Title: Projective Ribbon Permutation Statistics: a Remnant of non-Abelian Braiding in Higher Dimensions

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Abstract: TBA

Projective Ribbon Permutation Statistics: a Remnant of non-Abelian Braiding in Higher Dimensions

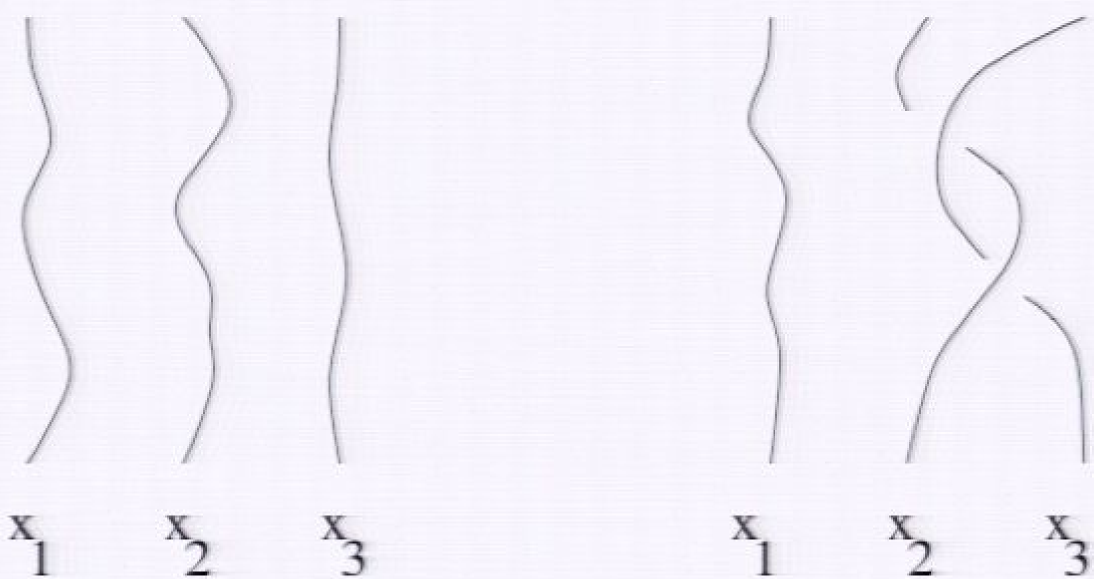
Chetan Nayak
Station Q

Michael Freedman, Matthew Hastings, Xiao-Liang Qi, Kevin Walker, Zhenghan Wang

arXiv:0042856

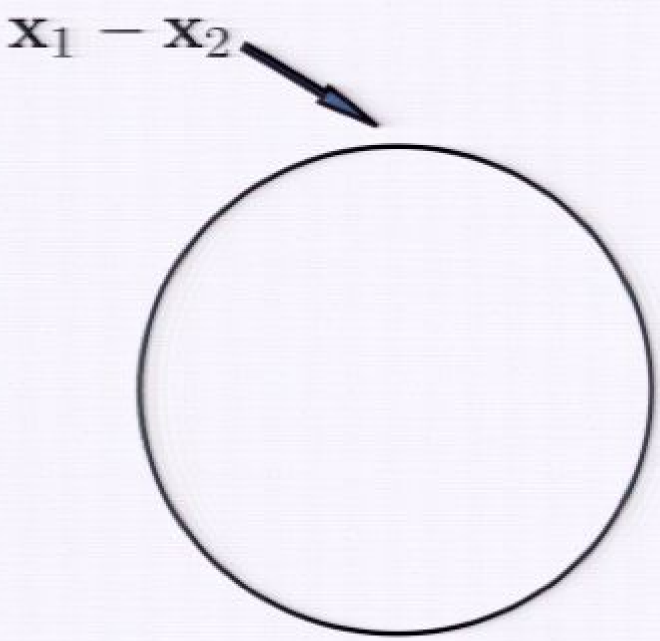
Exotic Statistics in 2D

- Quantum Mechanical Amplitude for particles to return at time t to their positions at t_0 .
- Feynman: sum over all histories, weighting each by e^{iS}
- In 2+1 dimensions, the space of histories is disconnected. Therefore, the relative phase is no longer dictated by the classical limit.

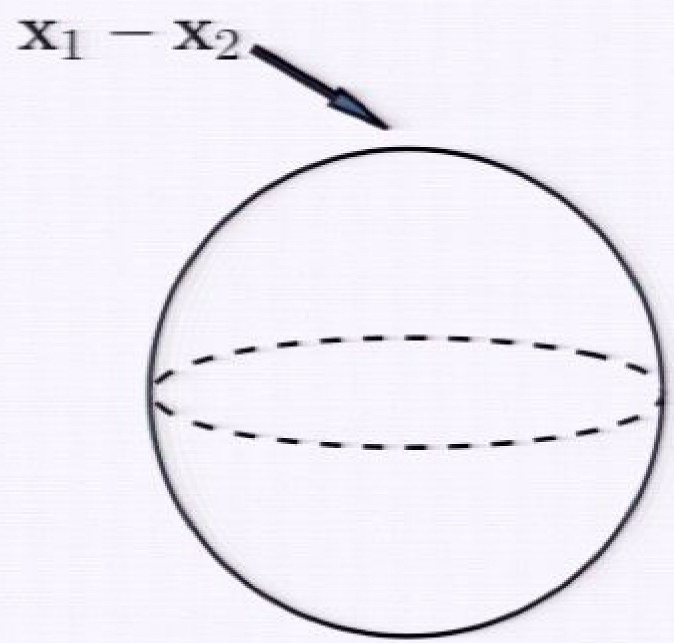


- Equivalently, the n-particle configuration space, $\mathcal{C}_n$, has non-trivial fundamental group, $\pi_1(\mathcal{C}_n)$ in 2D.

