta S = 0$$

- A and B are physically separate and have no common intersection, $\chi_A + \chi_B - \chi_{A \cup B} \neq 0$.
  The system physically splits in two disjoint parts $\Rightarrow \log L$ term in the entanglement entropy
- A and B share a common boundary $\Rightarrow \log L$ term whose coefficient is determined by the angles at the intersections

Universal Contributions to the Entanglement Entropy

- For regions $A \subseteq B$ the coefficient of the $\log L$ term vanishes since

$$\chi_A + \chi_B = \chi_{A \cup B} \Rightarrow \Delta S = 0$$

- A and B are physically separate and have no common intersection, $\chi_A + \chi_B - \chi_{A \cup B} \neq 0$.
  The system physically splits in two disjoint parts $\Rightarrow \log L$ term in the entanglement entropy
- A and B share a common boundary $\Rightarrow \log L$ term whose coefficient is determined by the angles at the intersections
- If the boundary of A is not smooth, the coefficient depends on the angles α_i for both regions

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

- If the coefficient of the logarithmic term vanishes, the finite term is universal, and determined by the contributions of the winding modes to the partition functions.

Physics Illinois

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

- If the coefficient of the logarithmic term vanishes, the finite term is universal, and determined by the contributions of the winding modes to the partition functions.
- It depends on the topology of the surface and on the properties of the CFT associated with the wave function

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

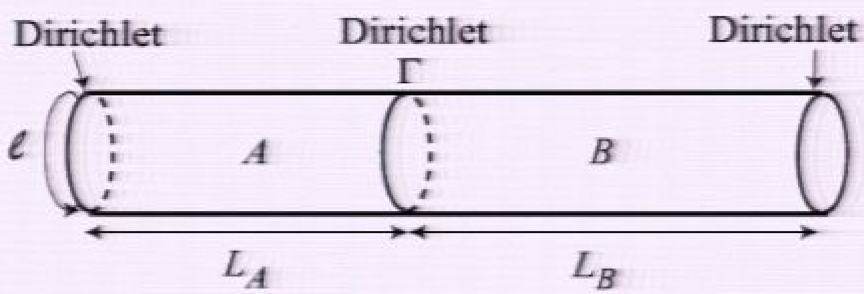
- If the coefficient of the logarithmic term vanishes, the finite term is universal, and determined by the contributions of the winding modes to the partition functions.
- It depends on the topology of the surface and on the properties of the CFT associated with the wave function
- For the Quantum Lifshitz universality class on a *cylinder* (with $L_{A,B} \gg \ell$) the $O(1)$ term equals $\log R$, where $R = \sqrt{2kr^2}$ is the compactification radius.

Physics Illinois

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

- If the coefficient of the logarithmic term vanishes, the finite term is universal, and determined by the contributions of the winding modes to the partition functions.
- It depends on the topology of the surface and on the properties of the CFT associated with the wave function
- For the Quantum Lifshitz universality class on a *cylinder* (with $L_{A,B} \gg \ell$) the $O(1)$ term equals $\log R$, where $R = \sqrt{2kr^2}$ is the compactification radius.

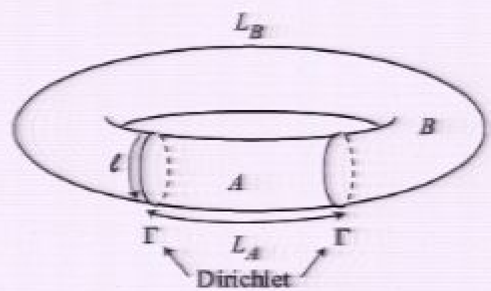


- For the RK quantum dimer model, $R = \sqrt{2}$. The finite term is $\log\sqrt{2}$; it different from its value $-\ln 2$ in the topological phase

Physics Illinois

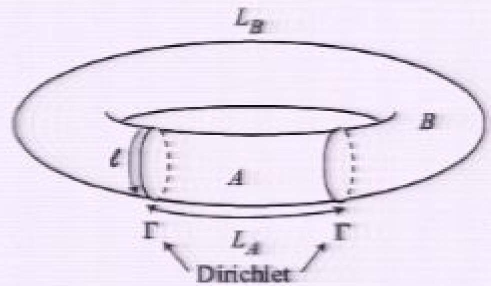
Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

- For a torus the finite term is $2\ln\left(\frac{R^2}{2}\right) \Rightarrow$ the finite term is a smooth function of the compactification radius!

