Title: Entanglement Spectra and Topological Insulators

Date: May 28, 2010 10:30 AM

URL: http://pirsa.org/10050082

Abstract: Recent work has explored some aspects of entanglement in topological insulators. Notably, the entanglement spectrum has been shown to mimic certain properties of the low-energy fermionic modes found on real spatial boundaries. I will discuss the many-body entanglement spectrum of topological insulators and show that it matches the expected CFT character structure that has been previously shown to hold in fractional quantum Hall effect ground states. I also present the analysis of a disorder-driven Anderson localization transition in a Chern-Insulator from an analysis of the entanglement spectrum. Interestingly, the disorder-averaged level-spacing statistics of the entanglement spectrum characterizes the system just as well as the statistics of the real energy spectrum, but with the advantage that only the ground state, and not the entire spectrum of excited states, needs to be calculated.

Entanglement Spectra and Topological Insulators

Taylor L. Hughes
UIUC

May-28-2010

In collaboration with B. A. Bernevig (Princeton), E. Prodan (Yeshiva)

Outline

- Introduction to topological insulators
  - I focus on (2+1)-d models of the quantum Hall effect and quantum spin Hall effect
- Discuss single-particle and many-particle entanglement spectra of these systems
- Consider entanglement spectra with disorder

What is a Topological Insulator?

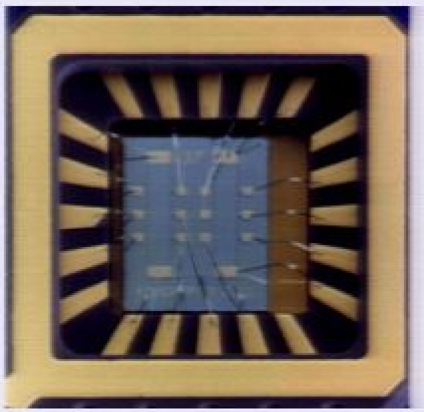
- Bulk of material is *completely* gapped
- On the boundaries there are gapless, protected fermionic modes (chiral, Dirac, Majorana, chiral-Majorana) which are *holographic*
- Bulk state characterized by a non-zero topological invariant
- May require an auxiliary symmetry to be a stable phase (T,C,…)
- Examples: IQHE, QAHE, QSH, 3d strong topological insulator, p+ip superconductor, d+id superconductor

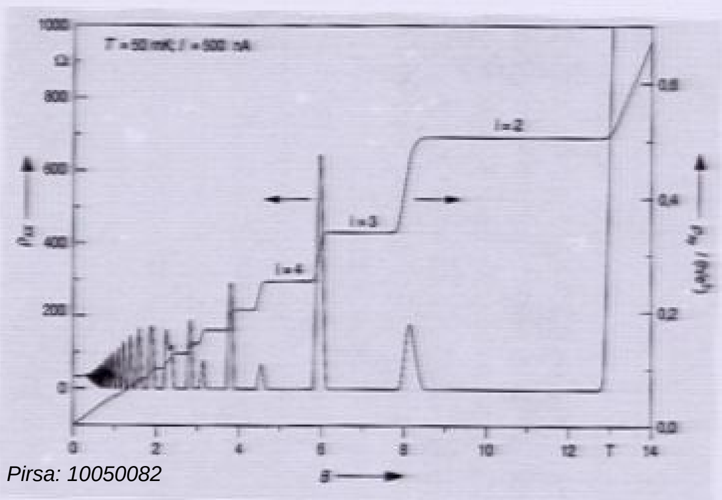
Periodic Table of Topological Insulators

Dim/Symmetry	BDI	D	DIII	AII	CII	C	CI	AI	A	AIII
(0+1)d	Z_2	Z_2	0	Z	0	0	0	Z	Z	0
(1+1)d	Z	Z_2	Z_2	0	Z	0	0	0	0	Z
(2+1)d	0	Z	Z_2	Z_2	0	Z	0	0	Z	0
(3+1)d	0	0	Z	Z_2	Z_2	0	Z	0	0	Z
(4+1)d	0	0	0	Z	Z_2	Z_2	0	Z	Z	0
(5+1)d	Z	0	0	0	Z	Z_2	Z_2	0	0	Z
(6+1)d	0	Z	0	0	0	Z	Z_2	Z_2	Z	0
(7+1)d	Z_2	0	Z	0	0	0	Z	Z_2	0	Z

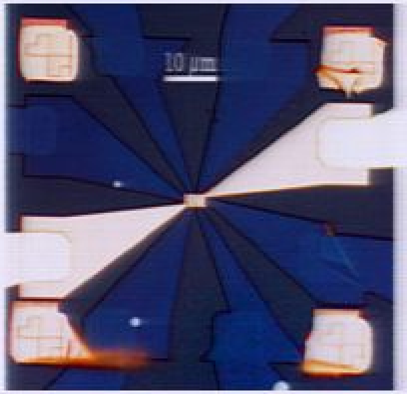
(2+1)-d Topological Insulators

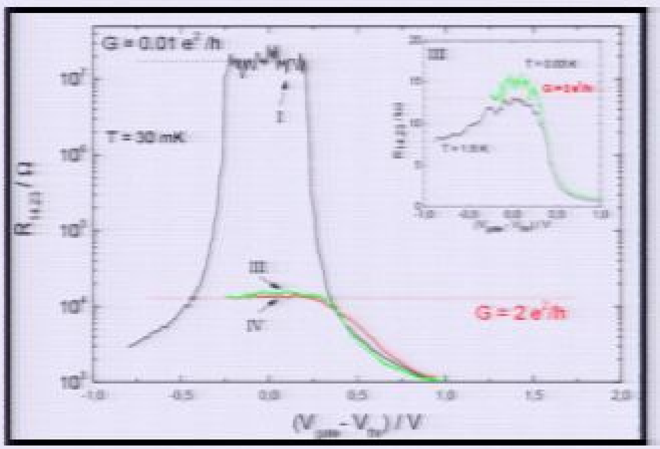
Quantum Hall





Quantum Spin Hall





(2+1)-d Topological Insulators

Quantum Hall

Bulk is gapped and described by an integer topological invariant.

Chiral Edge States on the Edge

Unitary Class (requires no special symmetries)

Quantum Spin Hall

(2+1)-d Topological Insulators

Quantum Hall

Bulk is gapped and described by an integer topological invariant.

Chiral Edge States on the Edge

Unitary Class (requires no special symmetries)

Quantum Spin Hall

Bulk is gapped and described by a Z_2 topological invariant.

Chiral and Anti-chiral Edge States on the Edge (a time-reversed pair)

Symplectic Class (Requires T symmetry)

(2+1)-d Topological Insulators

Quantum Hall

Quantum Spin Hall

$$H_{QH} = \begin{pmatrix} M & p_x - ip_y \\ p_x + ip_y & -M \end{pmatrix}$$

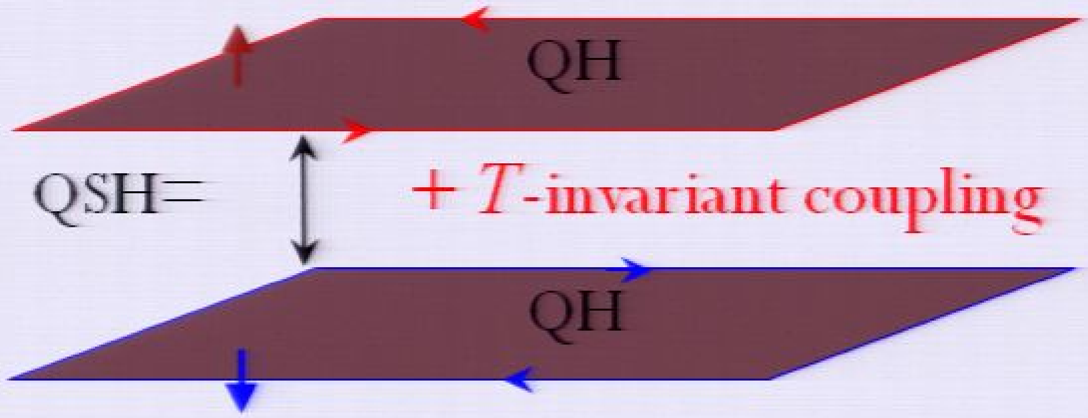
(2+1)-d Topological Insulators

Quantum Hall

$$H_{QH} = \begin{pmatrix} M & p_x - ip_y \\ p_x + ip_y & -M \end{pmatrix}$$

Quantum Spin Hall

$$H_{QSH} = \begin{pmatrix} M & p_x - ip_y & 0 & 0 \\ p_x + ip_y & -M & 0 & 0 \\ 0 & 0 & M & -p_x - ip_y \\ 0 & 0 & -p_x + ip_y & -M \end{pmatrix}$$



Topological Insulators and Entanglement

Topological Insulators and Entanglement

- Are TI's topologically ordered?
  - No. They have non-degenerate ground states, always. Thus there is no topological entanglement entropy.

Topological Insulators and Entanglement

- Are TI's topologically ordered?
  - No. They have non-degenerate ground states, always. Thus there is no topological entanglement entropy.
- Do TI's have interesting entanglement properties?
  - Yes. They exhibit non-trivial fermion entanglement spectra and cannot be deformed to have vanishing entanglement entropy.

Topological Insulators and Entanglement

- Are TI's topologically ordered?
  - No. They have non-degenerate ground states, always. Thus there is no topological entanglement entropy.

- Do TI's have interesting entanglement properties?
  - Yes. They exhibit non-trivial fermion entanglement spectra and cannot be deformed to have vanishing entanglement entropy.

In contrast, quantum double models (toric code) and string-net states *only* have entanglement entropy coming from "gauge" fluctuations of the string condensed ground state. There is no fermionic piece and the entanglment spectra are featureless.

Topological Insulators and Entanglement

- Are TI's topologically ordered?
  - No. They have non-degenerate ground states, always. Thus there is no topological entanglement entropy.

- Do TI's have interesting entanglement properties?
  - Yes. They exhibit non-trivial fermion entanglement spectra and cannot be deformed to have vanishing entanglement entropy.

In contrast, quantum double models (toric code) and string-net states *only* have entanglement entropy coming from "gauge" fluctuations of the string condensed ground state. There is no fermionic piece and the entanglment spectra are featureless.

Interestingly, the Kitaev honeycomb model has both, and they are completely decoupled from each other.

Topological Insulators and Entanglement

- While the entanglement properties of topological insulators do not contain the topological piece coming from gauge-like fluctuations, the fermionic contributions are very clear
- One additional advantage is that *everything* is calculable since we are dealing with free fermion systems

Topological Insulators and Entanglement

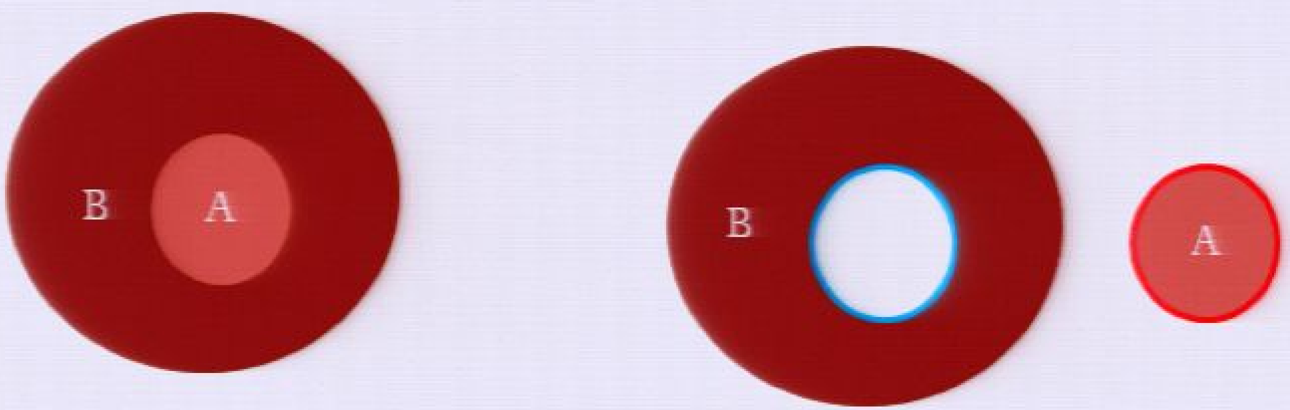
- While the entanglement properties of topological insulators do not contain the topological piece coming from gauge-like fluctuations, the fermionic contributions are very clear
- One additional advantage is that *everything* is calculable since we are dealing with free fermion systems

Heuristic Picture:

Topological Insulators and Entanglement

- While the entanglement properties of topological insulators do not contain the topological piece coming from gauge-like fluctuations, the fermionic contributions are very clear
- One additional advantage is that *everything* is calculable since we are dealing with free fermion systems

Heuristic Picture:



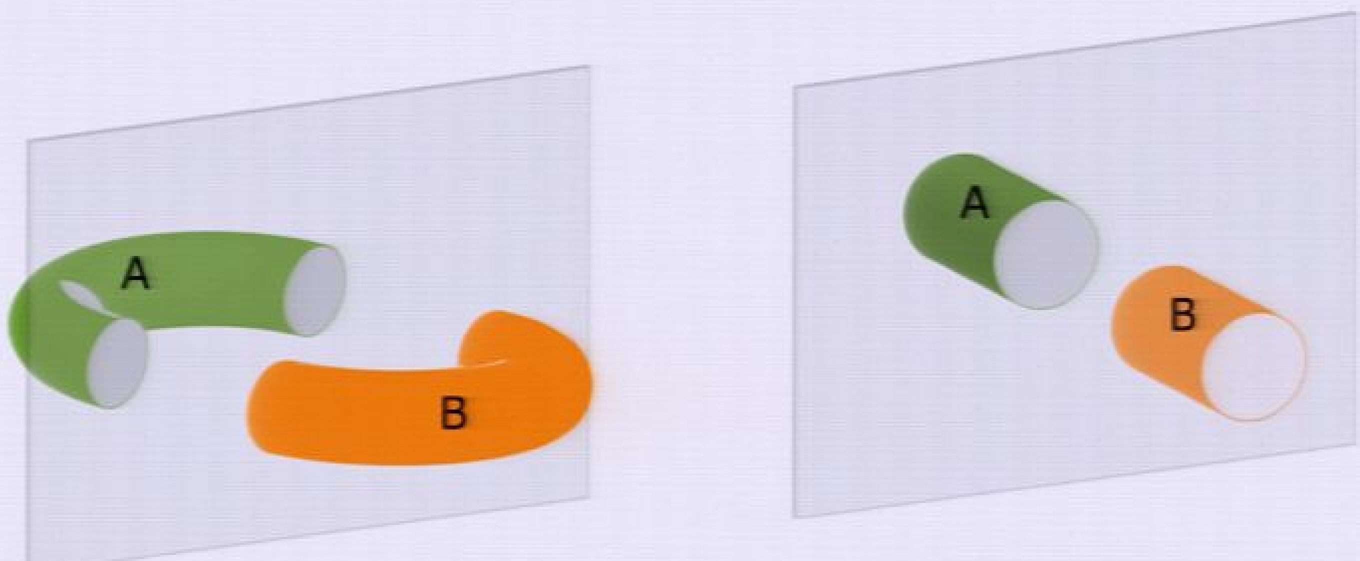
Topological Insulators and Entanglement

- While the entanglement properties of topological insulators do not contain the topological piece coming from gauge-like fluctuations, the fermionic contributions are very clear
- One additional advantage is that *everything* is calculable since we are dealing with free fermion systems

Heuristic Picture:

Topological Insulators and Entanglement

- While the entanglement properties of topological insulators do not contain the topological piece coming from gauge-like fluctuations, the fermionic contributions are very clear
- One additional advantage is that *everything* is calculable since we are dealing with free fermion systems



Entanglement Spectra of Free Fermions

- We can get the *single-particle* entanglment spectrum of any free-fermion Hamiltonian by using Peschel's method:

Entanglement Spectra of Free Fermions

- We can get the *single-particle* entanglment spectrum of any free-fermion Hamiltonian by using Peschel's method:

Given: $$\hat{\mathcal{H}} = -\hat{t}\sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$

Calculate: $$\hat{C}_{mn} = \langle 0|c_m^\dagger c_n|0\rangle$$

Entanglement Spectra of Free Fermions

- We can get the *single-particle* entanglment spectrum of any free-fermion Hamiltonian by using Peschel's method:

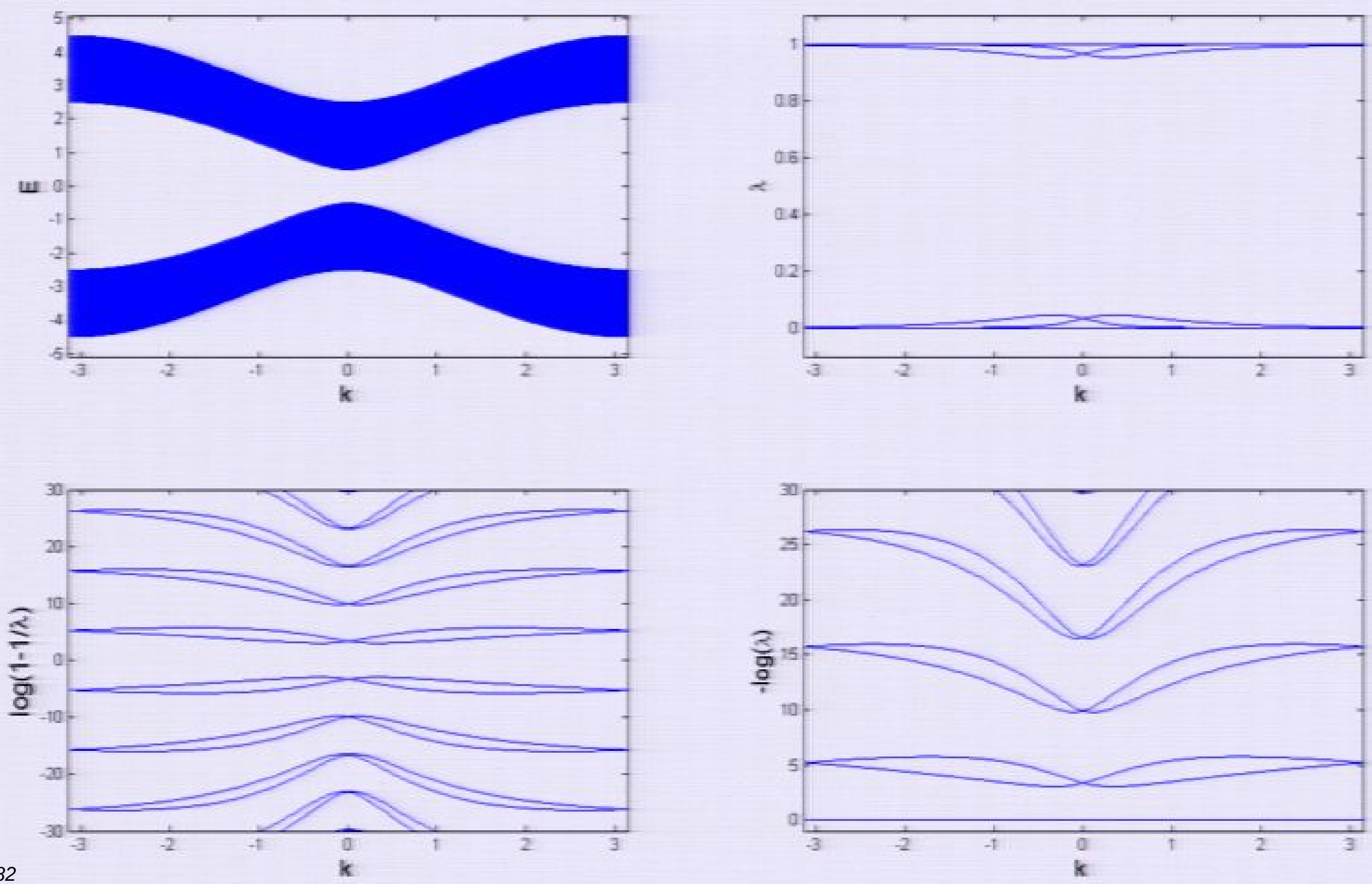
Given: $$\hat{\mathcal{H}} = -\hat{t}\sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$

