Title: Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order

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Abstract: Adiabatic evolutions connect two gapped quantum states in the same phase. We argue that the adiabatic evolutions are closely related to local unitary transformations which define a equivalence relation. So the equivalence classes of the local unitary transformations are the universality classes that define the different phases of quantum system. Since local unitary transformations can remove local entanglements, the above equivalence/universality classes correspond to pattern of long range entanglement, which is the essence of topological order. The local unitary transformation also allows us to define wave function renormalization, where a wave function can flow to a simpler one within the same equivalence/universality class. Using such a setup, we find conditions on the possible fixed-point wave functions where the local unitary transformations have finite dimensions. The solutions of the conditions allow us to classify this type of topological orders, which include all the string-net states.

# Local unitary transformation, long-range entanglement, wave function renormalization, and topological order

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Emergence and entanglement, PI;    May, 2010

Xie Chen, Zheng-Cheng Gu, XGW
arXiv:1004.3835

Xie Chen

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- **Describe topological order by what it is**:
- robust and topology-dependent ground state degeneracy Wen 89
- non-Abelian Berry phases of degenerate ground states induced by modular transformations Wen 90

  may completely characterize/define topological orders
- robust gapless edge excitations in FQH states Wen 91
- topological entanglement entropy Kitaev, Priskill 06; Levin, Wen 06

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Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, are in the same phase iff they are related through a local unitary (LU) evolution

$$|\Psi(1)\rangle = P\left(e^{-i\int_0^1 dg\, \tilde{H}(g)}\right)|\Psi(0)\rangle$$

where $\tilde{H}(g) = \sum_{\mathbf{i}} O_{\mathbf{i}}(g)$ and $O_{\mathbf{i}}(g)$ are local hermitian operators.

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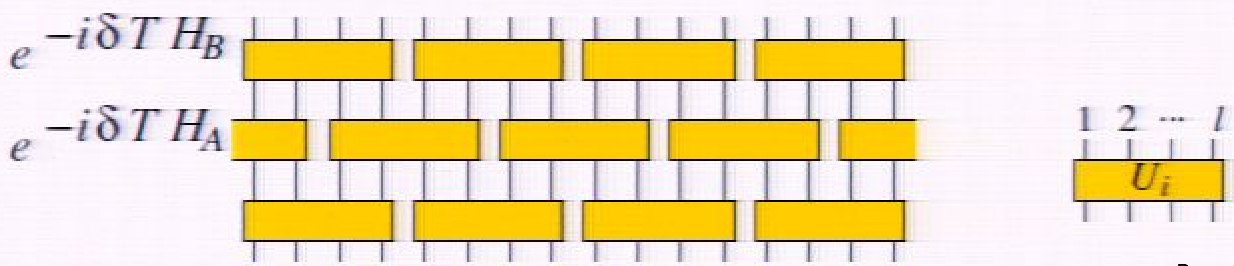

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- The trivial class: states equivalent to product state under LU transformations. They belong to the same class (phase) $\rightarrow$ short-range entanglement and trivial topological order.
- Non-trivial classes: states not equivalent to direct-product states. Those states form many different equivalent classes (phases) $\rightarrow$ many patterns of long-range entanglements and many different topological orders.
- In absence of symmetry:

**Quantum phases of matter**
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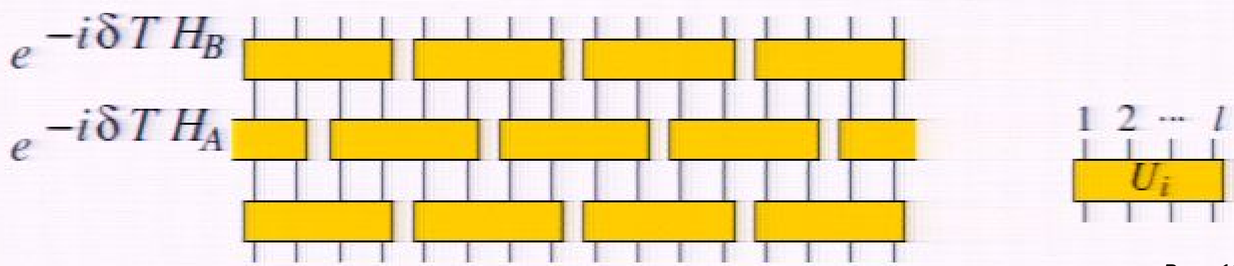

