Title: From few to many

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Abstract: I discuss a class of systems with a very special property: exact results for physical quantities can be found in the many-body limit in terms of the original (bare) parameters in the Hamiltonian. A classic result of this type is Onsager and Yang's formula for the magnetization in the Ising model. I show how analogous results occur in a fermion chain with strong interactions, closely related to the XXZ spin chain. This is done by exploiting a supersymmetry, and noting that certain quantites are independent of finite-size effects. I also discuss how these ideas are related to an interacting generalization of the Kitaev honeycomb model.

From few to many

P. Fendley

How do we deal with many degrees of freedom?

"Mathematically, the composition-temperature curve in a solid solution presents the same problem as the degree of order in a ferromagnetic with a scalar spin. B. Kaufman and I have recently solved the latter problem (unpublished) for a two-dimensional rectangular net with interaction energies J, J'. If we write $\sinh(2J/kT)\sinh(2J'/kT) = 1/k$, then the degree of order for $k < 1$ is simply $(1 - k^2)^{\frac{1}{8}}$."

L. Onsager, *Nuovo Cim. Suppl.* 2(9)(1949):261 (in Rushbrooke's article)

Solving a system with more than a few degrees of freedom is usually an impossible task.

A common approach is usually Landau theory, or more generally, effective field theory.

While almost always valuable qualitatively, and often valuable quantitatively, at and near many interesting critical points it often fails miserably.

Take the two-dimensional Ising model, or equivalently, the quantum Ising chain.

The Hilbert space is a chain of two-state systems, i.e. $(\mathbb{C}^2)^{\otimes N}$. The Hamiltonian includes a term which can flip the "spin", and a nearest-neighbor interaction term:

$$H = \sum_{j=1}^{L} \left(k\sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right) ,$$

where the σ_j^a are the Pauli matrices acting at site j, and the identity on the other sites.

For $k < 1$, the model should be ordered: neighboring spins want to line up, spontaneously breaking the $\mathbb{Z}_2$ spin-flip symmetry.

Landau theory is easy to apply: it predicts that near the critical point $k = 1$:

$$\langle \sigma_j^z \rangle \propto \sqrt{1-k}$$

But Onsager, Kaufman and Yang tell us that

$$\langle \sigma_j^z \rangle = (1-k^2)^{1/8}$$

exactly as $L \to \infty$!!!

Now we know a lot about how to understand behavior at critical points: renormalization group, epsilon expansion, conformal field theory, large N, integrability, …

But why is the result for the spontaneous magnetization in the Ising model both ridiculously simple and exact?

Outline:

1. What's so special about the 2d Ising model/1d quantum Ising chain?
2. The XYZ chain
3. Simple and exact formulas for the magnetization and gap by exploiting scale free behavior
4. Models with explicit supersymmetry
5. Completely unbelievable (but true) results

new work with C. Hagendorf

What's so special about the Ising chain?

Yes, I know it can be mapped onto free fermions. But:

The map from spins onto fermions is non-local. Thus the computation of the magnetization is still pretty complicated, as opposed to the computation of the free energy.

The analogous computation for dimers on the triangular lattice was only done a few years ago

Fendley, Moessner and Sondhi; Basor and Ehrhardt

So the question should be:

Are there any non-free fermion models with simple and exact formulas for expectation values?

Yes!

The N-state chiral Potts model is a parity-breaking $\mathbb{Z}_N$ generalization of the Ising model with some amazing properties

Howes, Kadanoff and den Nijs; von Gehlen and Rittenberg.

It is definitely not a free-fermion model (except for the Ising case $N = 2$).

Yet the order parameters for spontaneously breaking the $\mathbb{Z}_N$ symmetry are

$$\langle e^{2\pi i r \sigma_j / N} \rangle = (1 - k^2)^{r(N-r)/(2N^2)}$$

conjectured in 1989 by Albertini, McCoy, Perk and Tang, proved in 2005 by Baxter.

So the question should be:

What's so special about the chiral Potts models?