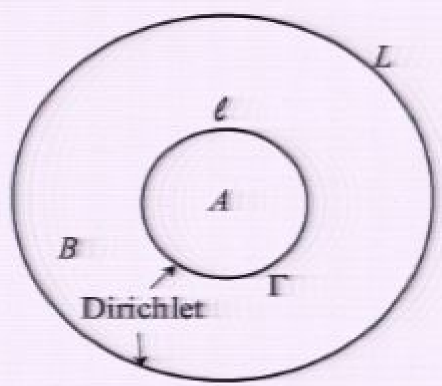


Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

- For a torus the finite term is $2\ln\left(\frac{R^2}{2}\right) \Rightarrow$ the finite term is a smooth function of the compactification radius!



- For a disk the result is $\frac{1}{2}\ln\left[\frac{1}{\pi}\ln\left(\frac{L}{\ell}\right)\right] + \ln R \Rightarrow$ it also depends on the *aspect ratio* L/ℓ as well as in the *shape of the observed region*



Universal Contributions to the Entanglement Entropy: General Case

Universal Contributions to the Entanglement Entropy: General Case

- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models

Universal Contributions to the Entanglement Entropy: General Case

- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models
- A 2D RCFT has a set of *primary fields* Φ_a have an OPE $\Phi_a \times \Phi_b = \sum_j N^j_{ab} \Phi_j$, and define a set of conformally invariant boundary conditions labelled by a

Universal Contributions to the Entanglement Entropy: General Case

- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models
- A 2D RCFT has a set of *primary fields* Φ_a have an OPE $\Phi_a \times \Phi_b = \sum_j N^j_{ab} \Phi_j$, and define a set of conformally invariant boundary conditions labelled by a
- An RCFT has a modular S-matrix which is related to the fusion coefficients:

$$N^j_{ab} = \sum_i \frac{S^i_j S^i_a S^b_i}{S^i_0}$$

Physics Illinois

Universal Contributions to the Entanglement Entropy: General Case

- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models
- A 2D RCFT has a set of *primary fields* Φ_a have an OPE $\Phi_a \times \Phi_b = \sum_j N^j_{ab} \Phi_j$, and define a set of conformally invariant boundary conditions labelled by a
- An RCFT has a modular S-matrix which is related to the fusion coefficients:

$$N^j_{ab} = \sum_i \frac{S^i_j S^i_a S^b_i}{S^i_0}$$

- On a cylinder with BC's a, b: $Z_{a/b} = \frac{\pi L c}{6\ell} + \ln g_{ab}$
  $g_{ab} = \sum_i N^i_{ab} S^0_i$, and $\ln g_{a,b}$ is the Affleck-Ludwig boundary entropy of the 2D Euclidean boundary CFT

Universal Contributions to the Entanglement Entropy: General Case

Universal Contributions to the Entanglement Entropy: General Case

- On a cylinder, the BC is $a = I$, the entanglement entropy is

$$S_{QCP} = -\log\left(Z_I^A Z_I^B / Z_{A\cup B}\right) = \mu\ell - \ln\left(\frac{g_{a0}g_{0b}}{g_{ab}}\right) + O(\ell^{-1})$$

Universal Contributions to the Entanglement Entropy: General Case

- On a cylinder, the BC is $a = I$, the entanglement entropy is

$$S_{QCP} = -\log\left(Z_I^A Z_I^B / Z_{A\cup B}\right) = \mu\ell - \ln\left(\frac{g_{a0}g_{0b}}{g_{ab}}\right) + O(\ell^{-1})$$

- The universal $O(1)$ term is determined by the full structure of the CFT associated with the wavefunction

