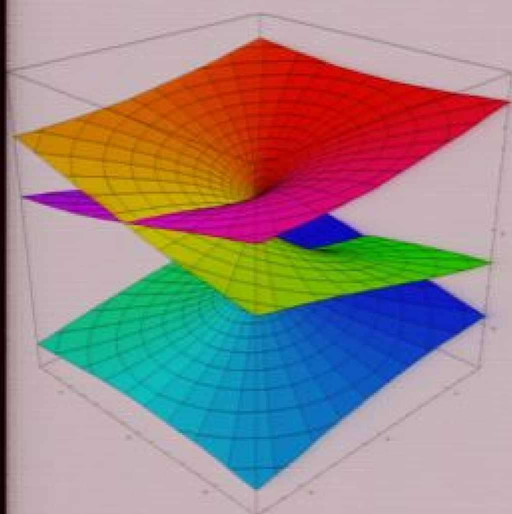
Title: Entanglement entropy in the O(N) model

Date: May 26, 2010 09:45 AM

URL: http://pirsa.org/10050069

Abstract: In recent years the characterization of many-body ground states via the entanglement of their wave-function has attracted a lot of attention. One useful measure of entanglement is provided by the entanglement entropy S.
The interest in this quantity has been sparked, in part, by the result that at one dimensional quantum critical points (QCPs) S scales logarithmically with the subsystem size with a universal coefficient related to the central charge of the conformal field theory describing the QCP. On the other hand, in spatial dimension d > 1 the leading contribution to the entanglement entropy scales as the area of the boundary of the subsystem. The coefficient of this "area law" is non-universal. However, in the neighbourhood of a QCP, S is believed to possess subleading universal corrections. In this talk, I will present the first field-theoretic study of entanglement entropy in dimension d > 1 at a stable interacting QCP - the quantum O(N) model. Our results confirm the presence of universal corrections to the entanglement entropy and exhibit a number of surprises such as different epsilon -> 0 limits of the Wilson-Fisher and Gaussian fixed points, violation of large N counting and subtle dependence on replica index.

Entanglement entropy in the O(N) model



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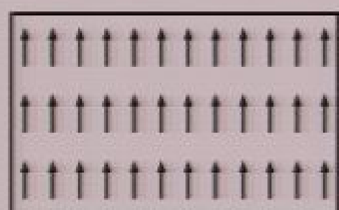
Phys. Rev. B **80**, 115122 (2009)

Emergence and Entanglement, Perimeter Institute

May 26, 2010

The venerable Landau-Ginzburg-Wilson paradigm

- Classical theory of phase transitions



ordered

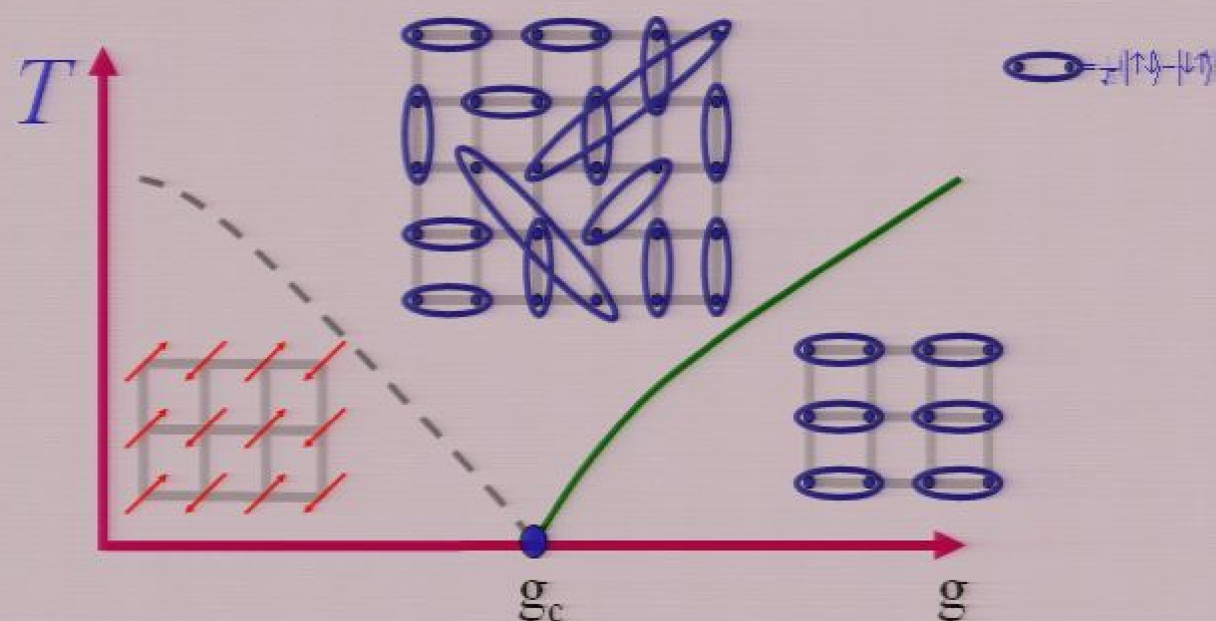
disordered

- spontaneous symmetry breaking
- local order parameter

→ universality class

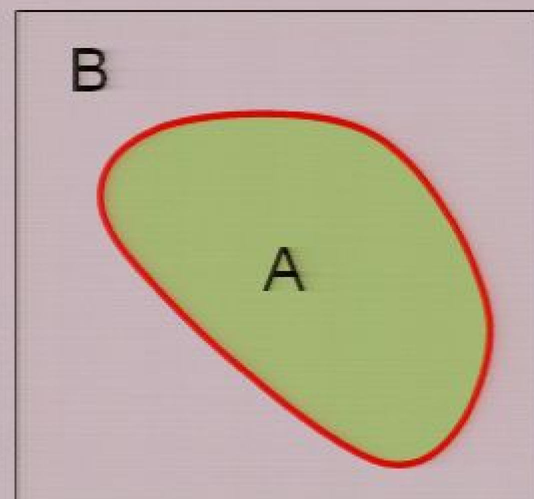
Quantum criticality

- Landau-Ginzburg paradigm describes some, but not all quantum phase transitions
- Quantum critical points realize some of the most non-classical states of matter
  - low energy excitations are neither particles nor waves
  - ground state wave-function is highly entangled