Calculate: $$\hat{C}_{mn} = \langle 0|c_m^\dagger c_n|0\rangle$$

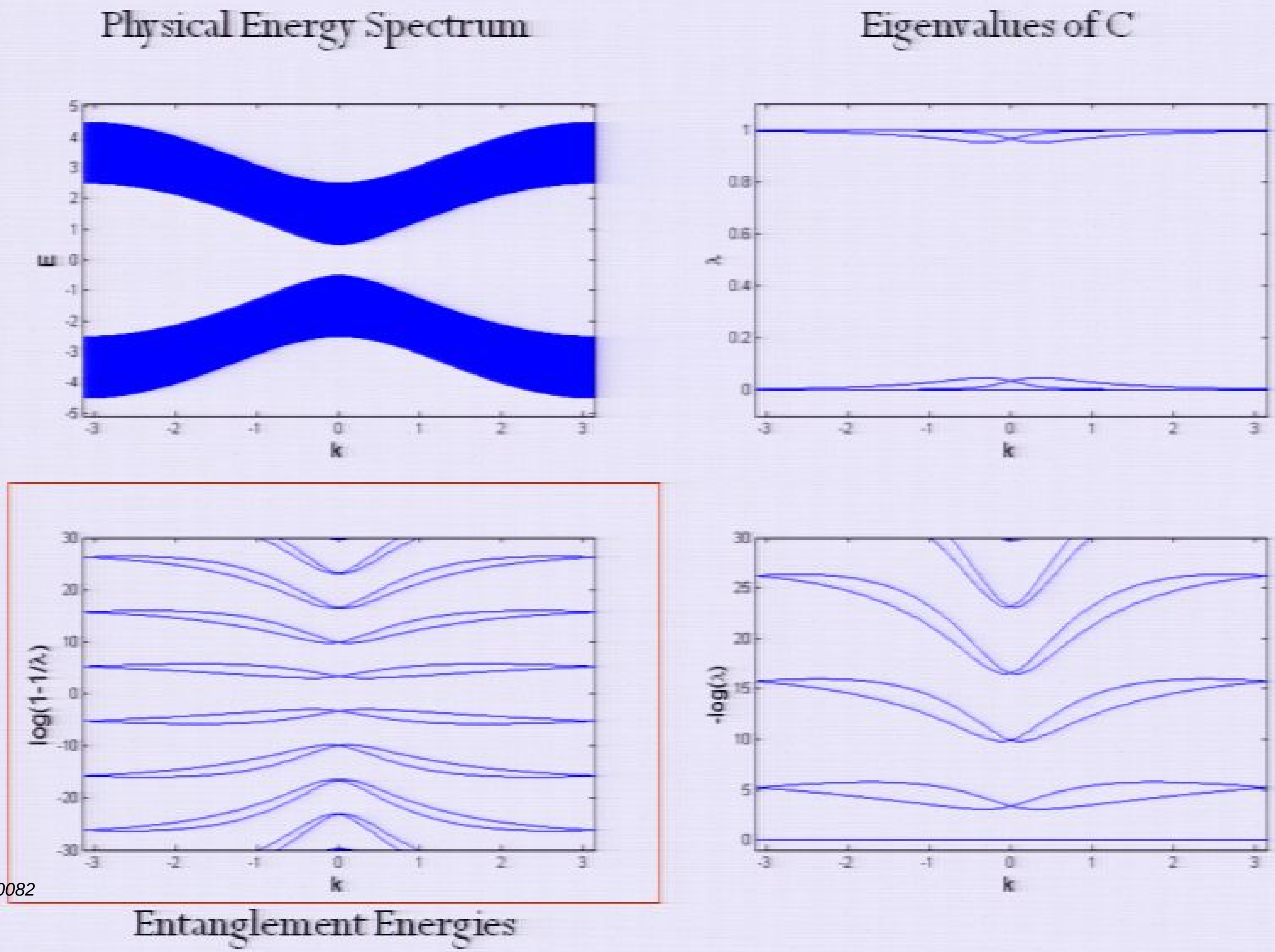
anglement "Hamiltonian": $$H = \ln[(1-C)/C]$$

Where C is the two-point correlation function restricted to sites in region A

Single-Particle Entanglement Spectrum



Single-Particle Entanglement Spectrum



Entanglement Spectra of Free Fermions

- We can get the *single-particle* entanglment spectrum of any free-fermion Hamiltonian by using Peschel's method:

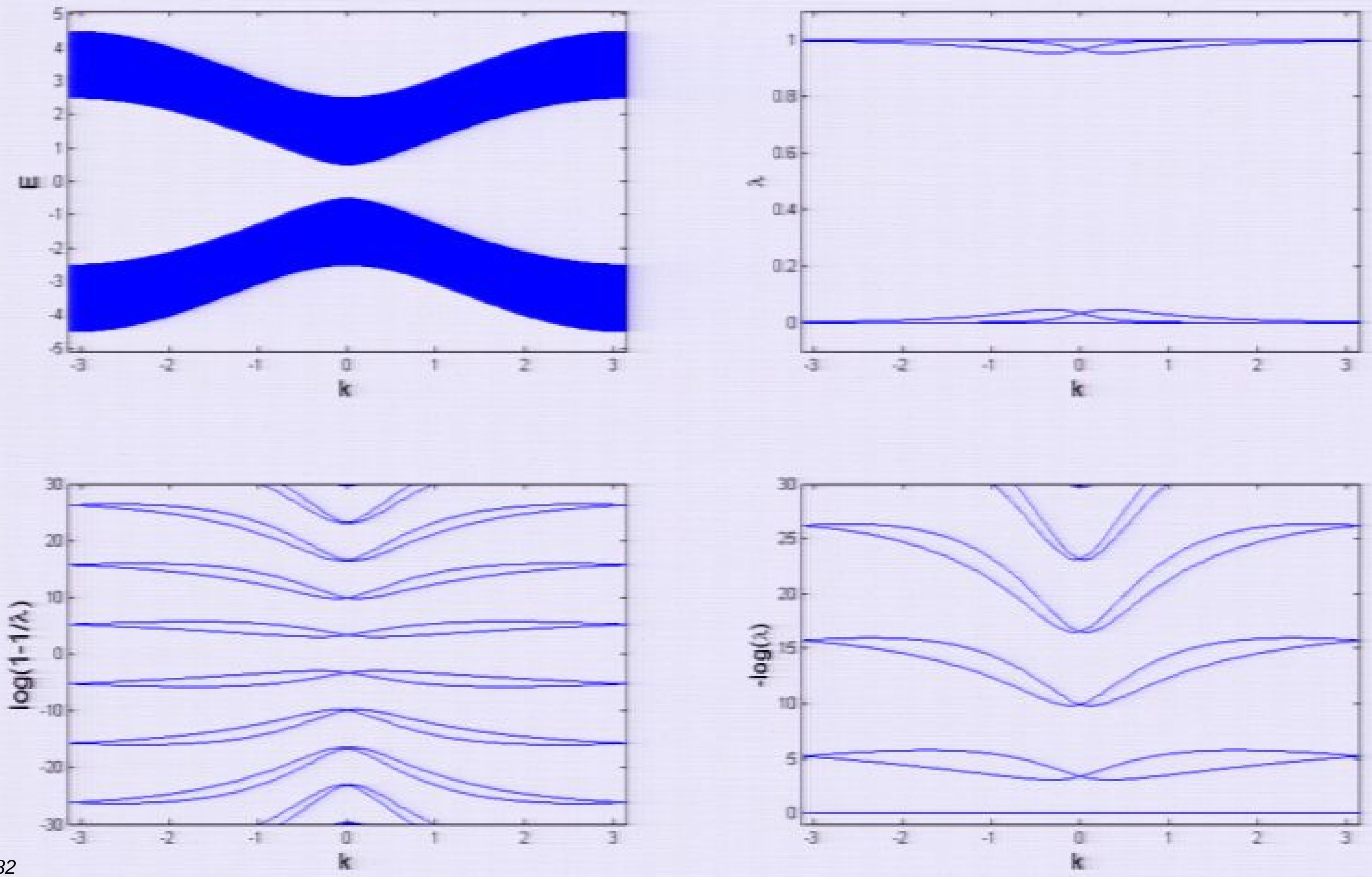
Given: $$\hat{\mathcal{H}} = -\hat{t}\sum_{n}(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$

Calculate: $$\hat{C}_{mn} = \langle 0|c_m^\dagger c_n|0\rangle$$

anglement "Hamiltonian": $$H = \ln[(1-C)/C]$$

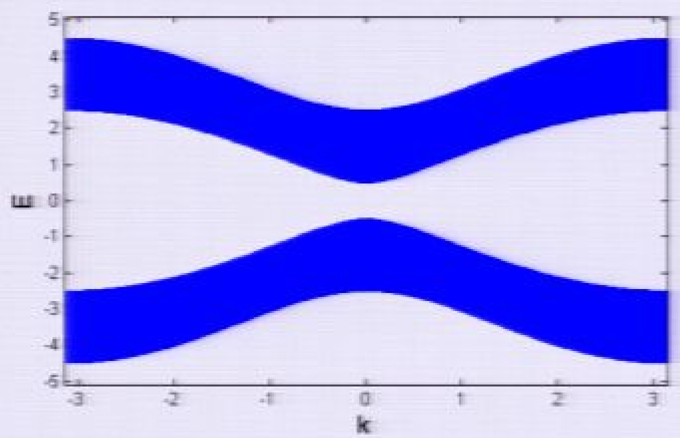
Where C is the two-point correlation function restricted to sites in region A

Single-Particle Entanglement Spectrum

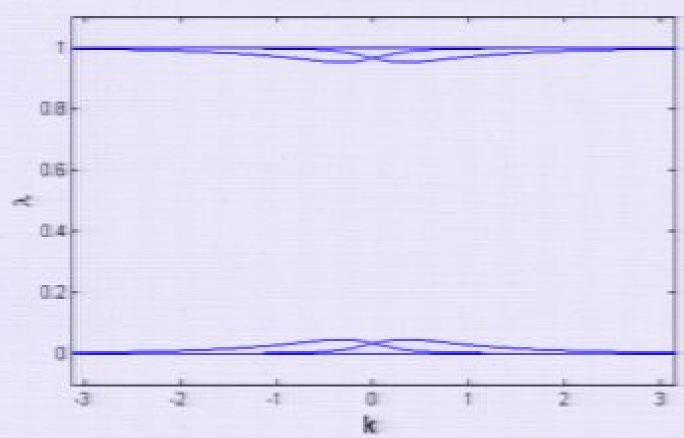


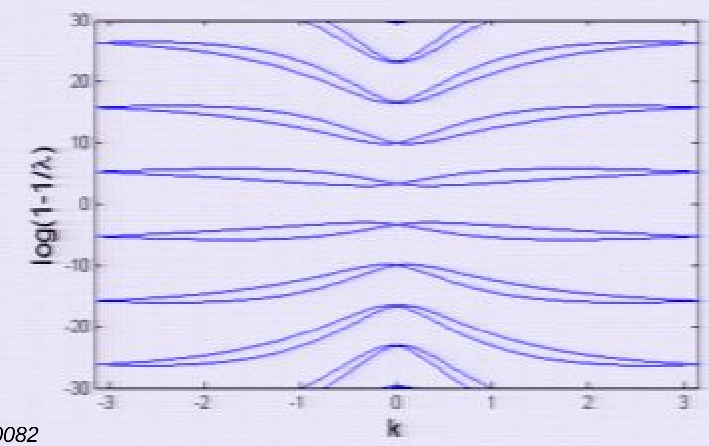
Single-Particle Entanglement Spectrum

Physical Energy Spectrum

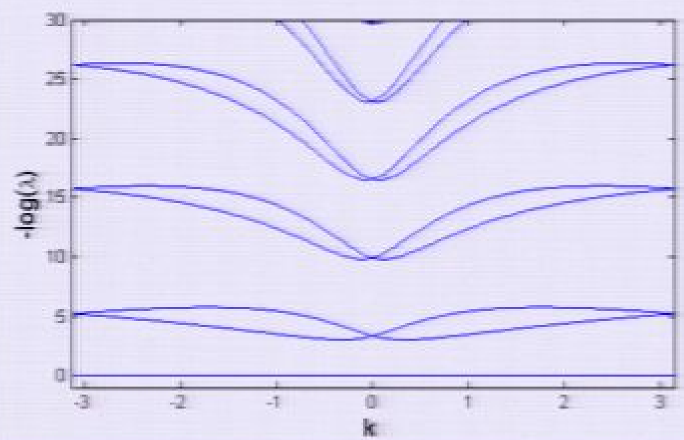


Eigenvalues of C





Entanglement Energies



-Log of eigenvalues of C

Many-body Entanglement Spectrum

- Take a many-body ground state ψ and perform a Schmidt decomposition

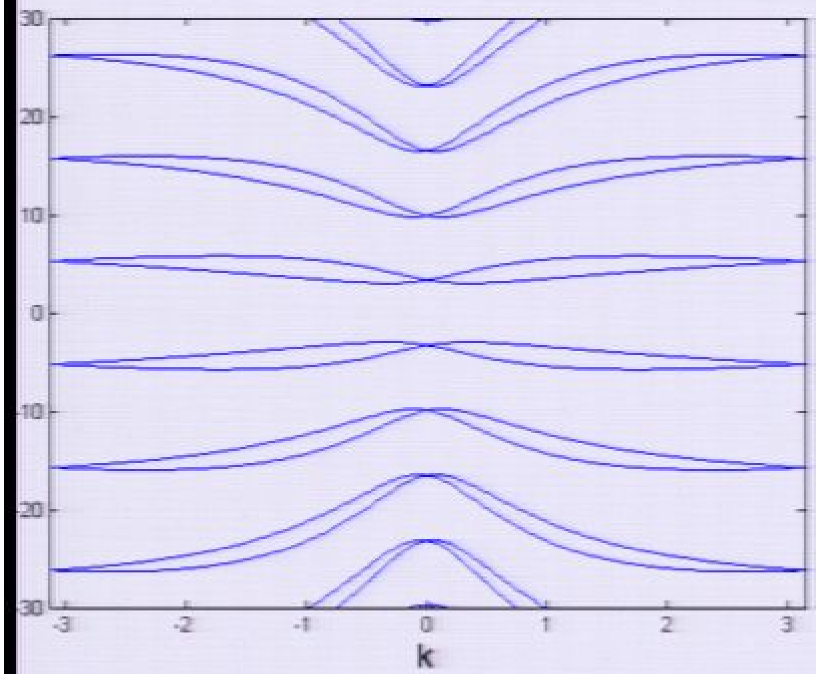
$$|\psi\rangle = \sum_i e^{-\frac{1}{2}\xi_i} |\psi_A^i\rangle \otimes |\psi_B^i\rangle$$

The set of ξ_i are the *many-body* entanglement "energies"

•For free fermions the ground state is a Slater determinant state but calculating many-body spectrum is still slow and requires a lot of memory.

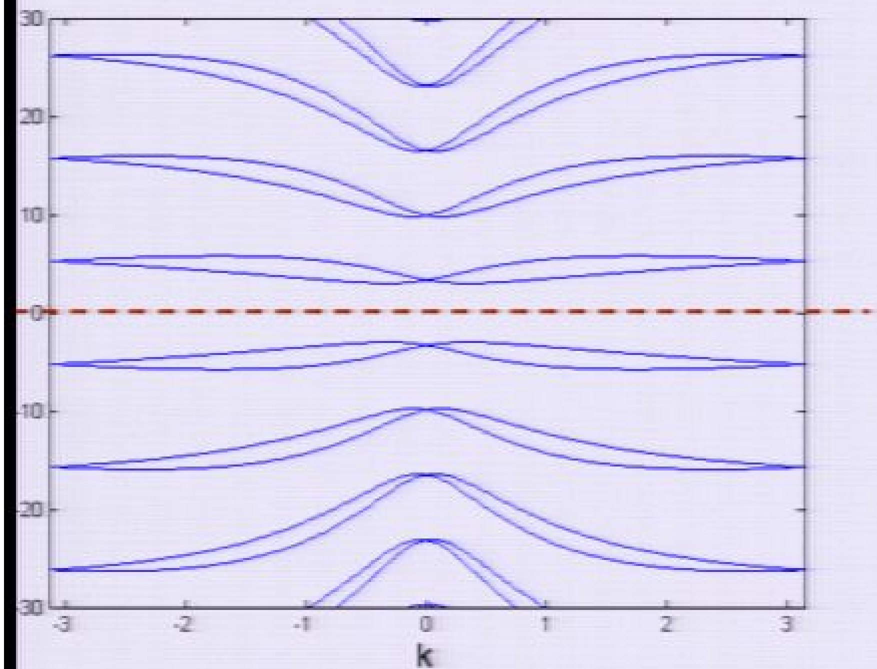
Many-body Entanglement Spectrum

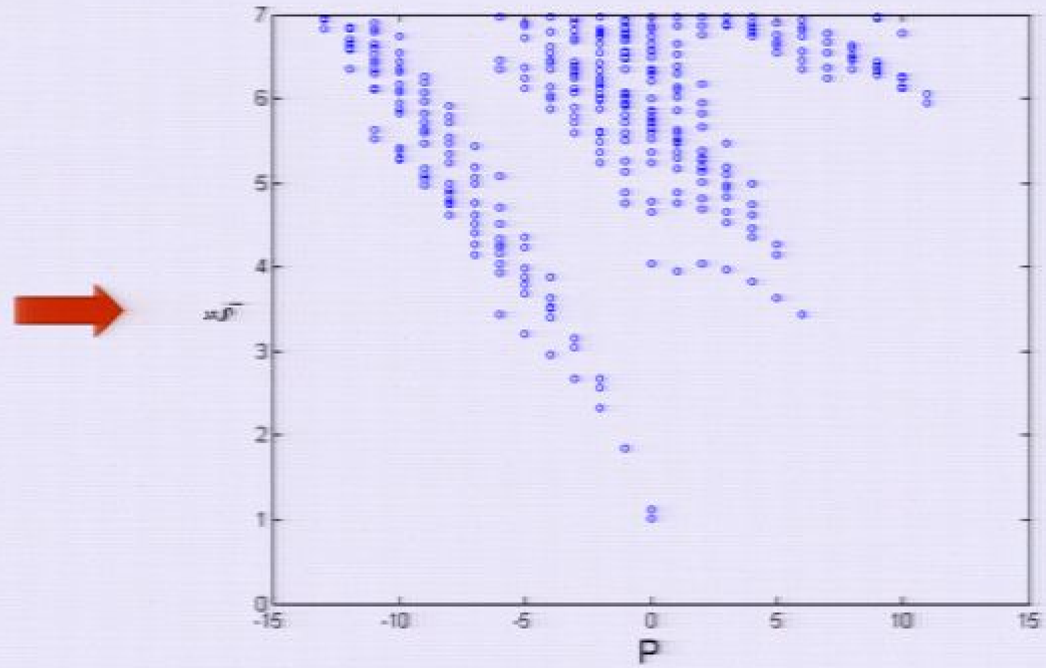
- There is a better way to get the many-body excitation spectrum analogous to filling the single-particle energy states



Many-body Entanglement Spectrum

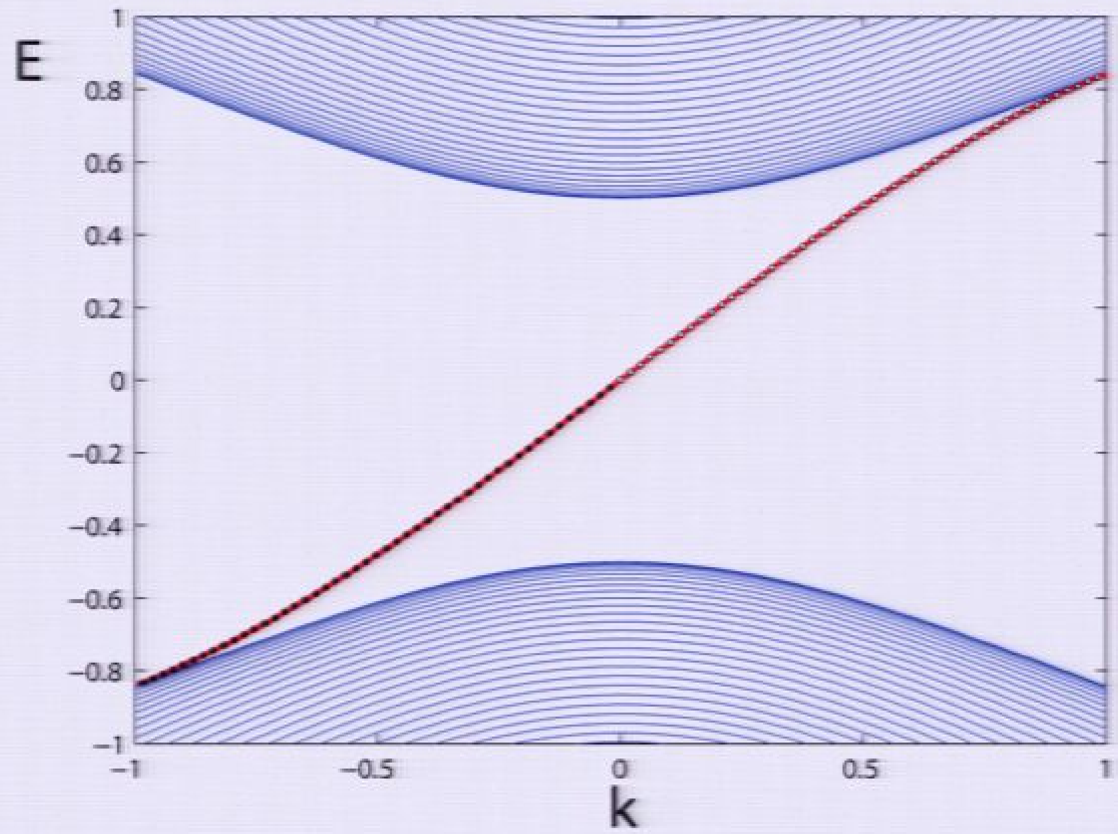
- There is a better way to get the many-body excitation spectrum analogous to filling the single-particle energy states





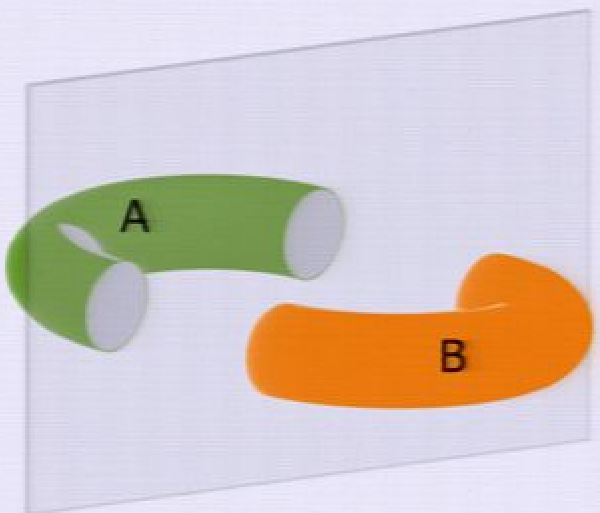
Energy Spectrum of the Chern Insulator

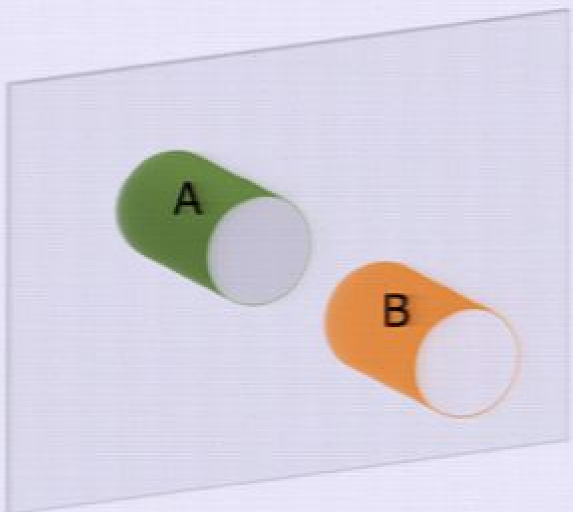
Solve for energy spectrum on a cylinder



Entanglement Spectrum of the Chern Insulator (Quantum Hall)