Universal Contributions to the Entanglement Entropy: General Case

- On a cylinder, the BC is $a = I$, the entanglement entropy is

$$S_{QCP} = -\log\left(Z_I^A Z_I^B / Z_{A\cup B}\right) = \mu\ell - \ln\left(\frac{g_{a0}g_{0b}}{g_{ab}}\right) + O(\ell^{-1})$$

- The universal $O(1)$ term is determined by the full structure of the CFT associated with the wavefunction
- For the case of an "Ising" wavefunction on a cylinder, $S_{QCP} = \ln 2$. For a quantum net model such as the "chromatic polynomial" Fibonacci wavefunction (Fendley and Fradkin, 2005), $S = -\ln\left(\sin(\pi/9)/15 + 3\sqrt{5}\right)$ (!)

Universal Contributions to the Entanglement Entropy: General Case

- On a cylinder, the BC is $a = I$, the entanglement entropy is

$$S_{QCP} = -\log\left(Z_I^A Z_I^B / Z_{A\cup B}\right) = \mu\ell - \ln\left(\frac{g_{a0}g_{0b}}{g_{ab}}\right) + O(\ell^{-1})$$

- The universal $O(1)$ term is determined by the full structure of the CFT associated with the wavefunction
- For the case of an "Ising" wavefunction on a cylinder, $S_{QCP} = \ln 2$. For a quantum net model such as the "chromatic polynomial" Fibonacci wavefunction (Fendley and Fradkin, 2005), $S = -\ln\left(\sin(\pi/9)/15 + 3\sqrt{5}\right)$ (!)
- A similar scaling of entanglement entropy holds for the conventional ϕ^4 field theory in the $N \to \infty$ limit (Metlitskii, Fuertes and Sachdev, 2009)

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$

Physics Illinois

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$
- QLM with compactification radius R, on a cylinder of length L and circumference ℓ
- Region A and region B have each length $L/2 \gg \ell$

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$
- QLM with compactification radius R, on a cylinder of length L and circumference ℓ
- Region A and region B have each length $L/2 \gg \ell$
- Fold about Γ: $2n$ cylinders of length $L/2$ and circumference ℓ with the fields are identified on Γ and free at the other end

Physics Illinois

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$

Physics Illinois

Universal Contributions to the Entanglement Entropy: General Case

Universal Contributions to the Entanglement Entropy: General Case

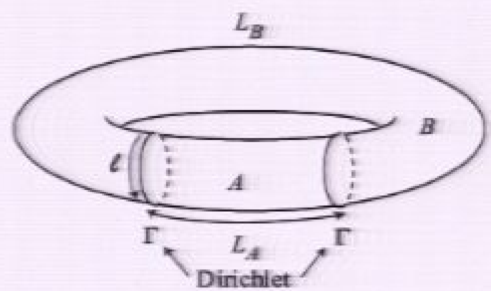
- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models
- A 2D RCFT has a set of *primary fields* Φ_a have an OPE $\Phi_a \times \Phi_b = \sum_j N_{ab}^j \Phi_j$, and define a set of conformally invariant boundary conditions labelled by a
- An RCFT has a modular S-matrix which is related to the fusion coefficients:

$$N_{ab}^j = \sum_i \frac{S_j^i S_a^i S_i^b}{S_0^i}$$

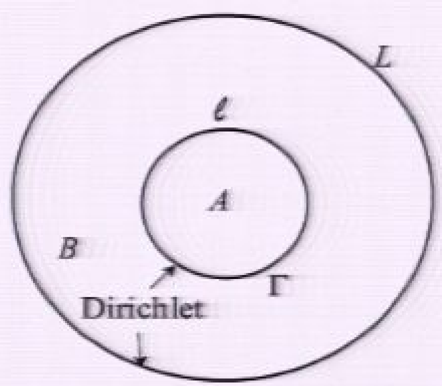
- On a cylinder with BC's a, b: $Z_{a/b} = \frac{\pi L c}{6\ell} + \ln g_{ab}$
  $g_{ab} = \sum_i N_{ab}^i S_i^0$, and $\ln g_{a,b}$ is the Affleck-Ludwig boundary entropy of the 2D Euclidean boundary CFT

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

- For a torus the finite term is $2\ln\left(\frac{R^2}{2}\right) \Rightarrow$ the finite term is a smooth function of the compactification radius!



