Title: Topological Quantum Information, Khovanov Homology and the Jones Polynomial

Date: May 13, 2010 10:00 AM

URL: http://pirsa.org/10050049

Abstract: In this talk (based on arXiv:1001.0354) we give a quantum statistical interpretation for the Kauffmann bracket polynomial state sum <K> for the Jones polynomial. We use this quantum mechanical interpretation to give a new quantum algorithm for computing the Jones polynomial. This algorithm is useful for its conceptual simplicity, and it applies to all values of the polynomial variable that lie on the unit circle in the complex plane. Letting C(K) denote the Hilbert space for this model, there is a natural unitary transformation U from C(K) to itself such that <K> = <F|U|F> where |F> is a sum over basis states for C(K). The quantum algorithm arises directly from this formula via the Hadamard Test. We then show that the framework for our quantum model for the bracket polynomial is a natural setting for Khovanov homology. The Hilbert space C(K) of our model has basis in one-to-one correspondence with the enhanced states of the bracket state summmation and is isomorphic with the chain complex for Khovanov homology with coefficients in the complex numbers. We show that for the Khovanov boundary operator d defined on C(K) we have the relationship dU + Ud = 0.
Consequently, the unitary operator U acts on the Khovanov homology, and we therefore obtain a direct relationship between Khovanov homology and this quantum algorithm for the Jones polynomial. The formula for the Jones polynomial as a graded Euler characteristic is now expressed in terms of the eigenvalues of U and the Euler characteristics of the eigenspaces of U in the homology. The quantum algorithm given here is inefficient, and so it remains an open problem to determine better quantum algorithms that involve both the Jones polynomial and the Khovanov homology.

ological Quantum Information, Khovanov Homology and the Jones Polynomial

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Arxiv: 1001.0354

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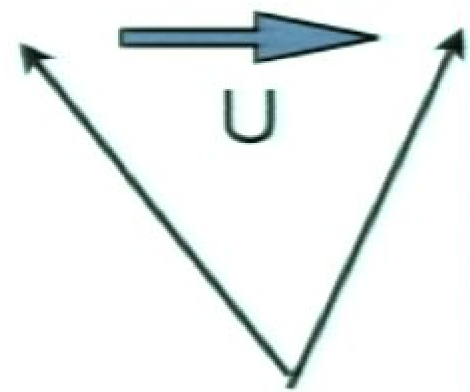
Quantum Mechanics in a Nutshell

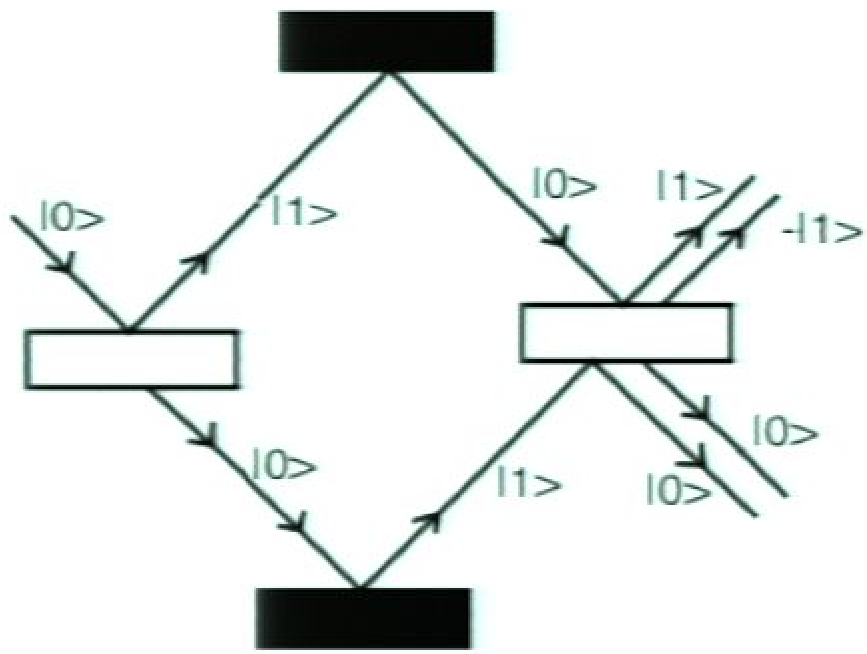
. A state of a physical system corresponds to a unit vector |S> in a complex vector space.

(measurement free) Physical processes re modeled by unitary transformations plied to the state vector: |S> -----> U|S>

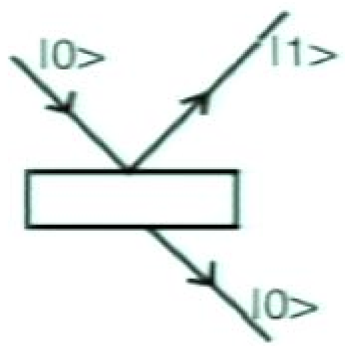


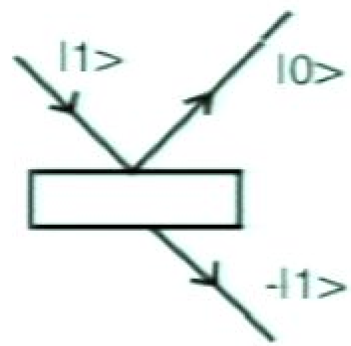
If $|S\rangle = z_1|1\rangle + z_2|2\rangle + \ldots + z_n|n\rangle$

in a measurement basis {|1>,|2>,...,|n>}, then measurement of |S> yields |i> with probability |z|^2.



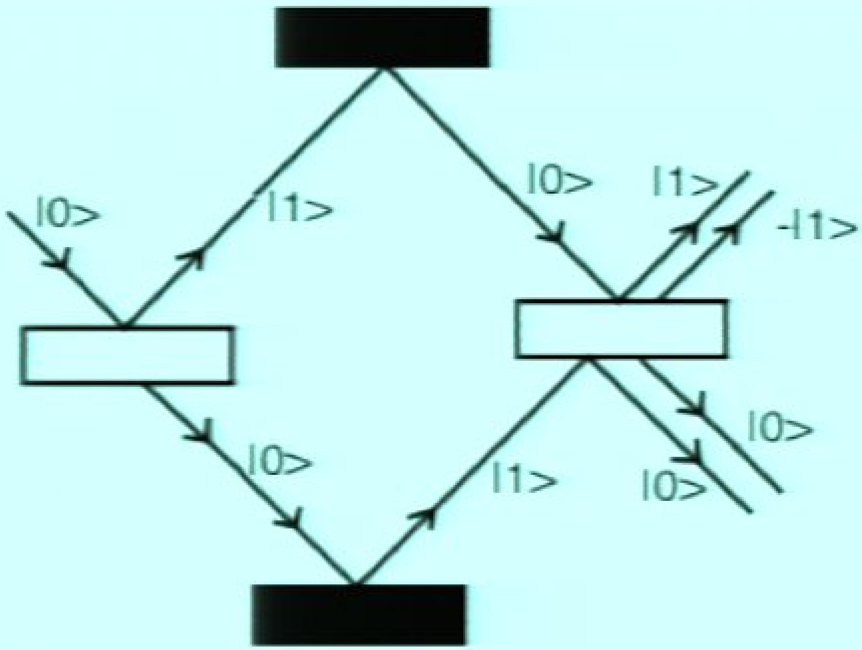
Mach-Zender Interferometer





$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \text{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



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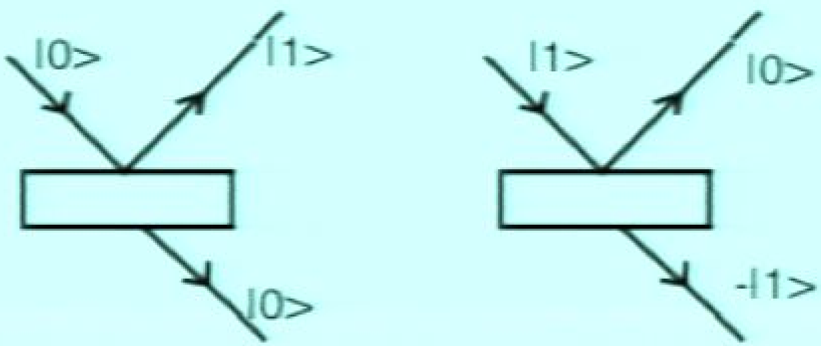

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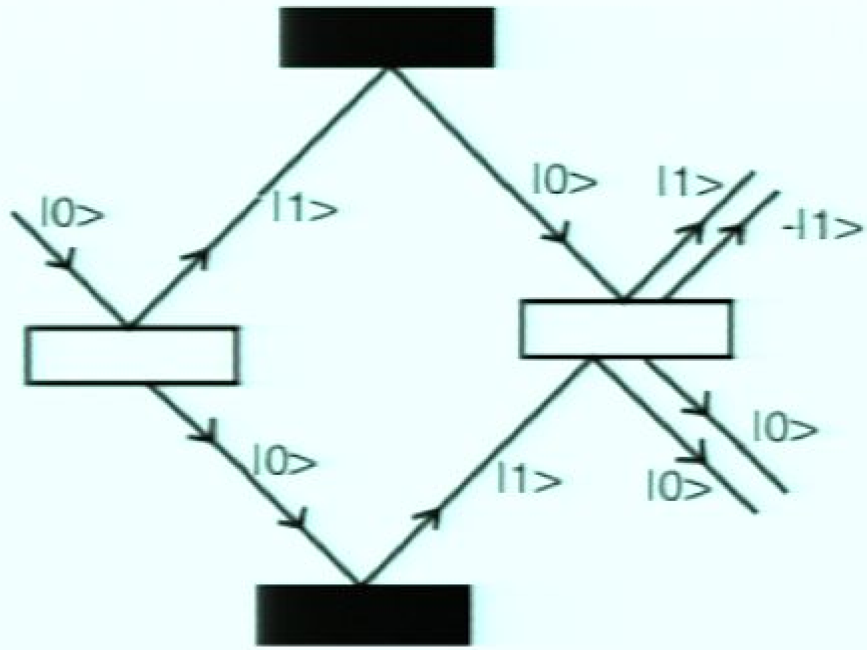