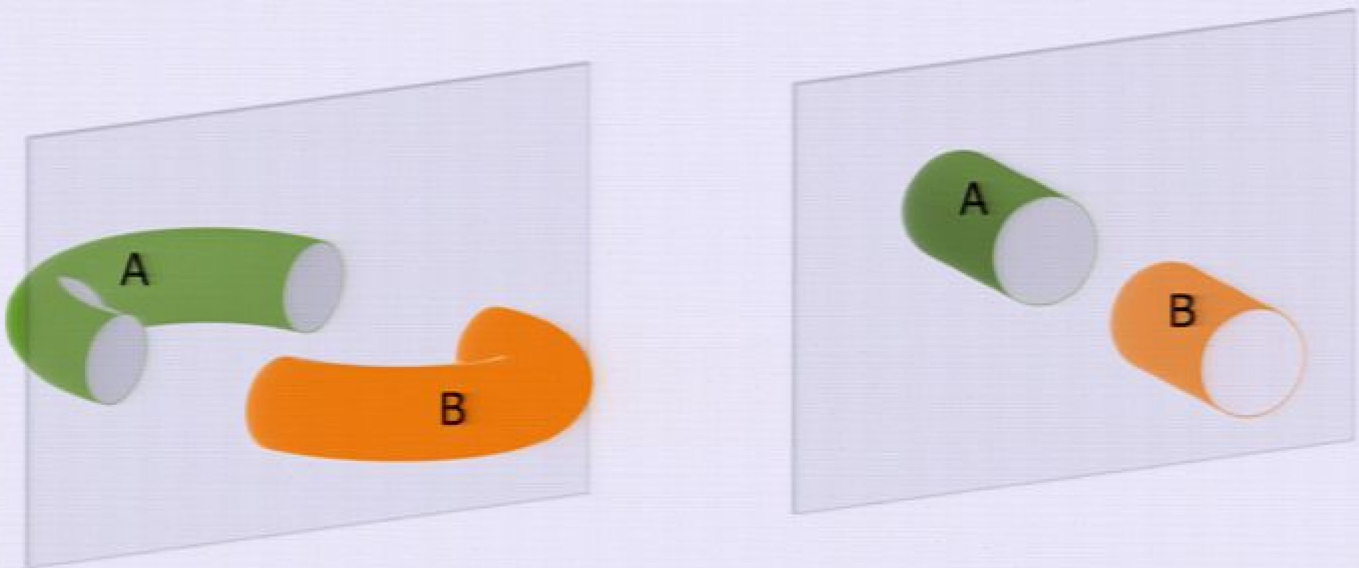
- We can make cuts on a cylinder or torus. A cylinder is clearer because there is only one set of low-energy "cut"-states





Entanglement Spectrum of the Chern Insulator (Quantum Hall)

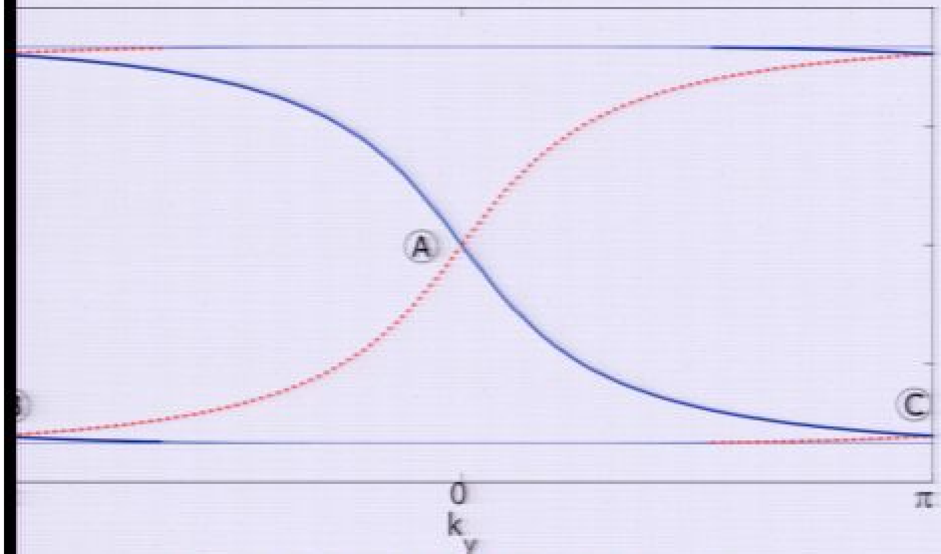
- We can make cuts on a cylinder or torus. A cylinder is clearer because there is only one set of low-energy "cut"-states

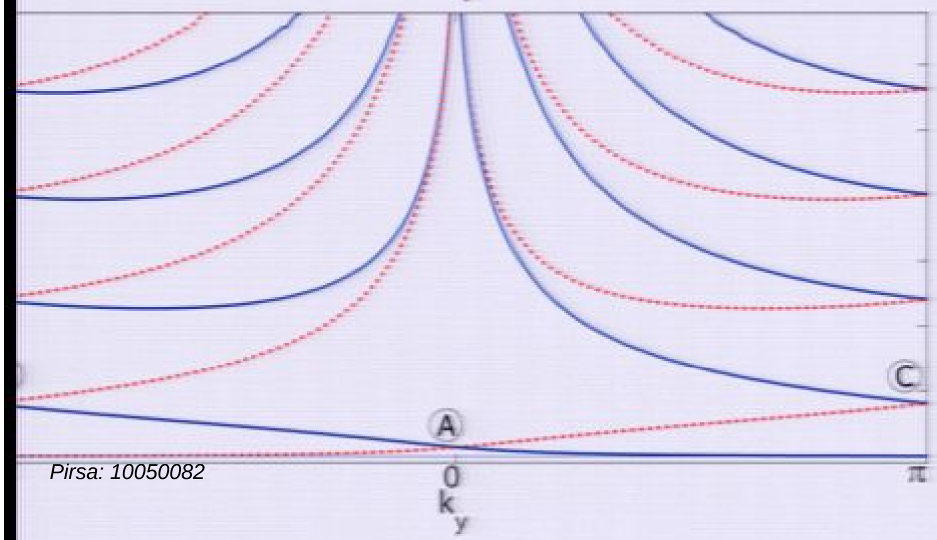


One might worry that the physical edge states of the topological insulator in the cylinder geometry would affect the entanglement. However this is not true. These states are exponentially localized as far away from the cut as one can get

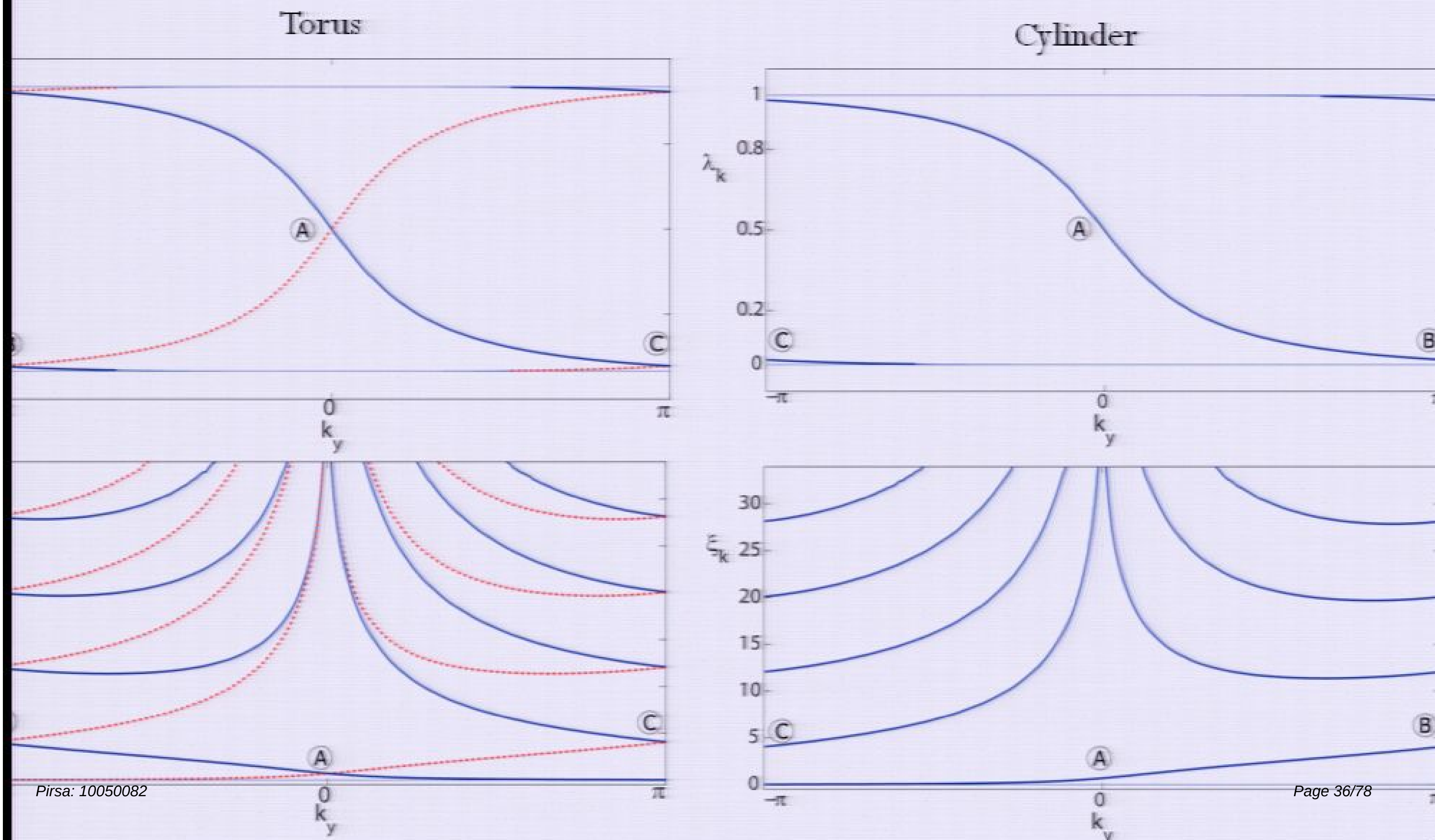
Entanglement Spectrum of the Chern Insulator (Quantum Hall)

Torus





Entanglement Spectrum of the Chern Insulator (Quantum Hall)

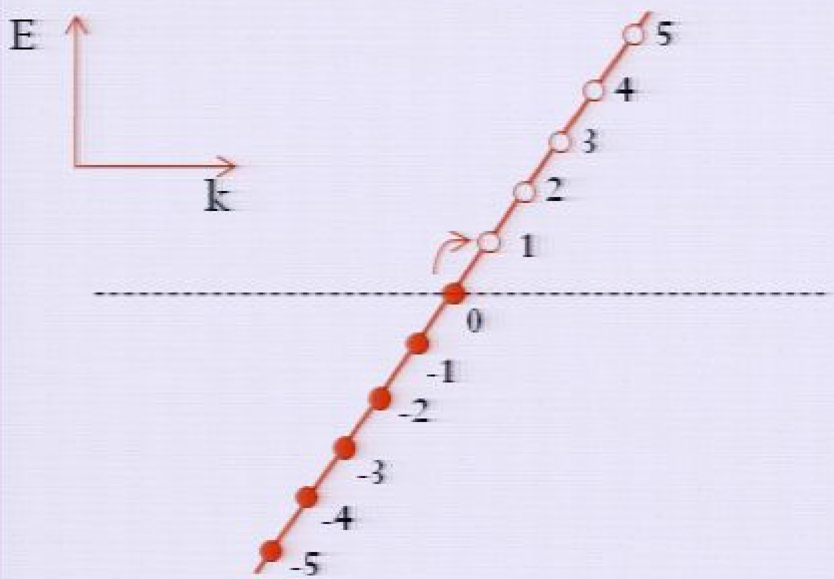


Many-body Entanglement Spectrum of Chern Insulator

- What should we expect? From Li and Haldane's analysis of the Moore-Read state we should expect to see the conformal character counting of the QH edge theory (i.e. a free chiral fermion)

Many-body Entanglement Spectrum of Chern Insulator

- What should we expect? From Li and Haldane's analysis of the Moore-Read state we should expect to see the conformal character counting of the QH edge theory (i.e. a free chiral fermion)

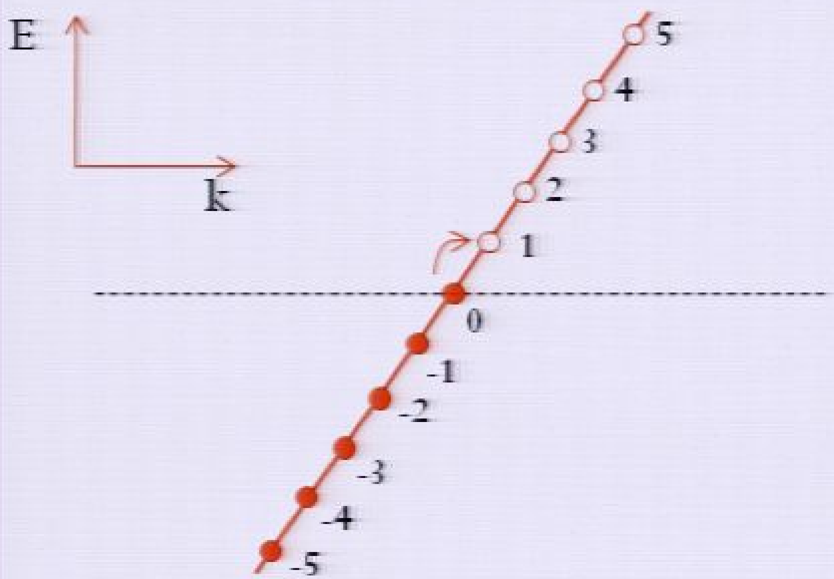


Counting of States

Level	States	#
1	$\vert\Omega\rangle$	1
2	$c_1^\dagger c_0\vert\Omega\rangle$	1
3	$c_2^\dagger c_0\vert\Omega\rangle$, $c_1^\dagger c_{-1}\vert\Omega\rangle$	2
4	$c_3^\dagger c_0\vert\Omega\rangle$, $c_2^\dagger c_{-1}\vert\Omega\rangle$, $c_1^\dagger c_{-2}\vert\Omega\rangle$	3
5	$c_4^\dagger c_0\vert\Omega\rangle$, $c_3^\dagger c_{-1}\vert\Omega\rangle$, $c_2^\dagger c_{-2}\vert\Omega\rangle$ $c_1^\dagger c_{-3}\vert\Omega\rangle$, $c_2^\dagger c_1^\dagger c_{-1} c_0\vert\Omega\rangle$	5

Many-body Entanglement Spectrum of Chern Insulator

- What should we expect? From Li and Haldane's analysis of the Moore-Read state we should expect to see the conformal character counting of the QH edge theory (i.e. a free chiral fermion)

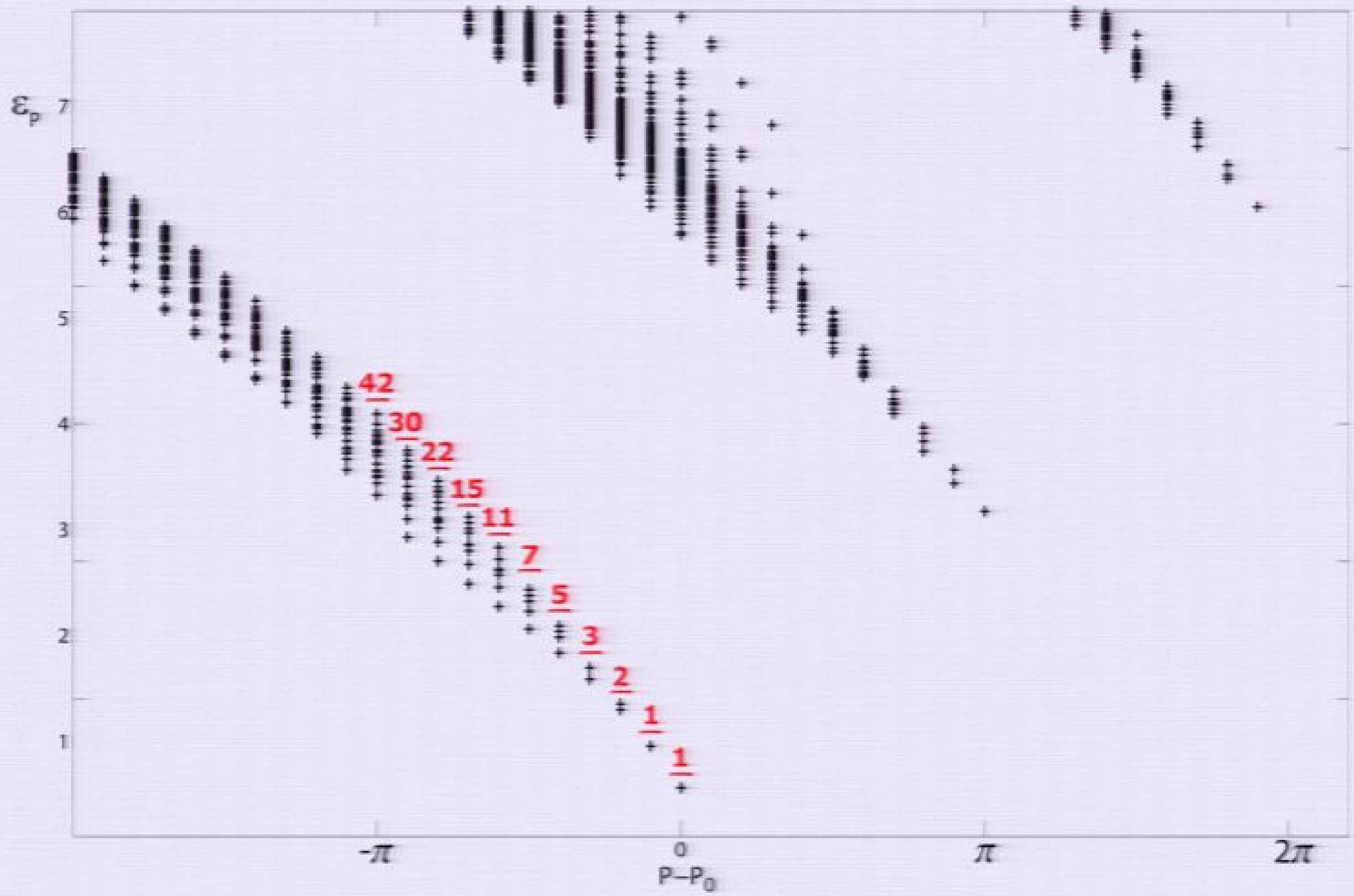


Counting of States

Level	States	#
1	$\|\Omega\rangle$	1
2	$c_1^\dagger c_0\|\Omega\rangle$	1
3	$c_2^\dagger c_0\|\Omega\rangle$, $c_1^\dagger c_{-1}\|\Omega\rangle$	2
4	$c_3^\dagger c_0\|\Omega\rangle$, $c_2^\dagger c_{-1}\|\Omega\rangle$, $c_1^\dagger c_{-2}\|\Omega\rangle$	3
5	$c_4^\dagger c_0\|\Omega\rangle$, $c_3^\dagger c_{-1}\|\Omega\rangle$, $c_2^\dagger c_{-2}\|\Omega\rangle$ $c_1^\dagger c_{-3}\|\Omega\rangle$, $c_2^\dagger c_1^\dagger c_{-1} c_0\|\Omega\rangle$	5

At each level all the states will be degenerate if spectrum is perfectly linear, for finite size systems there will be deviations from perfect degeneracy

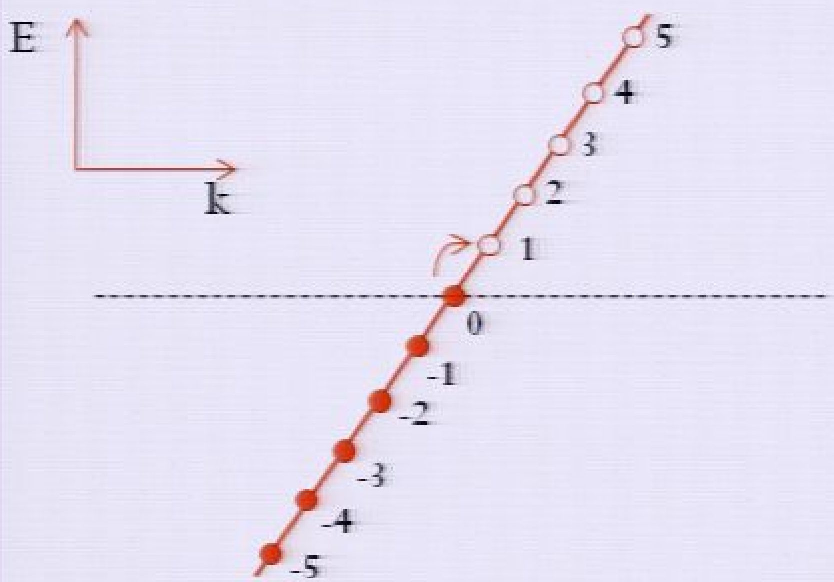
Many-body Entanglement Spectrum of Chern Insulator



see the correct counting up to 12 levels for this finite-sized system. Levels are only nearly-degenerate.
…ermodynamic limit levels will be degenerate and spectrum will become “gapless.”