- For a disk the result is $\frac{1}{2}\ln\left[\frac{1}{\pi}\ln\left(\frac{L}{\ell}\right)\right] + \ln R \Rightarrow$ it also depends on the *aspect ratio* L/ℓ as well as in the *shape of the observed region*

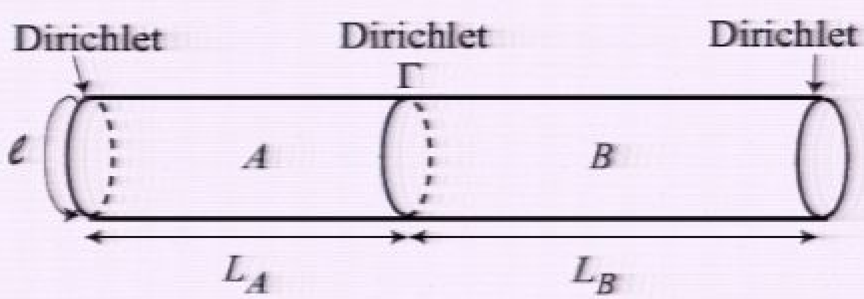


Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

Universal Contributions to the Entanglement Entropy: Quantum Lifshitz Case

with B. Hsu, M. Mulligan and E.-A. Kim

- If the coefficient of the logarithmic term vanishes, the finite term is universal, and determined by the contributions of the winding modes to the partition functions.
- It depends on the topology of the surface and on the properties of the CFT associated with the wave function
- For the Quantum Lifshitz universality class on a *cylinder* (with $L_{A,B} \gg \ell$) the $O(1)$ term equals $\log R$, where $R = \sqrt{2kr^2}$ is the compactification radius.



- For the RK quantum dimer model, $R = \sqrt{2}$. The finite term is $\log \sqrt{2}$; it different from its value $-\ln 2$ in the topological phase

Universal Contributions to the Entanglement Entropy: General Case

- It generalizes for a general conformally invariant wave function defined by the Gibbs weights of a 2D (Euclidean) RCFT, *e.g.* the Ising model, quantum loop and net models
- A 2D RCFT has a set of *primary fields* Φ_a have an OPE $\Phi_a \times \Phi_b = \sum_j N^j_{ab} \Phi_j$, and define a set of conformally invariant boundary conditions labelled by a

Universal Contributions to the Entanglement Entropy: General Case

Universal Contributions to the Entanglement Entropy: General Case

- On a cylinder, the BC is $a = I$, the entanglement entropy is

$$S_{QCP} = -\log\left(Z_I^A Z_I^B / Z_{A \cup B}\right) = \mu \ell - \ln\left(\frac{g_{a0} g_{0b}}{g_{ab}}\right) + O(\ell^{-1})$$

- The universal $O(1)$ term is determined by the full structure of the CFT associated with the wavefunction
- For the case of an "Ising" wavefunction on a cylinder, $S_{QCP} = \ln 2$. For a quantum net model such as the "chromatic polynomial" Fibonacci wavefunction (Fendley and Fradkin, 2005), $S = -\ln\left(\sin(\pi/9)/15 + 3\sqrt{5}\right)$ (!)
- A similar scaling of entanglement entropy holds for the conventional ϕ^4 field theory in the $N \to \infty$ limit (Metlitskii, Fuertes and Sachdev, 2009)