Entanglement entropy

- Goal: to characterize quantum critical points via entanglement properties of the many-body wave-function



$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_A = tr_B \rho$$

$$\rho_A = \sum_i p_i |i\rangle\langle i|$$ - mixed state

- Distribution of p_i - entanglement spectrum

- Entanglement entropy

$$S_A = -tr \rho_A \log \rho_A \qquad S_A = S_B$$

- inherently quantum mechanical, non-local observable

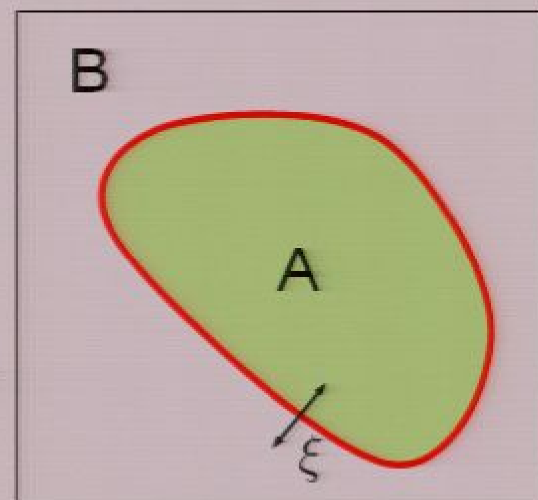
Entanglement entropy away from QCP

- Local Hamiltonian
- finite correlation length ξ

$$S \propto |\partial A|$$

$|\partial A|$ - area (length) of the boundary

- Area law: entanglement is local to the boundary



- For a generic state in the Hilbert space

$$S \sim \text{volume}(A)$$

Ground states of local Hamiltonians are very special!
(form set of measure zero in the Hilbert space)

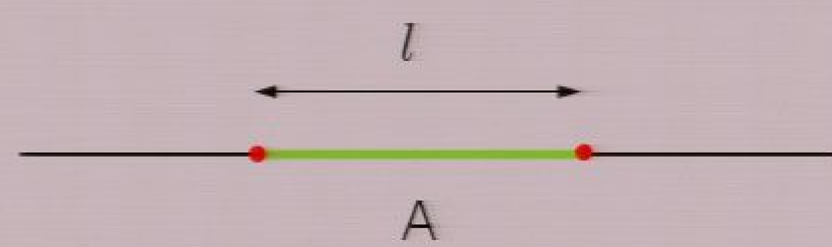
M. Srednicki (1993)

The entanglement entropy at 1d QCPs

- How does the entanglement entropy behave near a QCP?
- Problem completely solved for QCPs in d = 1 with z = 1

  - system described by a conformal field theory (CFT)

$$S = \frac{c}{3} \log l/a$$



c - central charge of the CFT

a - short distance cut-off

S - universal!

- Perturb the CFT slightly away from criticality

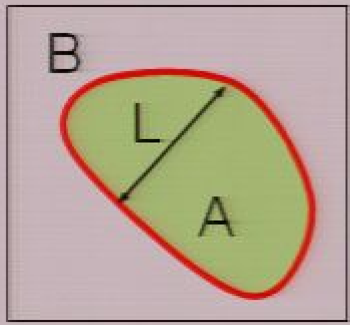
$$S = \frac{c}{3} \log \xi/a$$

C. Holzhey, F. Larsen and F. Wilczek (1994).
P. Calabrese and J. L. Cardy (2004).

Higher dimensional QCPs

- For QCPs described by local field theories

$$S = C\frac{|\partial A|}{a^{d-1}}$$ - area law

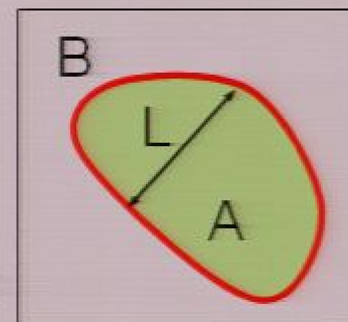


- Indistinguishable from a non-critical phase
- The coefficient C - non-universal ☹

Higher dimensional QCPs

- For QCPs described by local field theories

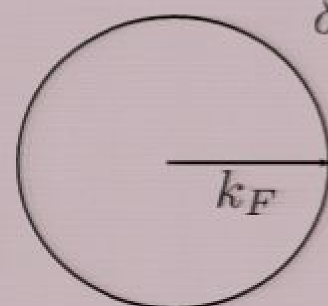
$$S = C\frac{|\partial A|}{a^{d-1}} \sim \frac{L^{d-1}}{a^{d-1}}$$