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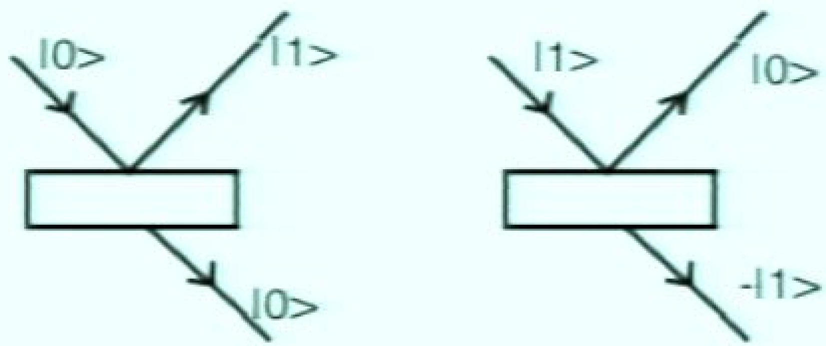

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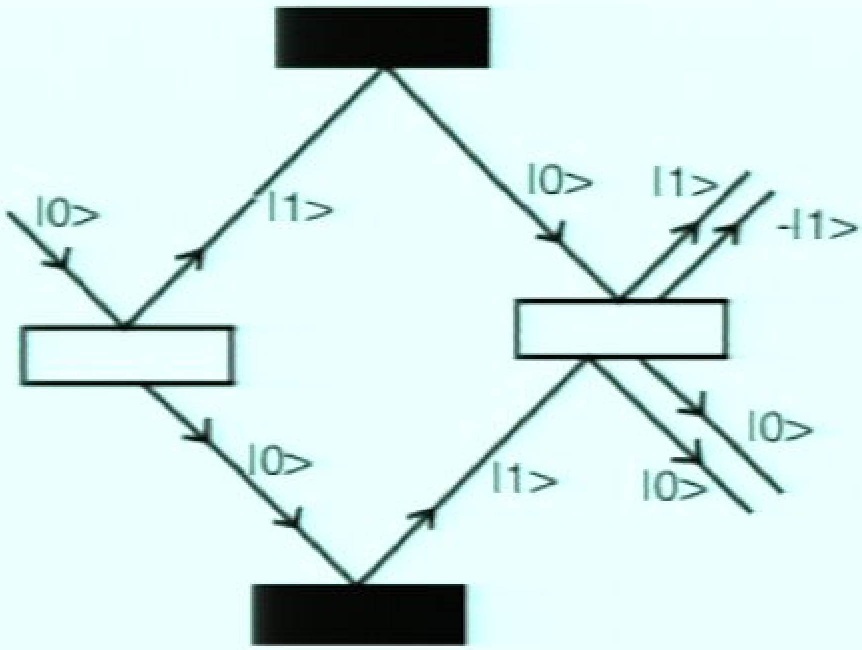

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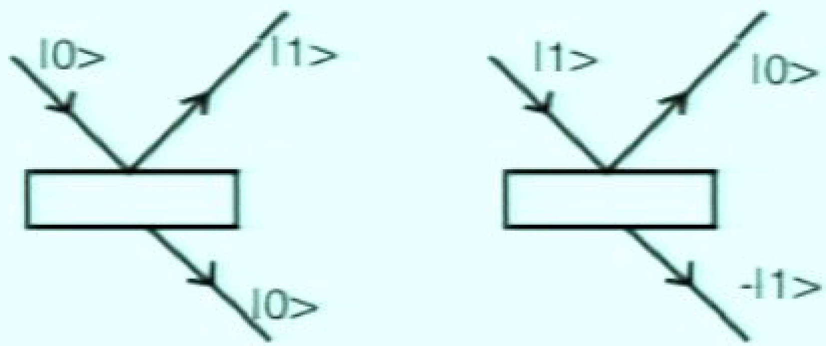

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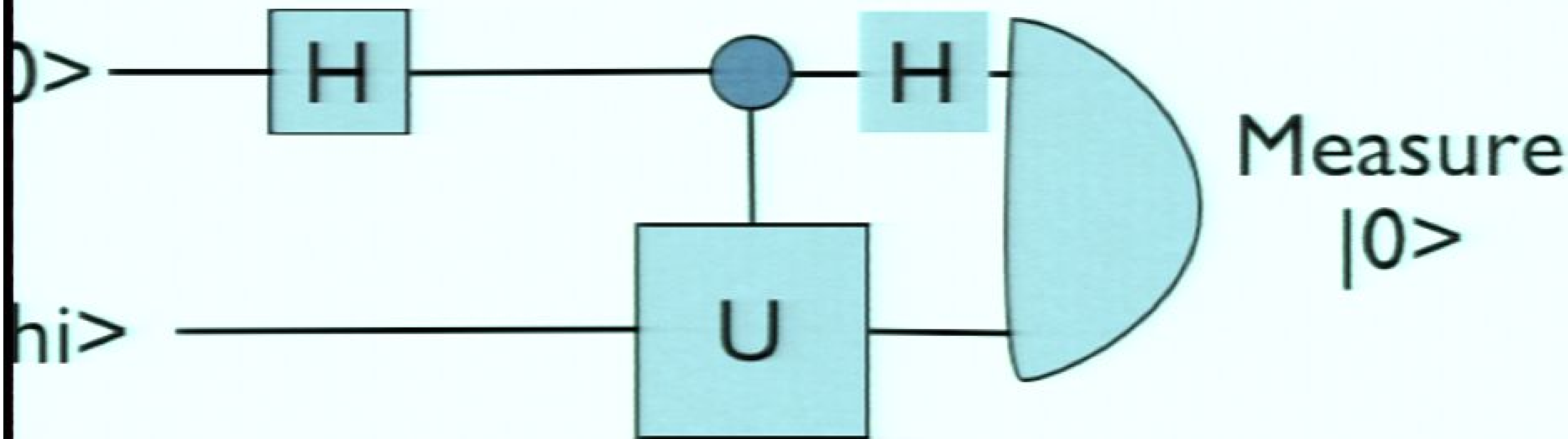
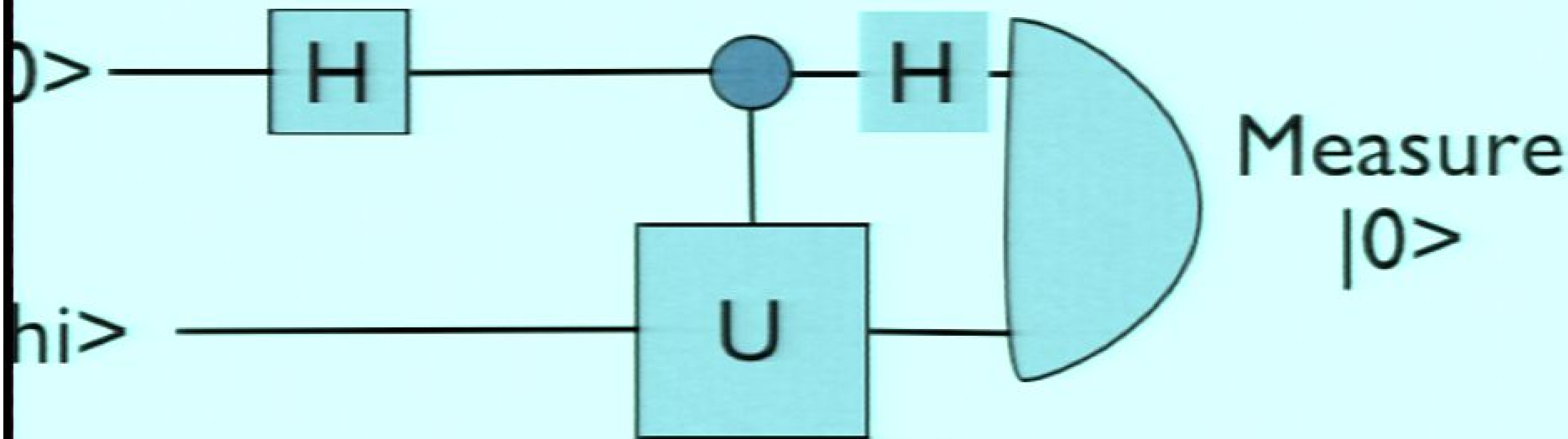
Hadamard Test



H = [1 1 ; 1 -1] /Sqrt(2)

|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2.