e.g. two hard-core particles in 2D



contrast: two hard-core particles in 3D

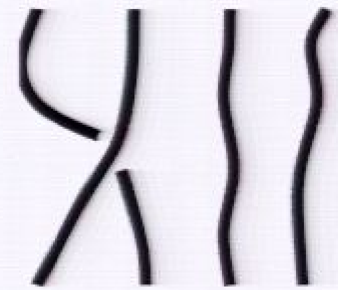


Center-of-mass degree of freedom, distance between the two particles unimportant topologically. Hard-core constraint will prove superfluous.

Braid Group

- For n particles in 2D, $\pi_1(\mathcal{C}_n) = \mathcal{B}_n$, the *Braid Group*.

  Generated by counter-clockwise exchanges σ_i satisfying



i $i{+}1$

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n-2 \end{aligned}$$

- In 3D, $\pi_1(\mathcal{C}_n) = S_n$, the *Permutation Group*.
  satisfies the relations above and one extra:

$$(\sigma_i)^2 = 1$$

Anyons

In 2D, the Braid Group allows for the possibility of anyons.

- 1D representations of the Braid Group are Abelian. They correspond to Abelian anyons (fermions, bosons, Laughlin quasiparticles, toric code electric/magnetic charges).
- Higher-dim. representations of the braid group correspond to non-Abelian anyons. Possible applications to quantum computing.

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If we also allow anyons to be created/annihilated in particle-anti-particle pairs, then we need a family of braid group representations.

Generalize pair creation/annihilation to fusion of pairs of anyons into other anyons, not just vacuum.
Family of braid group reps. with consistency between braiding and fusion -- braided fusion category.

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Ising Anyons

- Three particle types: $1, \sigma, \psi$ non-triv. fusion rule: $\sigma \times \sigma = I + \psi$
- Hilbert space of $2n$ Ising anyons:

Iajorana fermion
ero mode
ttached to each
 quasiparticle:

$$\gamma_i = \gamma_i^\dagger$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

for $i = 1, \ldots, 2n$

fusion: $i\gamma_i\gamma_j = \pm 1$

$2^{n\text{-}1}$ - dimensional.

Rep. of braid group generators:

$$\rho(\sigma_i) = e^{i\pi/8} e^{-\pi\gamma_i\gamma_{i+1}/4}$$

Parastatistics

- The permutation group also has higher-dimensional representations.
- However, higher-dim. reps. which are consistent with locality can be decomposed into the tensor product of local Hilbert spaces associated with each particle (Doplicher, Haag, Roberts '71).

  e.g. a 2^{2n}-dimensional representation of $2n$ particles can be understood as $2n$ bosons with with a hidden two-state system attached to each.

  The color quantum number of quarks was deduced by similar reasoning.

- In a system of non-Abelian anyons, it is not possible to decompose the Hilbert space into a product of local Hilbert spaces. e.g. for Ising anyons, $\sqrt{2}$ states per particle.

Teo-Kane Model

- Teo and Kane constructed a 3D model in which hedgehog singularities have Majorana fermion zero modes.

$$H = i\chi(\partial_i \gamma_i + n_i \Gamma_i)\chi$$

J. C. Y. Teo and C. L. Kane. Phys. Rev. Lett. **104**, 046401 (2010).

orbital, particle-hole, spin ⟶ χ: 8 components = 2 4-comp. Majorana spinors.

$$\vec{n} = (\mathrm{Re}\Delta, \mathrm{Im}\Delta, m)$$

⇧ (m): ordinary vs. top. insulator

γ_i : a set of 3 8x8 matrices

- There is a $2^{n\text{-}1}$-dimensional Hilbert space for 2n such hedgehogs.

- What happens when the hedgehogs are permuted?

  Looks like there should be a higher-dim representation.
  But the Hilbert space is not a tensor product of local Hilbert spaces ... conflict with Dopplicher et al.?

- Teo and Kane: exchanging two hedgehogs leads to

$$e^{i\theta}\, e^{-\pi\gamma_i\gamma_{i+1}/4}$$

- Two exchanges leaves the hedgehogs unmoved. Thus, even without moving the hedgehogs (just twisting them), we can enact

$$e^{2i\theta}\, e^{-\pi\gamma_i\gamma_{i+1}/2} = e^{2i\theta}\gamma_i\gamma_{i+1}$$

A simple explanation of these transformations later.

Raises Questions

- Is Teo and Kane's conclusion correct?
- What is the $2n$ hedgehog configuration space K_{2n}?
- What is its fundamental group, $\pi_1(K_{2n})$?
  i.e. what is the group whose representation was found by Teo and Kane?
- Why does it have a non-trivial non-Abelian representation? Are there others?
- Can we really have non-trivial particle statistics in 3D?