- Violations of the area law: systems with a Fermi surface

$$S \propto (k_F L)^{d-1}\log(k_F L)$$

$$\delta\theta \sim \frac{1}{k_F L}$$



$$N_m \sim (k_F L)^{d-1}$$

One-dimensional modes

$$\omega = v_F|k - k_F|$$

$$S_{1d} = \frac{c}{3}\log l/a, \quad c = \frac{1}{2}$$

$$S = N_m S_{1d}$$

D. Gioev and I. Klich (2006), Woolf (2006)

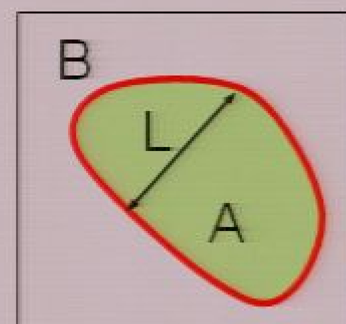
Beyond the leading term

- Any non-universal contributions to S must come from the boundary

$$S = g_{d-1}\frac{L^{d-1}}{a^{d-1}} + g_{d-2}\frac{L^{d-2}}{a^{d-2}} + \ldots + g_0 \log(L/a) + S_0(L/\xi)$$

$g_k L^k$ - integrals of local geometric quantities over the boundary

$$g_{d-1}L^{d-1} = C\int_{\partial A} 1 = C|\partial A|$$



$g_k,\ k > 0$ - non-universal

g_0 - universal

S_0 - universal function, up to additive contributions $\sim g_0[\partial A]$

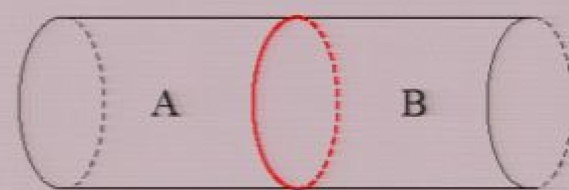
Beyond the leading term

- Any non-universal contributions to S must come from the boundary

$$S = g_{d-1}\frac{L^{d-1}}{a^{d-1}} + g_{d-2}\frac{L^{d-2}}{a^{d-2}} + \ldots + g_0 \log(L/a) + S_0(L/\xi)$$

- Straight closed boundary:

$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

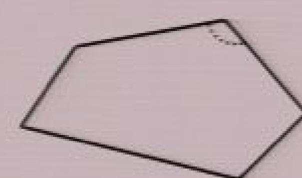


- also believed to be true for general smooth closed boundaries in d = 2

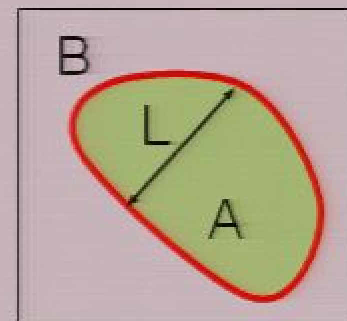
- $g_0 \neq 0$ for

  - boundaries with corners/endpoints in d = 2,

  - boundaries with intrinsic/extrinsic curvature in d = 3



Universal geometric correction



- Smooth, closed boundary 1 < d < 3

$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

- At QCP

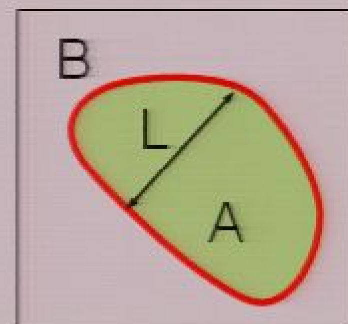
$$S = C\frac{|\partial A|}{a^{d-1}} + \gamma$$ - universal geometric constant

Akin to Privman-Fisher correction to free-energy: $F = Vf_\infty + U_0$

Only corrections to scaling, $L^{-\omega}$

- γ captures the non-local entanglement present at the QCP

Relation to massive topological phases



$$S = C\frac{L}{a} + \gamma \qquad d = 2$$

- $\gamma = -\log \mathcal{D}$ - universal constant

  $\mathcal{D} = 2$ - Kitaev model

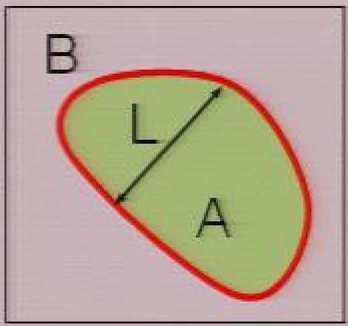
  $\mathcal{D} = \sqrt{q}$ - FQHE $\nu = \dfrac{1}{q}$

- γ - independent of geometry

- $\gamma = 0$ for non-topological gapped phases

A. Hamma, R. Ionicioiu, and P. Zanardi (2005)
A. Kitaev and J. Preskill (2006)
M. Levin and X.-G. Wen (2006)

Away from QCP



- Smooth, closed boundary 1 < d < 3

$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

- Away from QCP $(L \gg \xi)$

$$S = C(t)\frac{|\partial A|}{a^{d-1}} + r\frac{|\partial A|}{\xi^{d-1}}, \quad \xi \sim t^{-\nu} \qquad t = g - g_c$$

$C + C't$ $\qquad$ $t^{\nu(d-1)}$

$r -$ universal (up to definition of ξ)

Explicit calculations to date

- Free theories (straight boundaries)

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + m^2\phi^2 \qquad\qquad L = \bar{\psi}(i\partial_\mu\gamma_\mu - m)\psi$$

P. Calabrese and J. L. Cardy (2004), Fursaev (2006)
H. Casini and M. Huerta (2005), (2007), H. Casini, M. Huerta and L. Leitao (2009)

- Special multi-critical QCPs in d = 2 with z = 2 (e.g. RK point)

  - correlation functions reduce to those of a classical d = 2 CFT

E. Fradkin and J. E. Moore (2006),
B. Hsu, M. Mulligan, E. Fradkin, and Eun-Ah Kim(2009)

- Holographic calculations (AdS/CFT)