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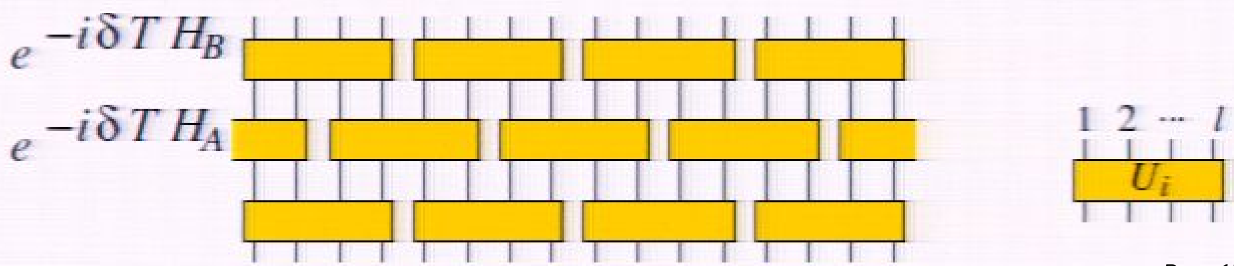

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# cal unitary trans. and wave function renormalization

- **The concept of support space**

The total wave function $\Phi(i, j, ..., a, b, ...)$ can be viewed as a wave function on a region $A$: $\Phi_{a,b,...}(i, j, ...)$, where $i, j, ...$ label states in the region $A$ and $a, b, ...$ label states out side of the region $A$.
$\Phi_{a,b,...}(i, j, ...)$ for all different $a, b, ...$ span the support space $\mathcal{V}_A^{sp}$ of the region $A$.
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- **The concept of generalized unitary transformation**

A generalized unitary transformation $U: \ \mathcal{V}_A \to \mathcal{V}_A^{sp}$ shrinks the degrees of freedom in $A$ *without loosing any quantum information*.
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  Answer:  (need to generalize tensor category theory)

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But this is a many-many to one labeling scheme.

Under the wave function renormalization, the wave function flows to simpler one within the same equivalent class.

- Use the fixed-point wave function: $\Phi \to \Phi_{\text{fix}}$ to label topological order.

Hopefully $\Phi_{\text{fix}}$ can give us a one-to-one labeling of topological order, and a classification of topological order.

## assify topological orders with fixed-point LU trans.

But using fixed-point wave function $\Phi_{\mathrm{fix}}$ to label topological orders has one problem:
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$$\Phi_1 \xrightarrow{U_1} \Phi_2 \xrightarrow{U_2} \Phi_3 \xrightarrow{U_3}$$

**The concept of fixed-point state**

- A fixed-point state is not one wave function, but a family of wave functions, $\Phi_n$, one wave function of each size/shape of lattice.
- "Fixed point" does not mean the fixed wave function.

It means a fixed relation between those wave functions,
– *fixed-point local unitary (LU) transformation,* $U_\infty$

$$\Phi(\text{lattice-2}) = U_\infty \Phi(\text{lattice-1}), \quad \Phi(\text{lattice-3}) = U_\infty \Phi(\text{lattice-2})$$

**Topological orders are classified by fixed-point local unitary transformations**

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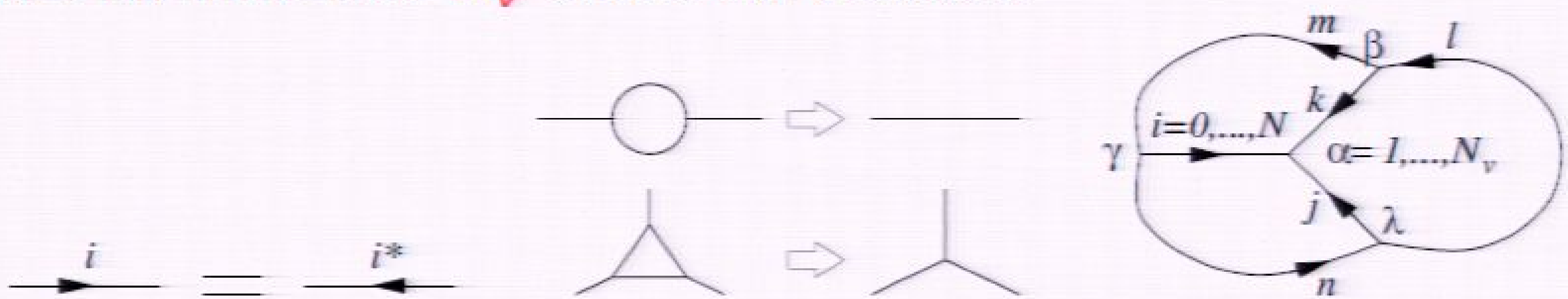
**Graphic state:**