Many-body Entanglement Spectrum of Chern Insulator

- What should we expect? From Li and Haldane's analysis of the Moore-Read state we should expect to see the conformal character counting of the QH edge theory (i.e. a free chiral fermion)

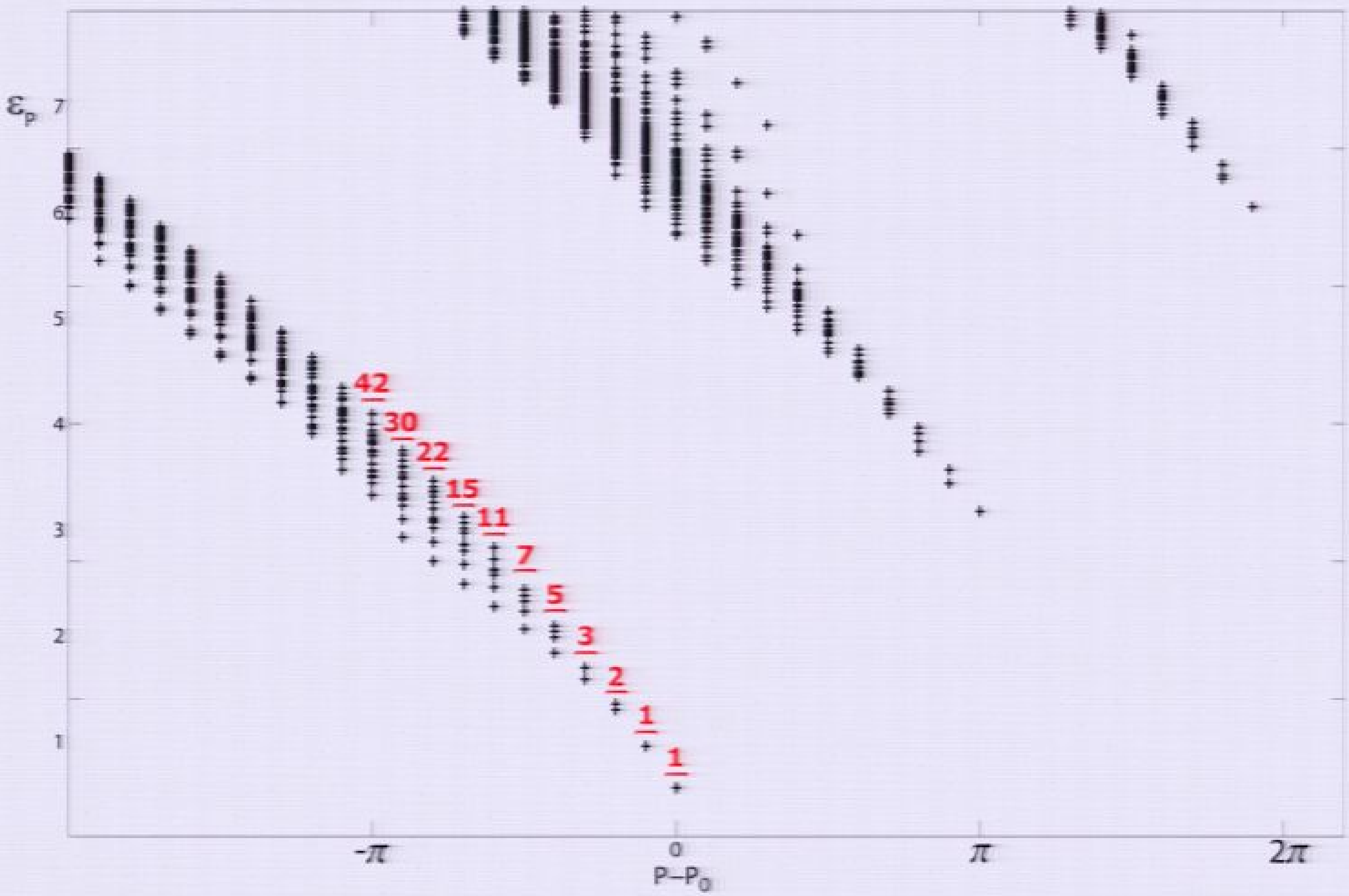


Counting of States

Level	States	#
1	$\|\Omega\rangle$	1
2	$c_1^\dagger c_0\|\Omega\rangle$	1
3	$c_2^\dagger c_0\|\Omega\rangle$, $c_1^\dagger c_{-1}\|\Omega\rangle$	2
4	$c_3^\dagger c_0\|\Omega\rangle$, $c_2^\dagger c_{-1}\|\Omega\rangle$, $c_1^\dagger c_{-2}\|\Omega\rangle$	3
5	$c_4^\dagger c_0\|\Omega\rangle$, $c_3^\dagger c_{-1}\|\Omega\rangle$, $c_2^\dagger c_{-2}\|\Omega\rangle$ $c_1^\dagger c_{-3}\|\Omega\rangle$, $c_2^\dagger c_1^\dagger c_{-1} c_0\|\Omega\rangle$	5

At each level all the states will be degenerate if spectrum is perfectly linear, for finite size systems there will be deviations from perfect degeneracy

Many-body Entanglement Spectrum of Chern Insulator



see the correct counting up to 12 levels for this finite-sized system. Levels are only nearly-degenerate.
ermodynamic limit levels will be degenerate and spectrum will become "gapless."

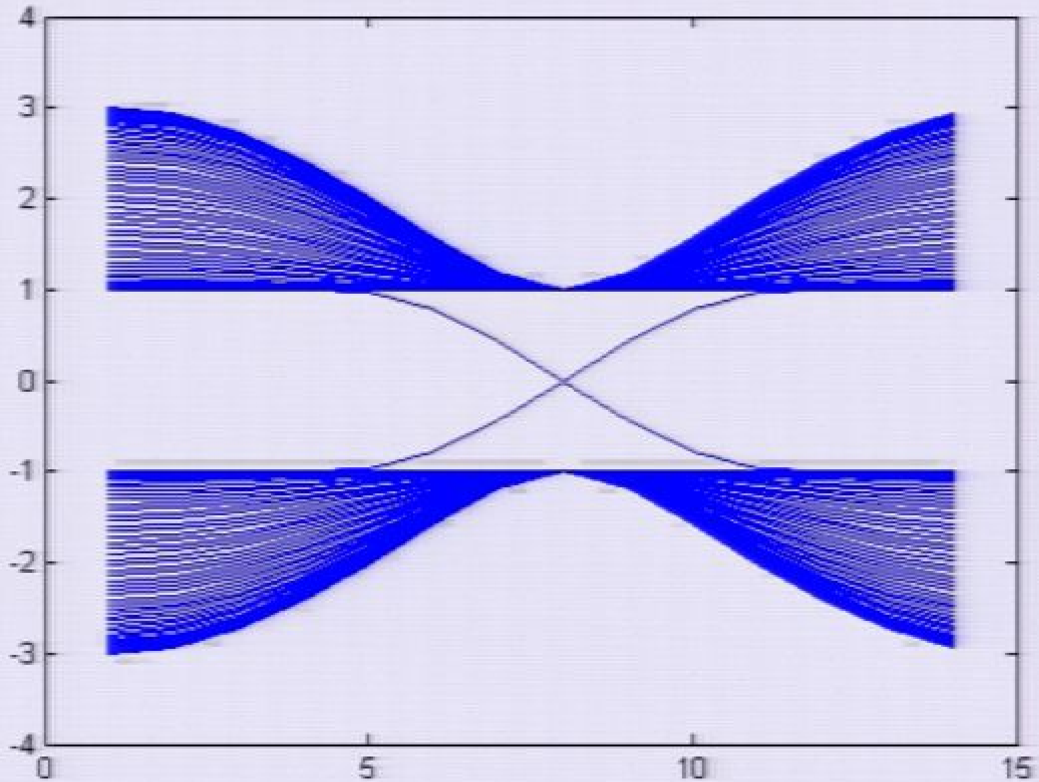
View of Phase Transitions From the Entanglement Spectrum

- For the topological insulator model (Dirac model) there is a phase transition when M switches sign

$$H_{QH} = \begin{pmatrix} M & p_x - ip_y \\ p_x + ip_y & -M \end{pmatrix}$$

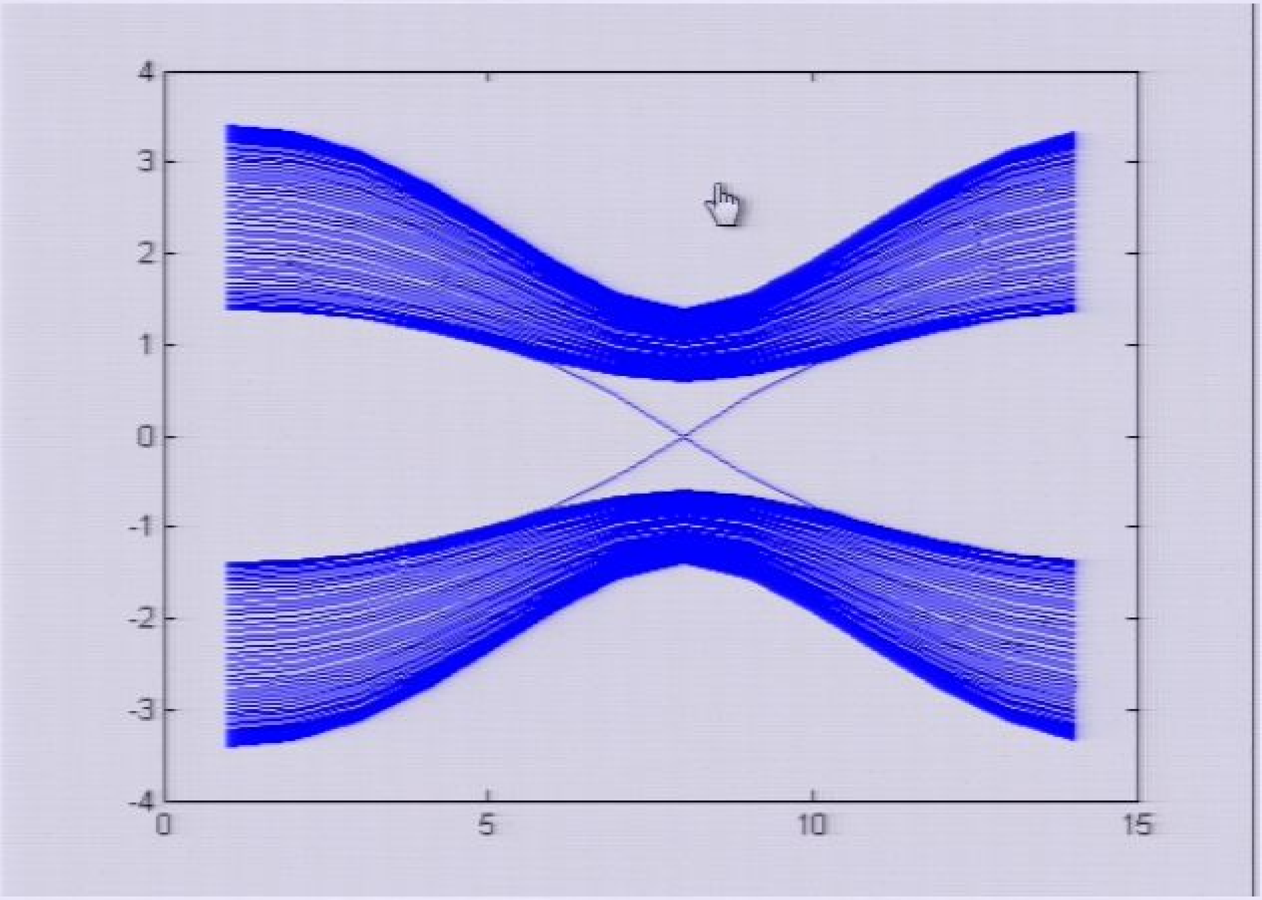
View of Phase Transitions From the Entanglement Spectrum

- For the topological insulator model (Dirac model) there is a phase transition when M switches sign



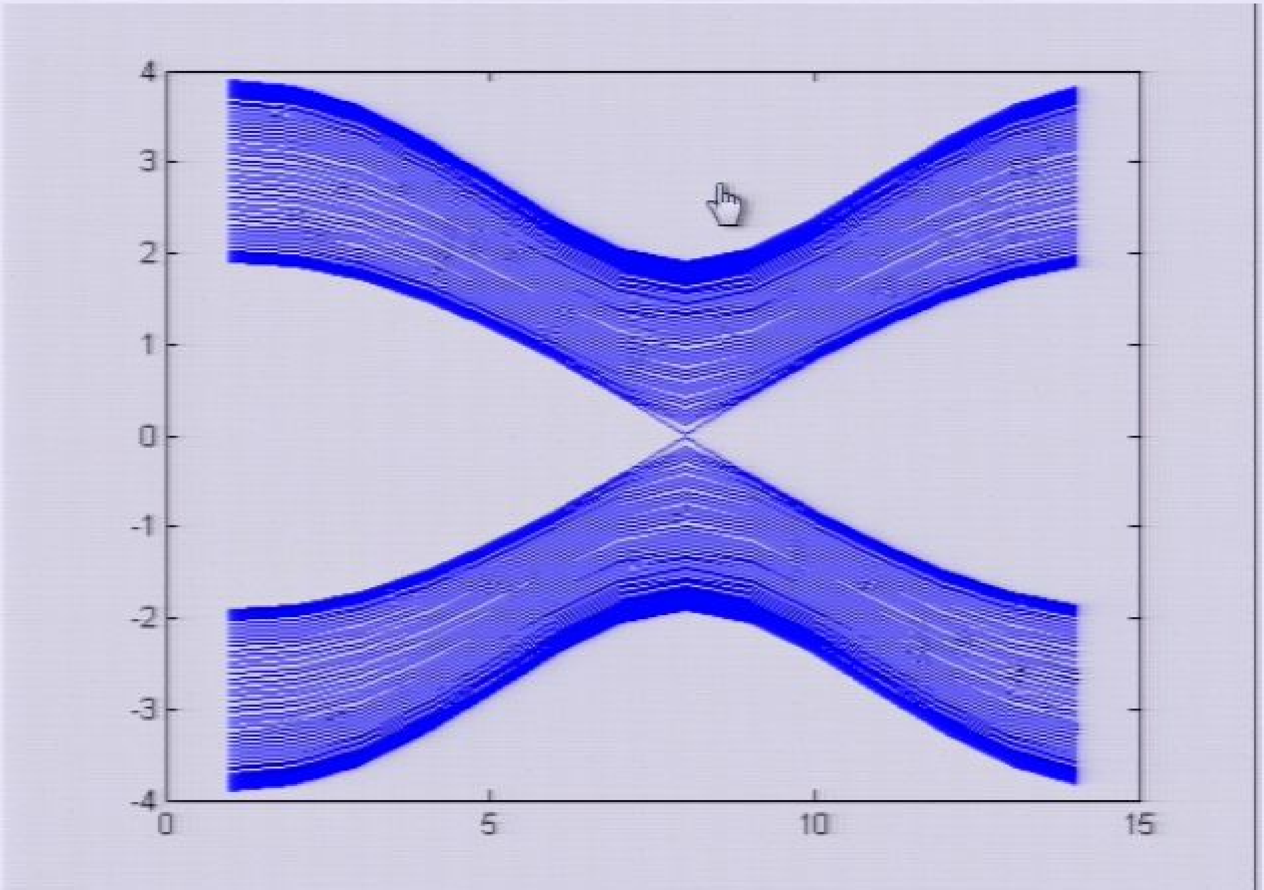
View of Phase Transitions From the Entanglement Spectrum

- For the topological insulator model (Dirac model) there is a phase transition when M switches sign



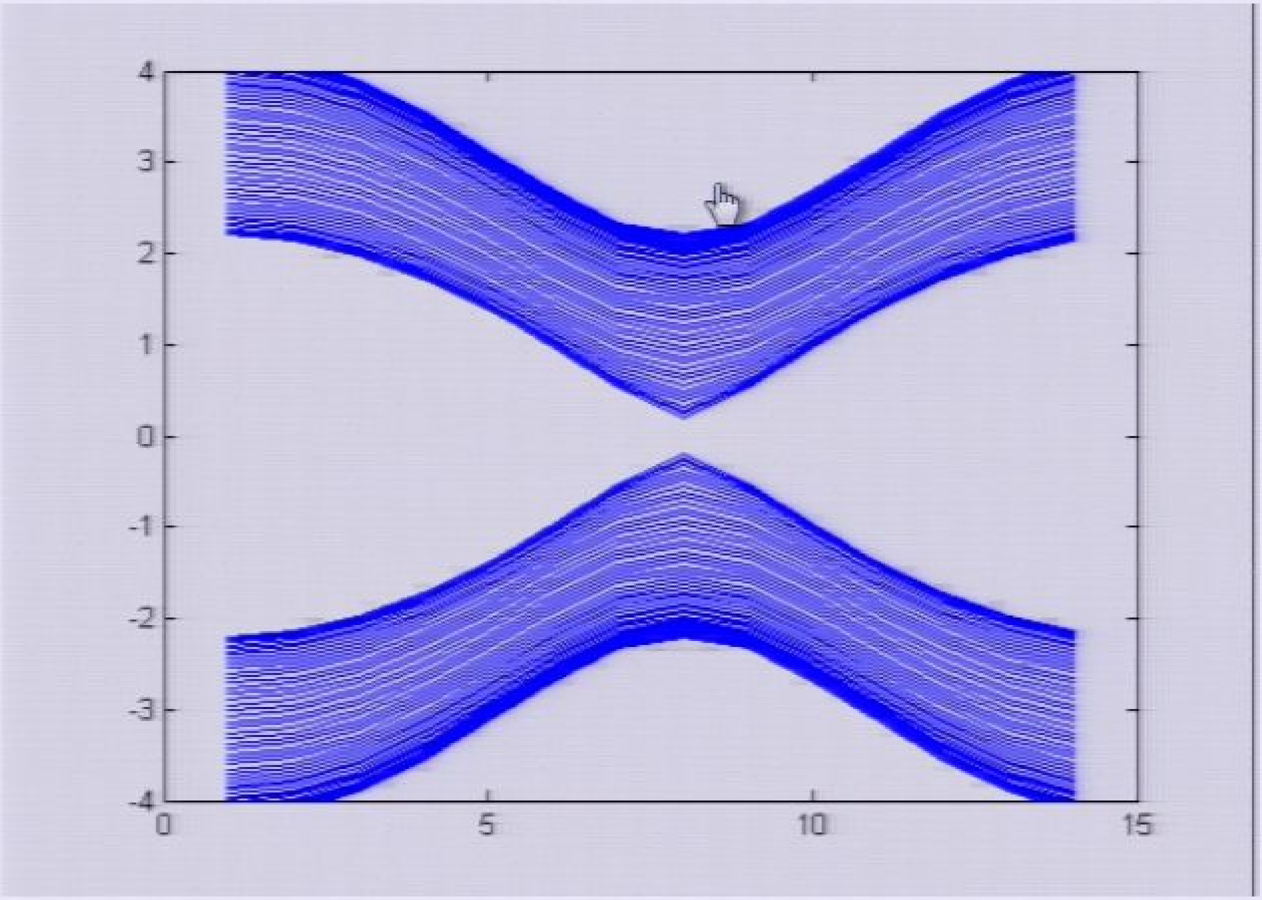
View of Phase Transitions From the Entanglement Spectrum

- For the topological insulator model (Dirac model) there is a phase transition when M switches sign