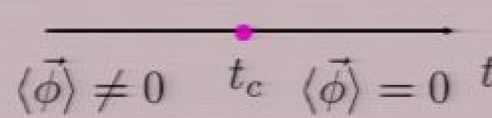
S. Ryu and T. Takayanagi (2006)

The quantum O(N) model

- Simplest interacting QCP in d = 2 (z = 1)

$$S = \int d^d x d\tau \left(\frac{1}{2}(\partial_\tau \vec{\phi})^2 + \frac{c^2}{2}(\nabla \vec{\phi})^2 + \frac{t}{2}\vec{\phi}^2 + \frac{u}{4}(\vec{\phi}^2)^2 \right)$$

$\vec{\phi}$ - N component order parameter



- Microscopic Hamiltonians:

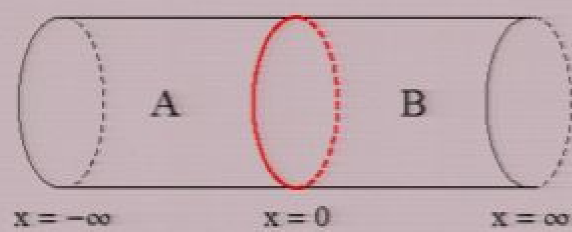
N = 1 – transverse field Ising model
N = 2 – Bose-Hubbard model at integer filling
N = 3 – bilayer S = ½ antiferromagnet on a square lattice

- Critical properties accessible using expansions in

$\epsilon = 3 - d, \qquad 1/N$

Entanglement entropy in the O(N) model

- Universal geometric correction at QCP



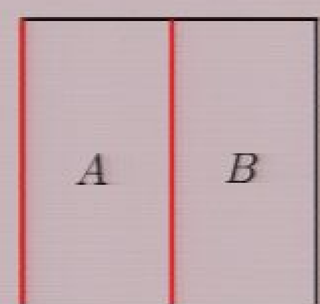
$$S = C\frac{L^{d-1}}{a^{d-1}} + \gamma$$

Boundary – d-1 dimensional torus of side L (circle in d = 2)

Twisted boundary conditions: $\phi(x + \hat{n}_i L) = e^{i\varphi}\phi(x)$

- Numerical evidence for finite γ at QCP in the transverse field Ising model (N = 1)

  - Tree tensor networks
  - Half-torus geometry

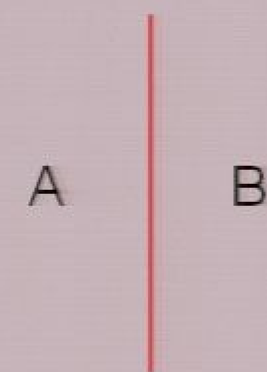


L. Tagliacozzo, G. Evenbly, G. Vidal (2009)

Entanglement entropy in the O(N) model

- Universal correlation length correction $(L \gg \xi)$

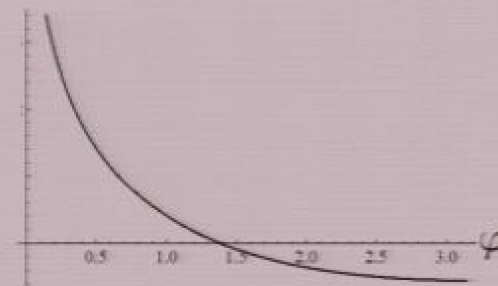
$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$



Use $\xi = \frac{c}{m}$, m - gap to lowest excitation ($O(N)$ - multiplet)

symmetry unbroken phase

Results: γ



- Result of expansion in $\epsilon = 3 - d$

$$\gamma = -\frac{N\epsilon}{6(N+8)}\left(\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi}, i\right)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Wilson} - \text{Fisher FP}$$

$$\gamma(\varphi = \pi) = -\frac{N\epsilon}{12(N+8)}\log 2$$

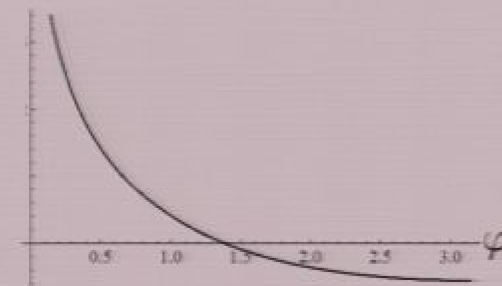
$$\gamma(\varphi \to 0) \to -\frac{N\epsilon}{6(N+8)}\log\varphi,$$ - breaks down for $\varphi \lesssim \epsilon^{1/2}$

$$\gamma(\varphi = 0) = -\frac{N\epsilon}{12(N+8)}\log\epsilon$$ - hypothesis

- Compare to Gaussian fixed point in : $d = 3 - \epsilon$

$$\gamma = -\frac{N}{6}\left(\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi}, i\right)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Gaussian FP}$$

Results: γ



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$$\gamma(\varphi = 0) = -\frac{N\epsilon}{12(N+8)}\log\epsilon$$ - hypothesis

- Compare to Gaussian fixed point in : $d = 3 - \epsilon$

Different $\epsilon \to 0$ limits

$$\gamma = -\frac{N}{6}\left(\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi}, i\right)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Gaussian FP}$$

Results: r

- Result of expansion in $\epsilon = 3 - d$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

$$r = -\frac{N}{144\pi}, \text{ Wilson-Fisher FP}$$

- Analytic correction: $\sim t$

  Singular correction in d = 2: $\sim t^{\nu}$ $\quad \nu < 1$ - wins!