- Fixed-point wave functions are defined on graphs, with $N+1$ states on links and $N_v$ states on vertices:



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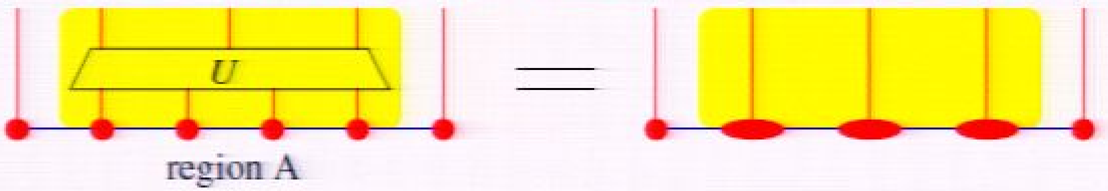

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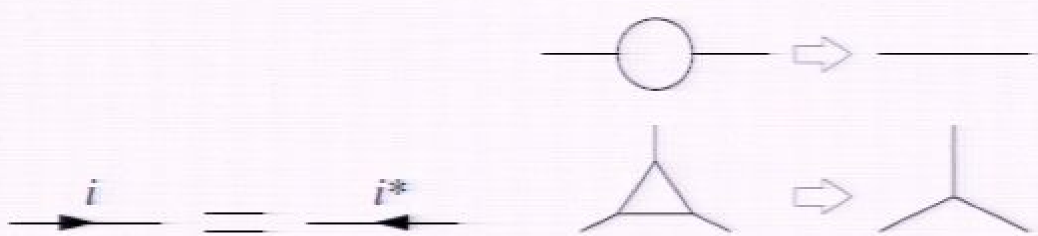
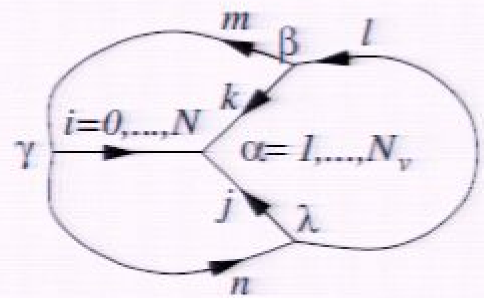



- **Support space and support dimension with boundary**:


$$\Phi\left( \cdots \right) = \Phi(\alpha,\beta,m,i,j,k,l;\Gamma) = \psi_{i,j,k,l,\Gamma}(\alpha,\beta,m)$$

Support space on $\alpha\beta m$: $V_{ijkl^*} = \{\psi_{i,j,k,l,\Gamma}(\alpha,\beta,m) | \text{fix } ijkl, \text{ vary } \Gamma\}$

Support dim.: $D_{ijkl^*} = \dim V_{ijkl^*}$ for the region bounded by $ijkl$.

- The support dimension of the $\Phi$ (  ) on a region bounded by links $i, j, k$: $D_{ijk} \leq N_v$



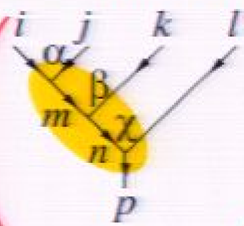
$\rightarrow$ shrink the range $\alpha = 1, ..., N_{ijk} = D_{ijk}$ (which depends on $ijk$).

- **Saturation condition**: For $\Phi =$ (  ) :



The support dimension $D_{ijkl^*}$
$=$ The number of $\alpha\beta m = \sum_m N_{jim^*} N_{kml^*}$

- Similar saturation condition for any "tree" region

$\Phi =$ (  ) $\qquad D_{ijklp^*} = \sum_{m,n} N_{jim^*} N_{mn^*k} N_{np^*l}$

# he first kind of wave function renormalization: F-move

- The fixed-point wave functions are related by a fixed LU trans.:

$$\Phi(\text{graph-2}) = U_\infty \Phi(\text{graph-1})$$

- $\Phi = \Bigg($ 

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dimension $D_{ijkl^*} = \tilde{D}_{ijkl^*} \to$

$$\sum_m N_{jim^*} N_{kml^*} = \sum_n N_{kjn^*} N_{l^*ni} \equiv N_{ijkl^*}$$

- The two wave functions are related by a LU trans. Leven, Wen, 04

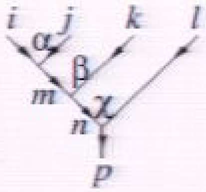
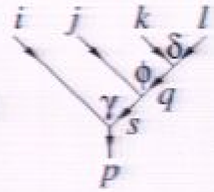
F-move: $\Phi \Bigg($ 