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$
- QLM with compactification radius R, on a cylinder of length L and circumference ℓ
- Region A and region B have each length $L/2 \gg \ell$
- Fold about Γ: $2n$ cylinders of length $L/2$ and circumference ℓ with the fields are identified on Γ and free at the other end
- Length of the cylinders: imaginary time. The partition functions become amplitudes between boundary states
- $$\mathrm{tr}\rho_A^n = R^{1-n}\left(\frac{\vartheta_3\left(0|\frac{-R^2}{2n\tau}\right)}{\eta(-1/2n\tau)}\right)\left(\frac{\eta(-1/2\tau)}{\vartheta_3\left(0|\frac{-1}{2\tau}\right)}\right)^n, \qquad 2\tau = iL/\ell$$
- $$\lim_{\ell/L\to 0} \mathrm{tr}\rho_A^n = R^{1-n}(1+\ldots)$$

Entanglement in FQH fluids and Chern-Simons theory

Entanglement in FQH fluids and Chern-Simons theory

with Shying Dong, Sean Nowling and Rob Leigh (2008)

- The FQH wave functions represent topological fluids with a finite correlation length $\xi \propto \ell$ (ℓ is the magnetic length).
- The entanglement entropy of FQH states has be computed numerically (K. Schoutens and coworkers, 2007).
- One can compute the entanglement entropy directly from the effective field theory of all FQH states: Chern-Simons gauge theory.
- This result can be applied directly to all known FQH states.

Physics Illinois

Entanglement in FQH fluids: Chern-Simons theory

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi}\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- States on a closed 2D surface: path integral over a 3D volume

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- States on a closed 2D surface: path integral over a 3D volume
- Chern-Simons states $\Leftrightarrow$ WZW conformal blocks
- The ground state degeneracy depends on the level k and on the topology of the surface
- The partition functions depend on the matrix elements of the modular S-matrix, *e.g.* the partition function on S^3 with a Wilson loop in representation ρ_j is

$$Z(S^3, \rho_j) = S_0^j$$

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

- States on a closed 2D surface: path integral over a 3D volume
- Chern-Simons states $\Leftrightarrow$ WZW conformal blocks
- The ground state degeneracy depends on the level k and on the topology of the surface
- The partition functions depend on the matrix elements of the modular S-matrix, *e.g.* the partition function on S^3 with a Wilson loop in representation ρ_j is

$$Z(S^3, \rho_j) = S_0^j$$

- Modular S-matrix: defines how the degenerate ground states on a torus transform under a modular transformation

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \dots, m-1$

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \ldots, m-1$

$$\mathcal{S}_{[n']}{}^{[n]} = \frac{1}{\sqrt{m}} e^{2\pi i n n'/m}$$

- For $SU(2)_k$, $j, j' = 0, 1/2, \ldots, k/2$

$$S_j^{(k)j'} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(2j+1)(2j'+1)}{k+2}\right)$$

Physics Illinois

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \dots, m-1$

$$\mathcal{S}_{[n']}{}^{[n]} = \frac{1}{\sqrt{m}} e^{2\pi i n n' / m}$$

- For $SU(2)_k$, $j, j' = 0, 1/2, \dots, k/2$

$$S_j^{(k)j'} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(2j+1)(2j'+1)}{k+2}\right)$$

- Quantum dimensions

$$d_j = \frac{S_0^j}{S_{00}}, \quad \mathcal{D} \equiv \sqrt{\sum_j |d_j|^2} = \frac{1}{S_{00}}$$

- The quantum dimensions d_j measure the rate of growth of the degenerate Hilbert spaces of particles labeled by representation ρ_j

Chern-Simons Surgeries

- If a 3-manifold M is the connected sum of two 3-manifolds M_1 and M_2 joined along an S^2, then

$$Z(M)Z(S^3) = Z(M_1)Z(M_2)$$

Physics Illinois

Chern-Simons Surgeries

- If a 3-manifold M is the connected sum of two 3-manifolds M_1 and M_2 joined along an S^2, then

$$Z(M)Z(S^3) = Z(M_1)Z(M_2)$$

- In particular, if M is M_1 and M_2 joined along n S^2's,

Physics Illinois

Entanglement and Chern Simons Theory: Results for the Sphere S^2

3-geometry is a ball.