- Compare to Gaussian fixed point in $d = 3 - \epsilon$:

$$r = -\frac{N}{24\pi\epsilon}, \text{ Gaussian FP}$$

Results: r

- Result of expansion in $\epsilon = 3 - d$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

Entanglement largest at the QCP!

$$r = -\frac{N}{144\pi}, \text{ Wilson-Fisher FP}$$

- Analytic correction: $\sim t$

  Singular correction in d = 2: $\sim t^{\nu}$ $\nu < 1$ - wins!

- Compare to Gaussian fixed point in $d = 3 - \epsilon$:

$$r = -\frac{N}{24\pi\epsilon}, \text{ Gaussian FP}$$

Renyi entropy

- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n$ $\qquad S = -tr_A \rho_A \log \rho_A = \lim_{n\to 1} S_n$

- In d = 1,

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log(l/a)$$

- In d > 1, area law + subleading universal corrections

$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$

$$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

Renyi entropy - results

- $$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$
  $$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

- Violation of (naïve) large N counting

  Expect: $r_n \sim O(N)$ Instead: $r_n \sim O(N^2)$ (but $r \sim O(N)$)

- Computed r_n in large-N expansion in d = 2

n	α_n
2	-0.16515
3	-0.26594
4	-0.32905
5	-0.36743

$$r_n = \frac{3\pi^2 N^2}{128} \frac{n\alpha_n^2}{n-1}$$

$$r_n \sim n-1, \quad n \to 1$$

- Can compare r_n to lattice Monte-Carlo simulations

P. V. Buividovich and M. I. Polikarpov (2008); M. Caraglio and F. Gliozzi (2008)
M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko (2010)

Renyi entropy - results

- $$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$
  $$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

- In dimension $d = 3 - \epsilon$, γ_n, r_n have a discontinuity at

  $$n = n^*,\ 1 < n^* \leq 1 + \frac{3}{4}\frac{N+2}{N+8}\epsilon$$

- New length scale emerges as $n \to n_*$, "phase transition"

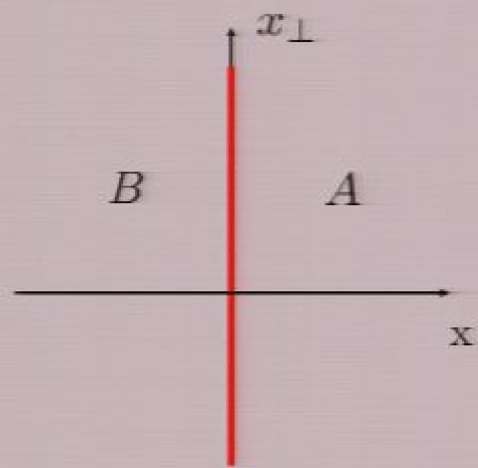
- Large N expansion shows that this phenomena occurs in 2.74 < d < 3 (likely absent in d = 2)

Replica Trick

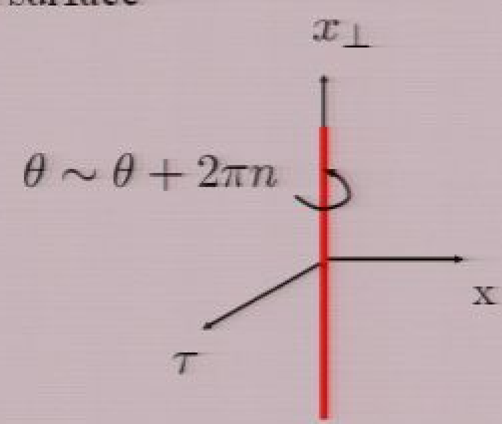
- $$S_n = \frac{1}{1-n} \log tr_A \rho_A^n \qquad S = -tr_A \rho_A \log \rho_A = \lim_{n \to 1} S_n$$

$$\boxed{tr_A \rho_A^n = \frac{Z_n}{Z^n}}$$

- Z_n - partition function on an n-sheeted Riemann surface



space



space-time

Conical singularity at

$(\mathrm{x}, \tau) = (0, 0)$

Replica Trick

- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n \qquad S = -tr_A \rho_A \log \rho_A = \lim_{n \to 1} S_n$

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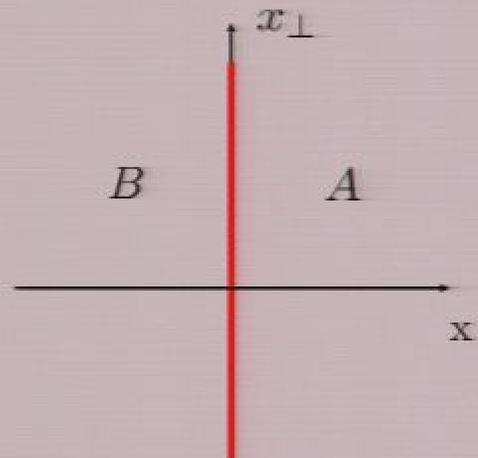
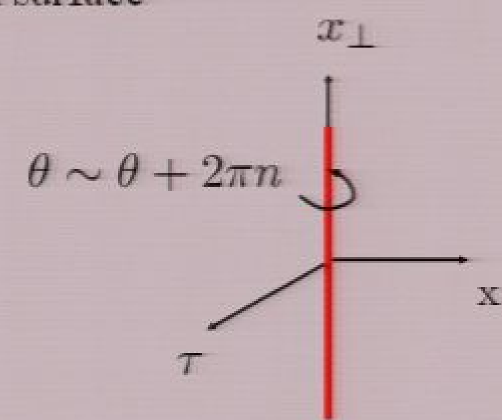
- Z_n - partition function on an n-sheeted Riemann surface