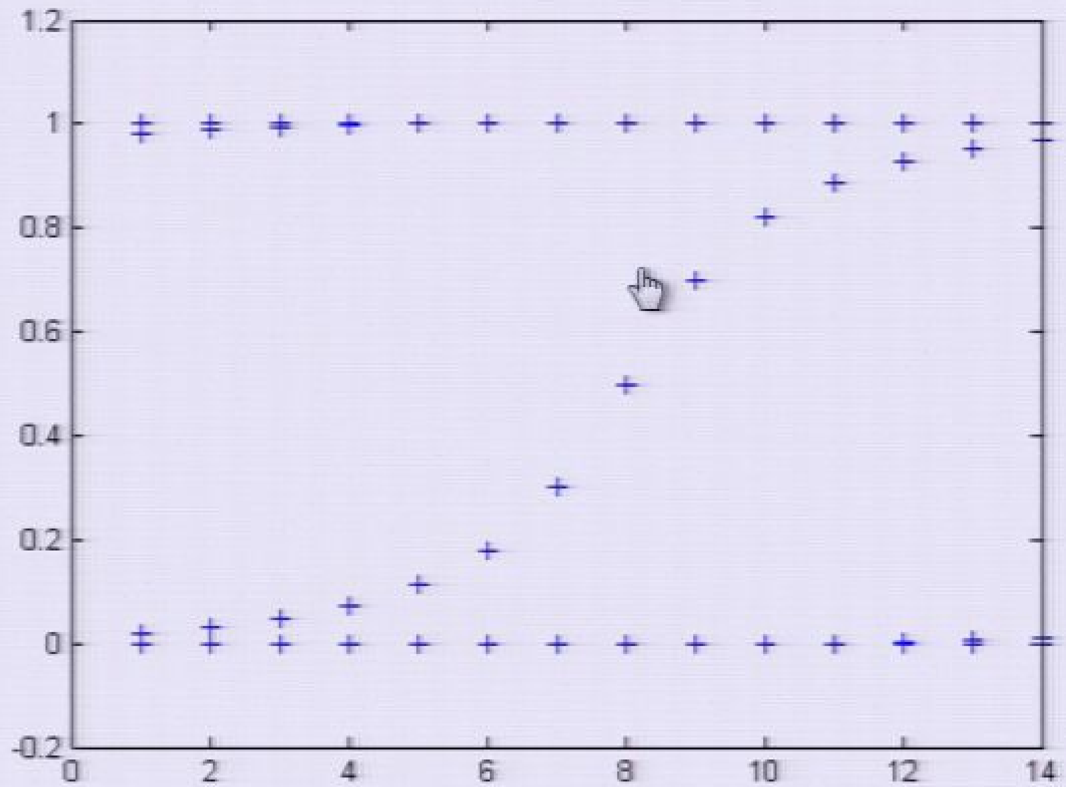


View of Phase Transitions From the Entanglement Spectrum

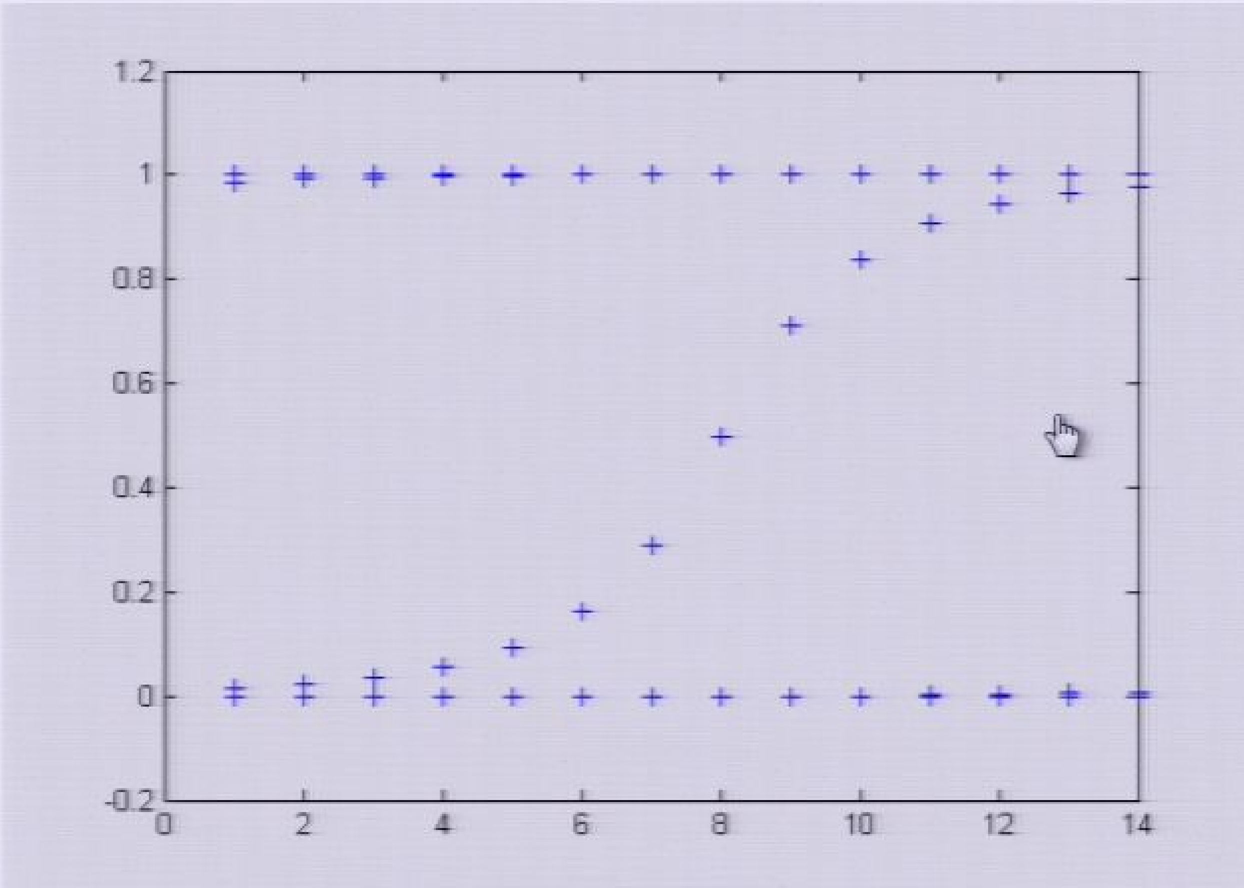
- For the topological insulator model (Dirac model) there is a phase transition when M switches sign



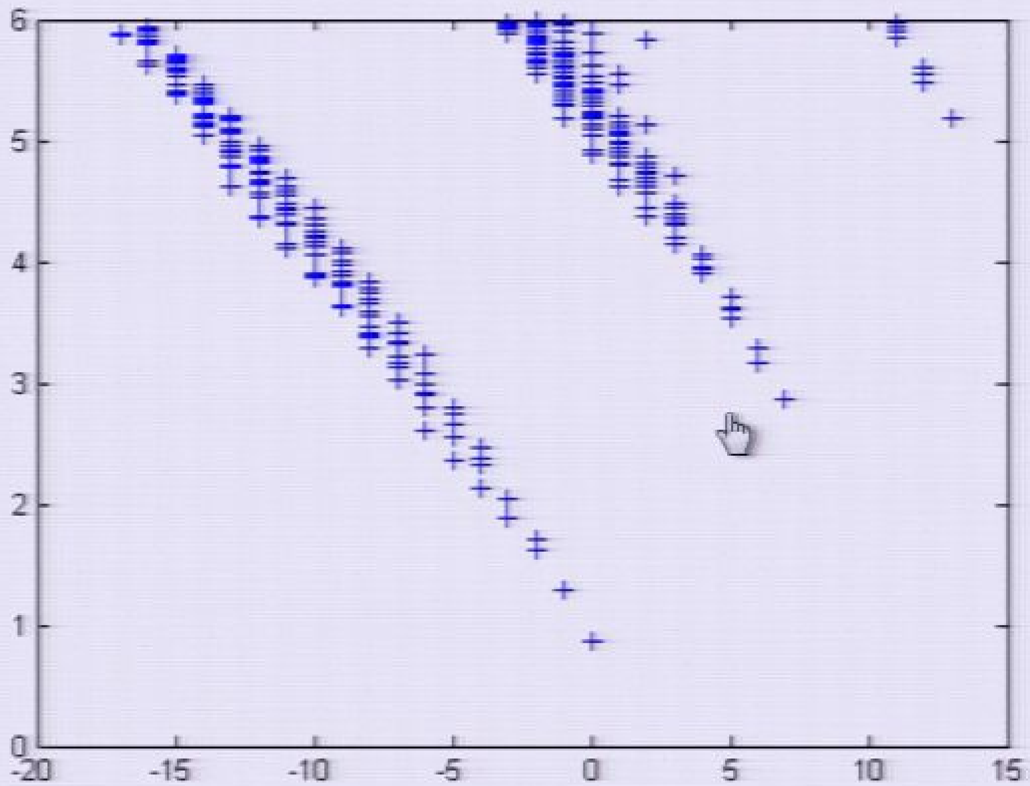
View of Phase Transitions From the Entanglement Spectrum



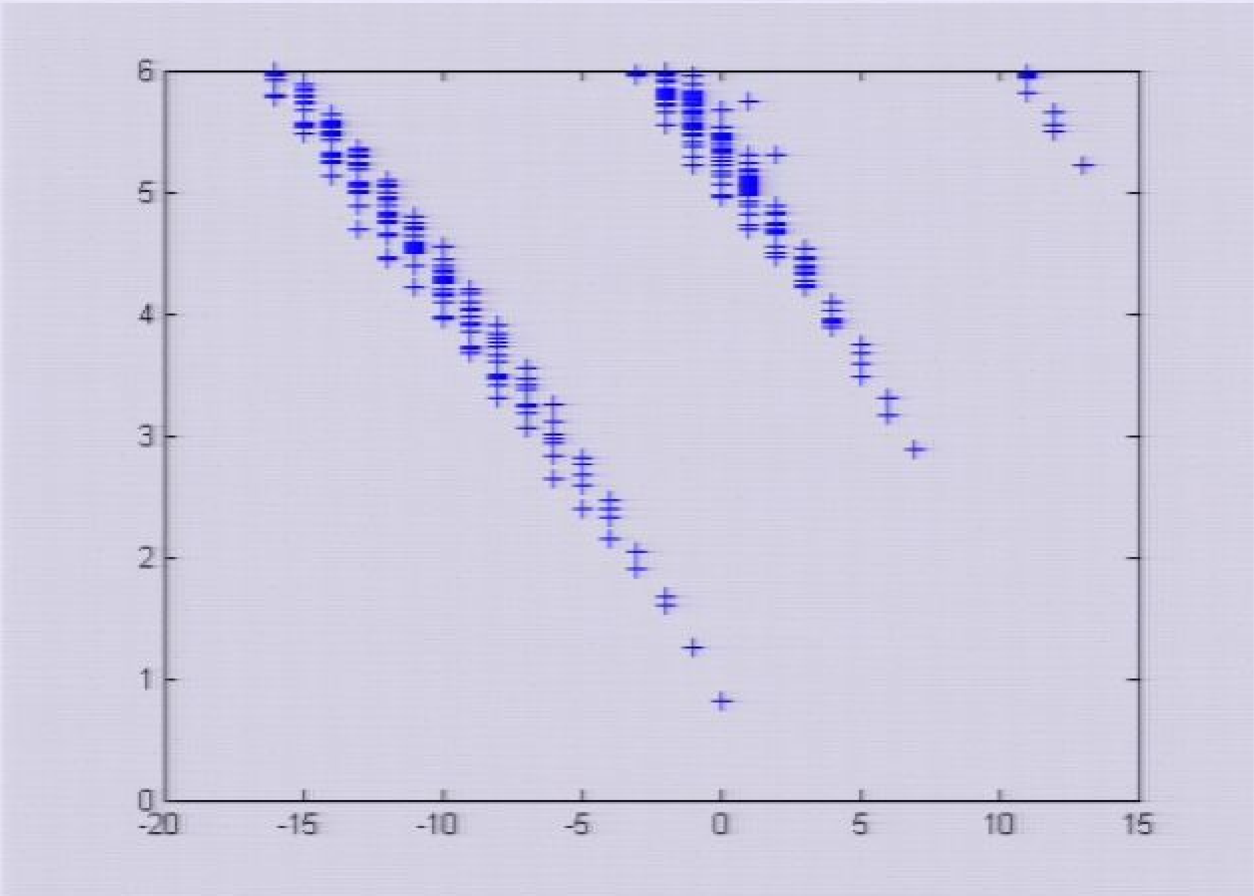
View of Phase Transitions From the Entanglement Spectrum



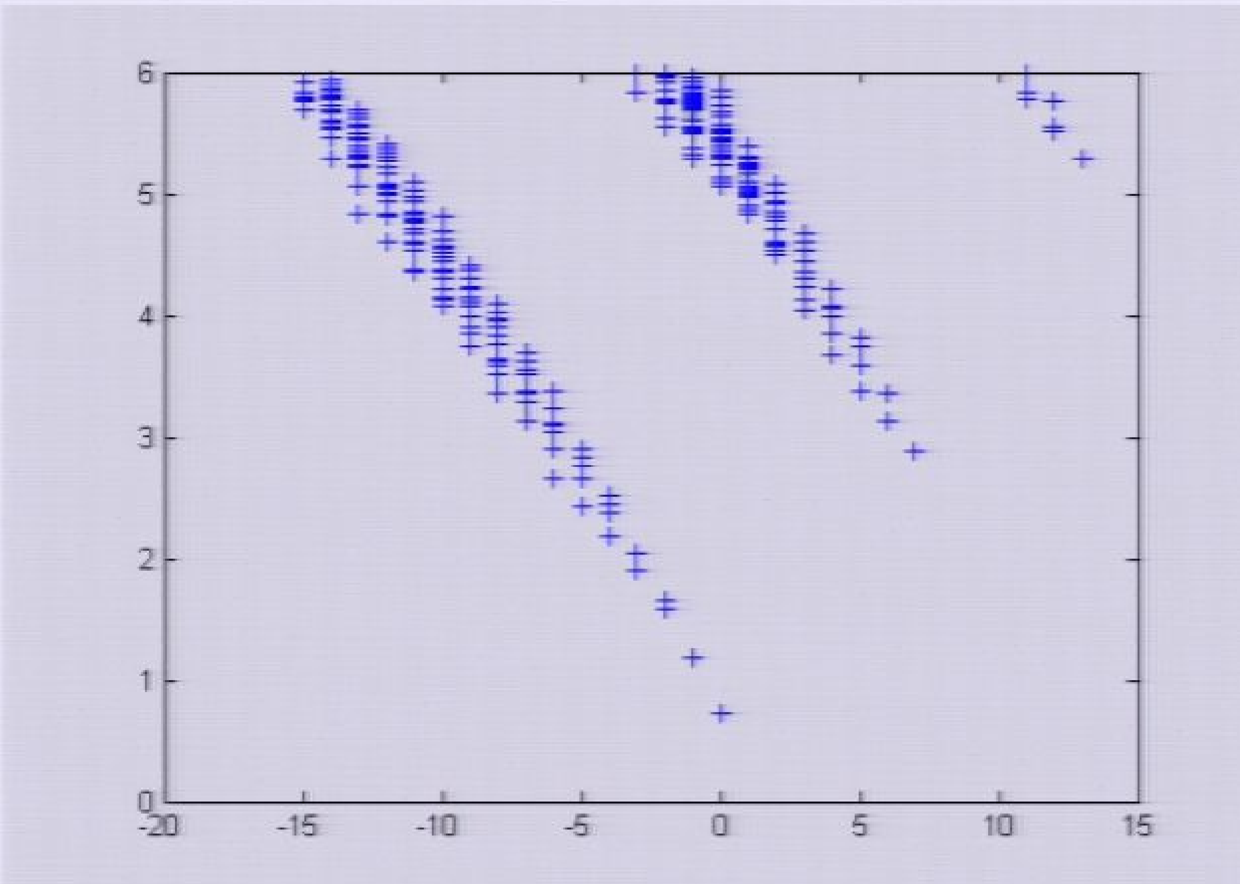
View of Phase Transitions From the Entanglement Spectrum



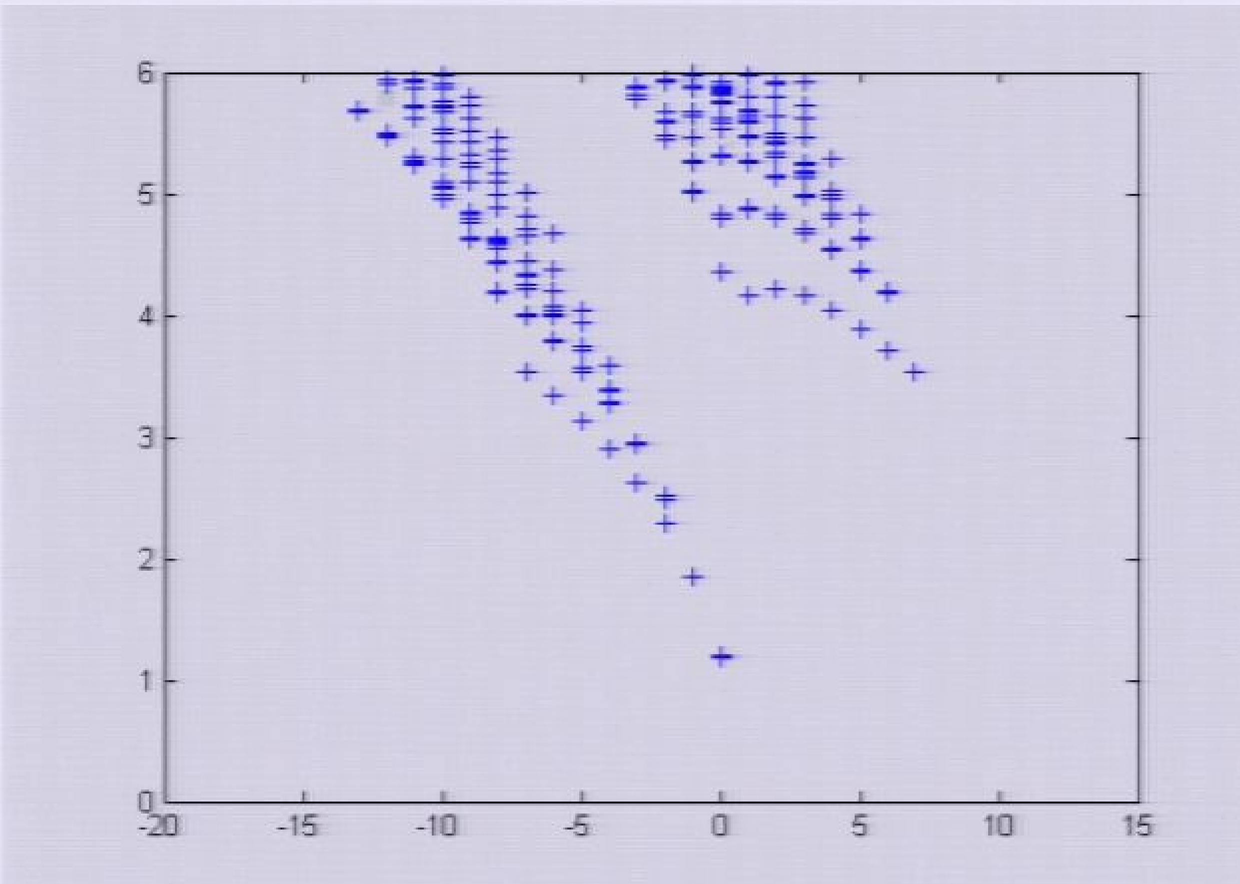
View of Phase Transitions From the Entanglement Spectrum



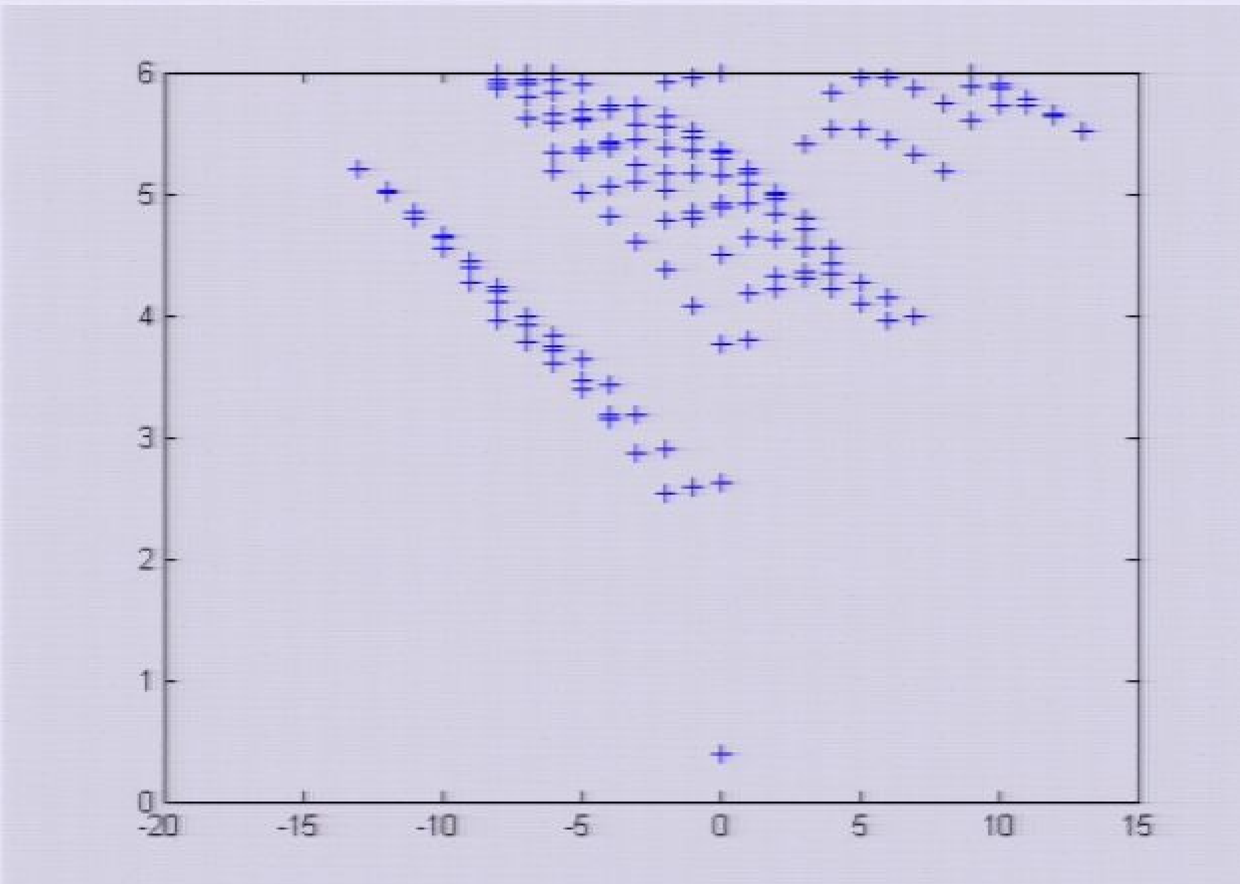
View of Phase Transitions From the Entanglement Spectrum