Hadamard Test



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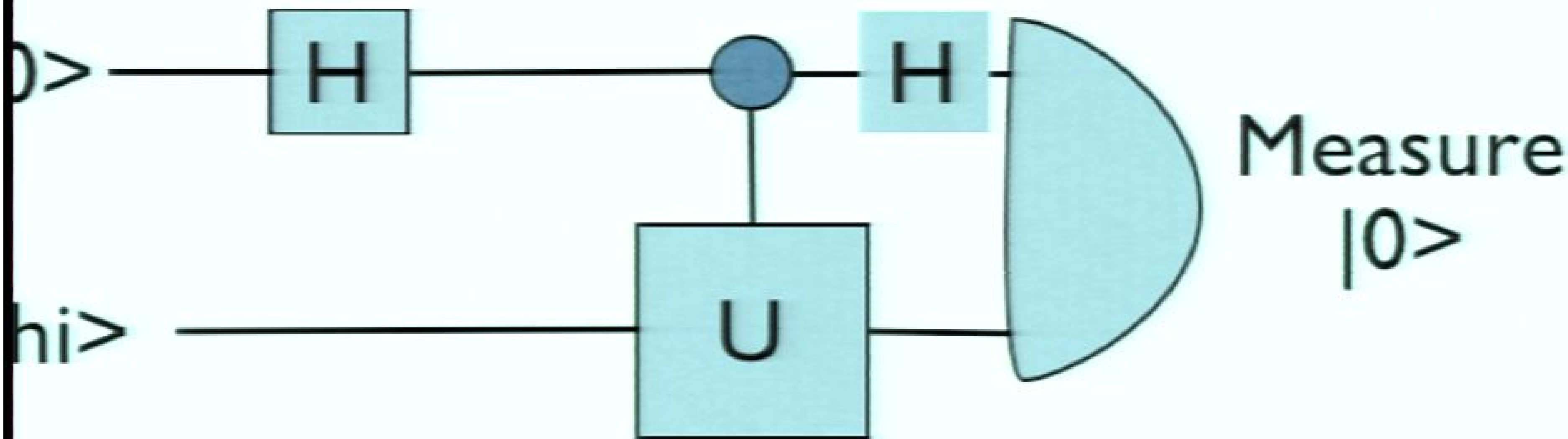

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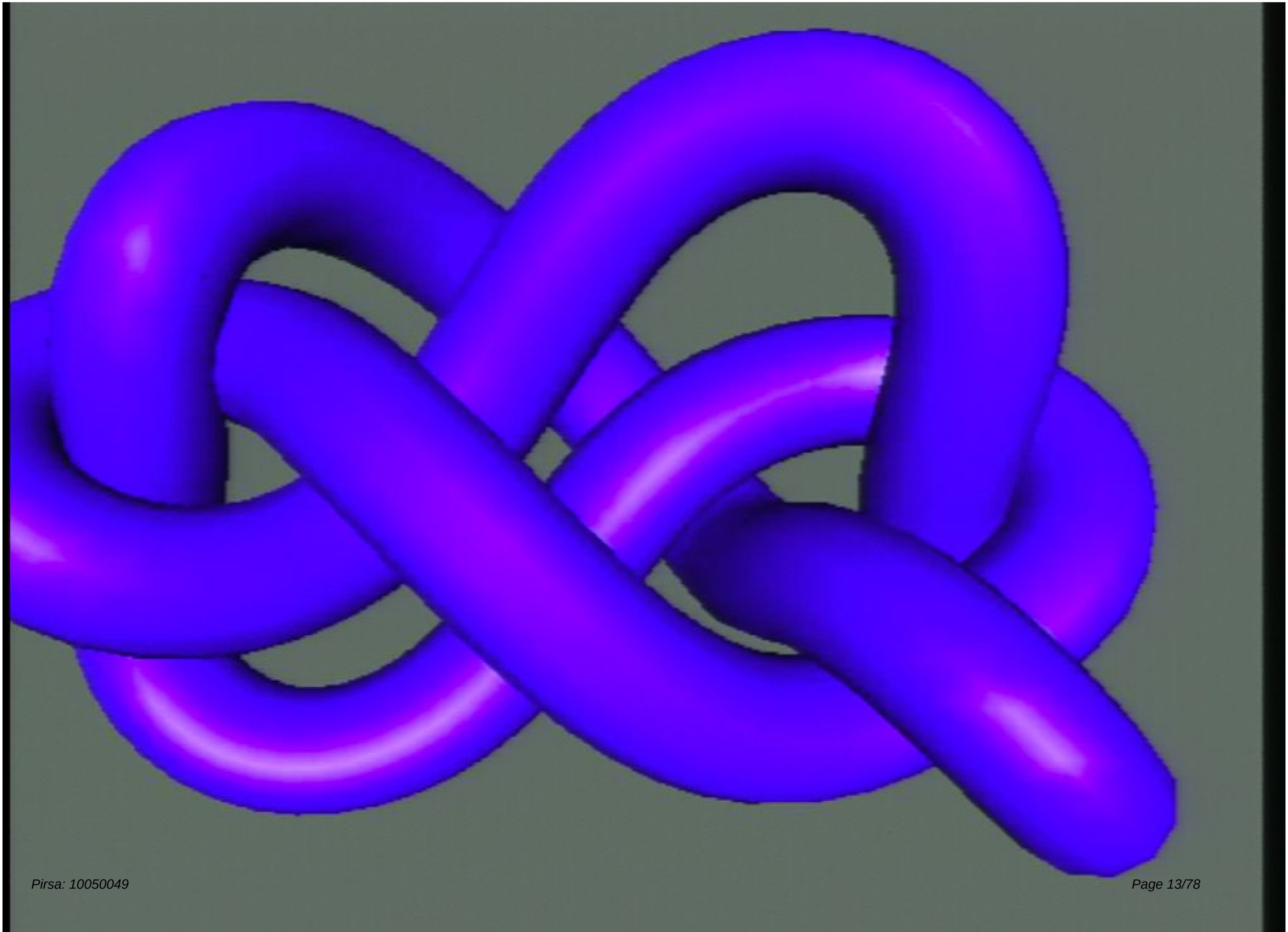
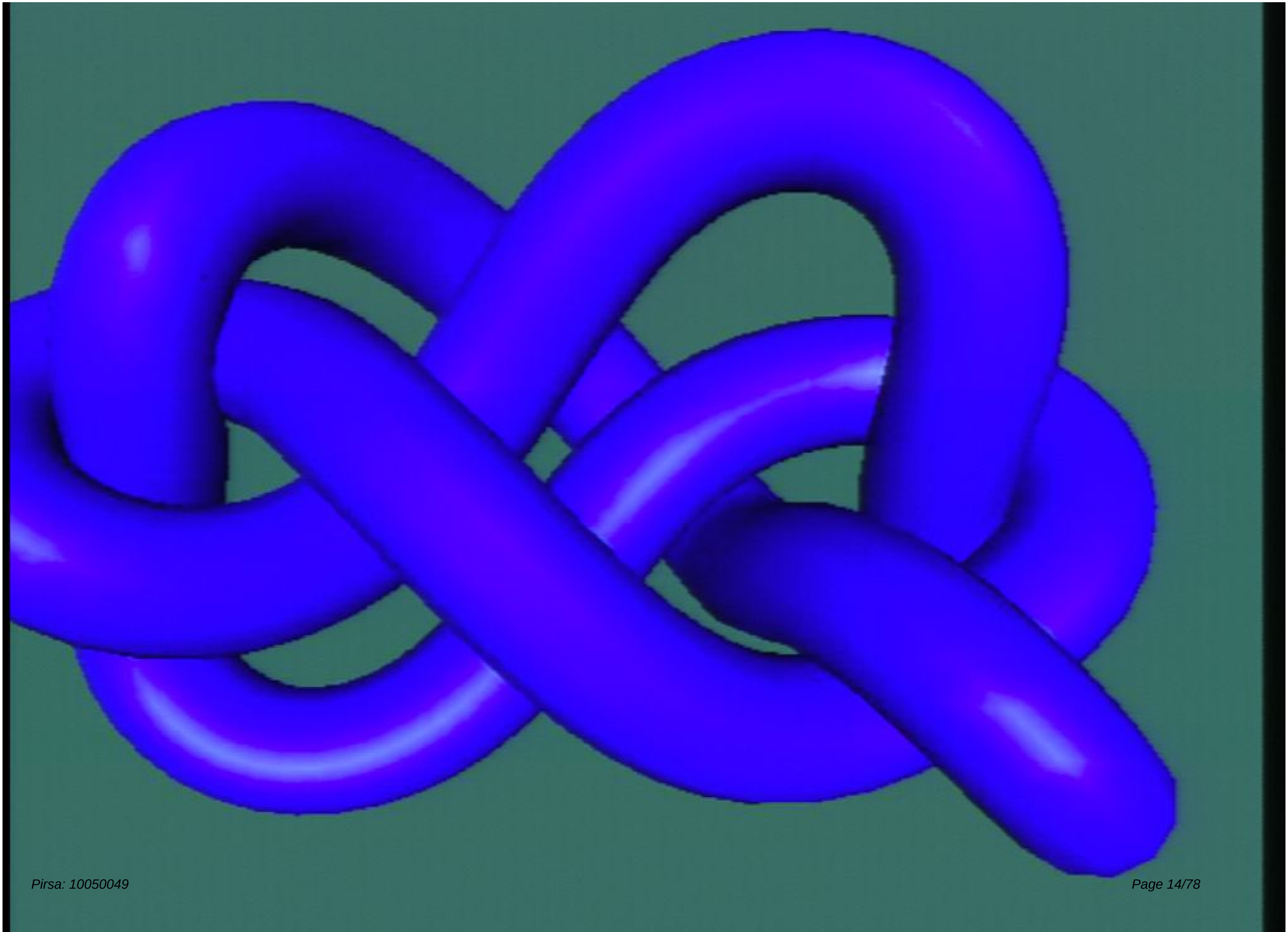
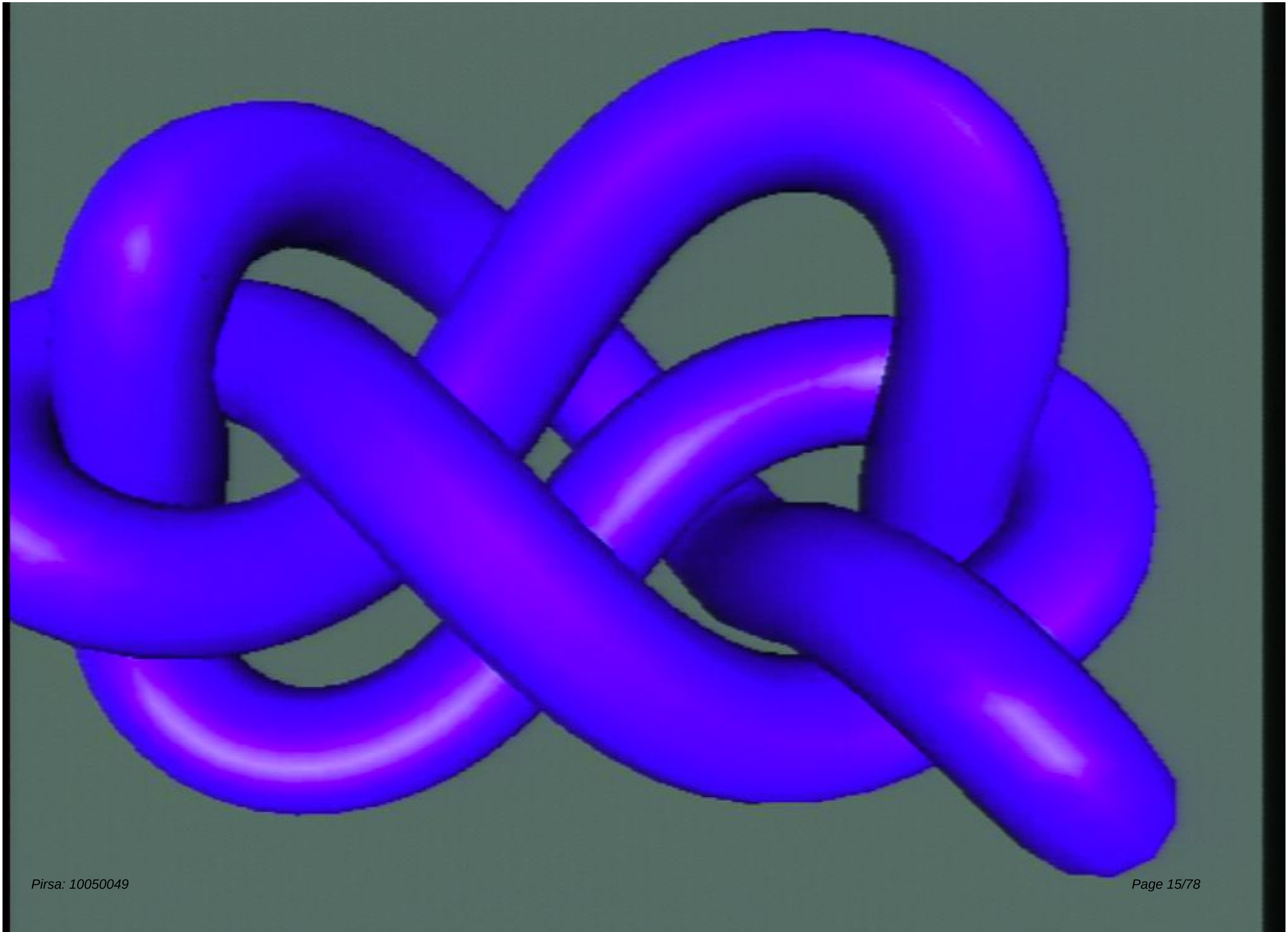
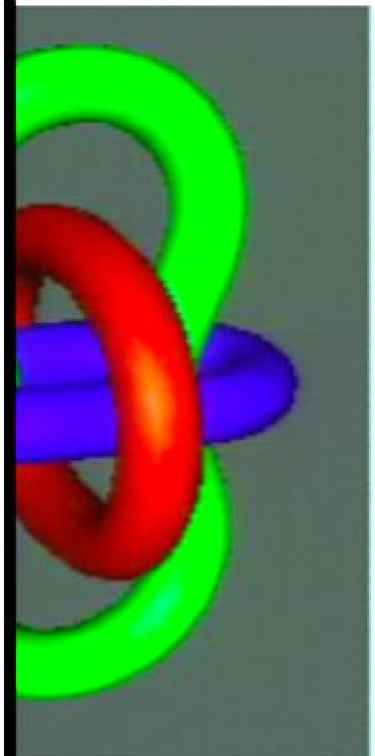
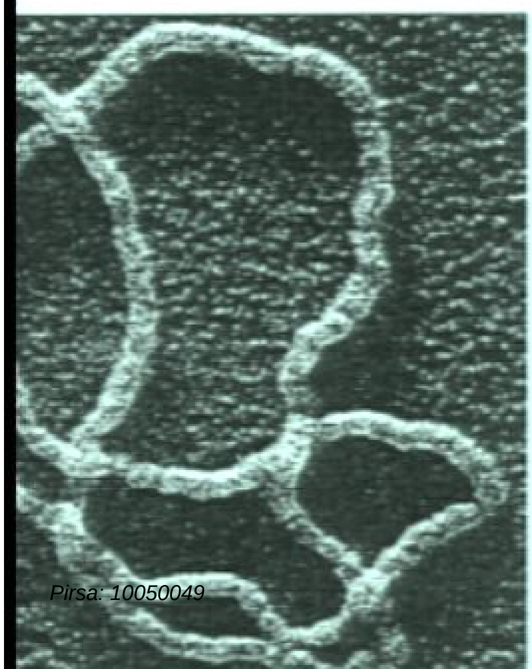