space

Conical singularity at

$(\mathrm{x}, \tau) = (0, 0)$



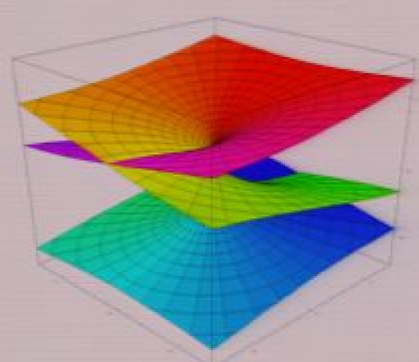
space-time

Riemann Surface

- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n \qquad S = -tr_A \rho_A \log \rho_A = \lim_{n\to 1} S_n$

$$tr_A \rho_A^n = \frac{Z_n}{Z^n}$$

- Z_n - partition function on an n-sheeted Riemann surface



$$(\mathrm{x}, \tau) = r(\cos\theta, \sin\theta)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dx_\perp^2$$

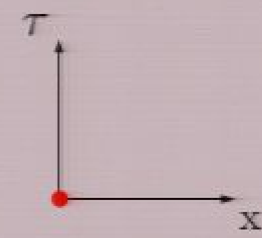
$$\theta \sim \theta + 2\pi n$$

Why is d = 1 easy?

- Conformal group in 1+1d consists of all analytic (or anti-analytic) mappings

  can map the n-sheeted Riemann surface to ordinary complex plane via

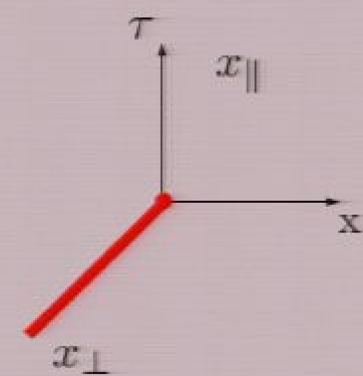
$$f : \mathbb{C}_n \to \mathbb{C}, \quad w = z^{1/n}$$



$$ds^2_{\mathbb{C}} = dw d\bar{w} = \left|\frac{dw}{dz}\right|^2 dz d\bar{z} = \left|\frac{dw}{dz}\right|^2 ds^2_{\mathbb{C}_n}$$

- This is no-longer a conformal symmetry for d > 1

$$ds^2_{\mathbb{C}} = dw d\bar{w} + dx^2_{\perp} = \left|\frac{dw}{dz}\right|^2 dz d\bar{z} + dx^2_{\perp}$$

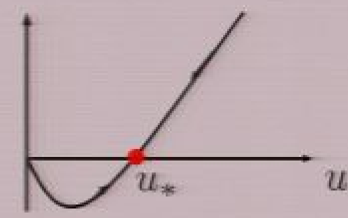


Have to work harder!

Explicit calculations on an n-sheeted surface

RG in the presence of conical singularity

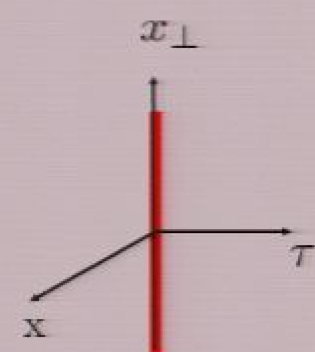
$$L = \frac{1}{2}(\partial_\tau \vec{\phi})^2 + \frac{1}{2}(\nabla \vec{\phi})^2 + \frac{t}{2}\vec{\phi}^2 + \frac{u}{4}(\vec{\phi}^2)^2$$



- Theory flows to Wilson-Fisher fixed point

$$\beta(u) = -\frac{du}{dl} = -\epsilon u + \frac{(N+8)}{8\pi^2} u^2$$

$$u_* = \frac{8\pi^2 \epsilon}{N+8}$$



- Boundary cannot renormalize bulk!
- New operators will be induced at the boundary!

$$S_b = \frac{c}{2} \int d^{d-1} x_\perp \, \phi^2(r = 0, x_\perp)$$

Boundary critical phenomena

- Analogy: classical boundary critical phenomena in O(N) model

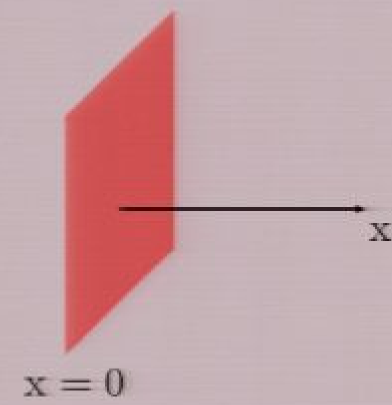
$$S_b = \frac{c}{2}\int d^{D-1}x_\perp \phi^2(\mathrm{x}=0, \vec{x}_\perp)$$

- Surface universality classes

$c_*^{+\infty}$ - ordinary

$c_*^{-\infty}$ - extra-ordinary

c_{sp} - special (multi-critical)