Physics Illinois

Entanglement and Chern Simons Theory: Results for the Sphere S^2

- S^2 with one $A - B$ boundary: The Hilbert space is one-dimensional. The two regions A and B are two hemispheres (disks). The 3-geometry is a ball.
- To construct $\mathrm{tr}\hat{\rho}_A^n$ we glue $2n$ such pieces together.

$$\frac{\mathrm{tr}\rho^n_{A(S^2,1)}}{\left(\mathrm{tr}\rho_{A(S^2,1)}\right)^n} = \frac{Z(S^3)}{\left(Z(S^3)\right)^n} = \left(Z(S^3)\right)^{1-n} = S_{00}^{1-n}$$

Entanglement and Chern Simons Theory: Results for the Sphere S^2

- S^2 with one $A-B$ boundary: The Hilbert space is one-dimensional. The two regions A and B are two hemispheres (disks). The 3-geometry is a ball.
- To construct $\mathrm{tr}\hat{\rho}_A^n$ we glue $2n$ such pieces together.

$$\frac{\mathrm{tr}\rho^n_{A(S^2,1)}}{\left(\mathrm{tr}\rho_{A(S^2,1)}\right)^n} = \frac{Z(S^3)}{\left(Z(S^3)\right)^n} = \left(Z(S^3)\right)^{1-n} = S_{00}^{1-n}$$

$$S_A^{(S^2,1)} = \ln S_{00} = -\ln \mathcal{D}, \qquad \text{Kitaev and Preskill, Levin and Wen}$$

- It also holds for surfaces with arbitrary topology if the region being observed A is trivial regardless of the pure state labeled by the representations ρ_j

Physics Illinois

Entanglement and Chern Simons Theory: Results for the Sphere S^2

- S^2 with one $A - B$ boundary: The Hilbert space is one-dimensional. The two regions A and B are two hemispheres (disks). The 3-geometry is a ball.
- To construct $\mathrm{tr}\hat{\rho}_A^n$ we glue $2n$ such pieces together.

$$\frac{\mathrm{tr}\rho^n_{A(S^2,1)}}{\left(\mathrm{tr}\rho_{A(S^2,1)}\right)^n} = \frac{Z(S^3)}{(Z(S^3))^n} = \left(Z(S^3)\right)^{1-n} = S_{00}^{1-n}$$

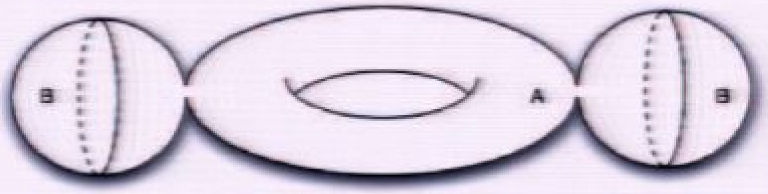
$$S_A^{(S^2,1)} = \ln S_{00} = -\ln \mathcal{D}, \qquad \text{Kitaev and Preskill, Levin and Wen}$$

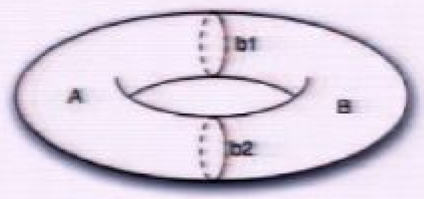
- It also holds for surfaces with arbitrary topology if the region being observed A is trivial regardless of the pure state labeled by the representations ρ_j
- For the case of a sphere S^2 and a disconnected connected region A with M boundaries we find $S_A^{(S^2,M)} = M \ln S_{00} = -M \ln \mathcal{D}$.

Entanglement and Chern Simons Theory: Results for the torus T^2 with more than one $A - B$ boundary

Entanglement and Chern Simons Theory: Results for the torus T^2 with more than one $A-B$ boundary