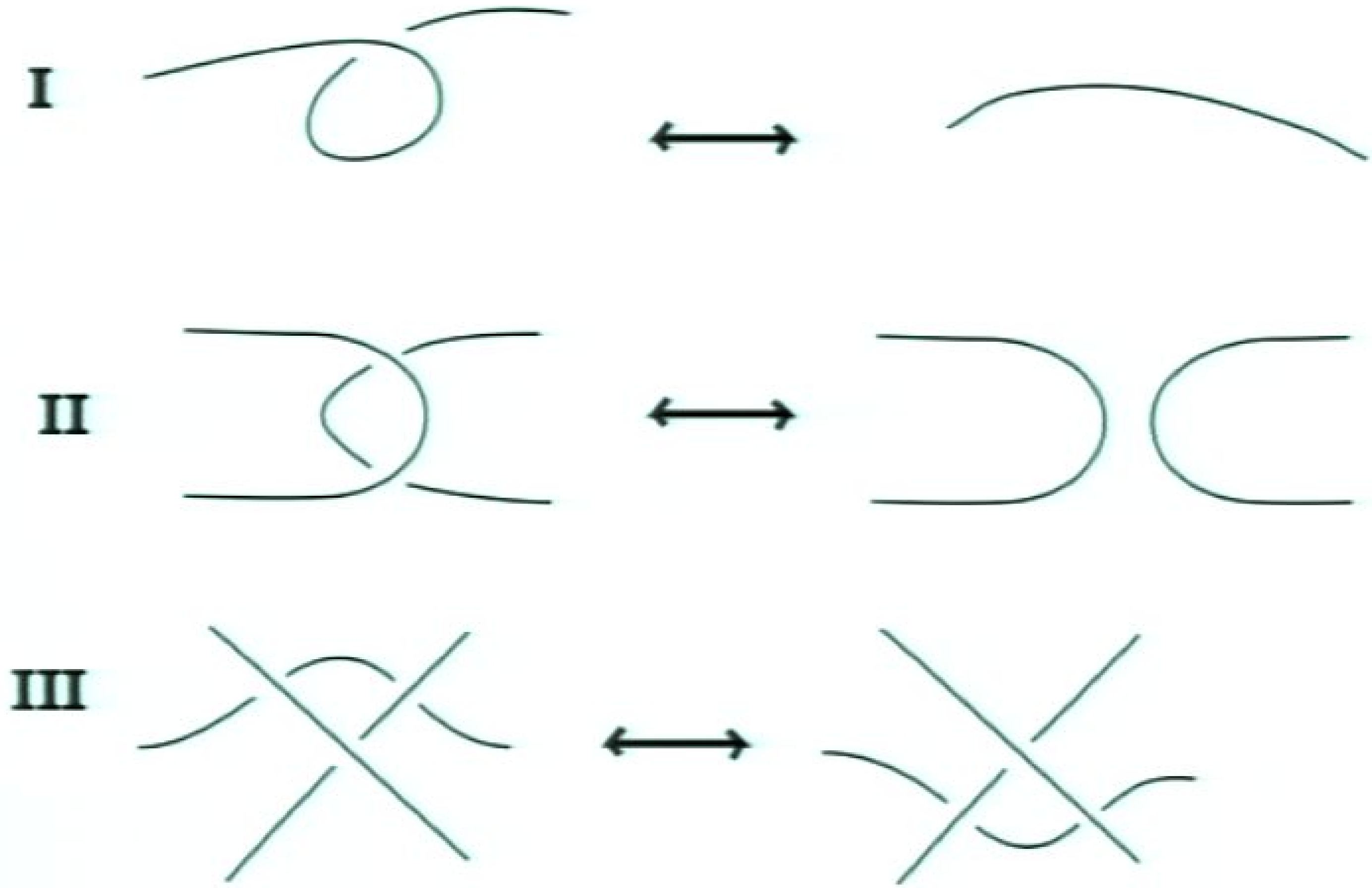


Figure 2 - The Reidemeister Moves.

Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

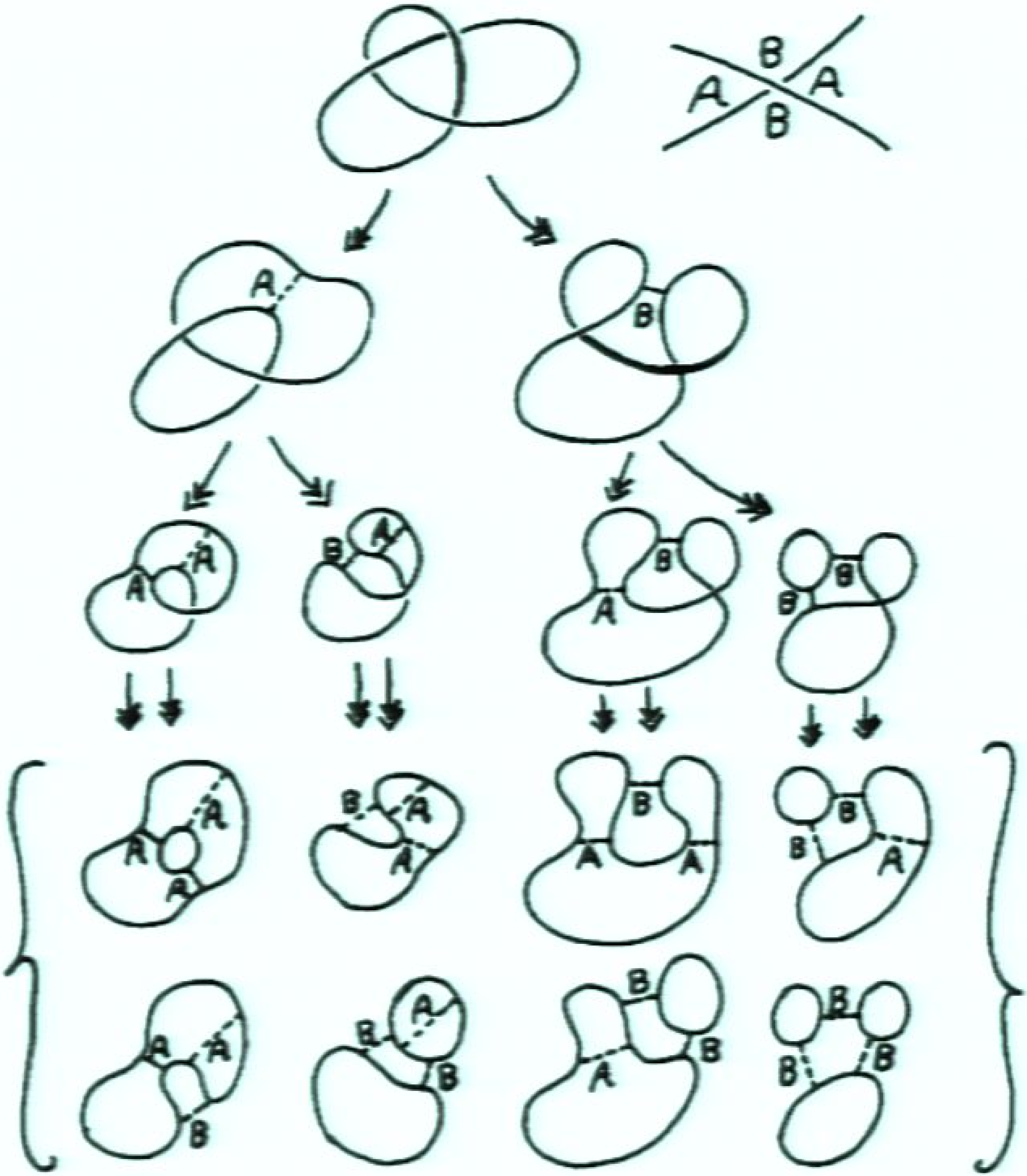
Bracket Polynomial Model for the Jones Polynomial

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$$\langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$

$\langle$  $\rangle = (-A^3) \langle \smile \rangle$

$\langle$  $\rangle = (-A^{-3}) \langle \smile \rangle$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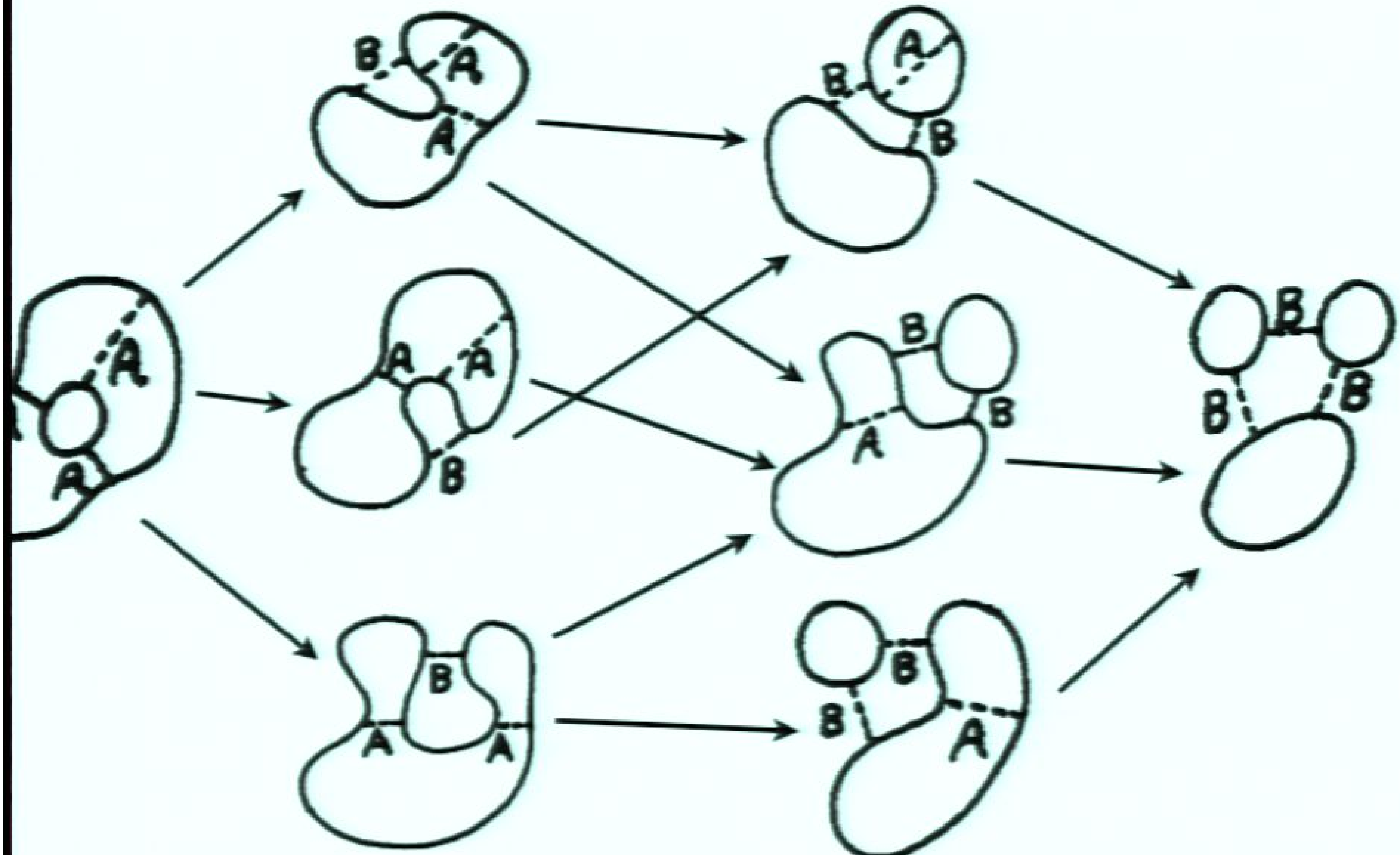
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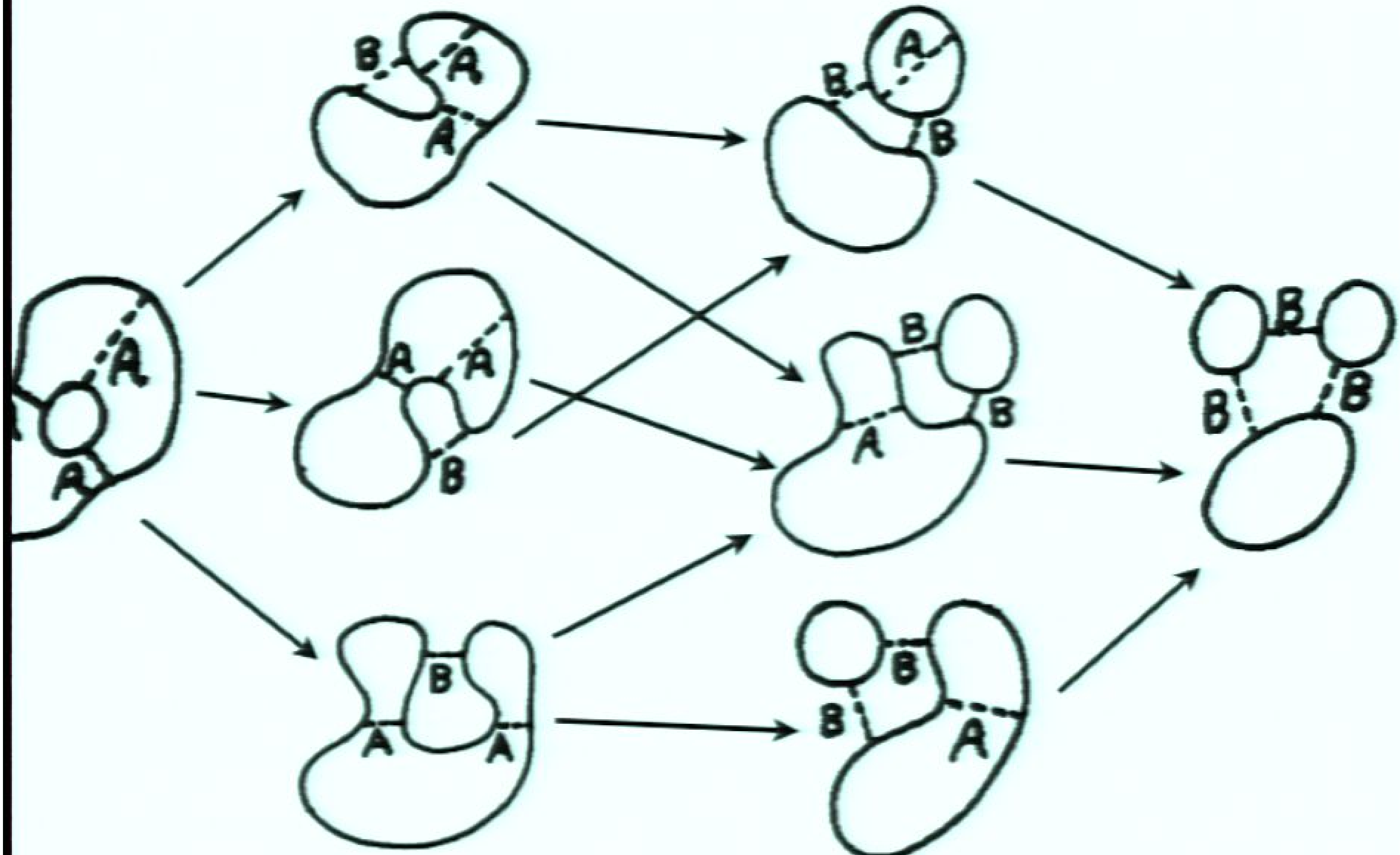