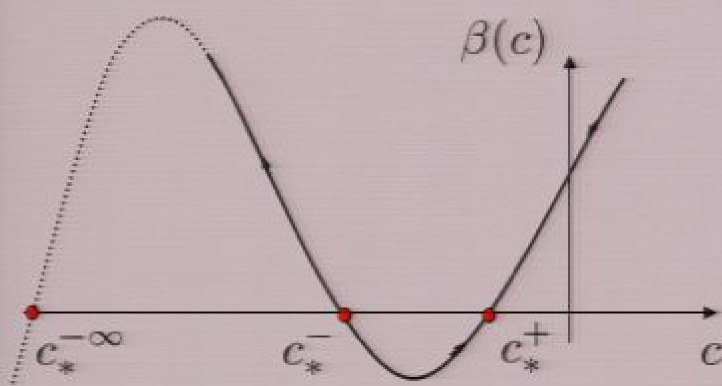


Flow of the boundary perturbation

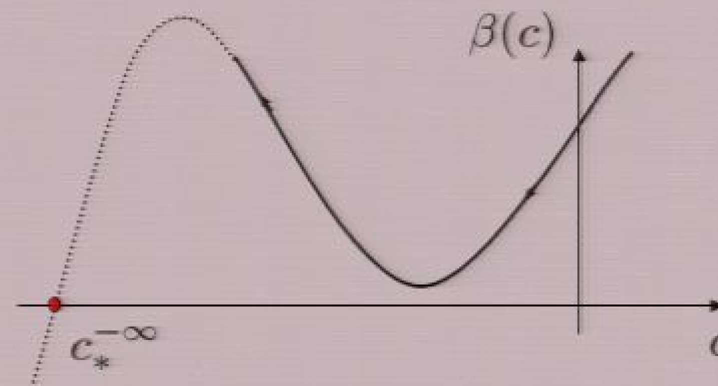
- A computation using ε-expansion gives:

$$\beta(c) = \frac{(N+2)u}{24\pi}\left(n - \frac{1}{n}\right) + \frac{(N+2)uc}{8\pi^2} + \frac{c^2}{2\pi n}$$

$n < n_c$



$n > n_c$



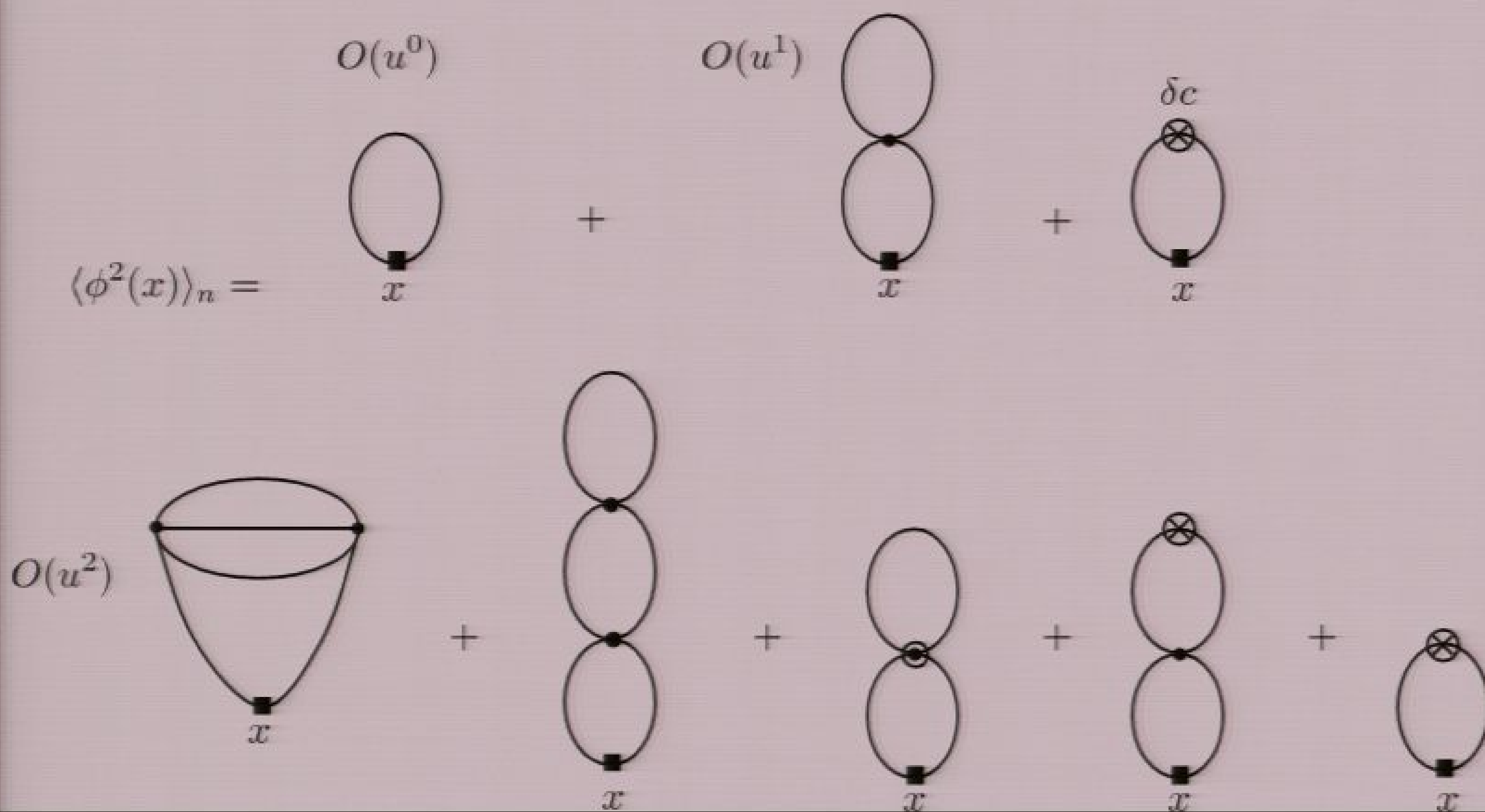
$$n_c = 1 + \frac{3}{4}\frac{N+2}{N+8}\epsilon$$

