Title: Emergent gauge theories in Quantum spin Hall superconductor Josephson arrays and cold atom system

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Abstract: We study a superconductor-ferromagnet-superconductor (SC-FM-SC) Josephson junction array deposited on top of a two-dimensional quantum spin Hall (QSH) insulator. The existence of Majorana bound states at the interface between SC and FM gives rise to charge-e tunneling, in addition to the usual charge-2e Cooper pair tunneling, between neighboring superconductor islands. Moreover, because Majorana fermions encode the information of charge number parity, an exact Z_2 gauge structure naturally emerges and leads to many new insulating phases, including a deconfined phase where electrons fractionalize into charge-e bosons and topological defects. We will also discuss the ultracold alkaline earth atoms trapped in optical lattice, which naturally has a SU(N) spin symmetry with N as large as 10. An SU(2)xU(1) gauge theory emerges in a large part of the phase diagram of this system.

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Cenke Xu

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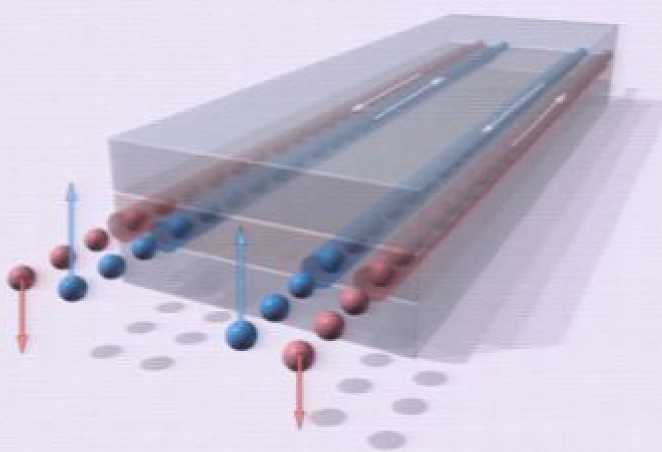

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Outline:

1, Introduction,

Quantum Spin Hall effect, edge states, theoretical model, Experimental realization, Jackiw-bound state

2, QSH-FM-SC Josephson array in 1d

Majorana bound state, single charge tunneling, map to Ising model and Z2 gauge field.

3, QSH-FM-SC Josephson array in 2d

Deconfinement of Z2 gauge field in 2d

4, Discussions, extensions

Cenke Xu, Liang Fu

PRB 81, 134435 (2010)

Introduction 1:
What is Quantum Spin Hall effect?

One sentence summary: $\nu = 1$ QH for spin up, $\nu = -1$ QH for spin down.

One equation summary $J_i^z = \sigma \epsilon_{zij} E_j$

Property 1, time reversal invariant, no need to turn on magnetic field, only need topological band structure.

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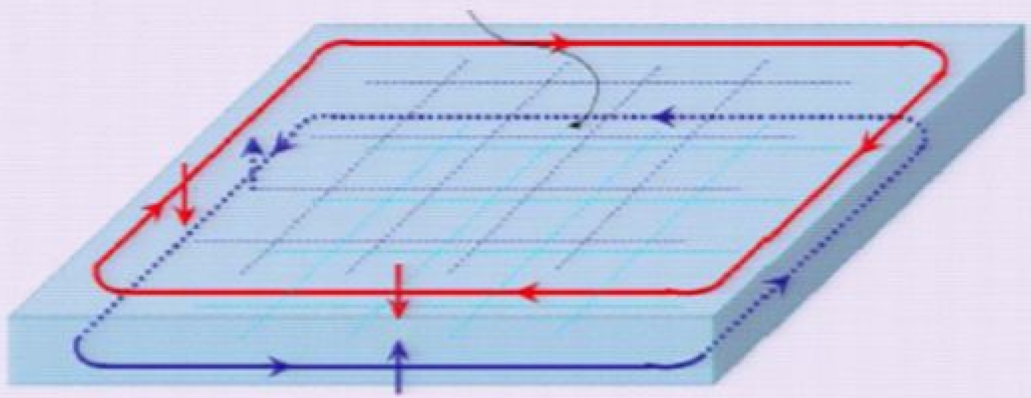
Property 2, helical edge states:



Counter-propagating spin-up and spin-down edge states

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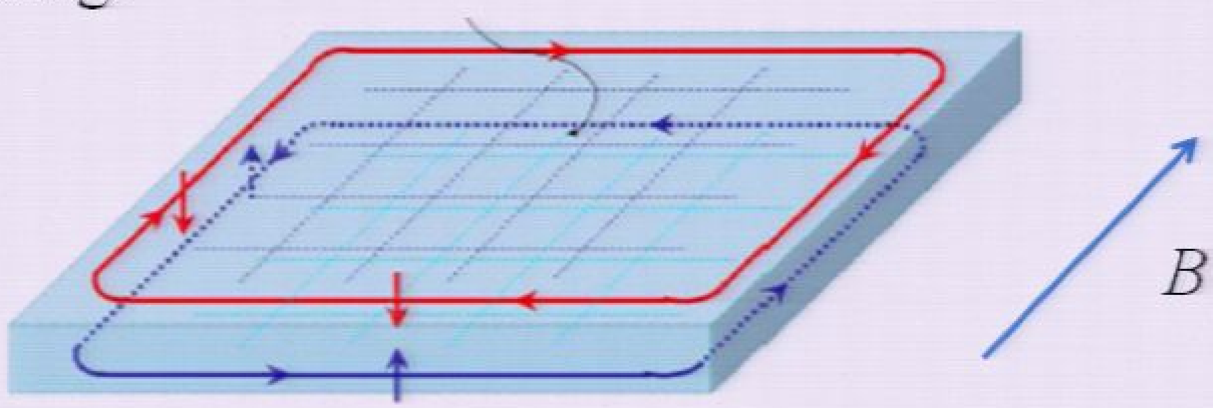
Property 3, is edge state as stable as QH?

Quantum Hall edge state is stable, because there is no way to back-scatter edge states.

Quantum spin Hall edge states, unstable if time-reversal symmetry is broken, for instance turning on transverse magnetic field, or magnetic impurity will induce back scattering.



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Property 3, is edge state as stable as QH?

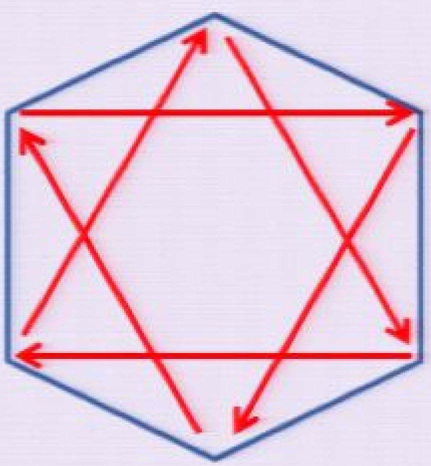
Quantum Hall edge state is stable, because there is no way to back-scatter edge states.

Quantum spin Hall edge states, if time-reversal symmetry is preserved, then single-particle back scattering is forbidden, but two-particle back scattering is allowed. So, less stable than quantum Hall state, but still OK.

Cenke Xu and Joel E. Moore, 2006
Congjun Wu *et. al.* 2006

Introduction 2:
Theoretical models:

From Haldane's model to Kane-Mele's model



$$H = \sum_{<j,k>} -tc_j^\dagger c_k + H.c. + \sum_{\ll j \to k \gg} t' i c_j^\dagger c_k + H.c.$$

Haldane, 1988

Introduction 2:
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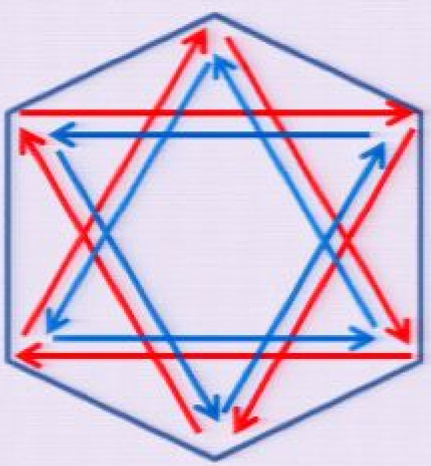
From Haldane's model to Kane-Mele's model



$$H = \sum_{<j,k>} -t c_j^\dagger c_k + H.c. + \sum_{\ll j \to k \gg} t' i c_j^\dagger c_k + H.c.$$