Topological Classification of Gapped Free-Fermion Hamiltonians

- If we don't forbid the gap from closing, then the space of Hamiltonians is contractible.
- If the gap remains open, then the space of free fermion Hamiltonians will be topologically non-trivial.
- The topology will depend on the symmetries imposed and the spatial dimension.
- 'Free-fermion' Hamiltonians includes: band insulators, superconductors, any ordered phase in which the low-energy effective field theory has gapped fermions with irrelevant interactions (and, possibly, irrel. interactions with gapped bosons, e.g. order parameter fluct.)
- Topological defects - non-trivial homotopy groups of the space of Hamiltonians.

Brief Review

- Two very different approaches to this topological classification.
  S. Ryu and collaborators: classifying protected edge/surface states.
  A. Kitaev: classifying Hamiltonians = classifying vector bundles over Brillouin zone (K-theory).
  Both lead to the same classification.

A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
A. Kitaev, arXiv.org:0901.2686 (unpublished).

- We will follow the latter which is more easily adapted for our purposes.

Basic Idea:

$$H = \sum_{i,j} i A_{ij} a_i a_j$$

2N Majorana fermions a_i

Don't assume charge cons. or any other symm., so the a_i are Majorana fermion operators $c_j = (a_{2j-1} + i a_{2j})/2$.

Zero-dim. system, so locality isn't encoded through restrictions on A_{ij}.

- Any real anti-symmetric matrix can be written in the form:

$$A = O^T \begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & 0 & -\lambda_2 & \\ & & \lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix} O$$

- Eigenvalues come in pairs $\pm\lambda_i$

  By assumption, all $\lambda_i > 0$. Can continuously deform A_{ij} so that all $\lambda_i = 1$.

- Thus,

$$A = O^T J O \qquad \text{with:} \quad J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

- Since $O \in \mathrm{O}(2N)$ satisfying $O^T J O = J$ is $\mathrm{U}(N) \subset \mathrm{O}(2N)$ we conclude that: $A = O^T J O \in \mathrm{O}(2N)/\mathrm{U}(N)$

- Equivalently, A_{ij} is a linear transformation on $\mathbb{R}^{2N}$

  It defines a *complex structure*, i.e. we can define multiplication of a vector $\vec{v} \in \mathbb{R}^{2N}$ by a complex scalar according to:

$$(a + ib)\vec{v} \equiv a\vec{v} + bA\vec{v}$$

The set of complex structures on $\mathbb{R}^{2N}$	=	all possible rotations in $\mathrm{O}(2N)$ of a given complex structure modulo those rotations of $\mathbb{C}^N$ which respect it, namely $\mathrm{U}(N)$

- Space of gapped free fermion Hamiltonians in 0D $\underset{\text{top. equiv.}}{\cong}$ $\mathrm{O}(2N)/\mathrm{U}(N)$

3D Systems

- Expand about the minima of the gap:

$$H = i\chi(\gamma_i \partial_i + M)\chi$$

$$i = 1, 2, 3$$

In more conventional notation:

$$H = i\chi(\alpha_i \partial_i - im\beta)\chi$$

Can be formalized through Kitaev's texture theorem; or suspension isomorphism in K-theory.

- Assume w.l.o.g. that all Majorana fermions have the same gap = 1. Then:

$$\{\gamma_i, M\} = 0$$

$$M^2 = -1$$

as usual:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

- Since $\gamma_1^2 = 1$, it is linear map which decomposes $\mathbb{R}^{2N}$ into $+1$ and -1 eigenspaces: $\mathbb{R}^{2N} = X_+ \oplus X_-$

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- Since $(\gamma_2\gamma_3)^2 = -1$ and $[\gamma_1, \gamma_2\gamma_3] = 0$, it is a complex structure on X_+, so that: $X_+ \cong \mathbb{C}^{N/2}$

i.e. for $\vec{v} \in X_+$, we can define $(a+bi)\vec{v} \equiv a\vec{v} + \gamma_2\gamma_3\vec{v}$

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- Since $(\gamma_2 M)^2 = 1$ and $[\gamma_1, \gamma_2 M] = 0$, we can decompose X_+ into the $+1$ and -1 eigenspaces of $\gamma_2 M$: $X_+ = Y \oplus \gamma_2\gamma_3 Y$

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  since $\{\gamma_2 M, \gamma_2\gamma_3\} = 0$

- Y is a real subspace of X_+

- Thus, the space of possible M is the space of possible real subspaces of $\mathbb{C}^{N/2}$ = all possible rotations of a fixed real subspace modulo rotations which keep it invariant.

$$M \in \mathrm{U}(N/2)/\mathrm{O}(N/2)$$

Periodic Table

dim.:	0	1	2	3	4	...
U(2), T, Q	$\mathbb{Z}\times\frac{O(N)}{O(N/2)\times O(N/2)}$	$U(N/2)/O(N/2)$	$Sp(N/4)/U(N/4)$	$Sp(N/8)$	$\mathbb{Z}\times\frac{Sp(N/8)}{Sp(N/16)\times Sp(N/16)}$	...
(2), T, Q, χ	$O(N/4)$	$\mathbb{Z}\times\frac{O(N/4)}{O(N/8)\times O(N/8)}$	$U(N/8)/O(N/8)$	$Sp(N/16)/U(N/16)$	$Sp(N/32)$	...
no symm.	$O(2N)/U(N)$	$O(N)$	$\mathbb{Z}\times\frac{O(N)}{O(N/2)\times O(N/2)}$	$U(N/2)/O(N/2)$	$Sp(N/4)/U(N/4)$	...
T only	$U(N)/Sp(N/2)$	$O(N)/U(N/2)$	$O(N/2)$	$\mathbb{Z}\times\frac{O(N/2)}{O(N/4)\times O(N/4)}$	...	
T and Q	$\mathbb{Z}\times\frac{Sp(N/2)}{Sp(N/4)\times Sp(N/4)}$	$U(N/2)/Sp(N/4)$	$O(N/2)/U(N/4)$	$O(N/4)$		
T, Q, χ	$Sp(N/4)$	$\mathbb{Z}\times\frac{Sp(N/4)}{Sp(N/8)\times Sp(N/8)}$	$U(N/4)/Sp(N/8)$	$O(N/4)/U(N/8)$		
	$\vdots$				$\ddots$	