- Along a parameter line (the "superintegrable" line), it possesses an unusual symmetry algebra, the Onsager algebra. Writing $H = H_0 + kH_1$, then

$$[H_0, [H_0, [H_0, H_1]]] = [H_0, H_1]$$

  This was how Onsager originally solved the Ising model! It allows the explicit construction of an infinite sequence of conserved quantities.

- The coupling of the chiral symmetry-breaking term in the field theory does not renormalize.

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- The coupling of the chiral symmetry-breaking term in the field theory does not renormalize.

Supersymmetric field theories have such properties. In particular, they often have non-renormalization theorems.

For example, in 1+1 dimensional N=(2,2) supersymmetric field theories, the potential energy receives no correction beyond naive scaling.

Thus if the potential has multiple minima, you can compute the kink energy exactly with a pedestrian computation.

So we should look for statistical-mechanical models that turn into supersymmetric field theories in their scaling limit.

One way to do this is to look at models with explicit supersymmetry on the lattice.

This we can do. However, there are more famous models that turn into supersymmetric field theories...

The XYZ spin chain

$$H = -\sum_{j=1}^{L} \left[J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right]$$

becomes a supersymmetric field theory in the scaling limit when

$$J_x J_y + J_x J_z + J_y J_z = 0$$

This is easy to prove at and near the critical point $J_x = J_y$: the field theory is that of a free massless boson at the supersymmetric radius. The operator perturbing away from criticality has the same dimension as the supersymmetry-preserving one in the field theory.

There are host of properties similar to those occurring in supersymmetric models.

Old Baxter result: ground state energy is $E_0 = -3(s^2+3)L$ as $L \to \infty$.

(Partially proved) conjecture of Stroganov: the shifted Hamiltonian $H - E_0$ has exactly zero energy at finite size when the number of sites is odd.

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We dub the XYZ chain on the supersymmetric line $J_x J_y + J_x J_z + J_y J_z = 0$ the sXYZ chain. It can be parametrized as

$$J_x = 2s(s-3), \qquad J_y = 2s(s+3), \qquad J_z = 9 - s^2$$

so that $s = \pm 1, \infty$ are critical.

The symmetries $s \to (3-s)/(s+1)$ and $s \to -s$ permute the couplings.

At $s = 0$, only the J_z term remains, with a negative coefficient. Thus the spins are ordered in this limit.

The magnetization

$$M_L(s) \equiv \langle \sigma_j \rangle$$

obeys $M_L(1) = 1$ in this limit.

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Expanding around the ordered limit (using Maple), the magnetization $M_L(s)$ with periodic boundary conditions on L sites:

$$
\begin{aligned}
M_5 &= 1 - 4\tilde{s}^2 - 12\tilde{s}^4 + 188\tilde{s}^6 - 844\tilde{s}^8 + 380\tilde{s}^{10} + \ldots \\
M_7 &= 1 - 4\tilde{s}^2 - 12\tilde{s}^4 - 52\tilde{s}^6 + 2516\tilde{s}^8 - 18004\tilde{s}^{10} + \ldots \\
M_9 &= 1 - 4\tilde{s}^2 - 12\tilde{s}^4 - 52\tilde{s}^6 - 284\tilde{s}^8 + 33516\tilde{s}^{10} + \ldots \\
M_{11} &= 1 - 4\tilde{s}^2 - 12\tilde{s}^4 - 52\tilde{s}^6 - 284\tilde{s}^8 - 1764\tilde{s}^{10} + \ldots
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where $\tilde{s} = s/3$.

The terms up to order L are independent of L !!!

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We say that that quantities are scale free when the first L terms in the expansion are independent of L.

The expansion

$$M_\infty(s) = 1 - 4\,\tilde{s}^2 - 12\,\tilde{s}^4 - 52\,\tilde{s}^6 - 284\,\tilde{s}^8 - 1764\,\tilde{s}^{10} + \dots$$

should be exact. We've obtained exact $L \to \infty$ results by solving at (very) finite L!