Haldane, 1988

$$H = \sum_{<j,k>} -t c_{j,\alpha}^\dagger c_{k,\alpha} + H.c. + \sum_{\ll j \to k \gg} t' i c_{j,\alpha}^\dagger S^z_{\alpha\beta} c_{k,\beta} + H.c.$$

Kane, Mele, 2005

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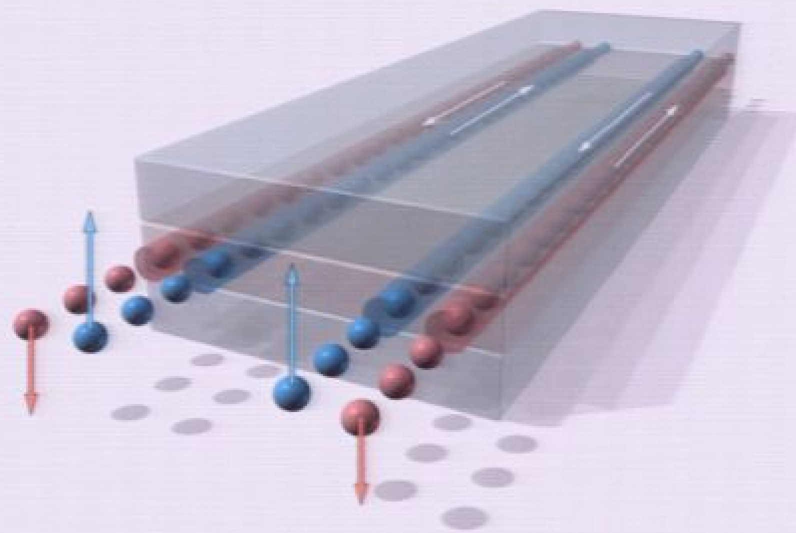
Introduction 3:
Experimental realization:

Graphene failed people, QSH realized in a totally different system: CdTe-HgTe-CdTe quantum well.



HgTe has spin-3/2 band (Γ_8 band) and spin-1/2 band (Γ_6 band). The $S^z = +3/2$ and $S^z = +1/2$ bands form $\nu = 1$ QH state, $S^z = -3/2$ and $S^z = -1/2$ bands form $\nu = -1$ QH state.

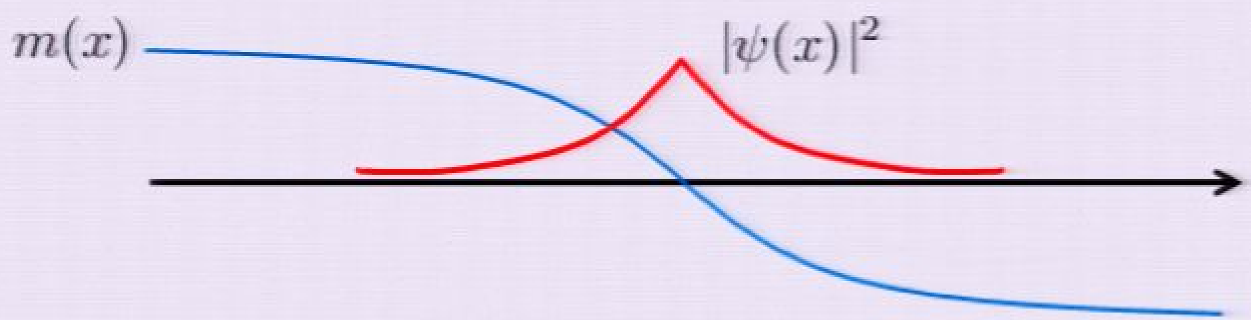
Bernevig, Hughes, Zhang, *Science, 314, 1757* (2006)

Koenig, *et.al.* 2008

Introduction 4:
Jackiw bound state:

1d Dirac fermion, a fermion zero mode localized at the mass domain wall.

$$H = \sigma^z p_x + m(x)\sigma^x$$



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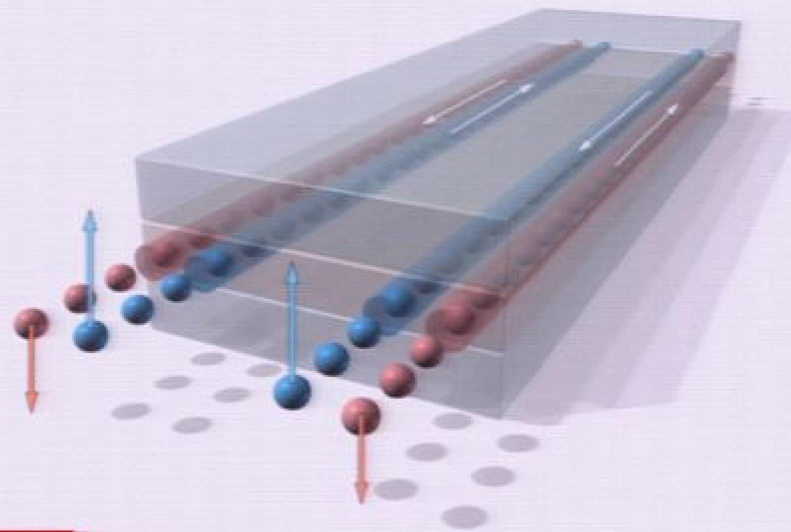
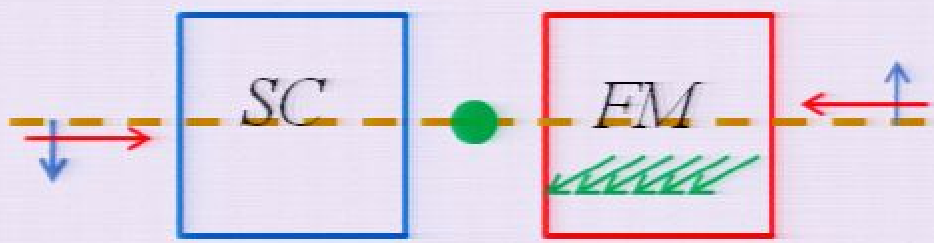
QSH-FM-SC Junction:



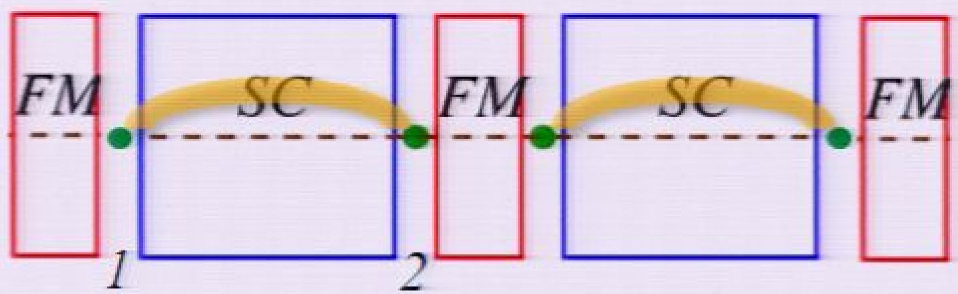
Take the edge states of the QSHI,
i.e. 1d helical fermi liquid:



Domain wall of the 1d Dirac mass gap localizes a Majorana fermion zero mode, Liang Fu and C. L. Kane, 2009

$$\gamma = e^{i\phi/2}(\alpha C_\uparrow + \beta C_\downarrow) + e^{-i\phi/2}(\alpha^* C_\uparrow^\dagger + \beta^* C_\downarrow^\dagger)$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



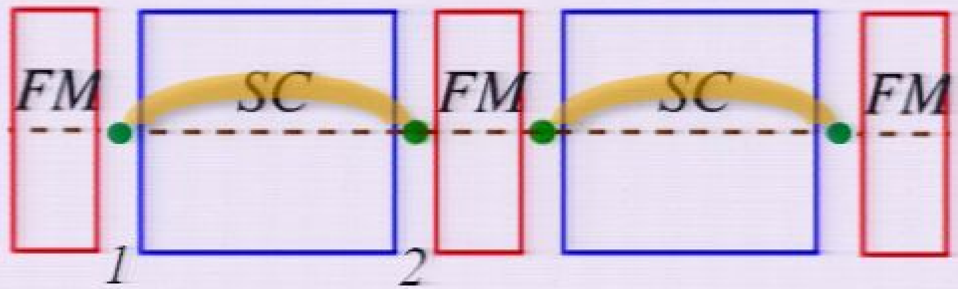
$C = \frac{\gamma_1 + i\gamma_2}{2}$ accommodates one fermion at zero energy

$2C^\dagger C - 1 = i\gamma_1\gamma_2 = \pm 1$ defines a qubit.