or N large (stable limit),
ott periodicity:

$$
\begin{aligned}
(16N)) &= \pi_{k-1}(O(16N)/U(8N)) \\
&= \pi_{k-2}(U(8N)/Sp(4N)) \\
&= \pi_{k-3}(\mathbb{Z}\times Sp(4N)/(Sp(2N)\times Sp(2N))) \\
&= \pi_{k-4}(Sp(2N)) \\
&= \pi_{k-5}(Sp(2N)/U(2N)) \\
&= \pi_{k-6}(U(2N)/O(2N)) \\
&= \pi_{k-7}(\mathbb{Z}\times O(2N)/(O(N)\times O(N))) \\
&= \pi_{k-8}(O(N))
\end{aligned}
$$

by inspection:

$$
\begin{aligned}
\pi_0(O(N)) &= \mathbb{Z}_2 \\
\pi_0(O(2N)/U(N)) &= \mathbb{Z}_2 \\
\pi_0(U(2N)/Sp(N)) &= 0 \\
\pi_0(\mathbb{Z}\times Sp(2N)/Sp(N)\times Sp(N)) &= \mathbb{Z} \\
\pi_0(Sp(N)) &= 0 \\
\pi_0(Sp(N)/U(N)) &= 0 \\
\pi_0(U(N)/O(N)) &= 0 \\
\pi_0(\mathbb{Z}\times O(2N)/(O(N)\times O(N))) &= \mathbb{Z}
\end{aligned}
$$

Topological Defects

- If the superconducting order parameter and other order parameters (CDW, etc.) as well as other terms in the free fermion Hamiltonian are allowed to vary spatially subject to the constraint that the gap not close locally except at the origin, then a topological defect may form, classified by:

$$\pi_2(\mathrm{U}(N)/\mathrm{O}(N))$$

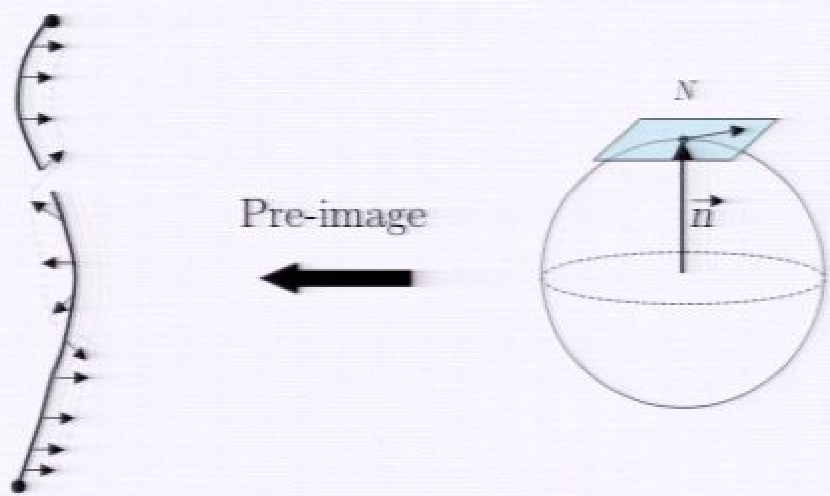
- From the previous slide, $\pi_2(\mathrm{U}(N)/\mathrm{O}(N)) = \mathbb{Z}_2$

  'Hedgehogs' in a 3D system with no symmetry carry a $\mathbb{Z}_2$ topological quantum number.

- Teo-Kane model with 'order parameter' $\vec{n} \in S^2$ is a special small N case. $\mathrm{U}(2)/\mathrm{O}(2) = \mathrm{U}(1) \times S^2$. We will focus on the large N case, which is more natural.

Heuristic Description of Multi-Defect Configuration Space

Pre-image of North pole = framed arcs connecting hedgehogs & loops.



The configuration of arcs summarizes the order parameter configuration between the defects.
In fact, it's the whole story topologically.

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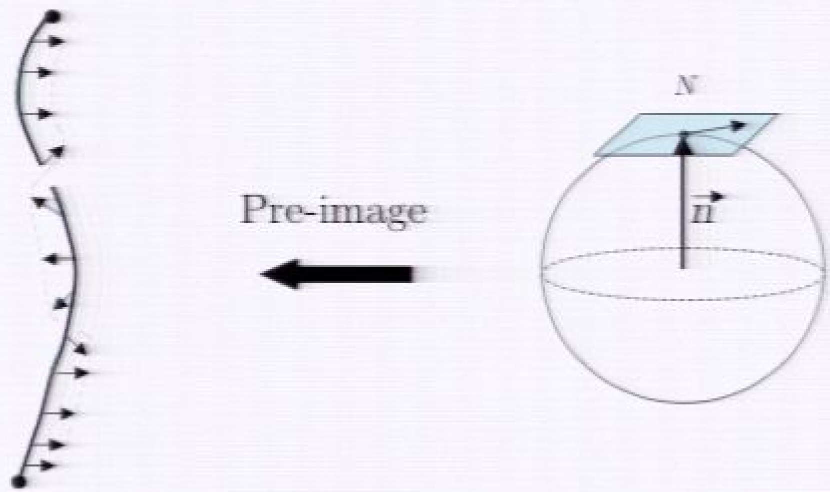

The configuration of arcs summarizes the order parameter configuration between the defects.
In fact, it's the whole story topologically.