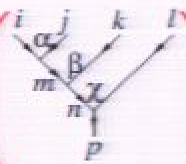
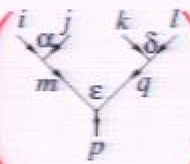
$\Bigg) = \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kjn^*}} \sum_{\delta=1}^{N_{nil^*}} F^{ijm,\alpha\beta}_{kln,\chi\delta} \Phi \Bigg($ 

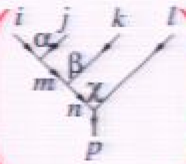
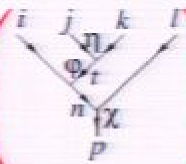
$\Bigg)$

The matrix $F^{ij}_{kl} \to (F^{ij}_{kl})^{m,\alpha\beta}_{n,\chi\delta}$ is unitary and has a dimension $N_{ijkl^*}$

# he pentagon identity

 can be trans. to  through two different paths:

$\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon}\,\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}\,\Phi\Big($  $\Big)$

$\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi} F^{ijm,\alpha\beta}_{knt,\eta\varphi}\,\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma}\,\Phi\Big($  $\Big)$

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi}\,\Phi\Big($  $\Big).$

The two paths should lead to the same LU trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} = \sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity.

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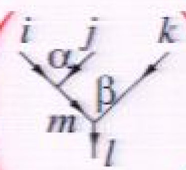
F-move: $\Phi\Bigg($  $\Bigg) = \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kjn^*}} \sum_{\delta=1}^{N_{nil^*}} F^{ijm,\alpha\beta}_{kln,\chi\delta} \Phi \Bigg($  $\Bigg)$





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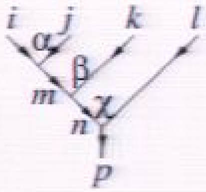
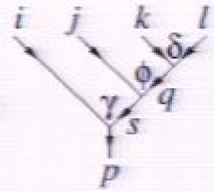
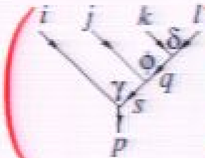
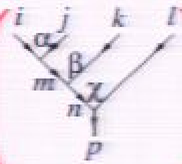
 can be trans. to  through two different paths:

$\Phi$ (  ) $= \sum_{q,\delta,\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} \Phi$ (  ) $= \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma} \Phi$ (  )

$\Phi$ (  ) $= \sum_{t,\eta,\varphi} F^{ijm,\alpha\beta}_{knt,\eta\varphi} \Phi$ (  ) $= \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} \Phi$ (  )

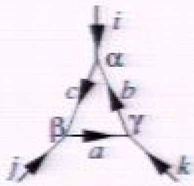
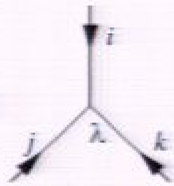
$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} \Phi$ (  ) .

The two paths should lead to the same LU trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} = \sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}$$

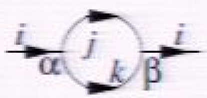
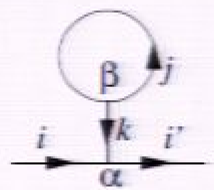
Such a set of non-linear algebraic equations is the famous pentagon identity.

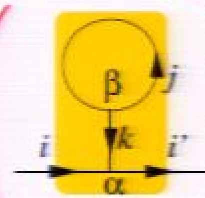
## he second kind of wave function renormalization: P-move

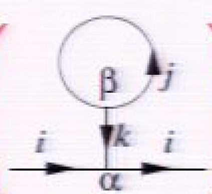
- First way:  to  which reduce the degrees of freedom.