The fermion number on each SC island can be either even or odd, no cooper pair breaking energy cost.

The Majorana zero modes enable single charge tunneling, no longer have to tunnel cooper pair.

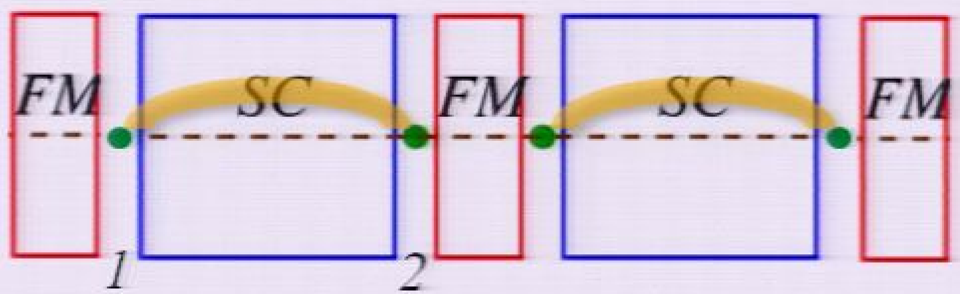
QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



Constraint: $i\gamma_{j,1}\gamma_{j,2} = (-1)^{n_j}$

$$-t_2 i\gamma_{j,2}\gamma_{j+x,1}\cos(\frac{\phi_j}{2} - \frac{\phi_{j+x}}{2})$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



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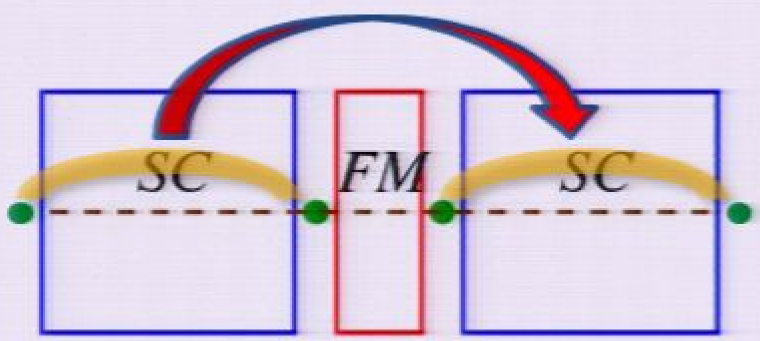

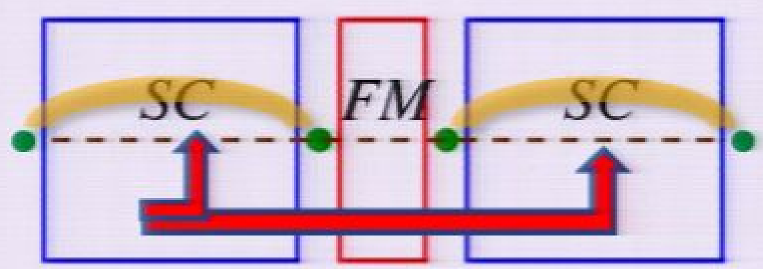

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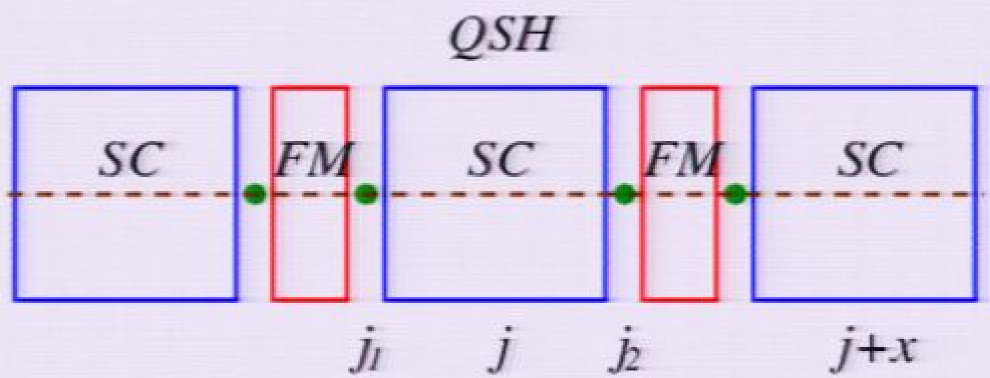
From junction to a lattice, 1d

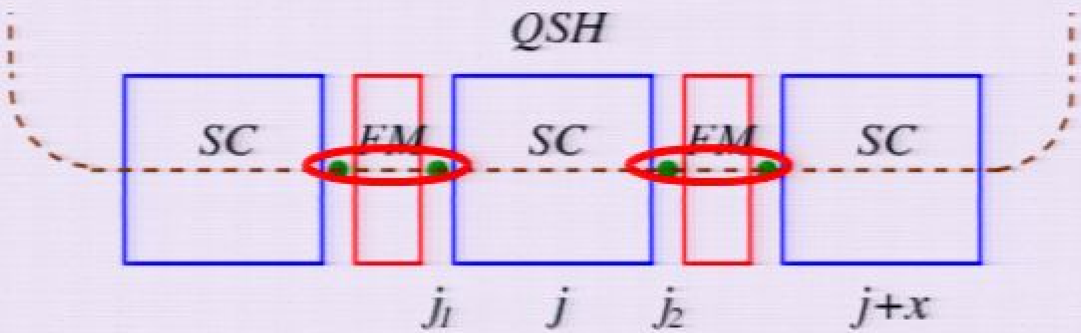


$$H = \sum_j -t_2 i\gamma_{j,2}\gamma_{j+x,1}\cos(\frac{\phi_j}{2} - \frac{\phi_{j+x}}{2}) + U(n_j - \bar{n})^2$$

$$i\gamma_{j,1}\gamma_{j,2} = (-1)^{n_j}$$

Ordinary Josephson array $\quad H = \sum_j -t_1\cos(\phi_j - \phi_{j+x}) + U(n_j - \bar{n})^2$

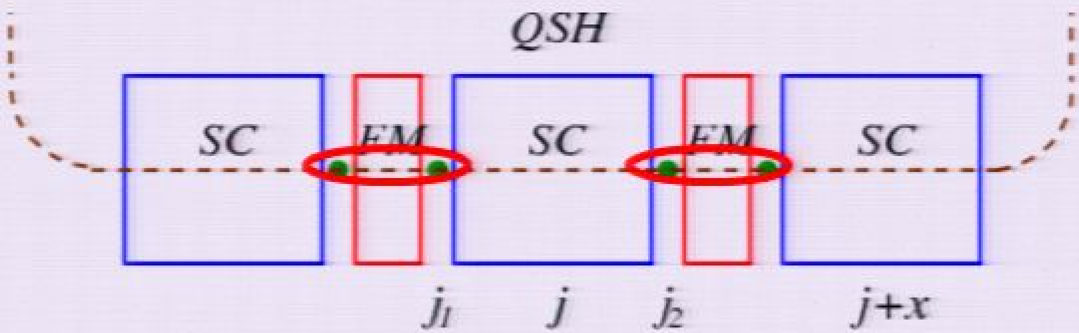
QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



$$H = \sum_j U(n_j - \bar{n})^2 - t_2 i\gamma_{j,2}\gamma_{j+x,1}\cos(\frac{\phi_j}{2} - \frac{\phi_{j+x}}{2})$$

Constraint $\quad i\gamma_{j,1}\gamma_{j,2} = (-1)^{n_j}$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

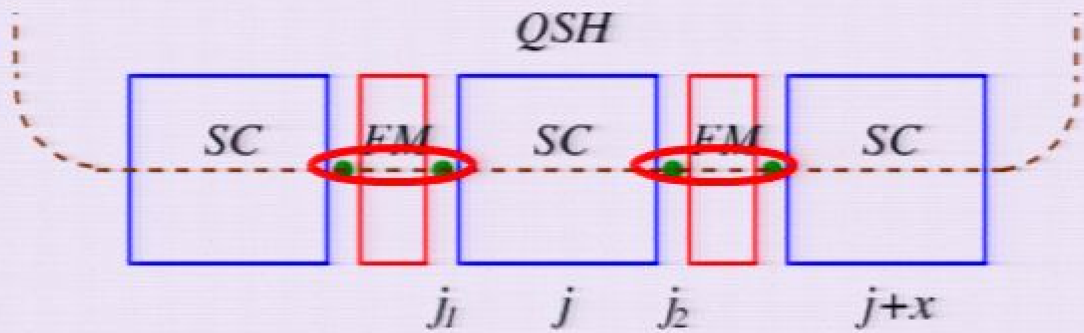


$$H = \sum_j U(n_j - \bar{n})^2 - t_2 i\gamma_{j,2}\gamma_{j+x,1}\cos\left(\frac{\phi_j}{2} - \frac{\phi_{j+x}}{2}\right)$$