- Configuration space = possible configs. of defects, arcs.

- Fundamental group = equivalence classes of loops in config. space
  = equiv. classes of defect motions with same initial and final configs.

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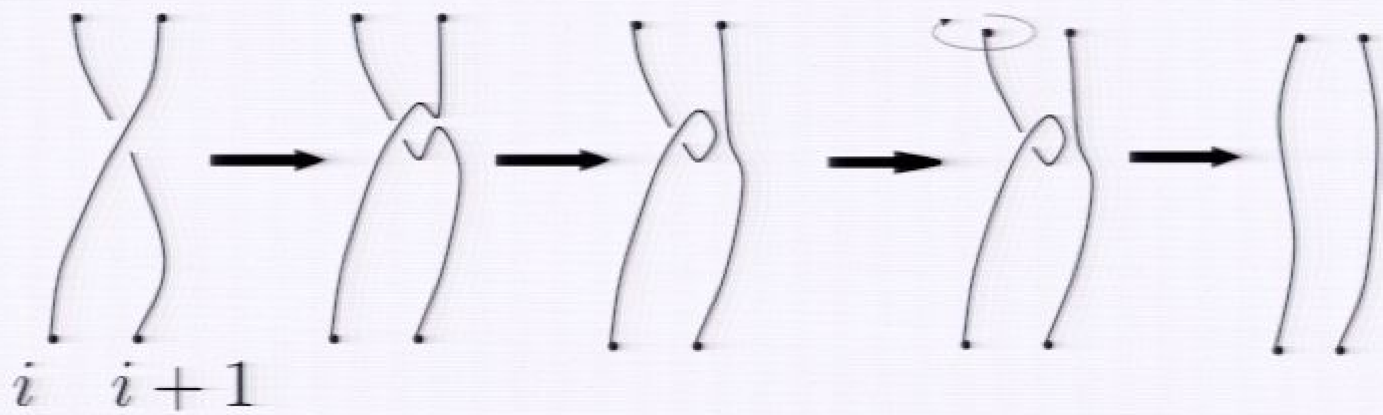
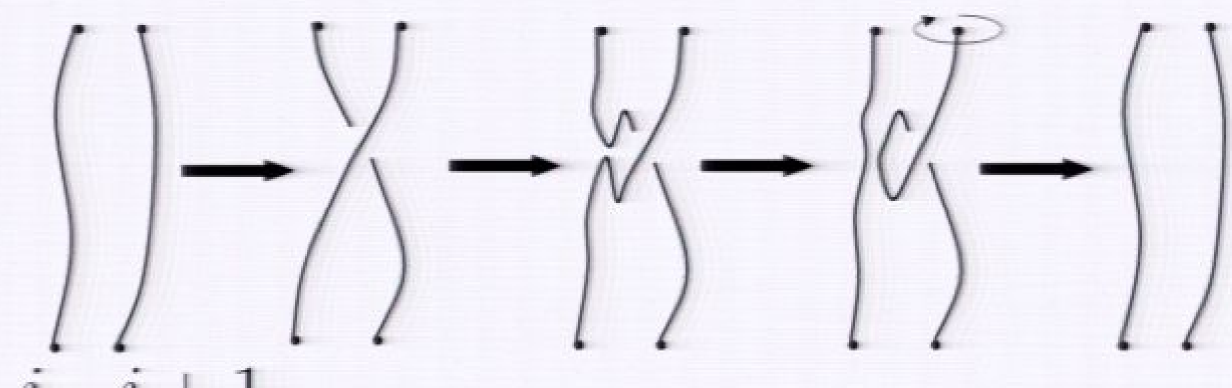
- Generators:

t_i :

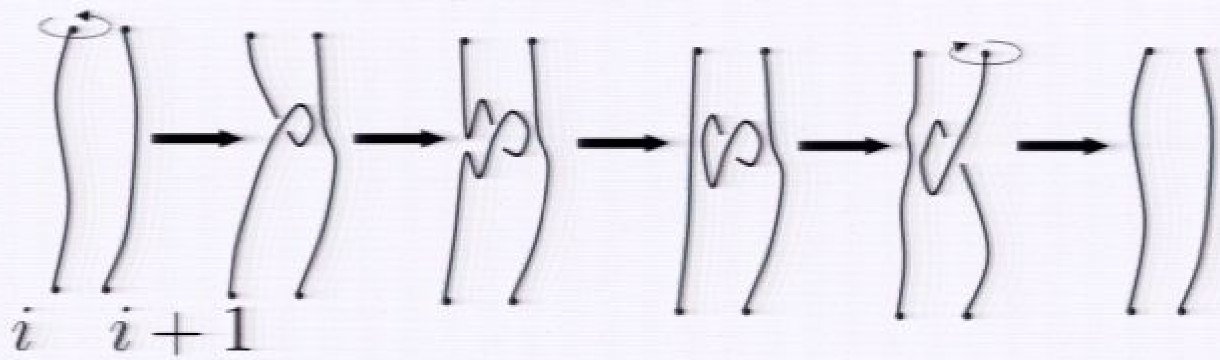


$(t_i)^{-1}$:



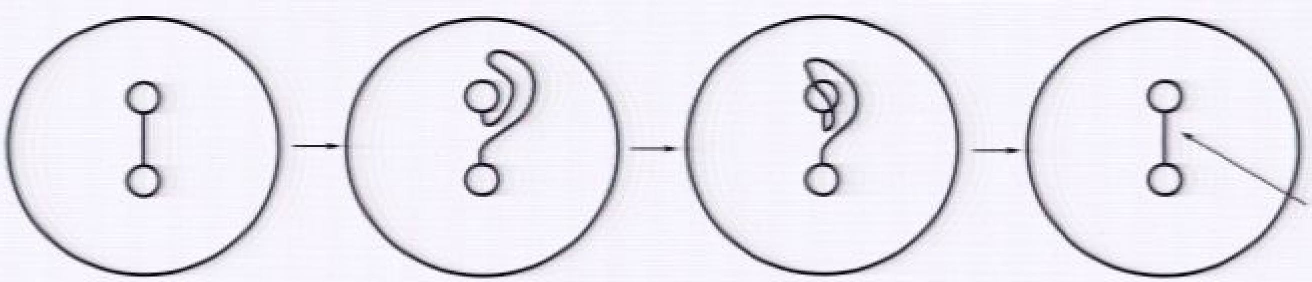
- Permutation squares to 1, but

$$(t_i)^2 =$$



Call this x_i defined by $x_i \equiv (t_i)^2$ or the above sequence.

- Then $(x_i)^2 = 1$ because a 4π twist can be undone.

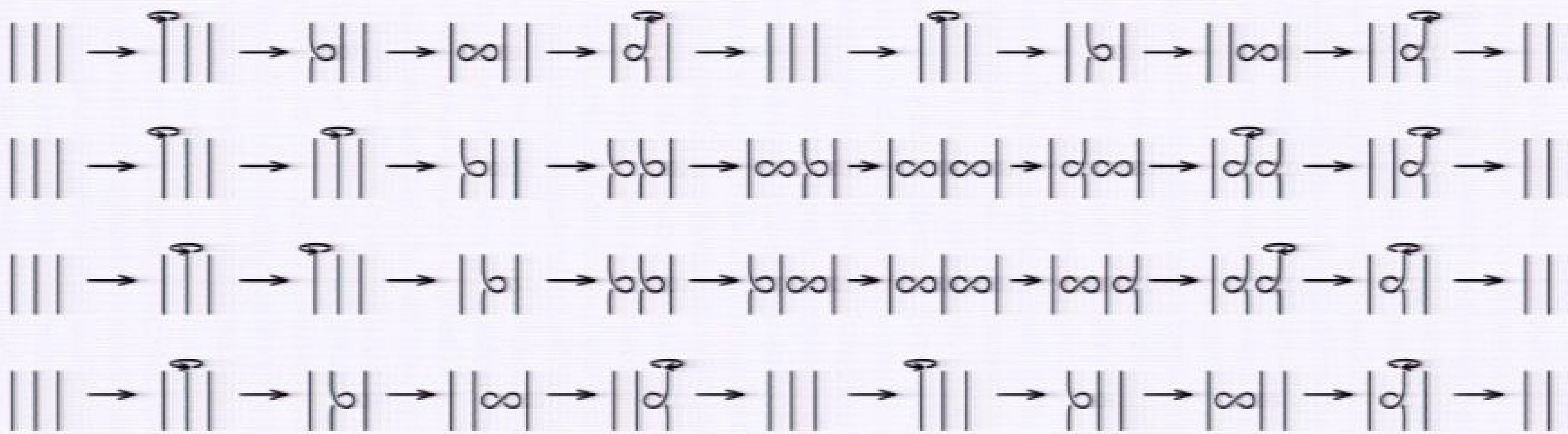


Therefore, $(t_i)^4 = 1$

- Clearly, distant t_i commute. Furthermore,

$$x_i x_{i+1} = x_{i+1} x_i$$

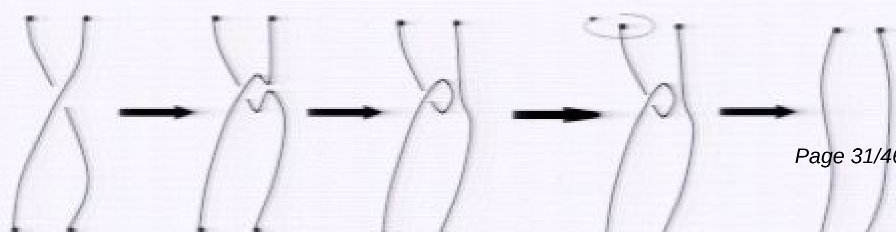
To verify:



- The x_i form $2n$-1 copies of $\mathbb{Z}_2$

They only twist the particles, don't permute them.