To sum this series, we do what any good combinatorialist would do: go to the On-Line Encyclopedia of Integer Sequences:

AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,3,13,71,441 Search Hints

Search: **1, 3, 13, 71, 441**

Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A162326 Let y = y(x) satisfy g(x,y(x)) = 0. The sequence a(n) is the number of terms in the expansion of the divided difference [x0,...,xn]y in terms of bivariate divided differences of g. +20 4

1, 3, 13, 71, 441, 2955, 20805, 151695, 1135345, 8671763, 67320573, 529626839, 4213228969, 33833367963, 273892683573, 2232832964895, 18314495896545, 151037687326755, 1251606057754605, 10416531069771111, 87029307323766681 (list; graph; listen)

OFFSET 1,2

FORMULA

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Let E = N x N \ {(0,0), (0,1)} be a set of pairs of natural numbers. The
  number of terms a(n) is the coefficient of x^n*y^{n-1} of the generating
  function
1 - log(1 - \sum_{(s,t) in E} x^s*y^{s+t-1}) = 1 + \sum_{q >= 1}
  (\sum_{(s,t) in E} x^s*y^{s+t-1})^q / q
```

EXAMPLE

```
Write [0...n]y for [x0,...,xn]y and [0...s,0...t]g for
  [x0,...,xs;y0,...,yt]g.
For n = 1 one finds 1 term, [01]y = -[01;1]g/[0;01]g.
For n = 2 one finds 3 terms, [012]y = -[012;2]g/[0;02]g +
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  def P(n, q): ....E = CartesianProduct(range(n+1), range(n+1)) ....E =
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  ....return sum([X^s * Y^(s+t-1) for (s, t) in E]) . R.<X, Y> =
  PolynomialRing(ZZ, 2) . n = 11 h = expand(1 + sum([((P(n, q))^q)/q for q
  in range(1, 2*n)])) for k in range(1, n+1): ....print k,
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```

CROSSREFS

Cf. A172003, which is a generalization to bivariate implicit functions.
Cf. A003262, which is the analogous sequence for implicit derivatives.

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CROSSREFS Cf. A172003, which is a generalization to bivariate implicit functions.

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So we have to think a little harder (although that looks an awful lot like a partition function...)

From Baxter we know that the dimension of the perturbing operator is $4/3$, and from standard Coulomb gas/CFT arguments, the magnetization operator has dimension 1/3. Indeed, the finite-size values at criticality fit nicely to $M_L(1) \approx .9552745\, L^{-1/3}(1 + O(L^{-2}))$.

Thus as $s \to 1^-$, $M_\infty(s)$ should vanish as

$$(1 - s)^\beta$$

with $\beta = (1/3)/(2 - 4/3) = 1/2$ (as opposed to $1/8$ for Ising).