Three universality classes: $c_*^{+}, c_*^{-}, c_*^{-\infty}$

Calculating β(c) to O(u²)

$$\beta(c_r) = A(u_r) + \eta_2(u_r)c_r$$

- Can determine $A(u_r)$ from $\langle \phi^2(x)_r \rangle_n$ at the critical point



To do list

- Symmetry broken phase

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}} \qquad \xi^{-1} \propto t^{\nu}$$

$$S_+ = C(t)\frac{L^{d-1}}{a^{d-1}} + B_+ t^{\nu(d-1)} L^{d-1}, \;\; t > 0$$

$$S_- = C(t)\frac{L^{d-1}}{a^{d-1}} + B_- |t|^{\nu(d-1)} L^{d-1}, \;\; t < 0$$

$\frac{B_-}{B_+}$ - universal

To do list

- Upper critical dimension $(d = 3)$ $\qquad \nu = \frac{1}{2}$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

$C + C't$ $\qquad\qquad t^{\nu(d-1)} \to t$

- Additional log is possible

$$S = C(t)\frac{L^2}{a^2} + b\frac{L^2}{\xi^2}\log \xi/a$$

$b-$ universal

Gaussian theory: $b = -\frac{N}{24\pi}$

Interacting theory?

To do list

- γ in large-N expansion (solve gap equation numerically)
- Corners/endpoints

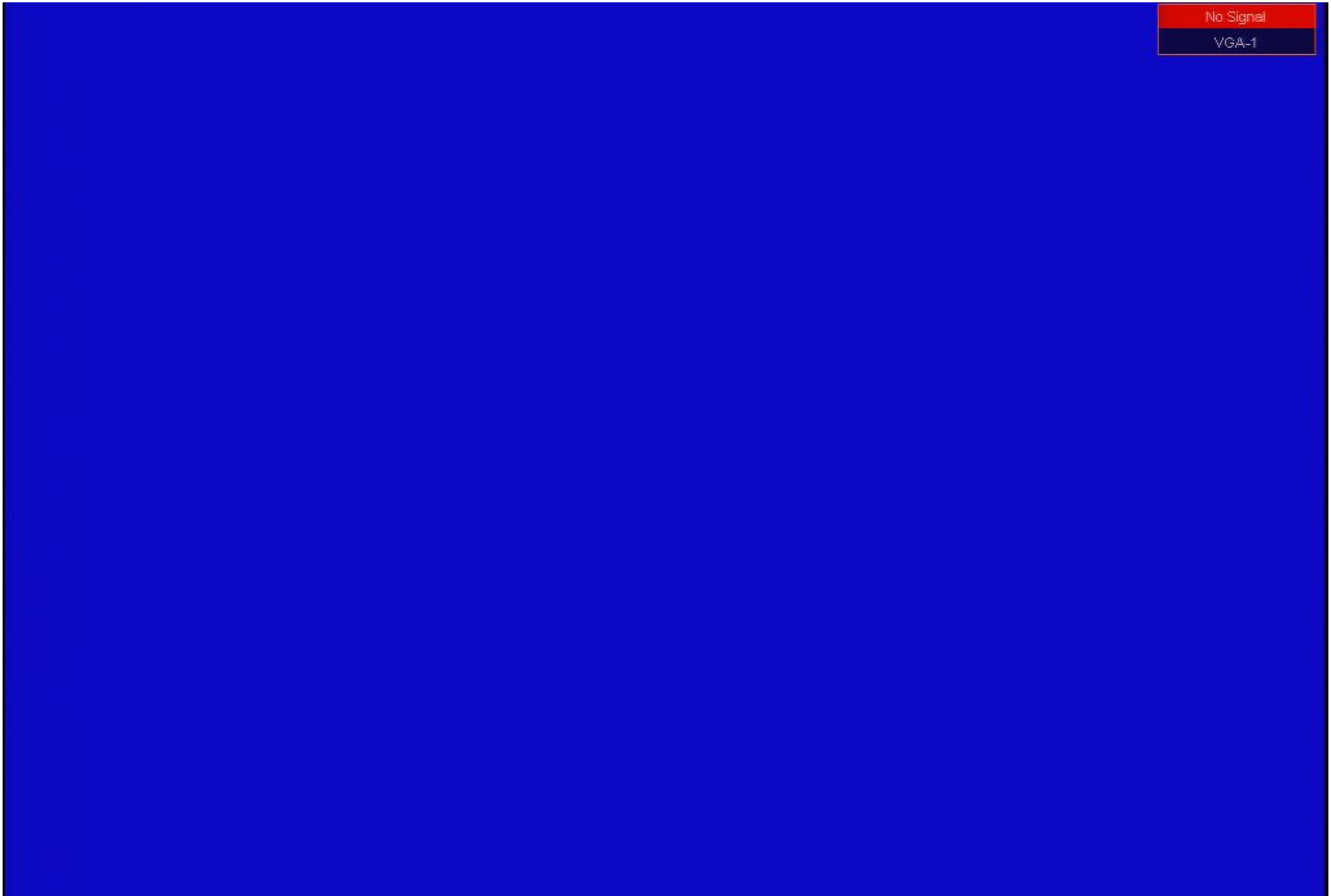
$$S = C\frac{L^{d-1}}{a^{d-1}} + c\log L/a$$

c - universal

Conclusions

- Demonstrated the presence of subleading universal corrections to entanglement entropy in the O(N) model
- Many unexpected results
    - violation of naïve large-N counting
    - non-analyticity of Renyi entropy as a function of replica index
- Results can be checked by lattice simulations

Thank you!


No Signal
VGA-1