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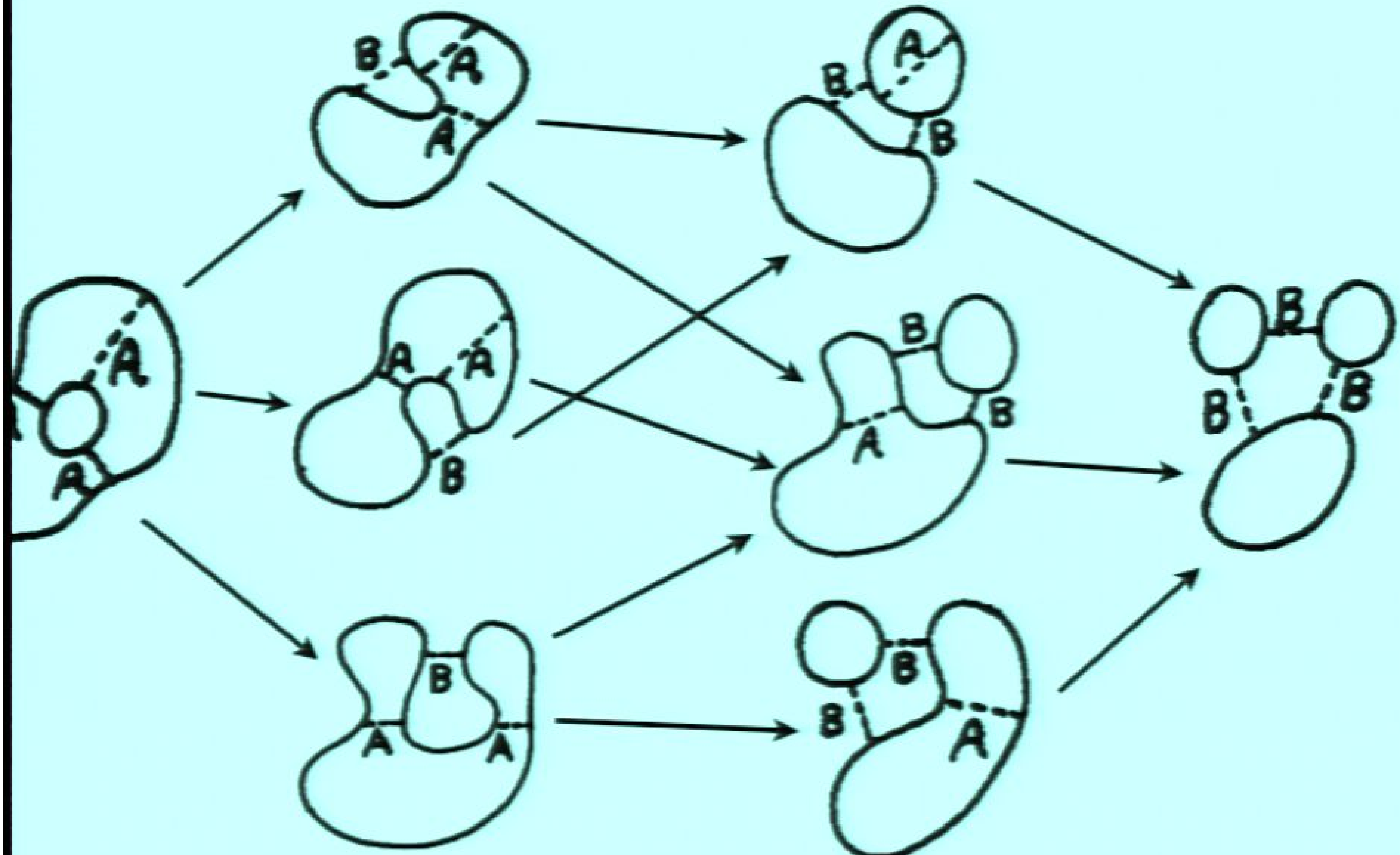

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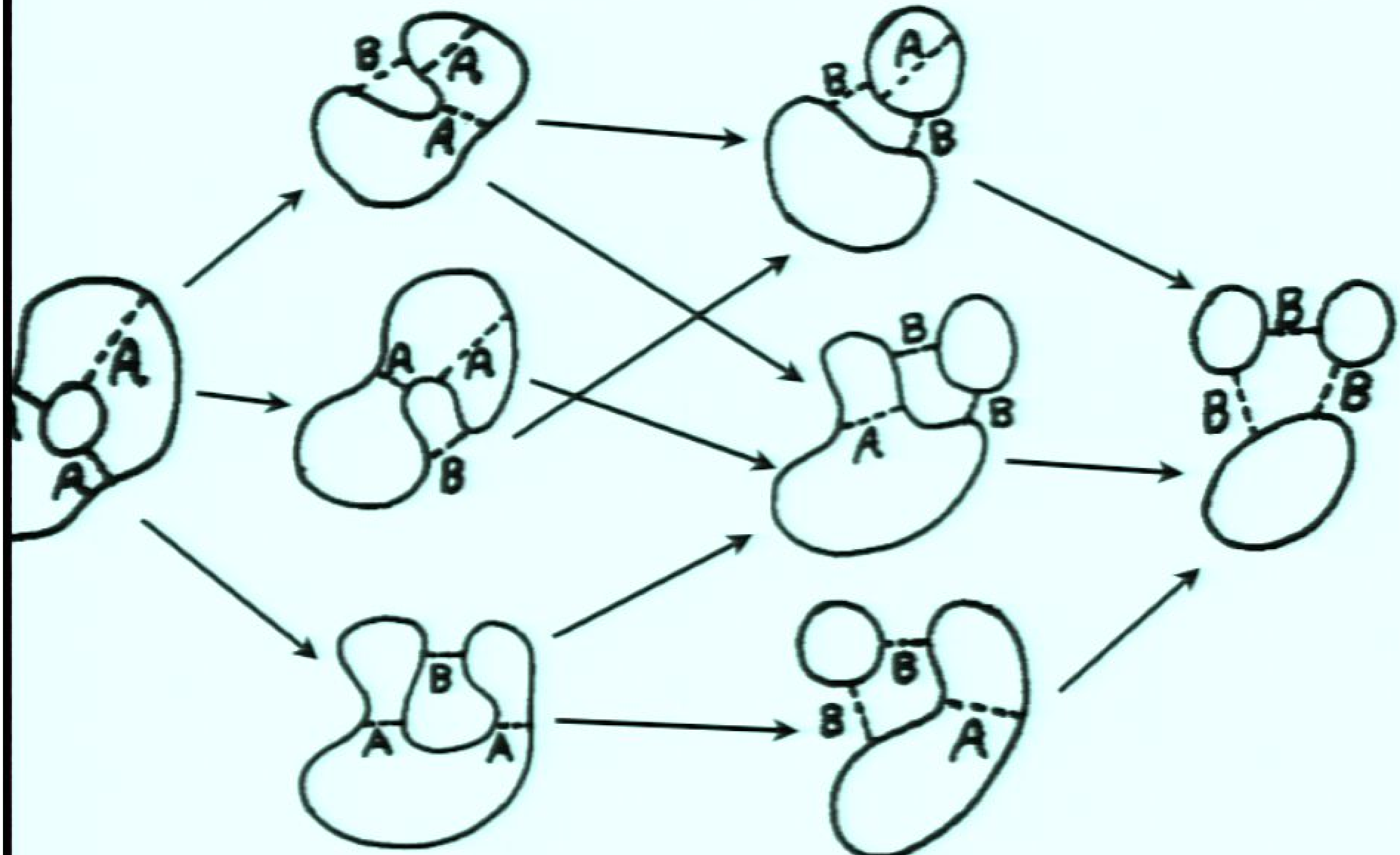