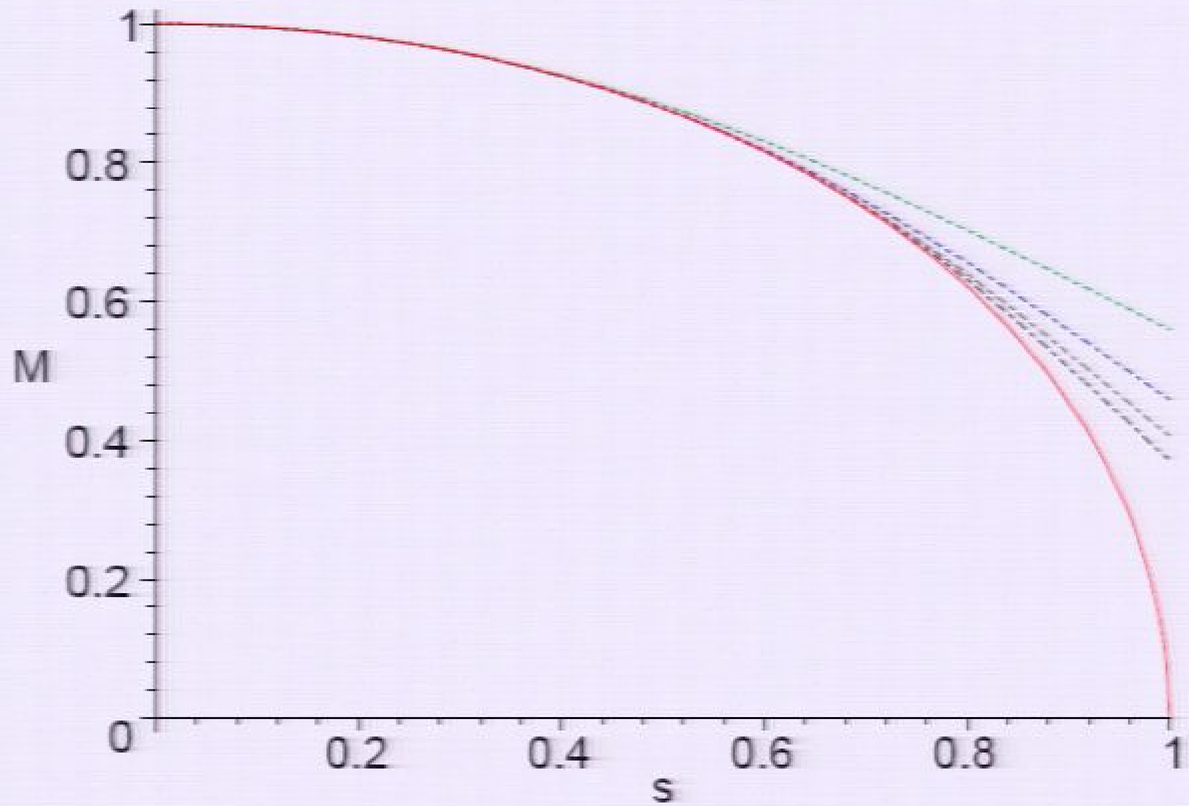
The expected square-root singularity at the critical point means that it might be a good idea to square the series for $M_L(s)$:

$$(M_L(s))^2 = 1 - 8\tilde{s}^2 - 8\tilde{s}^4 - 8\tilde{s}^6 - 8\tilde{s}^8 - 8\tilde{s}^{10} - \ldots + O(s^{L+1})$$

So our conjecture for the exact magnetization as $L \to \infty$:

$$M_\infty(s) = 3\left(\frac{1-s^2}{9-s^2}\right)^{1/2}$$

Not much more complicated than Onsager, Kaufman and Yang's formula!



The solid red curve is the conjecture for $M_\infty(s)$, while the dashed curves are for $L = 5, 9, 13, 17.$

Other quantities are scale free.

For example, for $s < 1$

$$\begin{aligned} \langle 0|\sigma_j^a \sigma_{j+1}^a|0\rangle &= 1 + 4\tilde{s}^2(-1 + \tilde{s}^2 + 3\tilde{s}^4 + 5\tilde{s}^6 + 7\tilde{s}^8 + \ldots) \\ &= 1 + 12\frac{s^2(s^2-3)}{(s^2-9)^2} + O(s^{L+1}) \end{aligned}$$

We can find this expectation value for $s > 1$ by expanding around $s = 3$, where $J_x = J_z = 0$. Letting $t = (3-s)/6$,

$$\begin{aligned} \langle 0|\sigma_j^a \sigma_{j+1}^a|0\rangle &= \frac{1}{2}(2t + 3t^2 + 4t^3 + 5t^4 + \ldots) \\ &= \frac{(s+9)(3-s)}{2(3+s)^2} + O(t^{L+1}) \end{aligned}$$

The one-kink gap also is scale free.