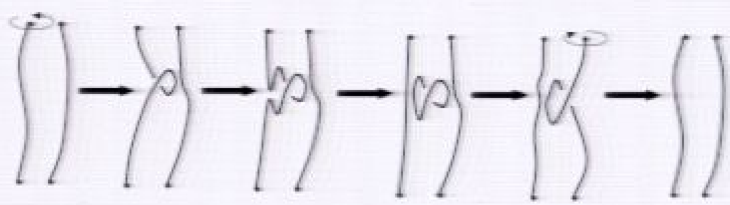
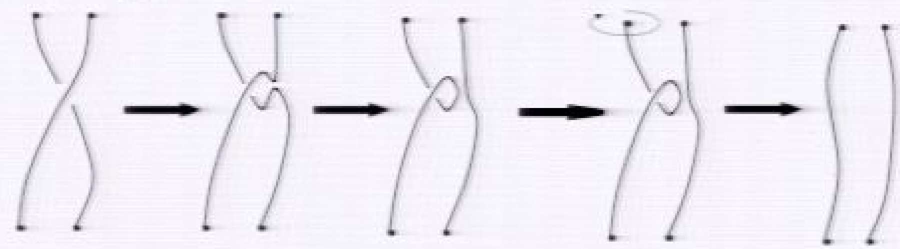
- The t_i both twist and permute particles.



- t_i and x_i need not commute: $t_i x_{i+1} = x_i x_{i+1} t_i$

  If a twist is followed by a permutation, then the twist acts on the permuted defects.
  This is the structure of a semi-direct product.

- The elements of $\pi_1(K_{2n})$ either (a) twist an even number of ribbons; (b) perform an even permutation; or (c) twist an odd number of ribbons and perform an odd permutation.

a)  (c) 

(b) $t_{i+1}(t_i)^{-1}$

$$\pi_1(K_{2n}) \approx E((\mathbb{Z}_2)^n \rtimes S_n)$$

even part

Outline of Full Calculation

- The configuration space of $2n$ defects, K_{2n}, is the total space of a *fibration*:

$$\begin{array}{c} \mathcal{M}_{2n} \to K_{2n} \\ \downarrow \\ X_{2n} \end{array}$$

config. space of 2n defects at fixed positions ($\mathcal{M}_{2n}$)

config. space of 2n point particles (X_{2n})

- From the long exact sequence associated with fibrations:

$$F \to E \to B \quad \Longrightarrow \quad \ldots \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \ldots$$

we conclude that $$\pi_1(\mathcal{M}_{2n}) \to \pi_1(K_{2n}) \to \pi_1(X_{2n}) \to 1.$$

$$\pi_1(X_{2n}) \cong S_{2n}$$

Hence, $\pi_1(K_{2n})$ is an extension of S_{2n}

- t_i and x_i need not commute: $t_i x_{i+1} = x_i x_{i+1} t_i$

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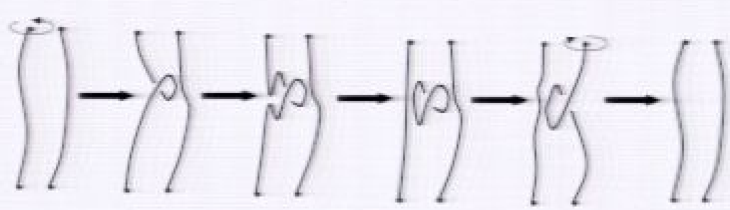
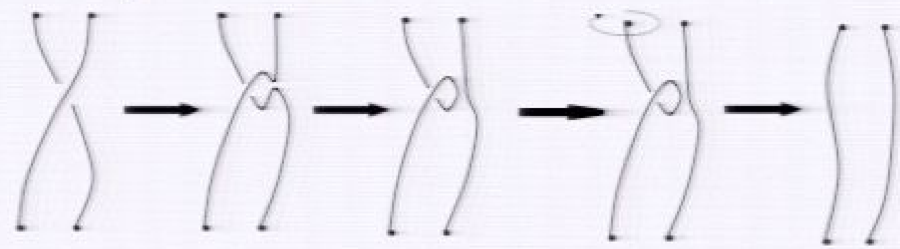
a)



(c)



(b) $t_{i+1}(t_i)^{-1}$

$$\pi_1(K_{2n}) \approx E((\mathbb{Z}_2)^n \rtimes S_n)$$

even part

Outline of Full Calculation

- The configuration space of $2n$ defects, K_{2n}, is the total space of a *fibration*:

$$\mathcal{M}_{2n} \to K_{2n}$$
$$\downarrow$$
$$X_{2n}$$

config. space of 2n defects at fixed positions ($\mathcal{M}_{2n}$)

config. space of 2n point particles (X_{2n})

- From the long exact sequence associated with fibrations:

$$F \to E \to B \quad \Rightarrow \quad \ldots \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \ldots$$

we conclude that $\pi_1(\mathcal{M}_{2n}) \to \pi_1(K_{2n}) \to \pi_1(X_{2n}) \to 1.$

$$\pi_1(X_{2n}) \cong S_{2n}$$

Hence, $\pi_1(K_{2n})$ is an extension of S_{2n}

- We introduce further fibrations to break the problem into pieces:

$$\begin{array}{ccc} R_{2n} & \to & K_{2n} \\ & & \downarrow \\ & & Y_{2n} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{N}_{2n} & \to & Y_{2n} \\ & & \downarrow \\ & & X_{2n} \end{array}$$

R_{2n}: of order parameter res interpolating een defects

Y_{2n}: configs. of infinitesimal spheres surrounding 2n points with maps to U/O

$\mathcal{N}_{2n}$: configs. of infinitesimal spheres surrounding 2n *fixed points* with maps to U/O