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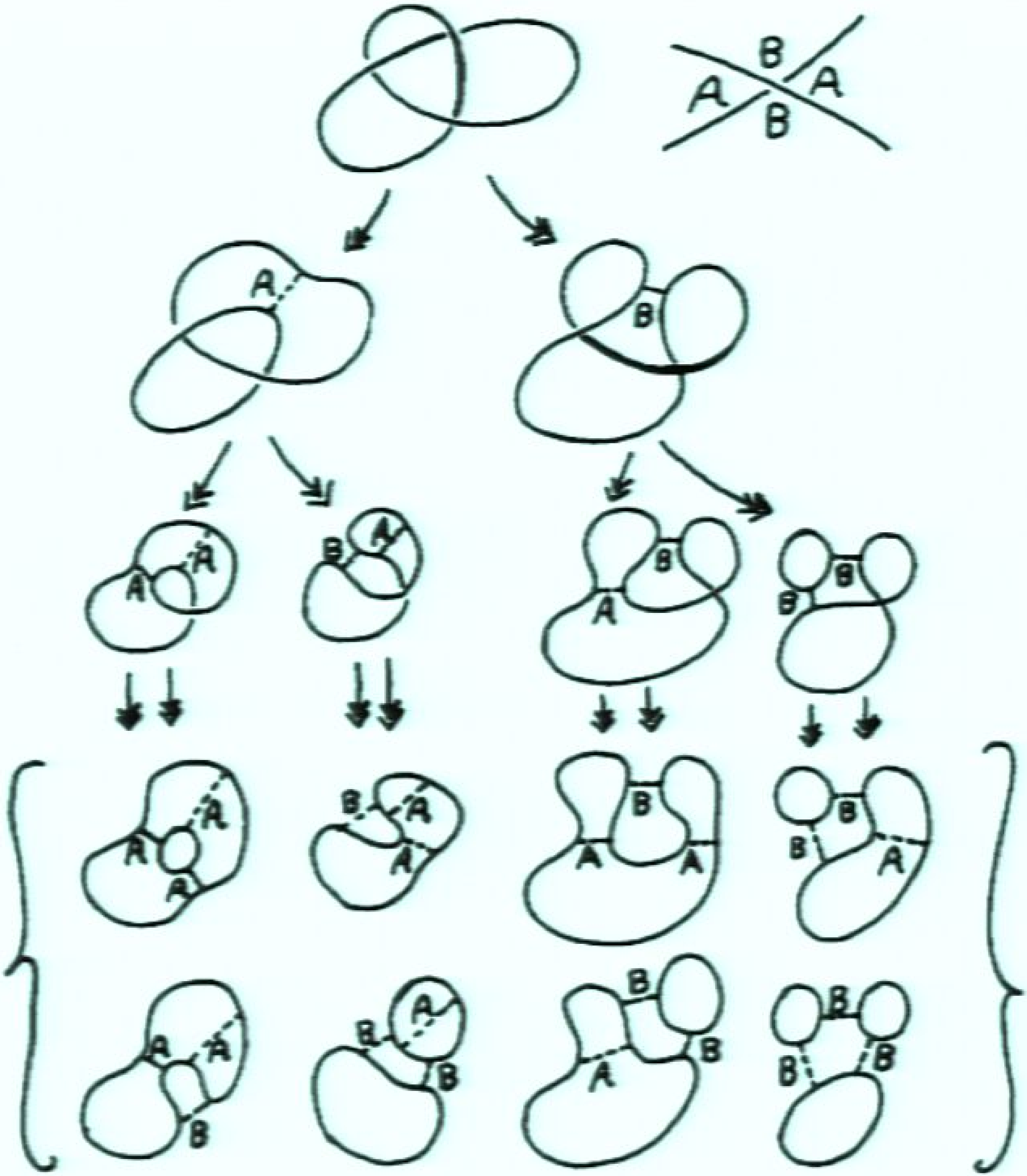
Reformulating the Bracket

Let c(K) = number of crossings on link K.

Form $A^{-c(K)}$ <K> and replace A by $-q^{-1}$.

Then the skein relation for <K> will be replaced by:

$$\langle \bigcirc \rangle = (q + q^{-1})$$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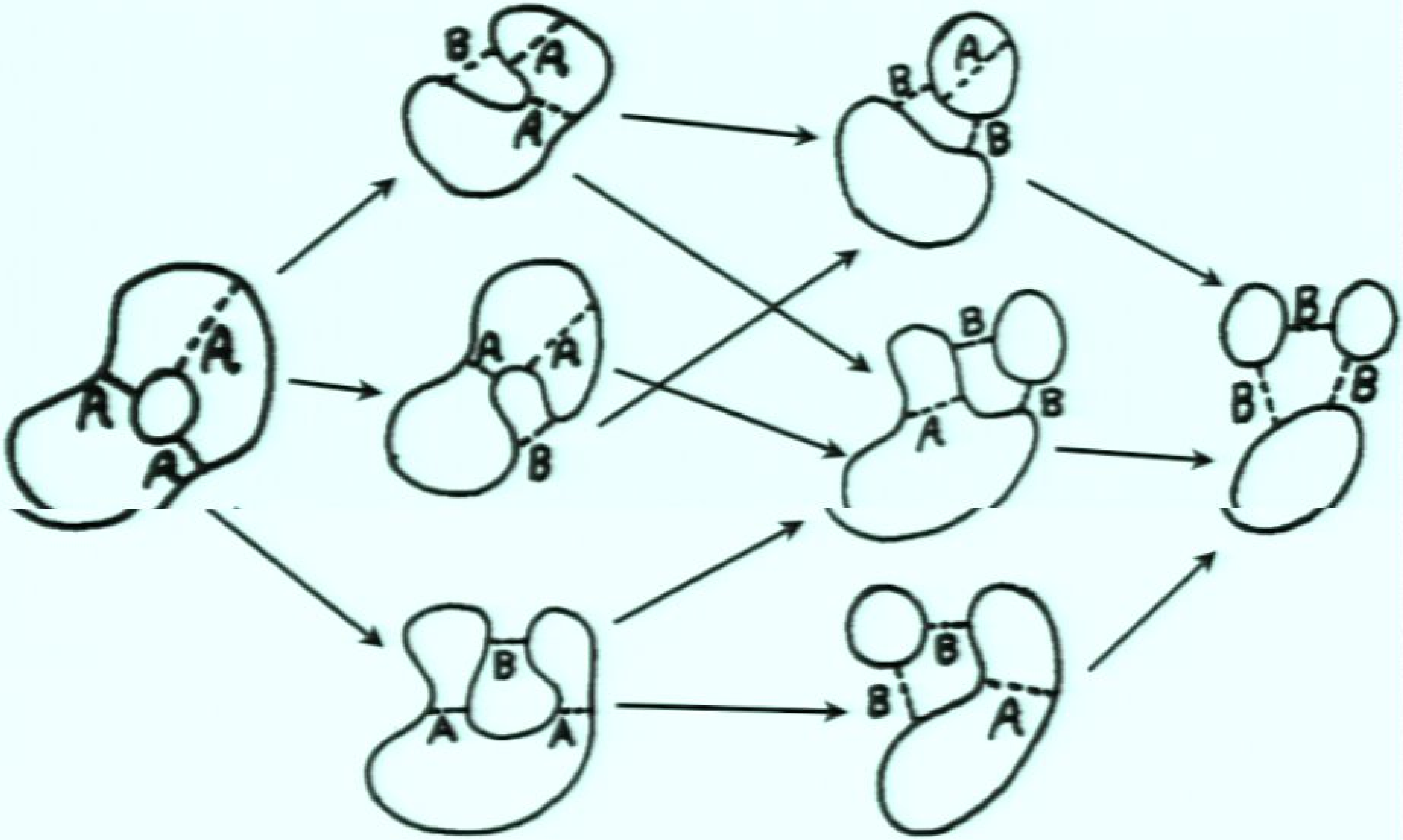
Bracket Polynomial Model for the Jones Polynomial