Constraint $\quad i\gamma_{j,1}\gamma_{j,2} = (-1)^{n_j}$

$$\sigma^x_{j,j+x} = \prod_{k\leq j} i\gamma_{k,1}\gamma_{k,2}, \quad \sigma^z_{j,j+x} = i\gamma_{j,2}\gamma_{j+x,1}$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



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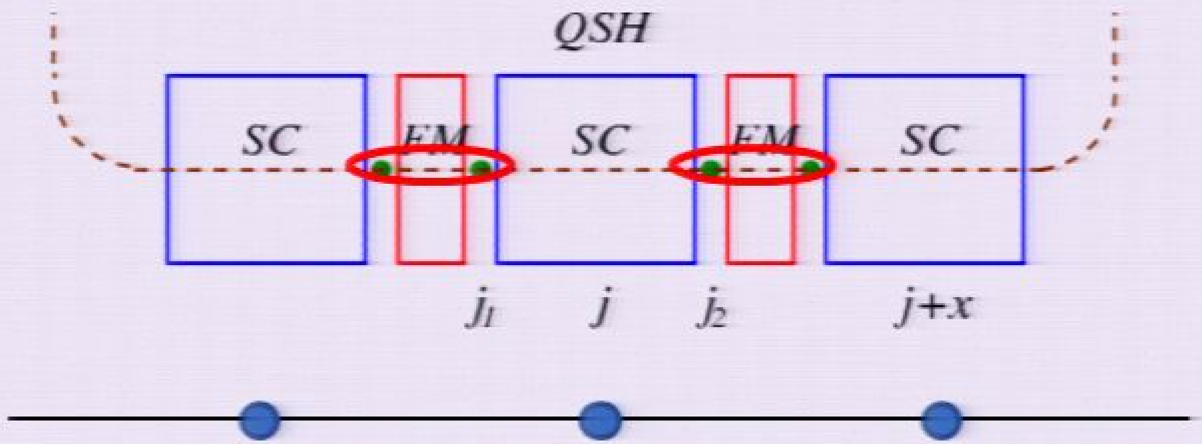
Constraint $\quad \sigma^x_{j-x,j}\sigma^x_{j,j+x}(-1)^{n_j} = 1$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field



$$H = \sum_j U(n_j - \bar{n})^2 - t_2 \sigma^z_{j,j+x} \cos\left(\frac{\phi_j}{2} - \frac{\phi_{j+x}}{2}\right)$$

$$\sigma^z_{j,j+x} = \tau^z_j \tau^z_{j+x} \qquad \theta_j = \frac{\phi_j}{2} + \pi \frac{1 - \tau^z_j}{2}$$

$$H = \sum_j U(n_j - \bar{n})^2 - t_2 \cos(\theta_j - \theta_{j+1})$$

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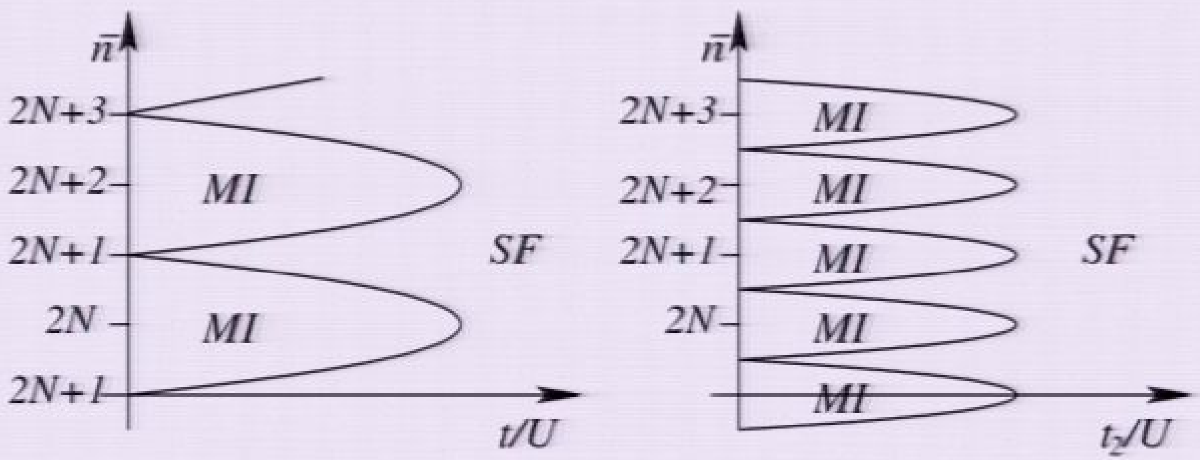
$$H = \sum_j U(n_j - \bar{n})^2 - t_2 \cos(\theta_j - \theta_{j+1})$$

Ordinary 1d charge-1 rotor model, by tuning t_2/U there is a KT transition between SF and MI at integer filling, all the scaling dimensions are known.



Charge-2 rotor Charge-1 rotor

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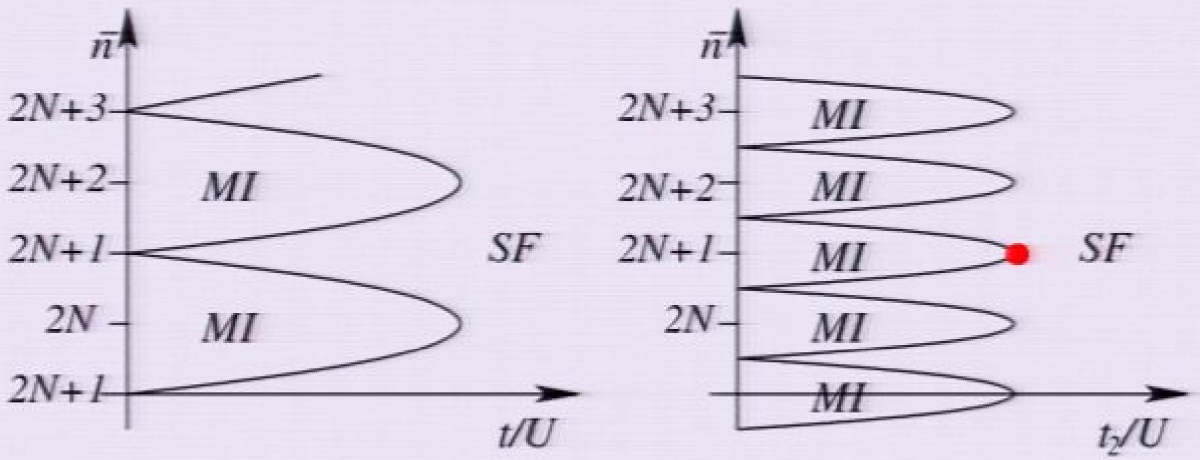
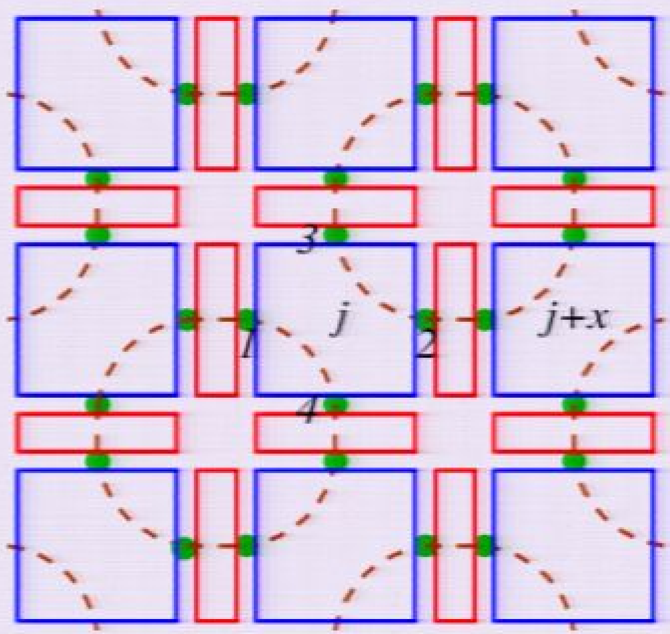