View of Phase Transitions From the Entanglement Spectrum



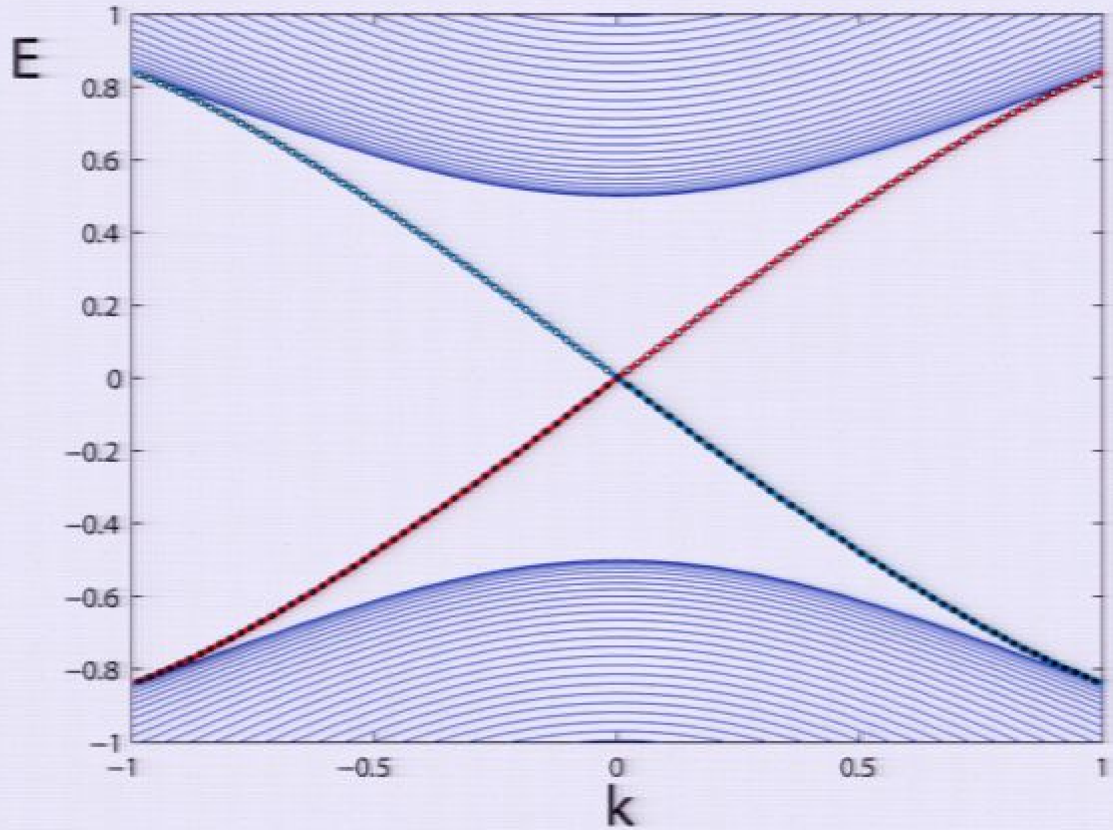
View of Phase Transitions From the Entanglement Spectrum



Energy Spectrum of QSH

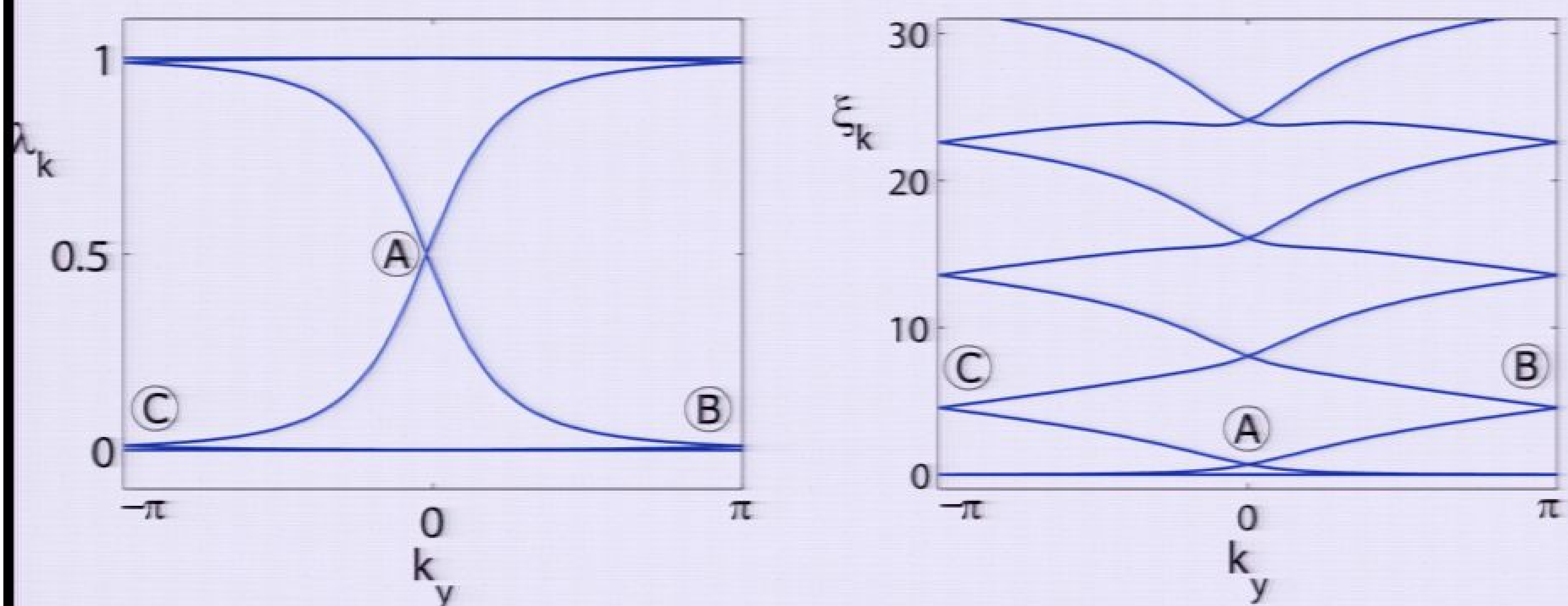
Solve for energy spectrum on a cylinder

Open BC's



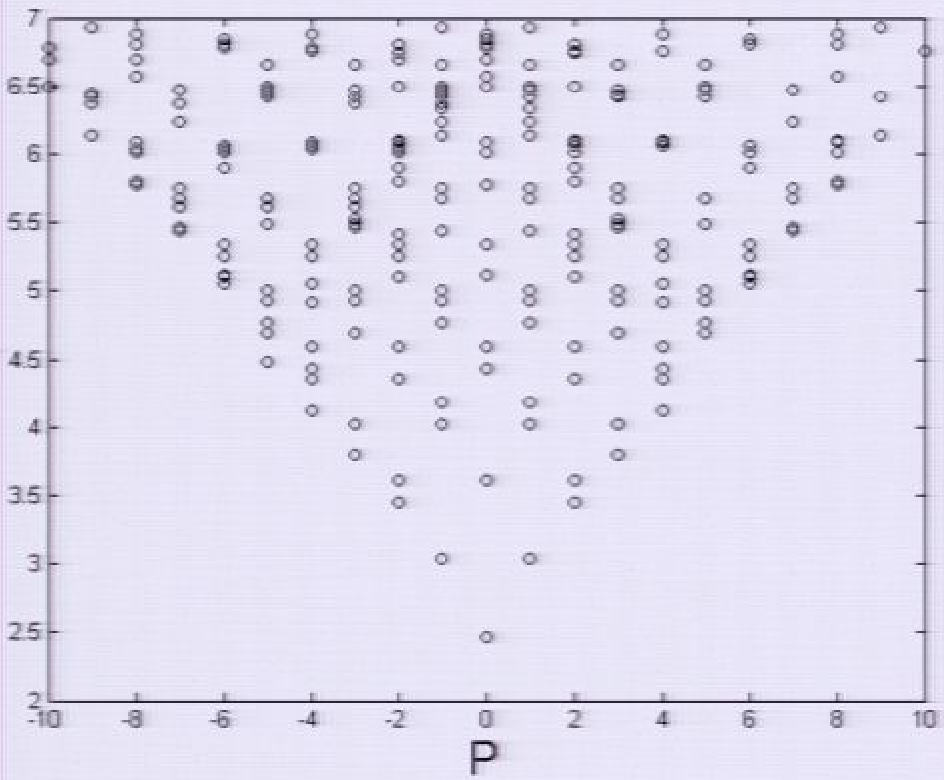
TRS protects the degeneracy

Entanglement Spectrum of QSH



This is for a cut on a cylinder, so there is only one pair of "cut" states. Note the clear partner switching spectral flow.

Many-body Entanglement Spectrum of QSH

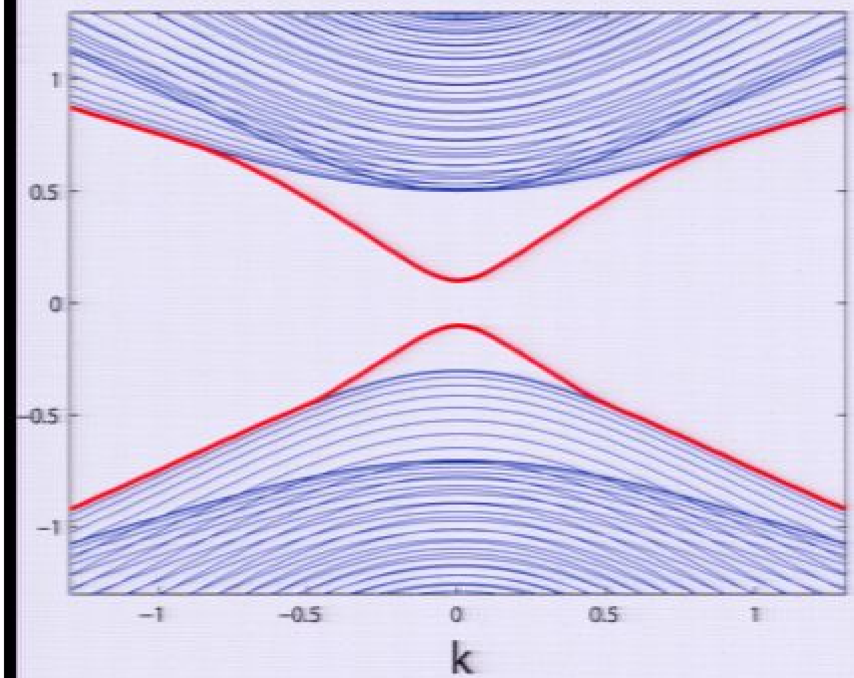


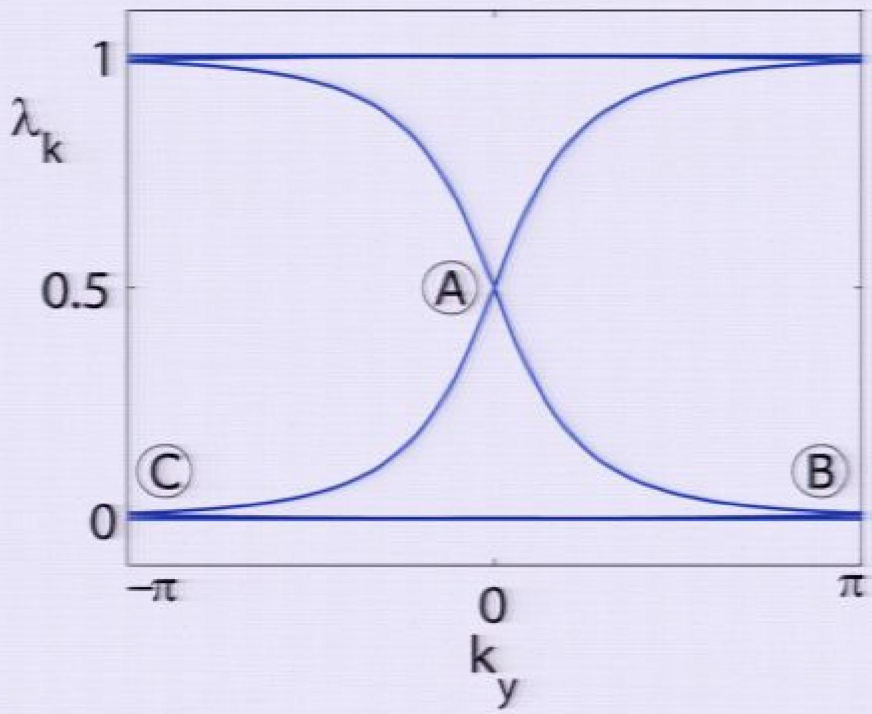
Counting is a similar to that of QH but more tedious because there are two branches of states.

Leads to many degeneracies. Entanglement ground state is doubly degenerate here because there are two cut-state "zero modes" and only one is initially filled. This double-degeneracy is a consequence of TRS.

Breaking Time-Reversal Symmetry

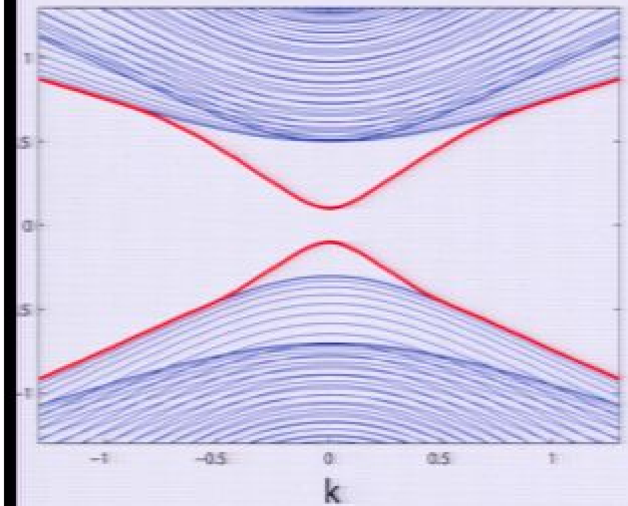
Z_2 Invariant no longer protected and a gap opens in the edge states

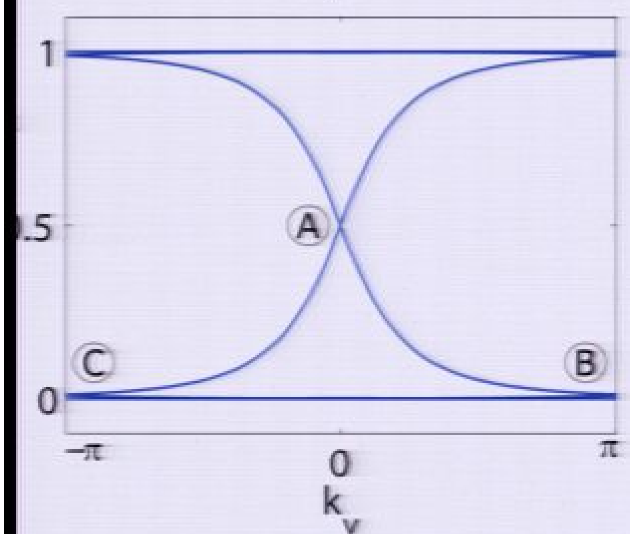


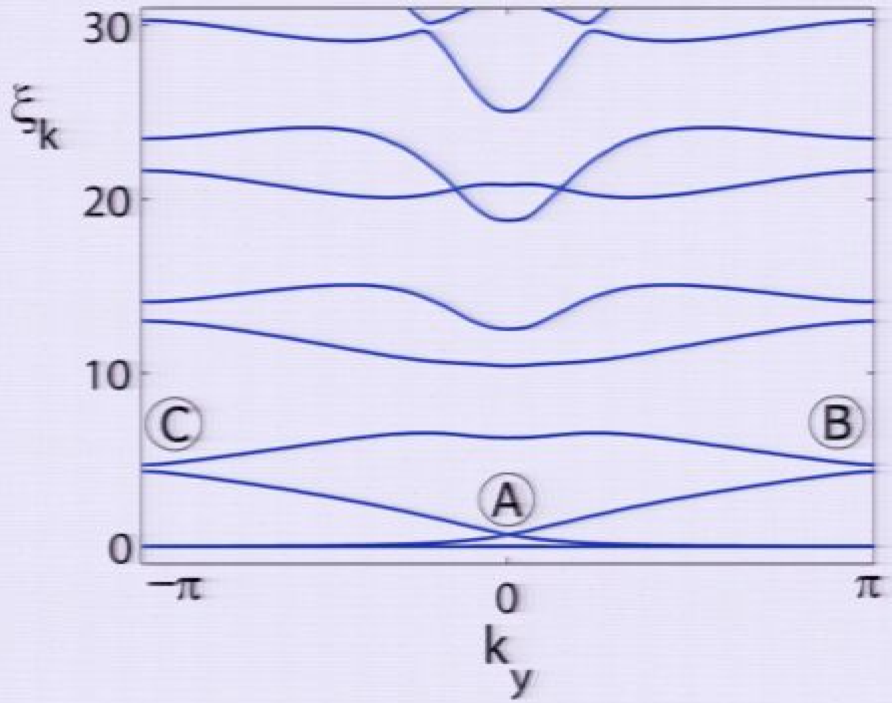


What is going on in the entanglment spectrum? It still looks gapless. The point is that the eigenvalues very close to 0 and 1 are telling you something and are important though they contribute little to the entropy.

Breaking Time-Reversal Symmetry







We clearly see here the signature of the TRS breaking, the spectral flow is cut-off. Every degeneracy protected by TRS is lifted *except* one.

Part 2: Entanglement Spectra in Disordered Topological Insulators

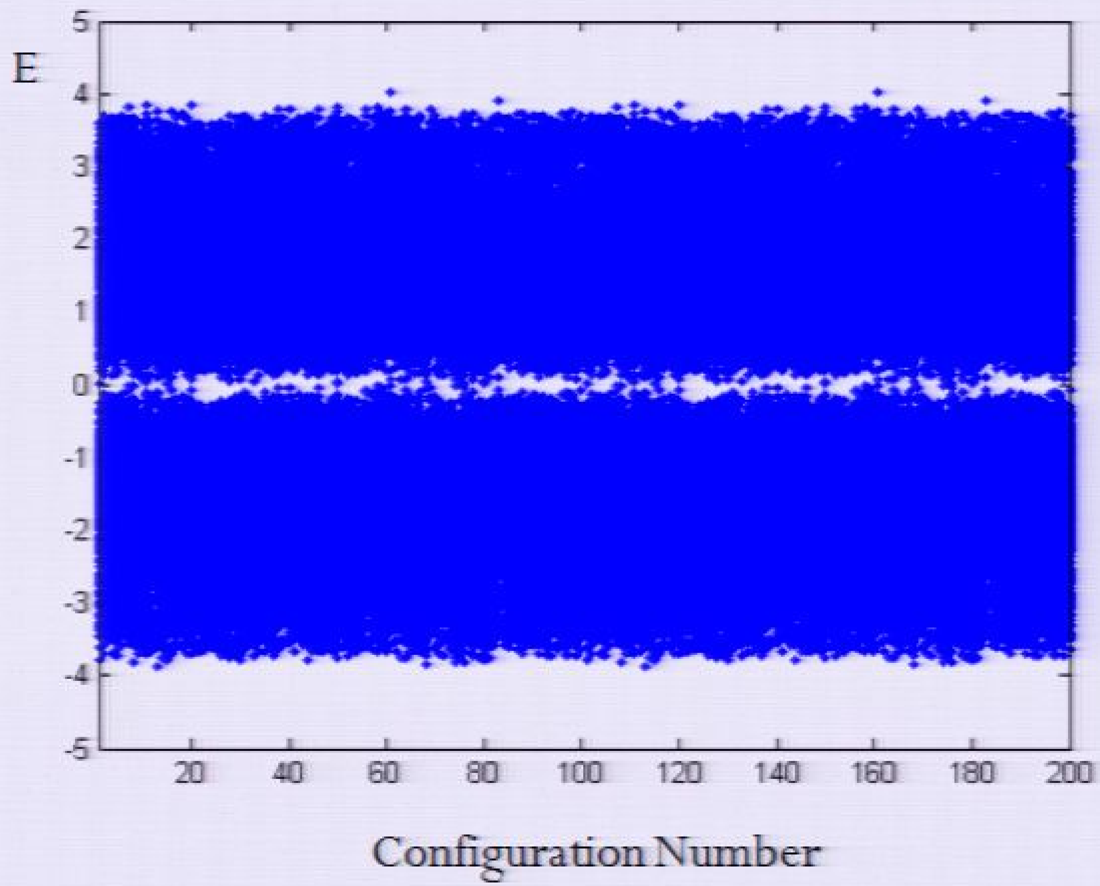
Anderson-Localization Transition

- In 2d, even if an electron is not classically localized, quantum interference effects due to the presence of disorder can localize the electronic state.
- Thus, while classically you might expect the material to be metallic, quantum effects can drive it into an insulating phase
- Given a fixed disorder strength, and a particular energy, one can ask a yes or no question: "Are the electronic states localized?"

Characterizing the Anderson Transition

- For solvable Hamiltonians (e.g. free fermions) it is not too difficult

olve for the energy spectrum for
nany independent random disorder
onfigurations.