X_{2n}: config. space of 2n point particles

- Direct computation: $\pi_1(\mathcal{N}_{2n}) \;=\; (\mathbb{Z}_2)^{2n}$

  Combined with $\pi_1(X_{2n}) \;=\; S_{2n}$, the homotopy long exact sequence implies that:

$$\pi_1(Y_{2n}) = (\mathbb{Z}_2)^{2n} \rtimes S_{2n}$$

- Direct computation: $\pi_0(R_{2n}) = \mathbb{Z}_2$, $\pi_1(R_{2n}) = (\mathbb{Z}_2)^{2n}$

- Thus, we have the short exact sequence:

$$\begin{array}{ccccccccc}
\pi_2(Y_{2n}) & \xrightarrow{\partial_2} & \pi_1(R_{2n}) & \longrightarrow & \pi_1(\widetilde{K}_{2n}) & \xrightarrow{\text{proj.}} & \pi_1(Y_{2n}) & \xrightarrow{\partial_1} & \pi_0(R_{2n}) \\
\wr\| & & \wr\| & & & & \wr\| & & \wr\| \\
1 & & (\mathbb{Z}_2)^{2n} & & & & (\mathbb{Z}_2)^{2n} \rtimes S_{2n} & & \mathbb{Z}_2
\end{array}$$

$$\ker(\partial_1) = E(\mathbb{Z}_2^{2n} \rtimes S_{2n})$$

from which we can conclude that

$$\pi_1(\tilde{K}_{2n}) = E(\mathbb{Z}_2^{2n} \rtimes S_{2n}) \ltimes \text{(kernel of the projection)}$$

⇑ shown by direct construction to be $\mathbb{Z}_2$

- Putting this all together:

$$\pi_1(\tilde{K}_{2n}) = \mathbb{Z}_2 \times E(\mathbb{Z}_2^m \rtimes S_m)$$

⇐ **'Ribbon Permutation Group'**

3D Majorana Fermion Statistics

- Quasiparticle statistics: representations of $\pi_1(K_{2n})$

- $\pi_1(K_{2n})$: generated by $t_1, \ldots, t_{2n-1}$ satisfying

$$(t_i t_{i+1}^{-1})^3 = (t_i^{-1} t_{i+1})^3 = 1, \text{ and } t_i^4 = 1$$

N.B.: the braidless ops. are $x_i \equiv (t_i)^2$

they satisfy: $x_i x_j = x_j x_i$ ⟵ $(\mathbb{Z}_2)^{2n-1}$ subgroup of $\pi_1(K_{2n})$

- Identify the rep. of these generators with Teo and Kane's unitary transf.:

$$\rho(t_i) = e^{i\theta}\, e^{\pi \gamma_i \gamma_{i+1}/4}$$

- But then, $\rho(x_i) = e^{2i\theta}\, e^{\pi\gamma_i\gamma_{i+1}/2} = e^{2i\theta}\gamma_i\gamma_{i+1}$

- Why is this surprising?

$$\rho(x_i)\rho(x_{i+1}) = -\rho(x_{i+1})\rho(x_i)$$

while $x_i x_{i+1} = x_{i+1} x_i$

- This is not a linear representation; it is a *projective representation*. The multiplication rule is only reproduced up to a phase.

- Projective statistics was first suggested by Wilczek (and criticized by Read), who suggested projective representations of the permutation group. This would violate locality.

F. Wilczek, hep-th/9806228

N. Read, Journal of Mathematical Physics **44**, 558 (2003).

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- Suppose perms. σ_1 and σ_3 anticommute even though 1,2 are far from 3,4.
  Let Alice prepare initial state $(1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$
  and perform $|\uparrow\rangle\langle\uparrow| \otimes \sigma_1 + |\downarrow\rangle\langle\downarrow| \otimes I$ At time $t = -\epsilon$
  and $|\uparrow\rangle\langle\uparrow| \otimes I + |\downarrow\rangle\langle\downarrow| \otimes \sigma_1$ At time $t = +\epsilon$.
  and, finally, σ_1 at time $t = 2$

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- Then, Alice will recover the initial state $(1/\sqrt{2})(|\uparrow\rangle+|\downarrow\rangle)$ at late times $t>2$.

Unless Bob (far away) performs σ_3 at times $t=0,1$,

in which case, Alice has $(1/\sqrt{2})(|\uparrow\rangle-|\downarrow\rangle)$

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- We could try gauging the order parameter in order to get rid of this long-ranged force. Hedgehog -> 't Hooft-Polyakov monopole.

$$H = i\chi(\partial_i\gamma_i + n_i\Gamma_i)\chi + \kappa(\partial_i n_j)^2 \rightarrow i\chi(D_i\gamma_i + n_i\Gamma_i)\chi + \kappa(D_i n_j)^2$$

But the resulting gauge theory suffers from Witten's SU(2) anomaly. Curing anomaly also eliminates Majorana zero modes.

Conclusions

- Defects in 3D systems can have Majorana zero modes. This actually occurs for configurations which are topologically non-trivial in the entire space of gapped fermion Hamiltonians which are quadratic in the low-energy limit.
- These defects have non-trivial 'statistics.'
When the defects are exchanged and the intervening field configuration is healed, the Majorana fermions transform non-trivially.
- The 'statistics' realizes a *projective representation* of the fundamental group of the multi-defect configuration space.
There is an extra, purely quantum-mechanical -1.
- It is not possible to realize this type of 'statistics' with weakly-coupled particles.
The defects have (at least) a linear confining force, and killing this confining force also kills the Majorana zero modes.