Charge-2 rotor Charge-1 rotor

$$c_{j,1} \sim e^{i\phi_j/2}\gamma_{j,1} \sim \prod_{k\leq j}\sigma^z_{k-x,k}\sigma^x_{j-x,j}e^{i\phi_j/2} \sim \exp[i\pi\sum_{k<j} n_k]e^{i\theta_j}$$

$$\langle C(0)C^\dagger(r)\rangle \sim (\frac{1}{r})^{K+\frac{1}{4K}} \qquad K_c = \frac{1}{4}$$

2d and deconfined phase of Z2 gauge field



$$\gamma_{j,1}\gamma_{j,2}\gamma_{j,3}\gamma_{j,4}(-1)^{n_j} = 1$$

$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 i\gamma_{j,a}\gamma_{j+\nu,b}\cos\left(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2}\right)$$

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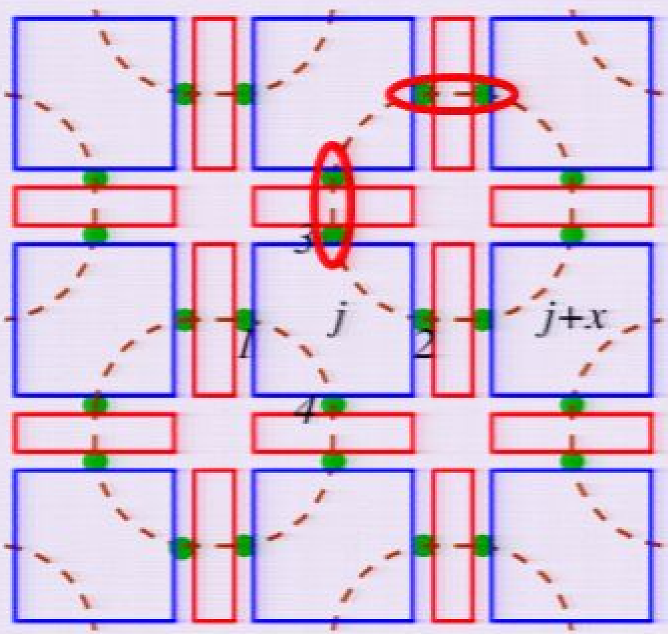
2d and deconfined phase of Z2 gauge field



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$$\downarrow$$

$$\sigma^x_{j,j-x}\sigma^x_{j,j+x}\sigma^x_{j,j-y}\sigma^x_{j,j+y} = (-1)^{n_j}$$

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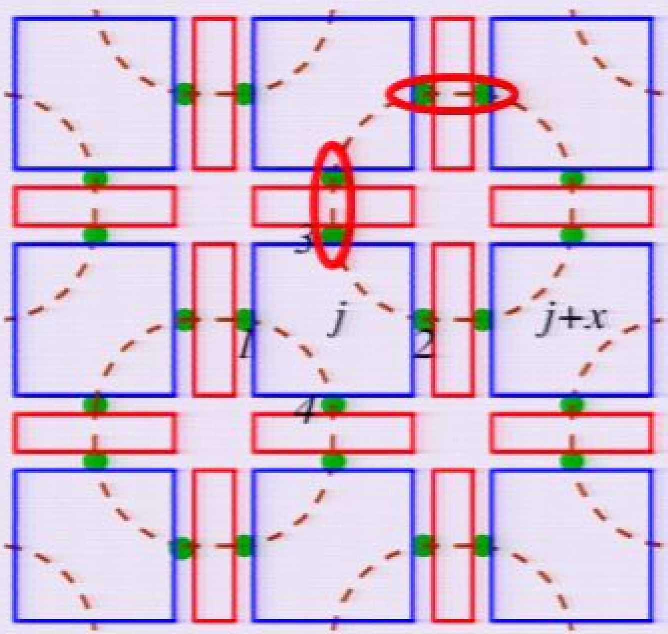
2d and deconfined phase of Z2 gauge field



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2d Z2 gauge field has nontrivial gauge dynamics!

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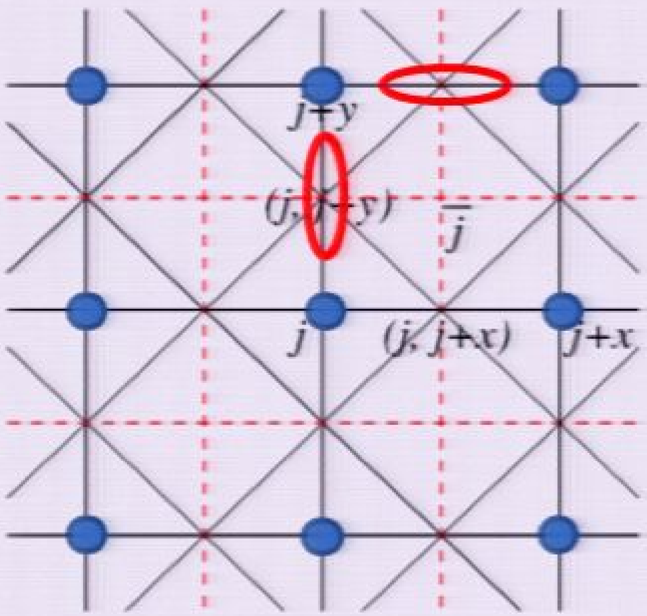
2d and deconfined phase of Z2 gauge field



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↓

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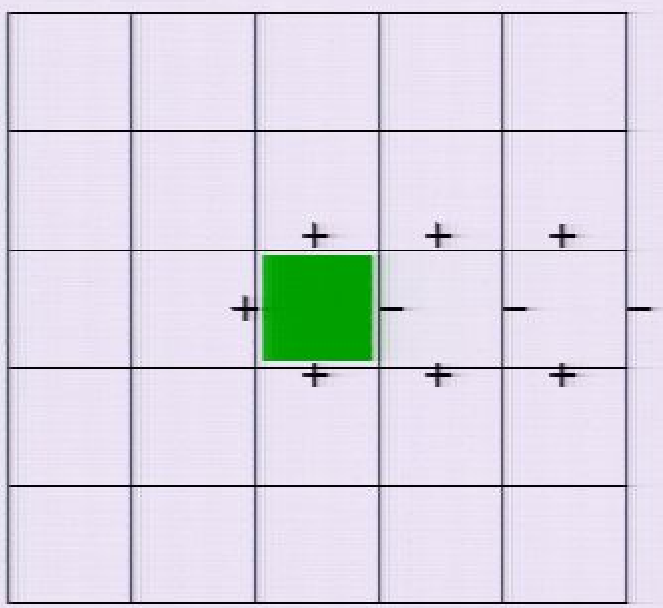
2d Z2 gauge field has nontrivial gauge dynamics!

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 \sigma^z_{j,j+\nu} \cos(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2})$$

In superfluid phase, 2π flux will be bound with a "vison"



$$\prod_{\square} \sigma^z = -1$$

Here vison is completely static, because it is locally conserved,

$\prod_{\square} \sigma^z$ Commutes with the Hamiltonian and constraint.