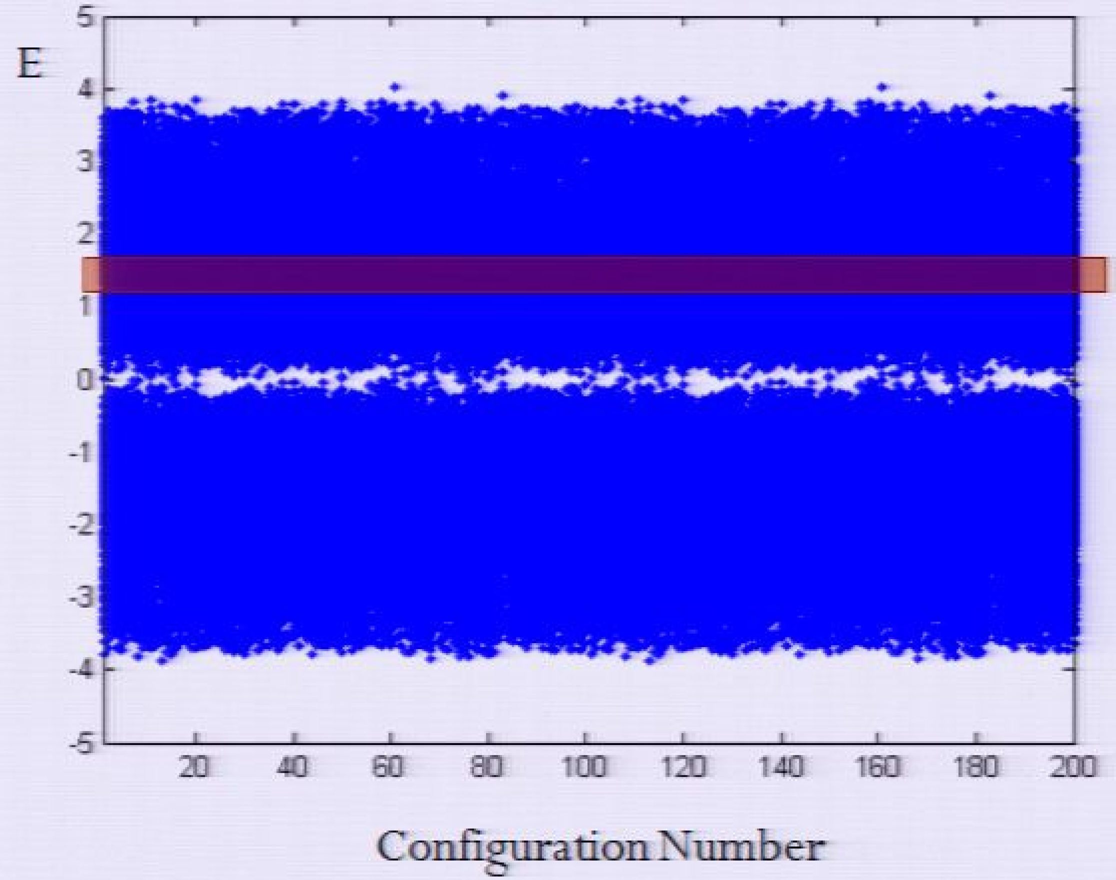


Characterizing the Anderson Transition

- For solvable Hamiltonians (e.g. free fermions) it is not too difficult

olve for the energy spectrum for nany independent random disorder onfigurations.

Calculate distribution of energy level pacings in an energy window



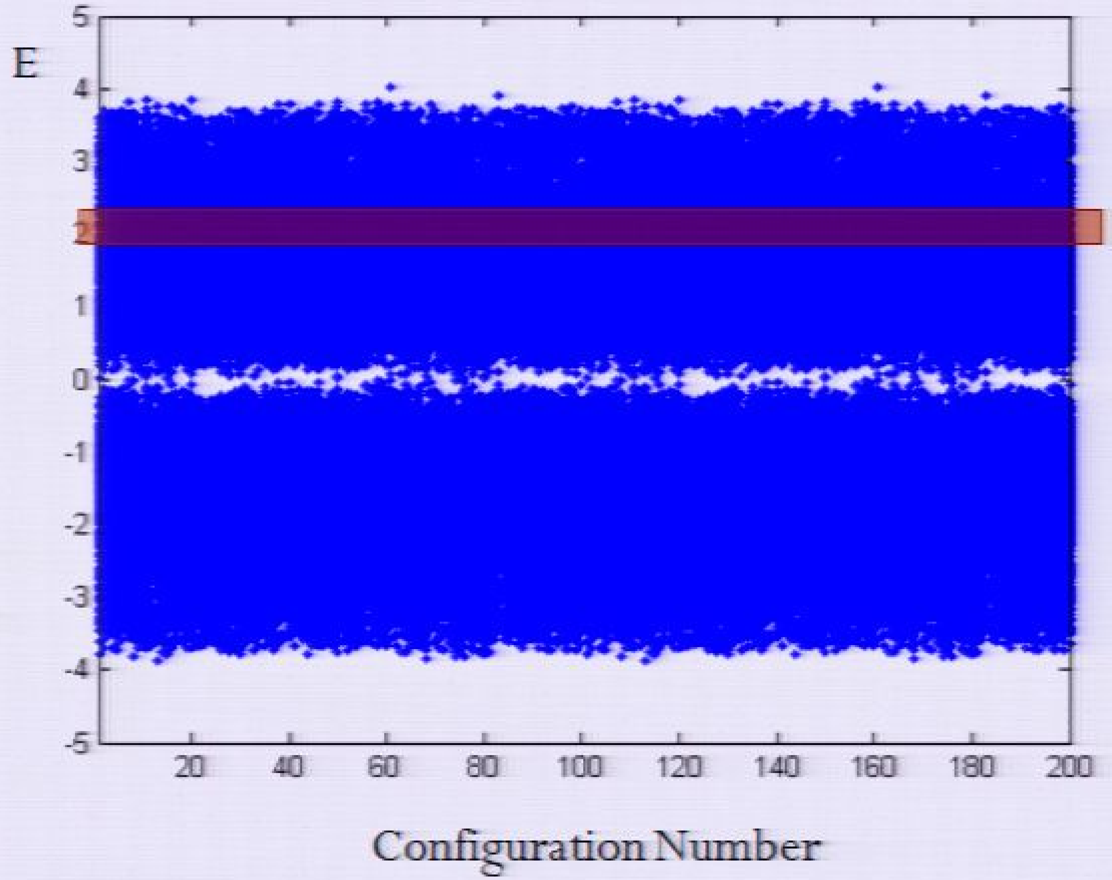
Characterizing the Anderson Transition

- For solvable Hamiltonians (e.g. free fermions) it is not too difficult

olve for the energy spectrum for nany independent random disorder onfigurations.

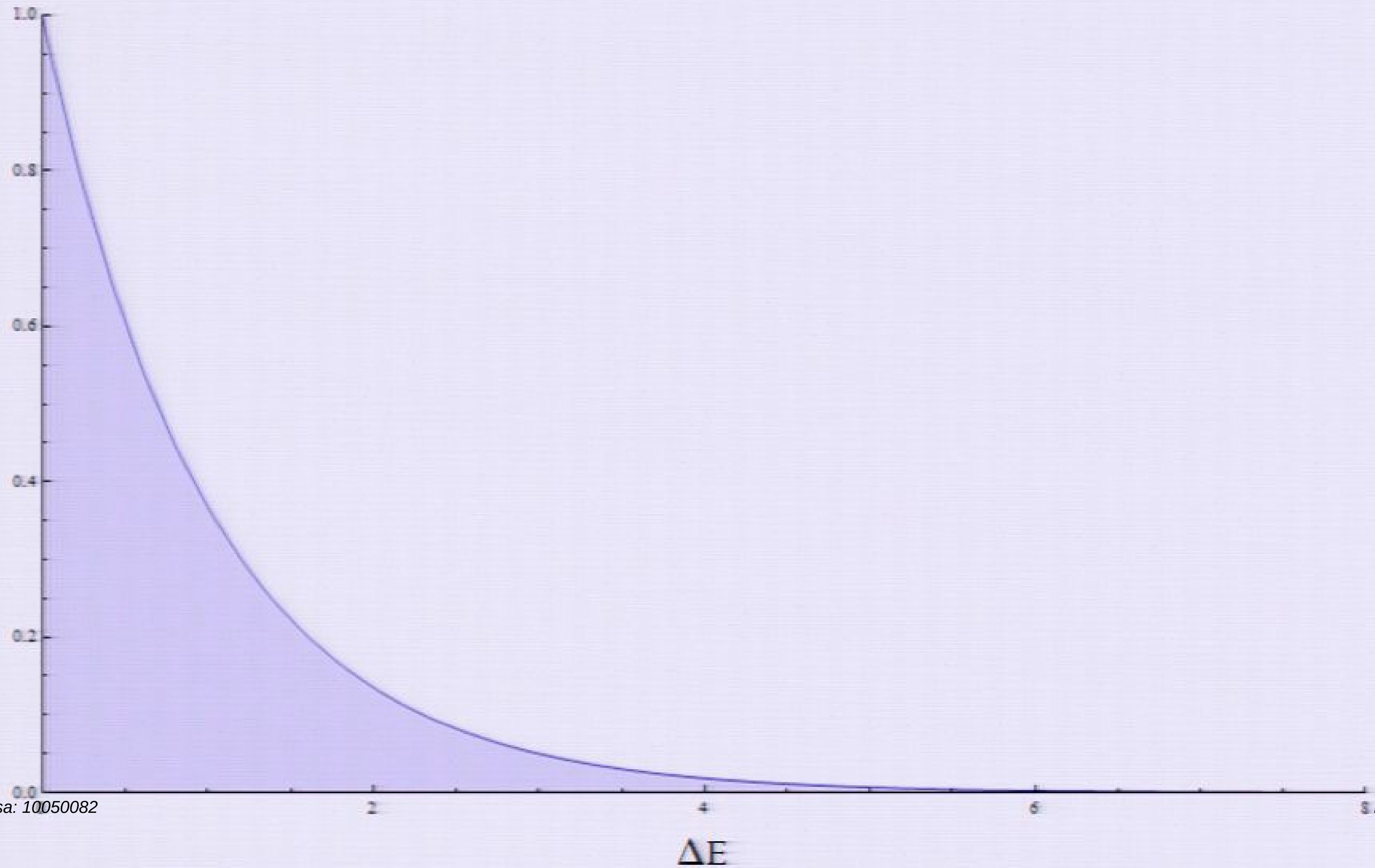
Calculate distribution of energy level pacings in an energy window

Repeat this process for different energy window locations and different disorder trengths.



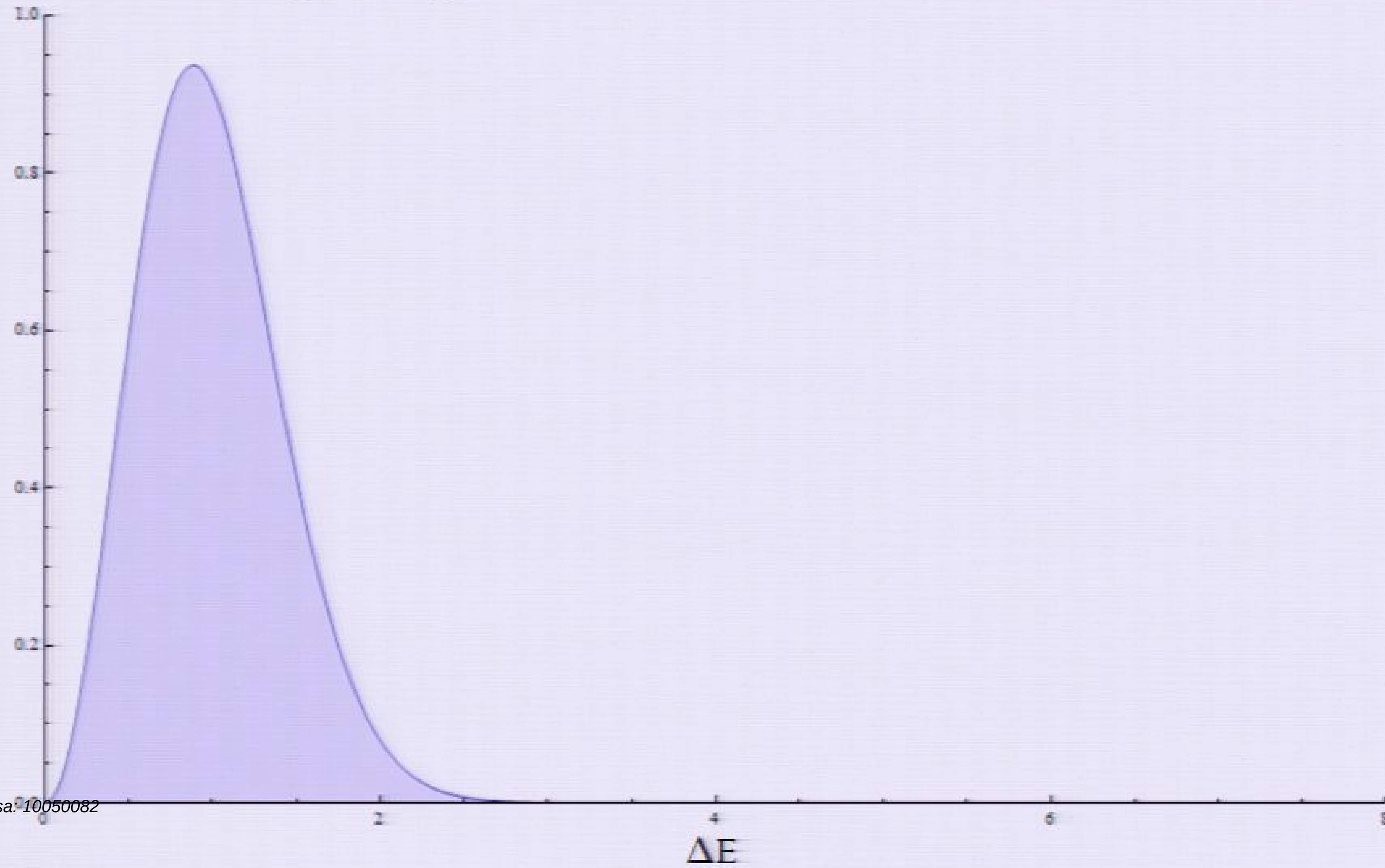
Energy Level Distributions

Localized states give a Poissonian-type Distribution (variance = 1):

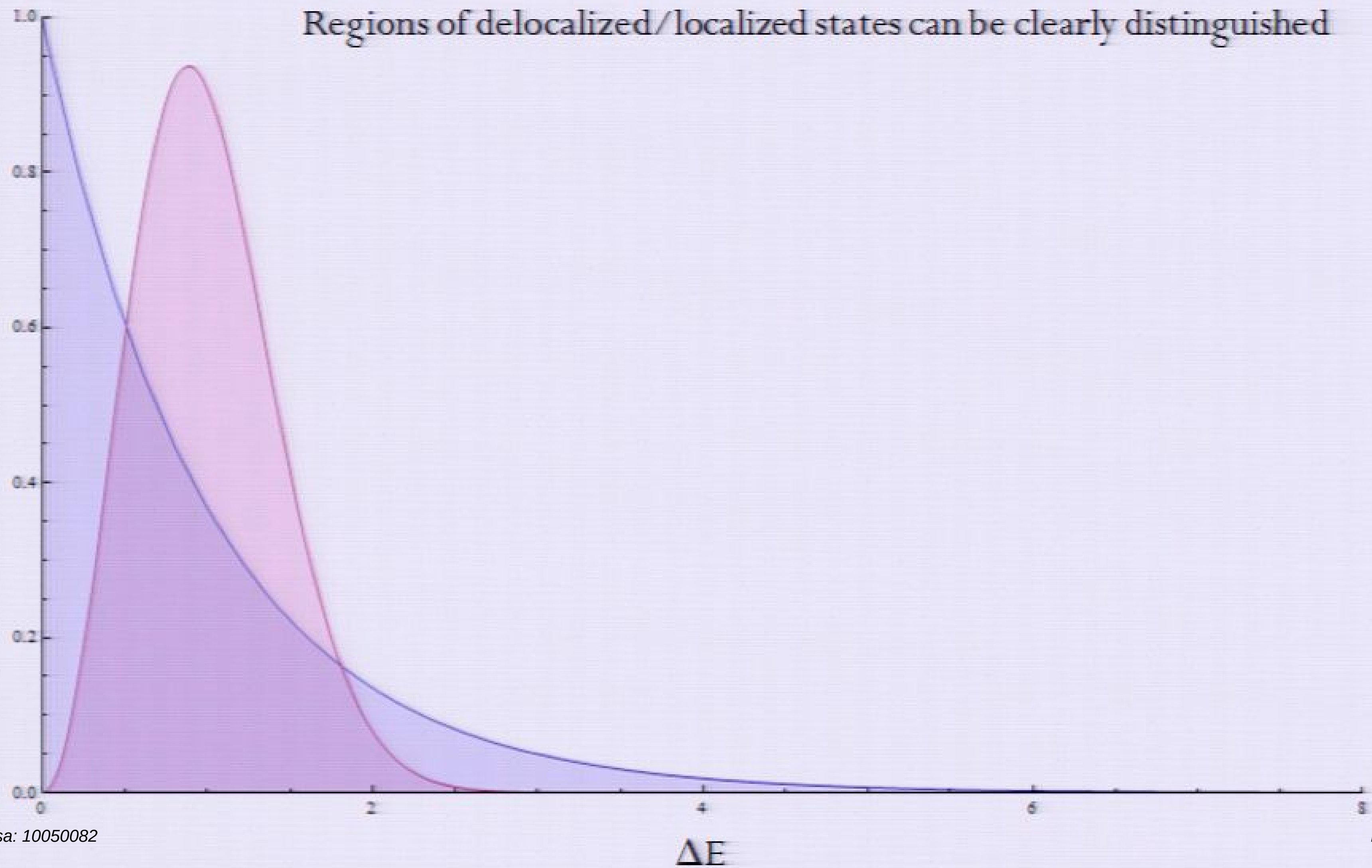


Energy Level Distributions

De-localized states give a Wigner-Surmise distribution for GUE (variance ~ 0.178):



Energy Level Distributions

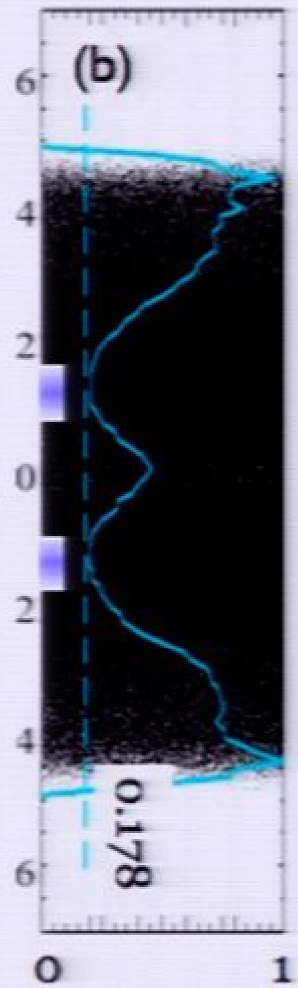


Disordered Chern Insulator

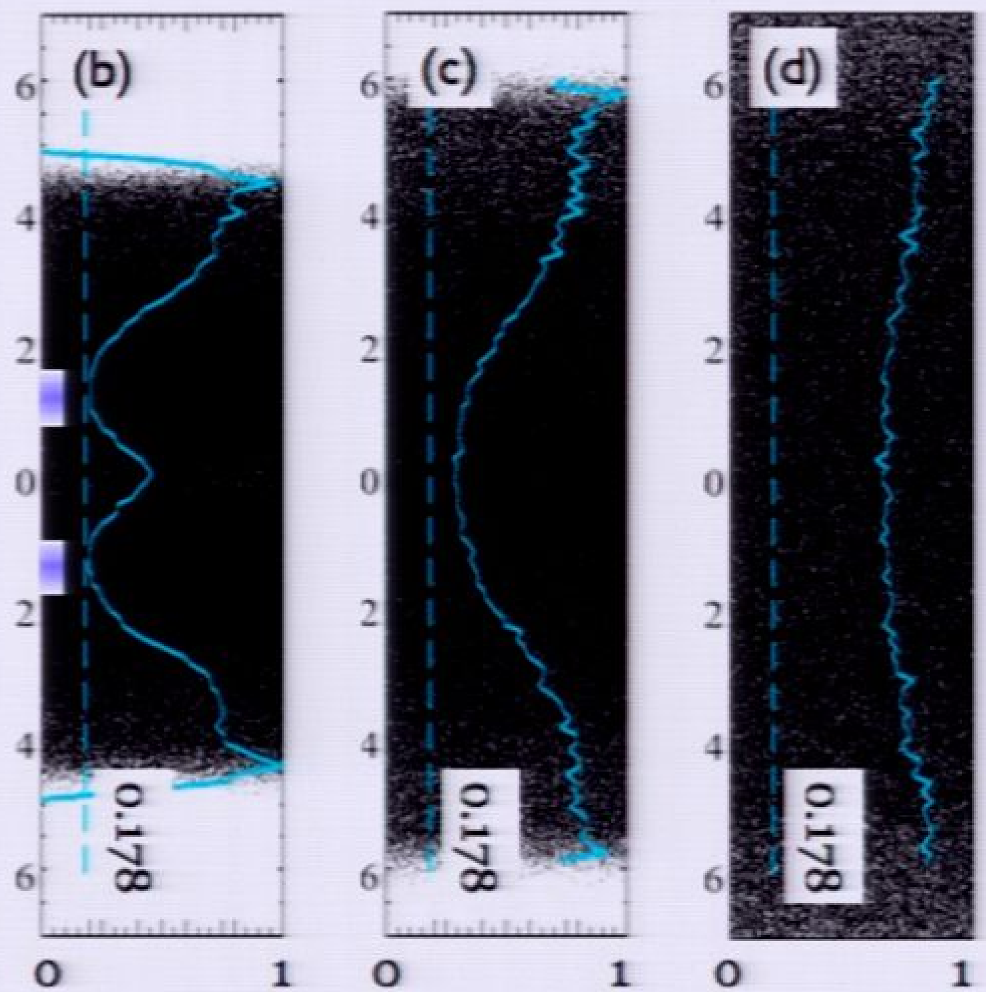
$H = H_0 + W(x, y)$

- Energy Spectra for white noise disorder (uniformly distributed)