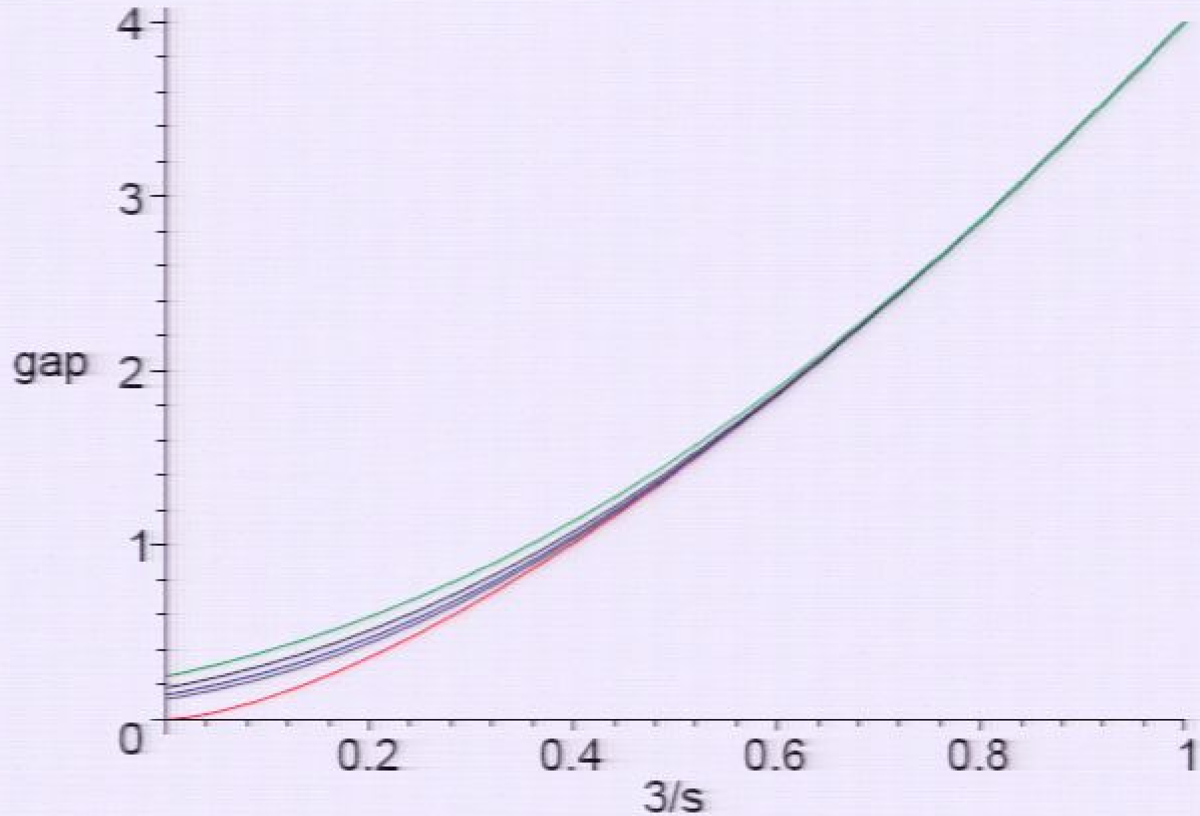
Since there is a spontaneously broken $\mathbb{Z}_2$ symmetry away from the critical points, think of the gapped excited states as kinks separating regions of the two ground states.

To define the gap to the one-kink state, we consider an even number of sites with twisted boundary conditions (aka a spin-flip defect)

We found the exact one-kink energy Δ_L for sizes up to $L = 10$. Expanding this in a power series around $s = 3$ in terms of $v = 1 - 3/s$, we find that

$$\begin{aligned} \Delta_L &= 4 - 6\,v + 3\,v^2/2 + v^3/4 + 3\,v^4/32 + \dots \\ &= 4\left(\frac{3}{s}\right)^{3/2} + O(v^{L/2}) \end{aligned}$$

At the critical point $s \to \infty$, the gap vanishes with exponent $\nu = 3/2$, exactly what one expects with dimension-4/3 thermal operator: $\nu = 1/(2 - 4/3) = 3/2$.



The solid red curve is the conjecture $4(3/s)^{3/2}$ while the others are for $L = 6, 8, 10, 12.$

So what's going on??!??!?

I haven't yet found the symmetry algebra and related conserved quantities analogously to free fermions or chiral Potts, but I expect that this is possible.

Studying chains with an explicit supersymmetry illuminates the situation:

$N = 2$ supersymmetry:

There are two conserved fermionic charges Q and $Q^\dagger$ obeying $(Q)^2 = (Q^\dagger)^2 = 0$. The Hamiltonian is

$$H = \{Q, Q^\dagger\}$$

Q increases fermion number by one, $Q^\dagger$ decreases it.