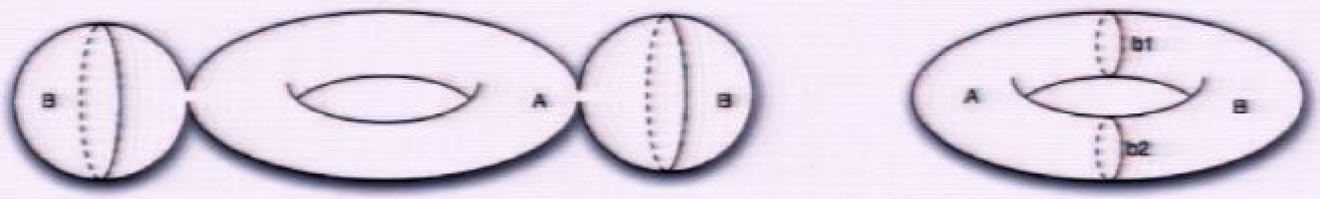
- For a torus T^2 split into two regions with more than one (say two) boundary, we have two cases,





Entanglement and Chern Simons Theory: Results for the torus T^2 with more than one $A - B$ boundary

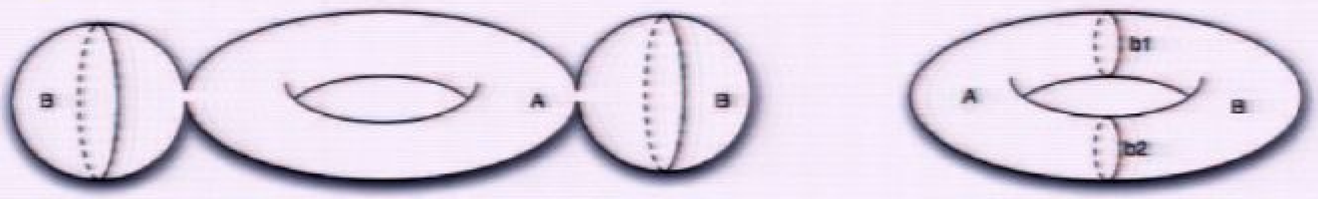
- For a torus T^2 split into two regions with more than one (say two) boundary, we have two cases,



- For the trivial state (no Wilson loop) the entropy is the same in both cases, $S_A(T^2, 2) = 2 \ln S_{00}$.

Entanglement and Chern Simons Theory: Results for the torus T^2 with more than one $A - B$ boundary

- For a torus T^2 split into two regions with more than one (say two) boundary, we have two cases,



- For the trivial state (no Wilson loop) the entropy is the same in both cases, $S_A(T^2, 2) = 2 \ln S_{00}$.
- If there is a Wilson loop with a non-trivial representation ρ_j, we obtain the same result for the case of the left. But, for the case of the right, for a Wilson loop in representation ρ we obtain instead,

Entanglement and Chern Simons Theory: Results for the Sphere S^2

- S^2 with one $A-B$ boundary: The Hilbert space is one-dimensional. The two regions A and B are two hemispheres (disks). The 3-geometry is a ball.
- To construct $\mathrm{tr}\hat{\rho}_A^n$ we glue $2n$ such pieces together.

$$\frac{\mathrm{tr}\rho^n_{A(S^2,1)}}{\left(\mathrm{tr}\rho_{A(S^2,1)}\right)^n} = \frac{Z(S^3)}{(Z(S^3))^n} = \left(Z(S^3)\right)^{1-n} = S_{00}^{1-n}$$

$$S_A^{(S^2,1)} = \ln S_{00} = -\ln \mathcal{D}, \qquad \text{Kitaev and Preskill, Levin and Wen}$$

- It also holds for surfaces with arbitrary topology if the region being observed A is trivial regardless of the pure state labeled by the representations ρ_j
- For the case of a sphere S^2 and a disconnected connected region A with M boundaries we find $S_A^{(S^2,M)} = M \ln S_{00} = -M \ln \mathcal{D}$.