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$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 \sigma^z_{j,j+\nu} \cos(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2})$$

2π vortex bound with a "vison",

Vison is completely static,

→ Only 4π vortex can proliferate and drive a SF-insulator transition

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 \sigma^z_{j,j+\nu} \cos(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2})$$

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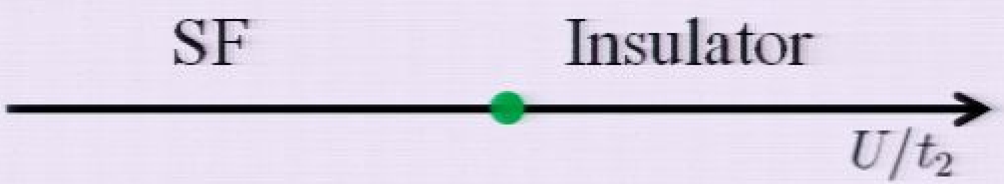


This is a special SF-insulator transition, cooper pair order parameter gains a very large anomalous dimension at the critical point:

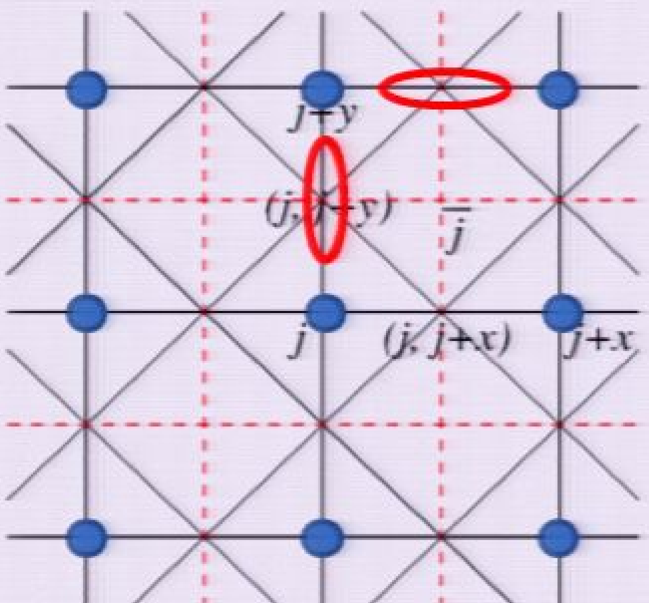
$$\langle \psi(0)\psi^*(r)\rangle \sim \frac{1}{r^{1+\eta}}$$

$$\eta = 1.47$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 \sigma^z_{j,j+\nu} \cos(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2})$$

In the insulator, phase variables are disordered. Integrating out phase variables, we obtain:



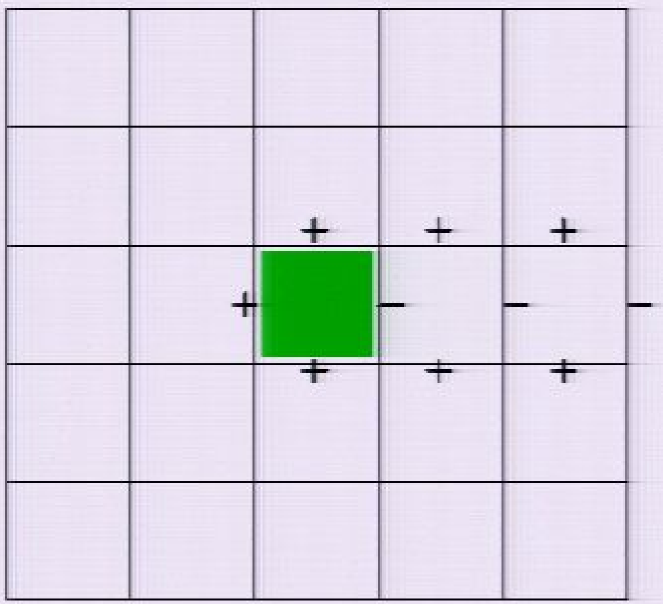
$$H_{\text{ring}} = \sum_j -K \sigma^z_{j,j+x} \sigma^z_{j+x,j+x+y} \sigma^z_{j+y,j+x+y} \sigma^z_{j,j+y}$$

This is "almost" a standard Z2 gauge field, but there is no string tension term σ^x

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

$$H_{\text{ring}} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y} - h\sigma^x_{i,i+\mu}$$

The string tension h term, enables the hopping of vison:



QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

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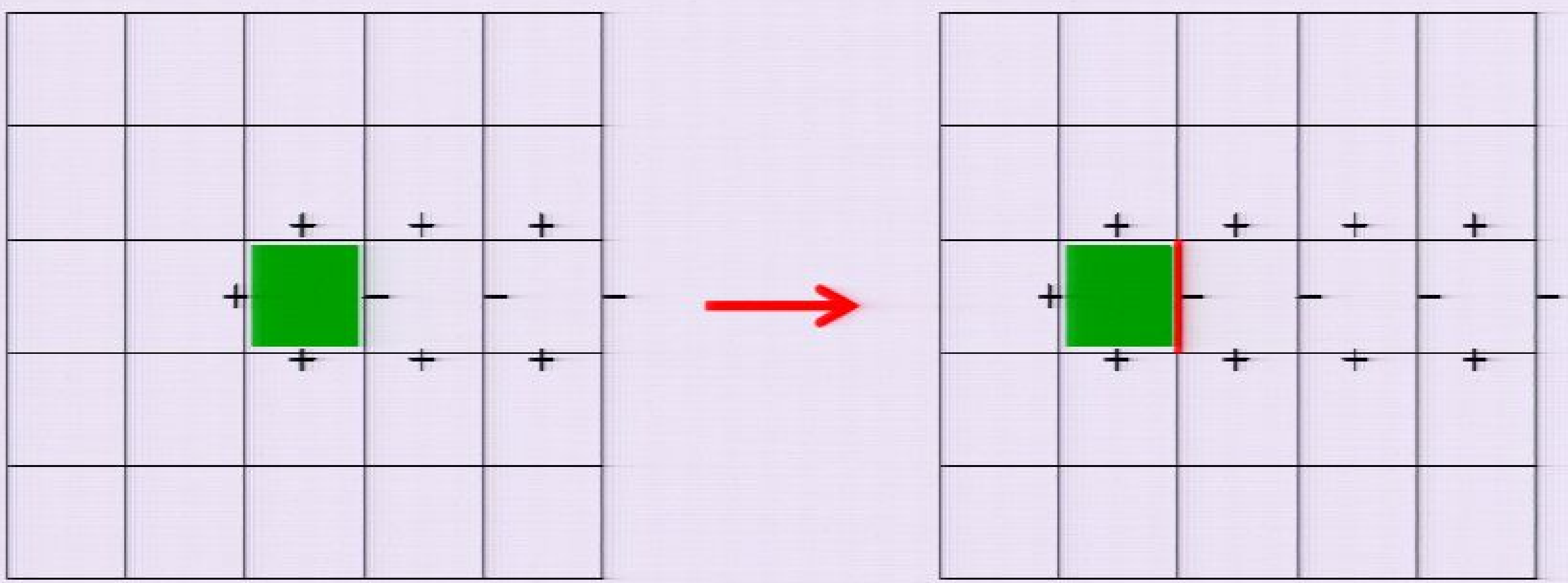

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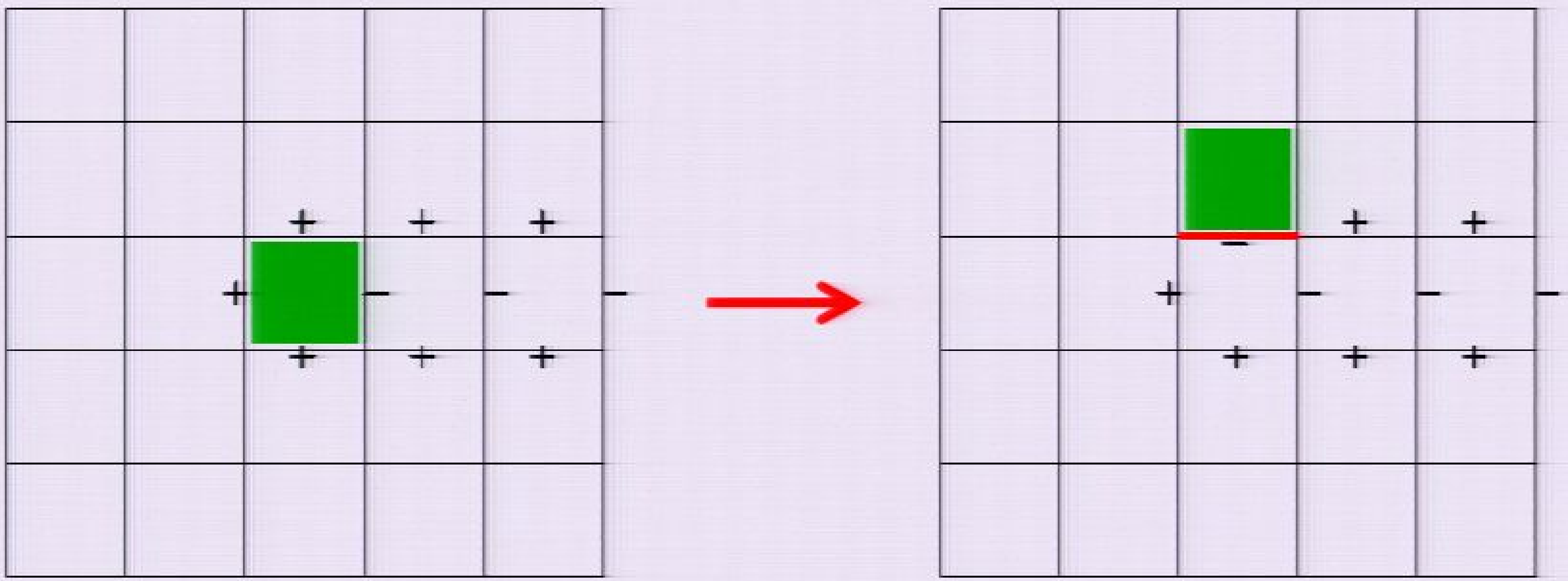
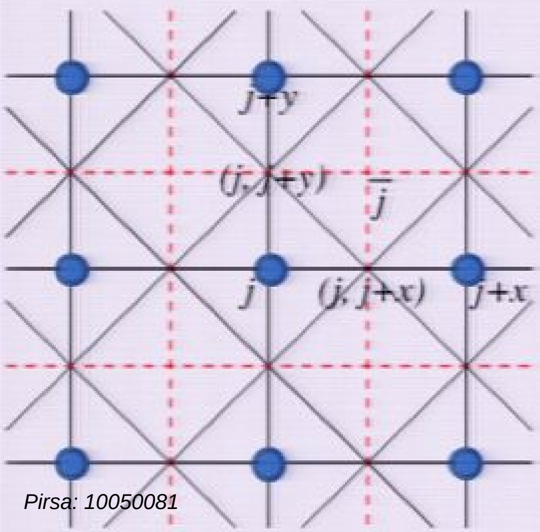


QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

$$H_{\text{ring}} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y} - h\sigma^x_{i,i+\mu}$$

When h is large enough, visons condense due to large "kinetic energy", the system enters a confined phase. The transition is characterized by Wilson loop:



$\mathcal{W} \sim \exp(-\mathcal{L})$ $\mathcal{W} = \langle \prod_C \sigma^z \rangle \sim \exp(-\mathcal{A})$

h/K

3d Ising transition, n even
or 3d XY transition, n odd

$$H_{\text{dual}} = \sum_{\bar{i}} -K\tau^x_{\bar{i}} - \sum_{\bar{i},\mu} h\tau^z_{\bar{i}}\tau^z_{\bar{i}+\mu}.$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

In our case, string tension is a very nonlocal product of Majorana fermions, therefore is forbiden, *i.e.* we are always in a Z2 deconfined phase.

$$H_{\text{ring}} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y}$$

$$+$$

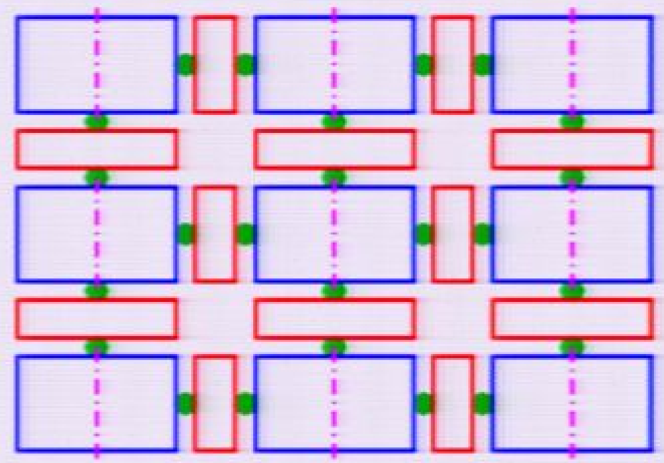
$$\sigma^x_{j,j-x}\sigma^x_{j,j+x}\sigma^x_{j,j-y}\sigma^x_{j,j+y} = (-1)^{n_j}$$

$$\downarrow$$

Toric code model

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

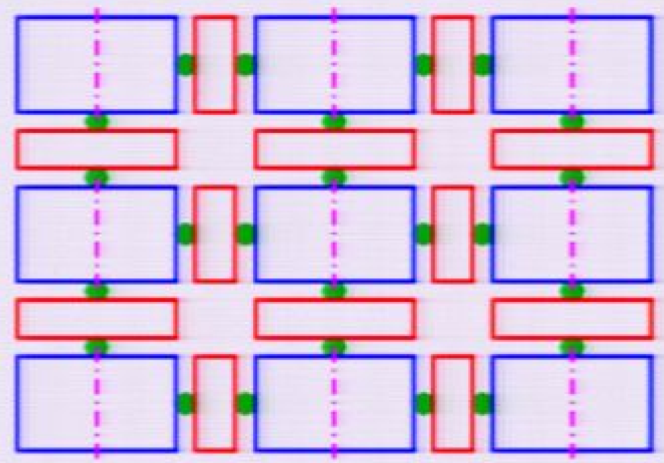
Turn on intra island tunneling, example 1:



$$Ji\gamma_{j,3}\gamma_{j,4}$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Turn on intra island tunneling, example 1:



$$Ji\gamma_{j,3}\gamma_{j,4}$$

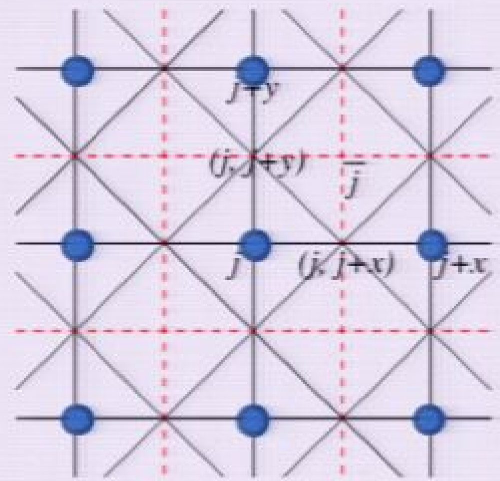
↓

$$H_{\rm ring} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y} - J\sigma^x_{i,i-\mu}\sigma^x_{i,i+\mu}$$

J will hop a pair of vison.

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

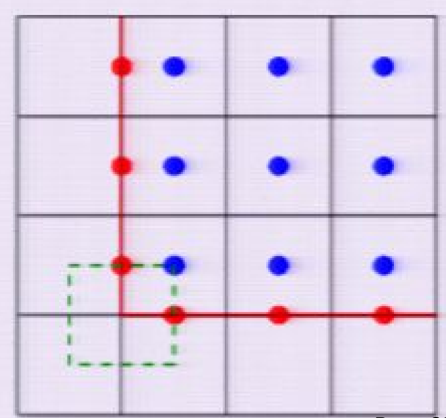
Modifications of the Z2 gauge theory:



$$H_{\rm ring} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y} - J\sigma^x_{i,i-\mu}\sigma^x_{i,i+\mu}$$

↕

$$H_{\rm dual} = \sum_{\bar{i}} -K\tau^x_{\bar{i}} - J\tau^z_{\bar{i}}\tau^z_{\bar{i}+x}\tau^z_{\bar{i}+y}\tau^z_{\bar{i}+x+y}$$

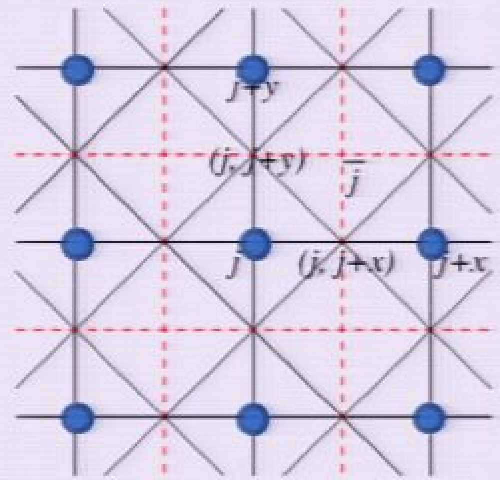


Exactly self-dual, so if there is a transition, will occur precisely at $J = K$.