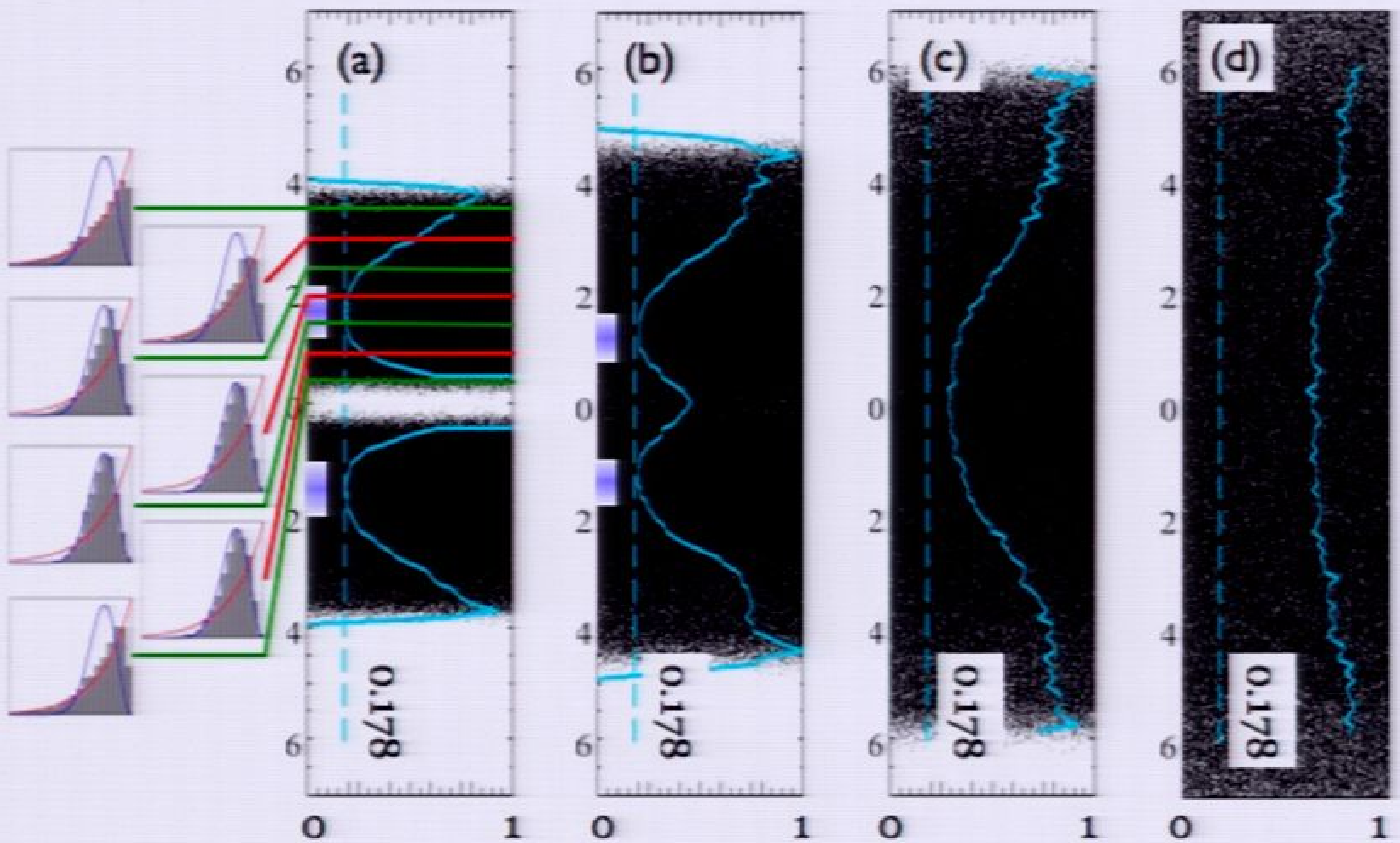
Disordered Chern Insulator



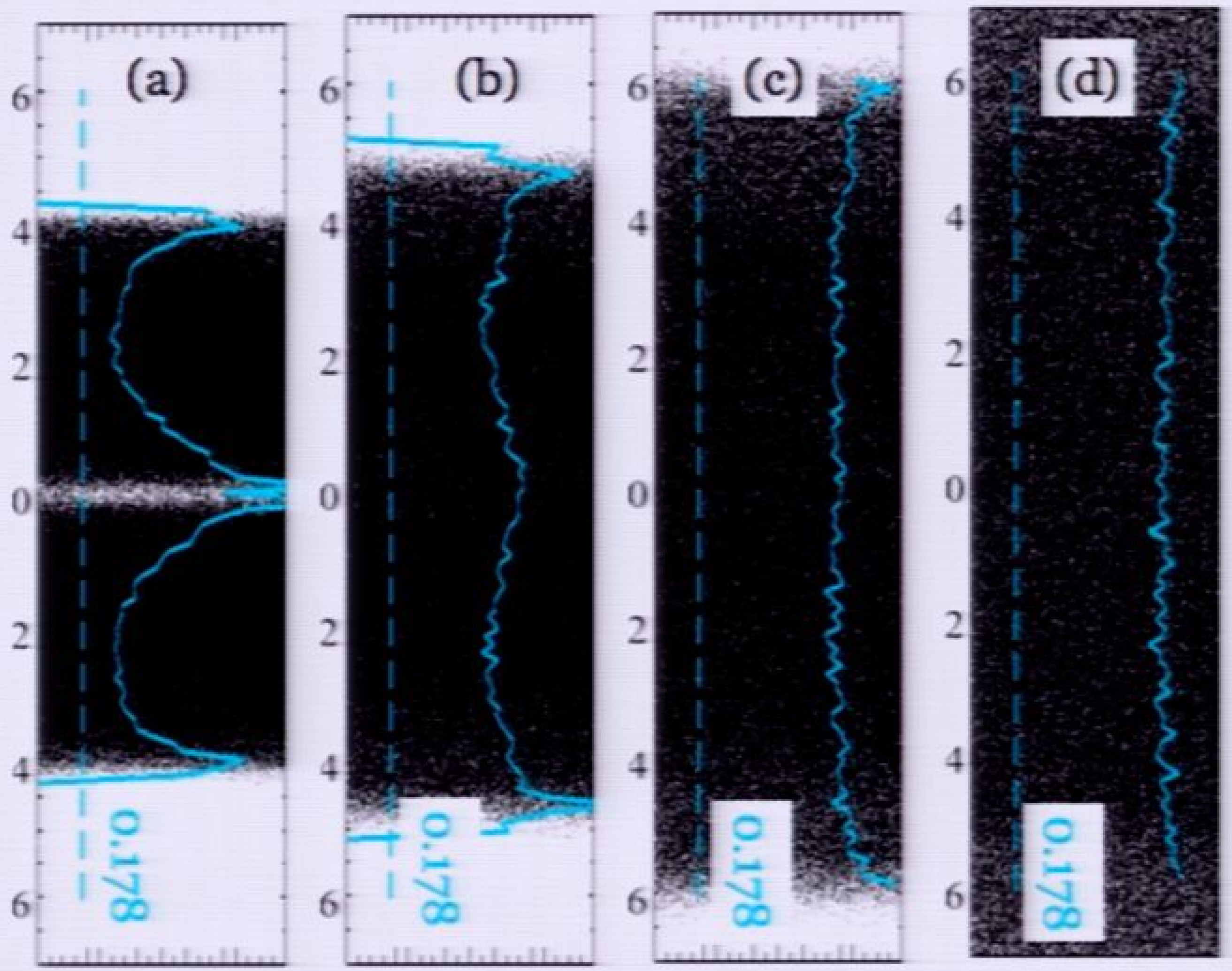
Disordered Chern Insulator

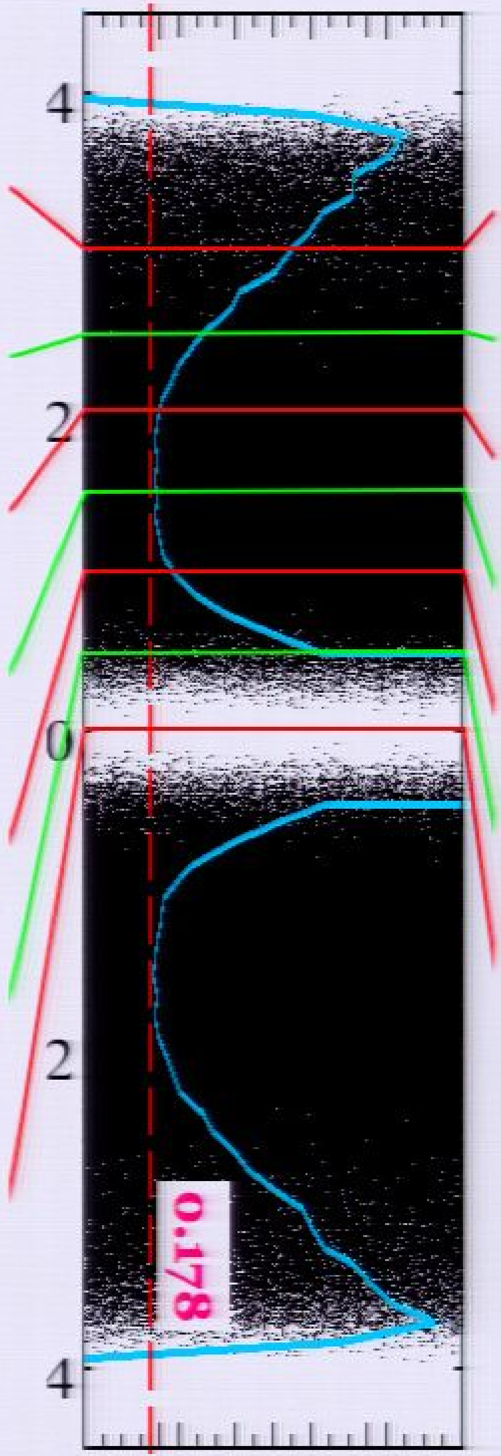


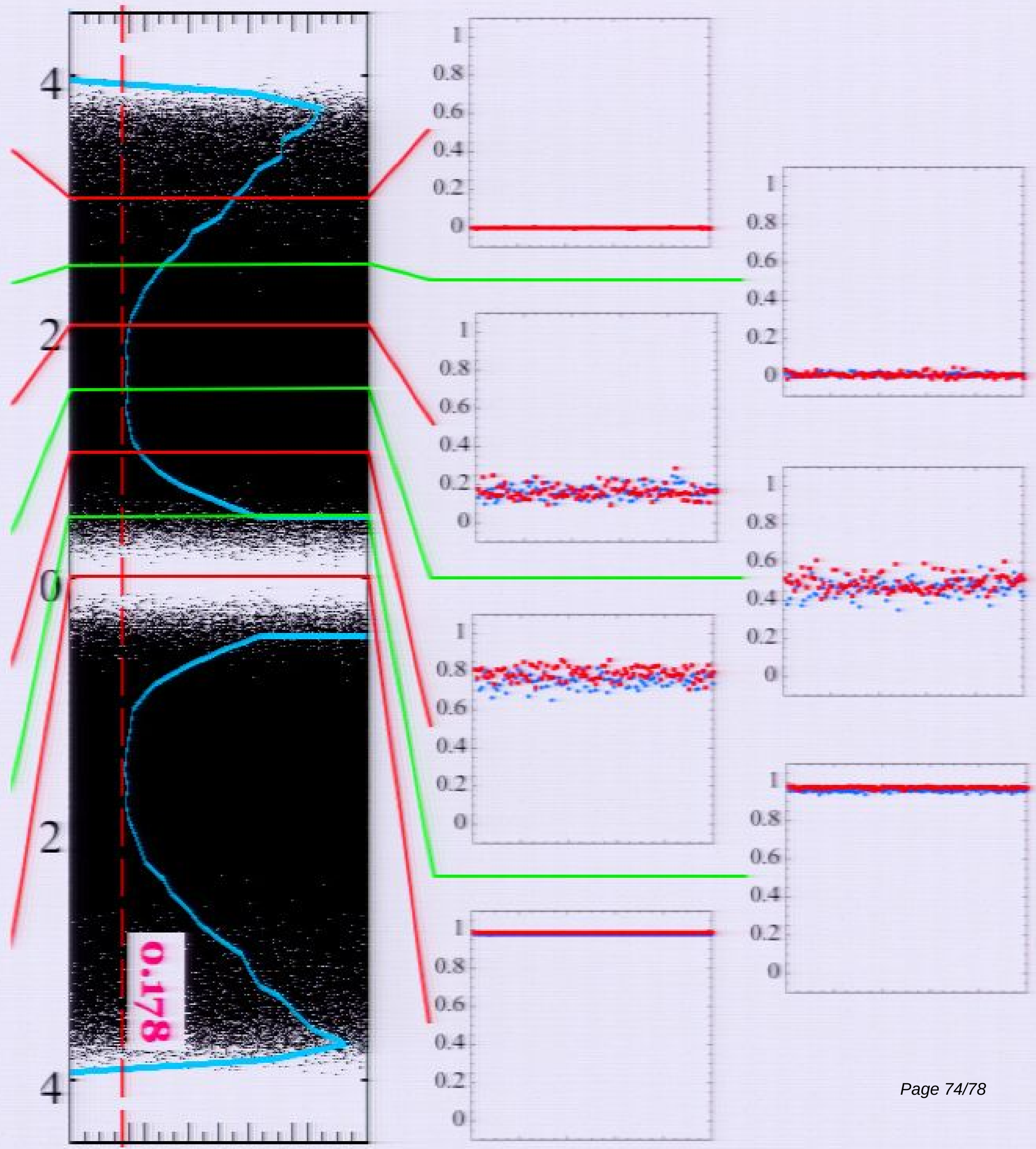
Disordered Chern Insulator



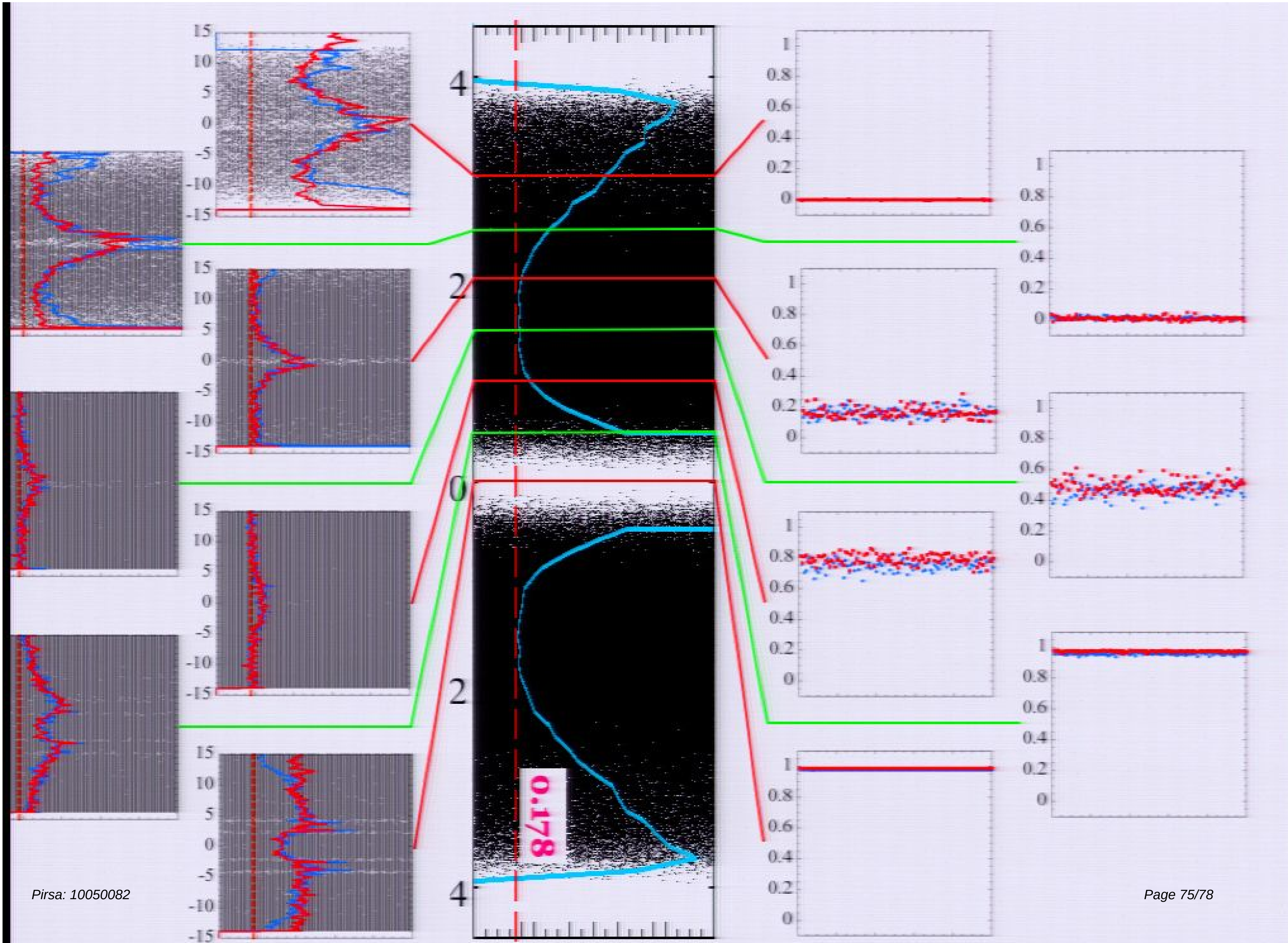
Disordered Trivial Insulator



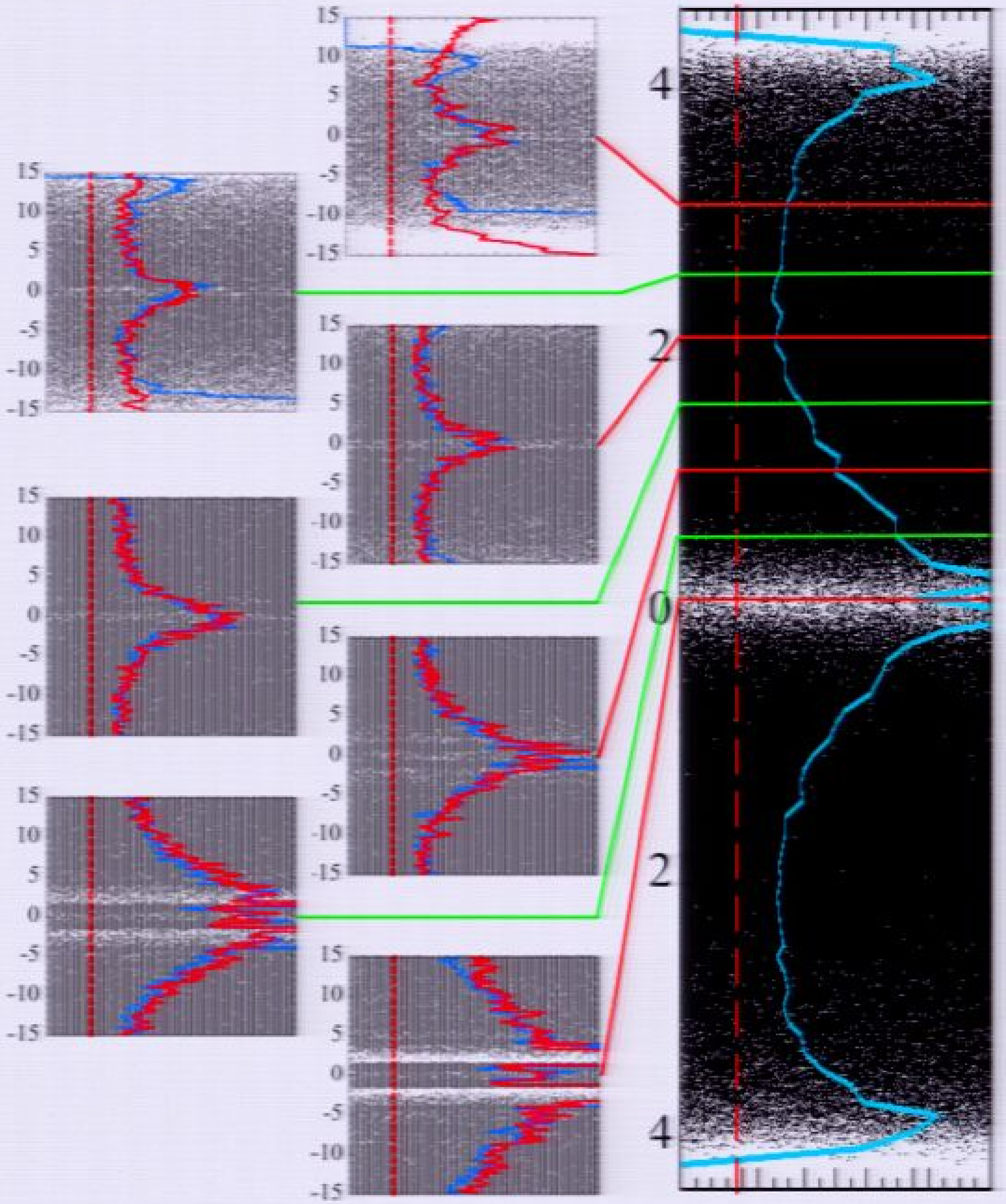

4
2
0
2
4
0.178



4
2
0
2
4
0.178
1
0.8
0.6
0.4
0.2
0



Entanglement of Trivial Insulator



Conclusions

- Using topological insulators as solvable test cases we see the entanglement spectrum exhibits many features that help to characterize the ground-state.
- The entanglement spectrum may serve as a tool to attack the many-body localization problem since we only need the ground state and not the entire excitation spectrum.

Conclusions

- Using topological insulators as solvable test cases we see the entanglement spectrum exhibits many features that help to characterize the ground state.
- The entanglement spectrum may serve as a tool to attack the many-body localization problem since we only need the ground state and not the entire excitation spectrum.

Click to add notes