Chern-Simons Surgeries

- If a 3-manifold M is the connected sum of two 3-manifolds M_1 and M_2 joined along an S^2, then

$$Z(M)Z(S^3) = Z(M_1)Z(M_2)$$

- In particular, if M is M_1 and M_2 joined along n S^2's,

$$Z(M) = \frac{Z(M_1)Z(M_2)}{Z(S^3)^n}$$

Physics Illinois

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \dots, m-1$

$$\mathcal{S}_{[n']}{}^{[n]} = \frac{1}{\sqrt{m}} e^{2\pi i n n'/m}$$

- For $SU(2)_k$, $j, j' = 0, 1/2, \dots, k/2$

$$S_j^{(k)j'} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(2j+1)(2j'+1)}{k+2}\right)$$

- Quantum dimensions

$$d_j = \frac{S_0^j}{S_{00}}, \quad \mathcal{D} \equiv \sqrt{\sum_j |d_j|^2} = \frac{1}{S_{00}}$$

- The quantum dimensions d_j measure the rate of growth of the degenerate Hilbert spaces of particles labeled by representation ρ_j

Physics Illinois

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \ldots, m-1$

$$\mathcal{S}_{[n']}{}^{[n]} = \frac{1}{\sqrt{m}} e^{2\pi i n n'/m}$$

- For $SU(2)_k$, $j, j' = 0, 1/2, \ldots, k/2$

$$S_j^{(k)j'} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(2j+1)(2j'+1)}{k+2}\right)$$

- Quantum dimensions

$$d_j = \frac{S_0^j}{S_{00}}, \quad \mathcal{D} \equiv \sqrt{\sum_j |d_j|^2} = \frac{1}{S_{00}}$$

Properties of conformal blocks

- For $U(1)_m$, $n = 0, \ldots, m-1$

$$\mathcal{S}_{[n']}{}^{[n]} = \frac{1}{\sqrt{m}} e^{2\pi i n n'/m}$$

- For $SU(2)_k$, $j, j' = 0, 1/2, \ldots, k/2$

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- States on a closed 2D surface: path integral over a 3D volume
- Chern-Simons states $\Leftrightarrow$ WZW conformal blocks
- The ground state degeneracy depends on the level k and on the topology of the surface
- The partition functions depend on the matrix elements of the modular S-matrix, *e.g.* the partition function on S^3 with a Wilson loop in representation ρ_j is

$$Z(S^3, \rho_j) = S_0^j$$

- Modular S-matrix: defines how the degenerate ground states on a torus transform under a modular transformation

Physics Illinois

Entanglement in FQH fluids: Chern-Simons theory

- We computed the entanglement entropy for a level k Chern-Simons theory on a smooth manifold with any number of handles, using Witten 1989 for the Chern-Simons theory (1989),

$$S(A) = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- States on a closed 2D surface: path integral over a 3D volume

Physics Illinois

Entanglement in FQH fluids and Chern-Simons theory

with Shying Dong, Sean Nowling and Rob Leigh (2008)

- The FQH wave functions represent topological fluids with a finite correlation length $\xi \propto \ell$ (ℓ is the magnetic length).
- The entanglement entropy of FQH states has be computed numerically (K. Schoutens and coworkers, 2007).
- One can compute the entanglement entropy directly from the effective field theory of all FQH states: Chern-Simons gauge theory.

Physics Illinois

Universal Contributions to the Entanglement Entropy as a Boundary CFT Problem

with Benjamin Hsu (2010)

- $$\mathrm{tr}\rho_A^n = \frac{\lim_{\lambda\to\infty} Z[\lambda, n]}{\lim_{\lambda\to 0} Z[\lambda, n]}$$
- QLM with compactification radius R, on a cylinder of length L and circumference ℓ
- Region A and region B have each length $L/2 \gg \ell$
- Fold about Γ: $2n$ cylinders of length $L/2$ and circumference ℓ with the fields are identified on Γ and free at the other end
- Length of the cylinders: imaginary time. The partition functions become amplitudes between boundary states
- $$\mathrm{tr}\rho_A^n = R^{1-n}\left(\frac{\vartheta_3\left(0|\frac{-R^2}{2n\tau}\right)}{\eta(-1/2n\tau)}\right)\left(\frac{\eta(-1/2\tau)}{\vartheta_3\left(0|\frac{-1}{2\tau}\right)}\right)^n, \qquad 2\tau = iL/\ell$$
- $$\lim_{\ell/L\to 0} \mathrm{tr}\rho_A^n = R^{1-n}(1+\dots)$$