All one needs to do is find a Q which squares to zero to create a supersymmetric lattice model.

This supersymmetry has many deep consequences.

- The energy E is never negative.
- An $E = 0$ ground state is annihilated by both Q and $Q^\dagger$.
- All other states form boson-fermion doublets under the supersymmetry.
- A lower bound on the number of $E = 0$ ground states (the Witten index) often can be computed exactly.

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A too-simple model with supersymmetry has

$$Q = \sum_i c_i^\dagger$$

where $c_i^\dagger$ creates a (spinless) fermion at lattice site i.

$$Q^2 = \sum_{i,j} c_i^\dagger c_j^\dagger = \sum_{i<j} \{c_i^\dagger, c_j^\dagger\} = 0$$

The Hamiltonian is trivial:

$$H = \{Q, Q^\dagger\} = \sum_{i,j} \{c_i, c_j^\dagger\} = \sum_{i,j} \delta_{ij} = N$$

for N lattice sites.

A model in the same universality class as the sXYZ model comes by from considering the "hard-core" fermion chain, where fermions are forbidden to be on adjacent sites:

$$Q = \sum_i \lambda_i (1 - n_{i-1}) c_i^\dagger (1 - n_{i+1})$$

where $n_i = c_i^\dagger c_i$.

For any choice of the λ_i, $Q^2 = 0$, and

$$\begin{aligned} H &= \sum_{j=1}^{3f} [(1 - n_{j-1})(\lambda_j^* \lambda_{j+1} c_j^\dagger c_{j+1} + h.c.)(1 - n_{j+2}) \\ &\quad + |\lambda_j|^2 (1 - n_{j-1})(1 - n_{j+1})] \end{aligned}$$

i.e. a hopping term and a finite-tuned potential.

One might expect that the ground states are the three "Néel" states with

. . . ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ . . .

This is partially right: for any choice of the λ_i, the ground states have f fermions on $3f$ sites, but there are only two of them

Fendley, Schoutens and de Boer

To favor or disfavor putting particles on every third site, we stagger the model by choosing $\lambda_{3i} = \lambda_{3i+1} = 1$, $\lambda_{3i-1} = z$.

There is one ground state for each parity, which we label $|\pm\rangle$. The analogs of the magnetization are then the staggered densities $D^{\pm}(z) = \langle \pm | c^{\dagger}_{3i-1} c_{3i-1} | \pm \rangle$

Then the now-usual miracle happens:

$$\begin{aligned} D^{+} + D^{-} &= 1 - 3z^2 + 3z^4 - 3z^6 + 3z^8 - \ldots \\ &= \frac{8 - 2z^2}{8 + z^2} + O(z^{2f}), \\ D^{+} - D^{-} &= 1 - 5z^2 - 3z^4 - 29z^6 - 131z^8 - \ldots \\ &= \frac{8\sqrt{1 - z^2}}{8 + z^2} + O(z^{2f}) \end{aligned}$$

A model in the same universality class as the sXYZ model comes by from considering the "hard-core" fermion chain, where fermions are forbidden to be on adjacent sites:

$$Q = \sum_i \lambda_i (1 - n_{i-1}) c_i^\dagger (1 - n_{i+1})$$

where $n_i = c_i^\dagger c_i$.

For any choice of the λ_i, $Q^2 = 0$, and

$$H = \sum_{j=1}^{3f} [(1 - n_{j-1})(\lambda_j^* \lambda_{j+1} c_j^\dagger c_{j+1} + h.c.)(1 - n_{j+2}) + |\lambda_j|^2 (1 - n_{j-1})(1 - n_{j+1})]$$

i.e. a hopping term and a finite-tuned potential.

To favor or disfavor putting particles on every third site, we stagger the model by choosing $\lambda_{3i} = \lambda_{3i+1} = 1$, $\lambda_{3i-1} = z$.