Cenke Xu, Joel Moore, 2003

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

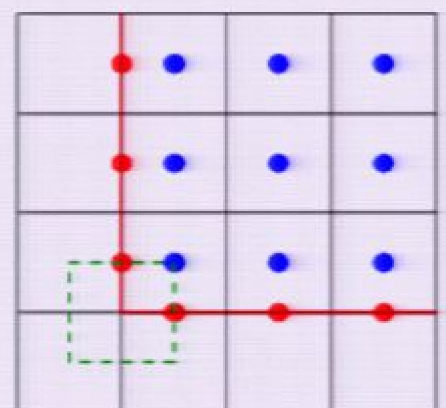
Modifications of the Z2 gauge theory:



$$H_{\text{ring}} = \sum_j -K\sigma^z_{j,j+x}\sigma^z_{j+x,j+x+y}\sigma^z_{j+y,j+x+y}\sigma^z_{j,j+y} - J\sigma^x_{i,i-\mu}\sigma^x_{i,i+\mu}$$

$$\updownarrow$$

$$H_{\text{dual}} = \sum_{\bar{i}} -K\tau^x_{\bar{i}} - J\tau^z_{\bar{i}}\tau^z_{\bar{i}+x}\tau^z_{\bar{i}+y}\tau^z_{\bar{i}+x+y}$$



$\mathcal{W} \sim \exp(-\mathcal{N}_c)$ $\qquad$ $\mathcal{W} = \langle \prod_C \sigma^z \rangle \sim \exp(-\mathcal{A})$

1 $\qquad J/K$

1st order transition

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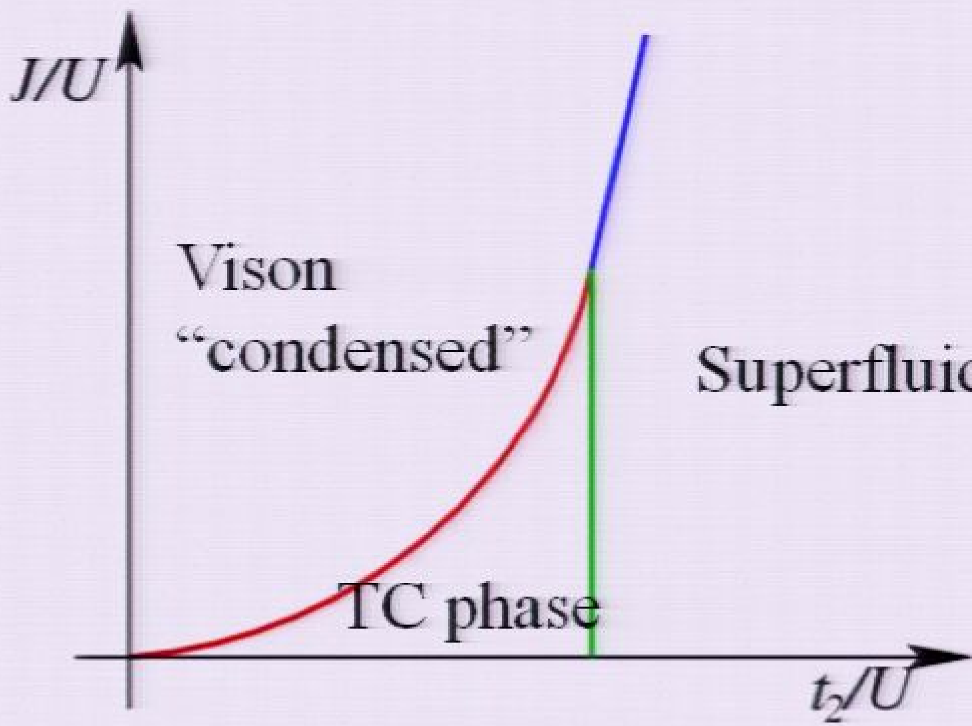
$$H = \sum_j U(n_j - \bar{n})^2 - \sum_{\nu=x,y} t_2 \sigma^z_{j,j+\nu} \cos(\frac{\phi_j}{2} - \frac{\phi_{j+\nu}}{2}) \;\; + \; J i \gamma_{j,3}\gamma_{j,4}$$



For integer filling

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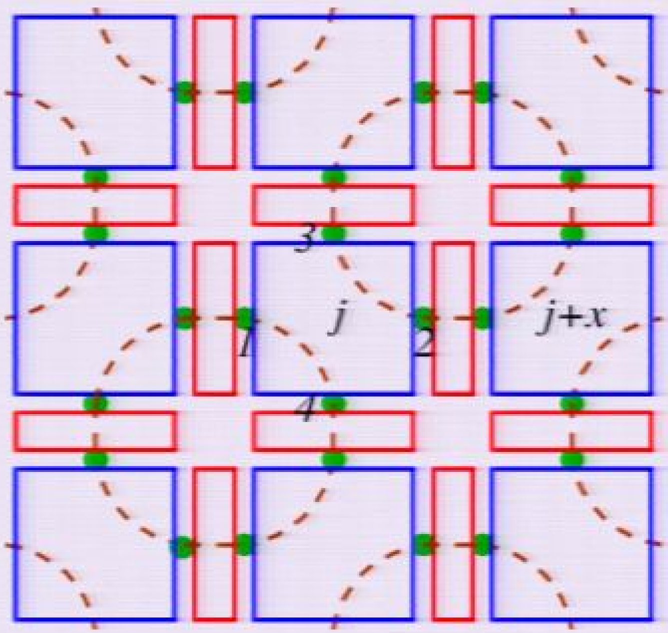
Another representation:



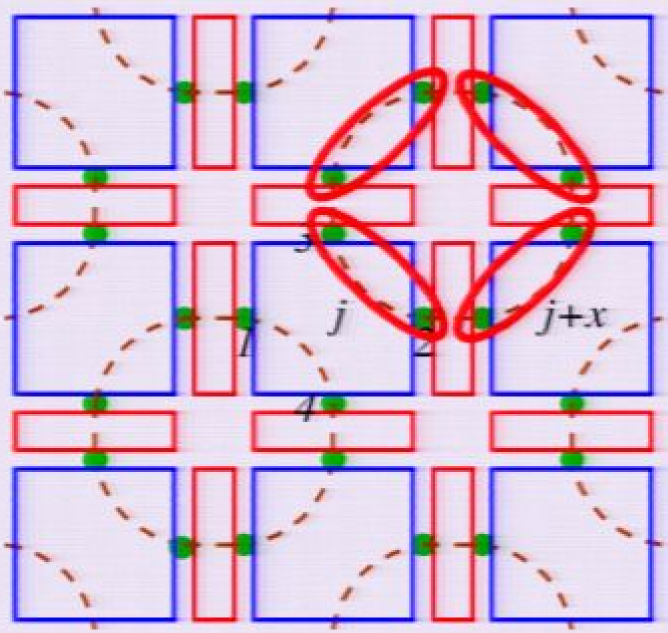
$$\mu_j^x = i\gamma_{j,1}\gamma_{j,3} = i\gamma_{j,2}\gamma_{j,4},$$

$$\mu_j^y = i\gamma_{j,3}\gamma_{j,2} = i\gamma_{j,1}\gamma_{j,4},$$

$$\mu_j^z = i\gamma_{j,1}\gamma_{j,2} = i\gamma_{j,4}\gamma_{j,3}.$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Another representation:



$$\mu_j^x = i\gamma_{j,1}\gamma_{j,3} = i\gamma_{j,2}\gamma_{j,4},$$

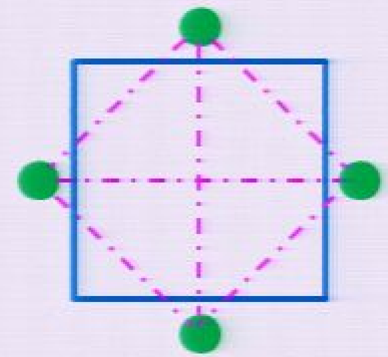
$$\mu_j^y = i\gamma_{j,3}\gamma_{j,2} = i\gamma_{j,1}\gamma_{j,4},$$

$$\mu_j^z = i\gamma_{j,1}\gamma_{j,2} = i\gamma_{j,4}\gamma_{j,3}.$$

$$\prod_{\square}\sigma^z = \prod^{8}\gamma = \mu_j^x\mu_{j+x}^y\mu_{j+y}^x\mu_{j+x+y}^y$$

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Turn on intra-island tunneling



$$H = \sum_j -K\mu_j^x \mu_{j+x}^y \mu_{j+y}^x \mu_{j+x+y}^y + J_j^x \mu_j^x + J_j^y \mu_j^y + J_j^z \mu_j^z.$$

$J^x = J^y = J^z = 0$ Reduces to Wen's model with topological order. Wen, 2003

An interesting three dimensional phase diagram to study.

QSH-FM-SC Josephson Arrays and emergent Z2 gauge field

Summary:

1, the Majorana bound state assisted Josephson junction, enables single charge tunneling.

2, the Josephson lattice realizes deconfined phase of Z2 gauge field naturally.

Future:

construct more novel topological orders?