But notice that, through the F-move, 

- Second way:

 to  Levin, Wen, 04 or  to  Koenig, Reichardt, Vidal, 09

The support dimension of $\Phi$ (  ) : $D_{ii'^*} = \delta_{ii'}$

$\Phi$ (  ) $= P_i^{kj,\alpha\beta}\Phi$ (  ) ,

$$\sum_{\alpha=1}^{N_{kii^*}} \sum_{\beta=1}^{N_{j^*jk^*}} P_i^{kj,\alpha\beta}(P_i^{kj,\alpha\beta})^* = 1$$

# onsistent conditions between $F^{ijm,\alpha\beta}_{kln,\chi\delta}$ and $P^{kj,\alpha\beta}_{i}$

From

$\Phi$ (  ) $= \displaystyle\sum_{n=0}^{N}\sum_{\chi,\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta}\,\Phi$ (  )

$\rightarrow \quad P^{jp,\alpha\eta}_{i}\delta_{im}\Phi$ (  ) $= \displaystyle\sum_{n,\chi,\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta} P^{jp,\chi\eta}_{k^*}\delta_{kn}\Phi$ (  )

we find more non-linear equation

$$P^{jp,\alpha\eta}_{i}\delta_{im}\delta_{\beta\delta} = \sum_{\chi=1}^{N_{kjk^*}} F^{ijm,\alpha\beta}_{klk,\chi\delta} P^{jp,\chi\eta}_{k^*}$$

for all $k, i, l$ satisfying $N_{kil^*} > 0$

# classification of topological orders

The data $N_{ijk}$, $F^{ijm,\alpha\beta}_{kln,\chi\delta}$, $P^{kj,\alpha\beta}_{i}$ classify the fixed-point LU transformation and topological orders. They satisfy

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- $P^{jp,\alpha\eta}_{i} \delta_{im} \delta_{\beta\delta} = \sum_{\chi} F^{ijm,\alpha\beta}_{klk,\chi\delta} P^{jp,\chi\eta}_{k^*}$ for all $k, i, l$ with $N_{kil^*} > 0$.

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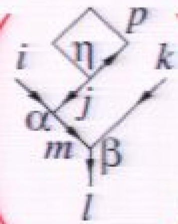
$\Phi$ (  ) $= \sum_{n=0}^{N} \sum_{\chi,\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta}\, \Phi$ (  )

$\to \quad P^{jp,\alpha\eta}_{i} \delta_{im} \Phi$ (  ) $= \sum_{n,\chi,\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta} P^{jp,\chi\eta}_{k^*} \delta_{kn} \Phi$ (  )

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# mple solutions of the non-linear equations

We choose $N_{000} = N_{110} = N_{101} = N_{011} = 1$, other $N_{ijk} = 0$, and find two sets of solutions

$$F^{000}_{000} = 1$$

$$F^{000}_{111} = (F^{011}_{100})^* = (F^{101}_{010})^* = F^{110}_{001} = 1$$

$$F^{011}_{011} = (F^{101}_{101})^* = 1$$

$$F^{110}_{110} = \eta = \pm 1$$

Both solutions are closed-string states:

Freedman, Nayak, Shtengel, Walker, Wang, 04; Levin, Wen, 04

- $\eta = 1$: $\Phi(\text{loops}) = 1$

Effective theory: $Z_2$ gauge theory

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Effective theory: $U(1) \times U(1)$ Chern-Simons gauge theory with semion excitations

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$$F^{110}_{110} = \gamma$$

$$F^{110}_{111} = (F^{111}_{110})^* = \sqrt{\gamma}$$

$$F^{111}_{111} = -\gamma, \qquad \gamma = (\sqrt{5} - 1)/2$$

$N = 1$ string-net state Freedman, Nayak, Shtengel, Walker, Wang, 04; Levin, Wen, 04
Low energy effective theory: $SO(3) \times SO(3)$ Chern-Simons theory with non-Abelian statistics.

## mmary

- Local unitary transformation defines quantum phases:
  *Equivalent classes of LU transformation =*
  *universality classes of quantum phases.*
- With no symmetry,
  all short-range entangled states belong to the same phase
  $\rightarrow$ the trivial topological order
  Other states can belong to the different classes
  $\rightarrow$ many patterns of long-range entanglement
  and many different non-trivial topological orders.
  **Topological orders = patterns of long-range entanglement**
- The data $N_{ijk}, F^{ijm,\alpha\beta}_{kln,\chi\delta}, P^{kj,\alpha\beta}_{i}$ that satisfy a set of non-linear equations classify a kind of topological orders.
  - All the topological properties can be calculated from the data.
  - Exact soluble Hamiltonian can be constructed from the data.
  - A generalization of string-net approach.

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$$\rightarrow \quad P^{jp,\alpha\eta}_{i} \delta_{im} \Phi\left( \cdots \right) = \sum_{n,\chi,\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta} P^{jp,\chi\eta}_{k^*} \delta_{kn} \Phi\left( \cdots \right)$$



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