There is one ground state for each parity, which we label $|\pm\rangle$. The analogs of the magnetization are then the staggered densities $D^{\pm}(z) = \langle\pm|c^{\dagger}_{3i-1}c_{3i-1}|\pm\rangle$

Then the now-usual miracle happens:

$$
\begin{aligned}
D^{+} + D^{-} &= 1 - 3z^2 + 3z^4 - 3z^6 + 3z^8 - \dots \\
&= \frac{8 - 2z^2}{8 + z^2} + O(z^{2f}). \\
D^{+} - D^{-} &= 1 - 5z^2 - 3z^4 - 29z^6 - 131z^8 - \dots \\
&= \frac{8\sqrt{1 - z^2}}{8 + z^2} + O(z^{2f})
\end{aligned}
$$

One might expect that the ground states are the three “Néel” states with

. . . ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● ○ ○ . . .

This is partially right: for any choice of the λ_i, the ground states have f fermions on $3f$ sites, but there are only two of them

Fendley, Schoutens and de Boer

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Then the now-usual miracle happens:

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The series expansion around $z = \infty$ gives

$$\begin{aligned} D^+ + D^- &= \frac{2}{z^2} - \frac{6}{z^4} + \frac{26}{z^6} - \frac{134}{z^8} + \frac{762}{z^{10}} - \frac{4614}{z^{12}} + \dots \\ &= \frac{4}{z^2 + z\sqrt{8+z^2} + 2} + O(z^{-4f}) \end{aligned}$$

This is ugly, so it suggests we define $S = 3z/\sqrt{z^2+8}$, to get

$$D^+ + D^- = \frac{2}{3}\frac{3-S}{S+1}$$

If this isn't amazing enough...

The XYZ model has a duality $s \to (3-s)/(s+1)$ that exchanges ordered and disordered phases. The field theory does as well; it corresponds to changing the sign of the off-critical perturbation.

No duality is obvious in the supersymmetric model. Nevertheless,

$$D^{+}(S)\,D^{-}(S) = D^{+}(\widehat{S})\,D^{-}(\widehat{S})$$

for $\widehat{S} = (3-S)/(S+1)$, even at finite size!!

But wait, there's more!

Let's look at the explicit zero-energy ground states. For sXYZ with 7 sites, the coefficients the parity- and translation-invariant states:

$$[1/\sqrt{7}(4\tilde{s}^4+3\tilde{s}^2+1), \tilde{s}(4\tilde{s}^4+3\tilde{s}^2+1), \tilde{s}^3(7\tilde{s}^2+1), \tilde{s}^3(7\tilde{s}^2+1),$$
$$\tilde{s}^2(2+5\tilde{s}^2+\tilde{s}^4), \tilde{s}^4\sqrt{2}(5+3\tilde{s}^2), \tilde{s}^2(4\tilde{s}^4+3\tilde{s}^2+1), \tilde{s}^4(7\tilde{s}^2+1), \tilde{s}^3(5+3\tilde{s}^2)]$$

with the first one the state with all spins up (the completely magnetized state).

Now look at the supersymmetric model with 12 sites. The coefficient of the completely staggered state (particle on every third site):

$$(-1+\tilde{S}^2)^2(1+3\tilde{S}^2+4\tilde{S}^4)$$

True in general!

But wait, there's still more!

In a remarkable series of papers, Bazhanov and Mangazeev showed that (at least for the same small systems we are studying here) these magic polynomials are related to the tau function of the Painlevé VI non-linear differential equation.

Using this, they find a recursion relation for these polynomials $\psi_n(s)$:

$$\begin{aligned} 2s(s-3)(3s-1)^2\partial_s^2 \log\psi_n \quad &+ \quad 2(s-1)^2(3s-1)\partial_s \log\psi_n \\ +72(2n+1)^2\frac{\psi_{n+1}\psi_{n-1}}{\psi_n^2} \quad &= \quad 9[4(3n+1)(3n+2)+(3s-1)n(5n+3)] \end{aligned}$$

Many, many open questions...

- Can we prove it in less than 15 years?
- Such models seem to be the simplest possible generalizations of free fermions in 1d. Can we make a Kitaev honeycomb model out of these?
- In some ways, these seem simpler than Ising. Could that be?
- Is there a direct map between the sXYZ chain and the supersymmetric model away from the critical point also?
- What's with Painlevé VI?