$$\langle K \bigcirc \rangle = (-A^{2} - A^{-2})\langle K \rangle$$



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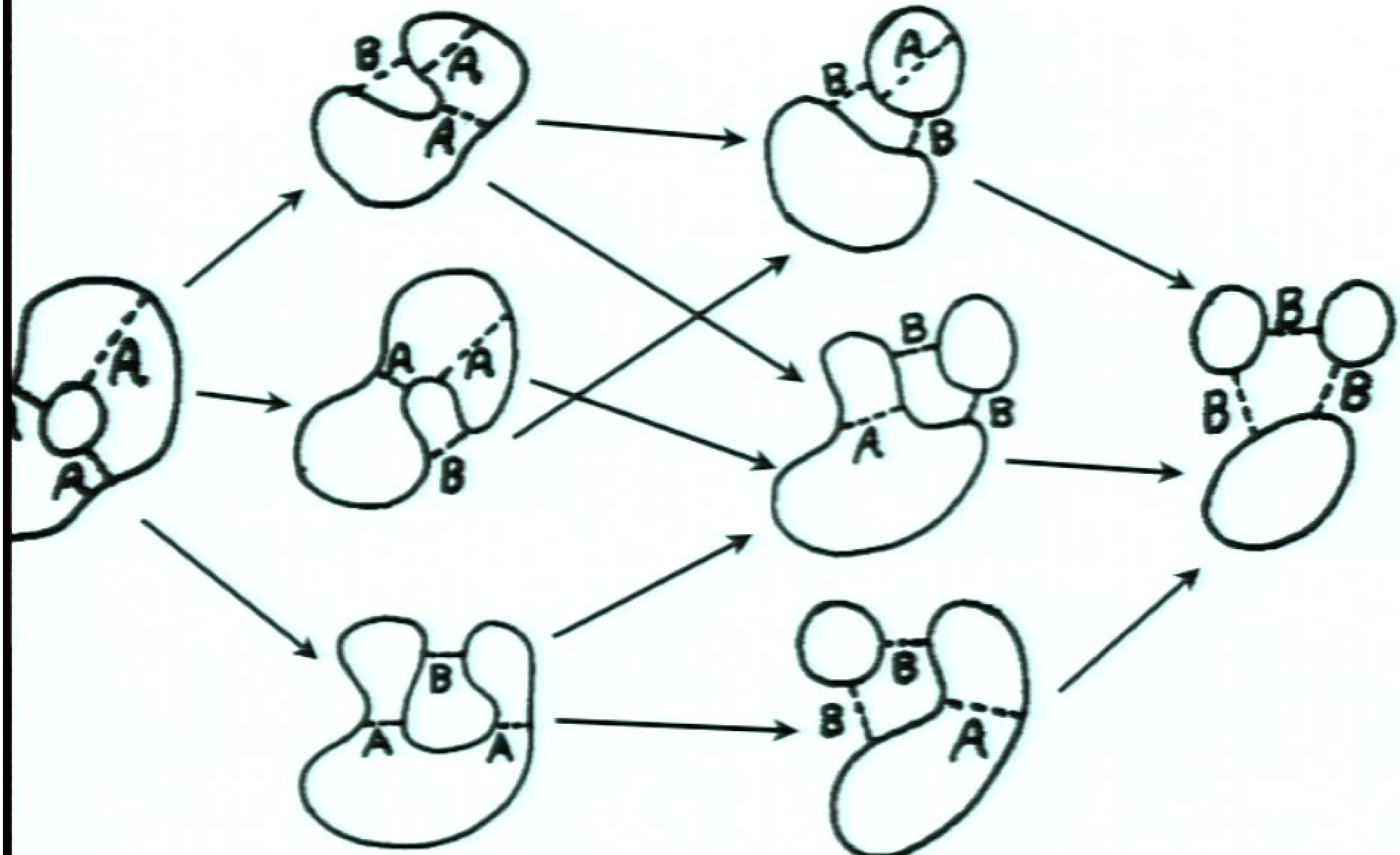

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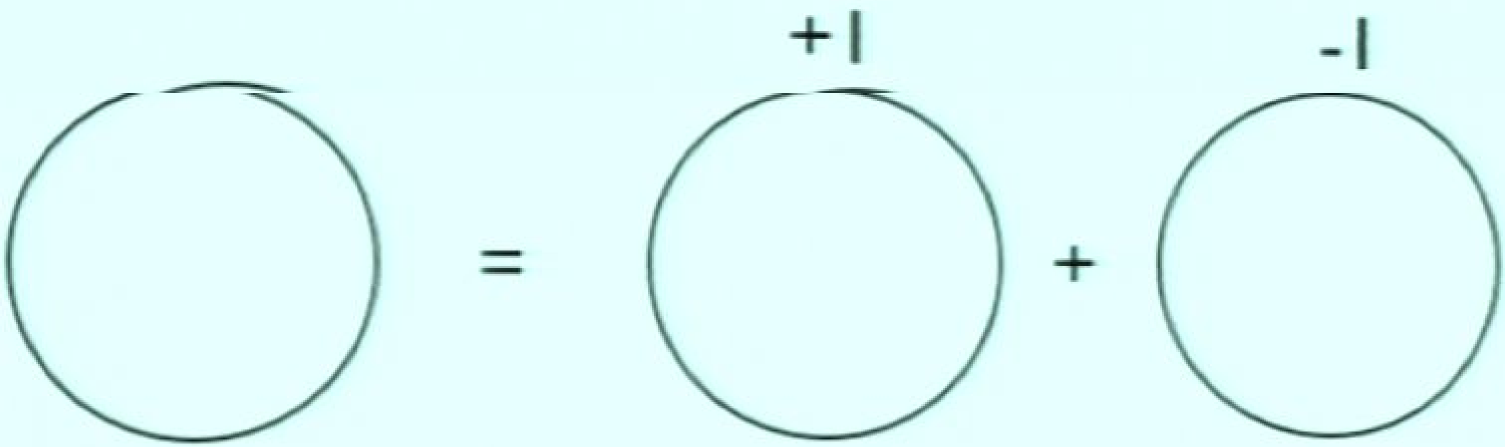
Use enhanced states by labeling each loop with +1 or -1.



$$\longleftrightarrow \quad q + q^{-1}$$

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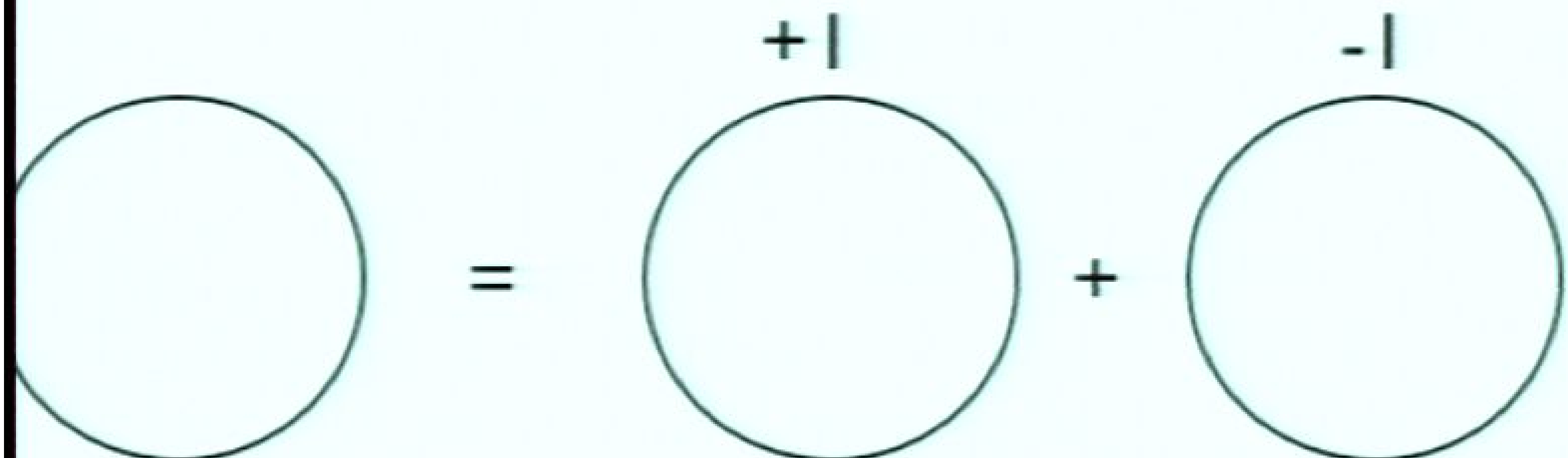
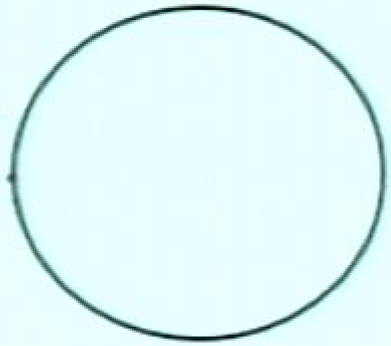


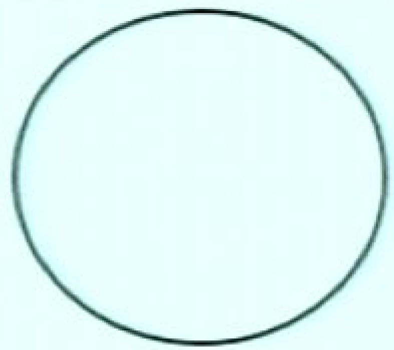
$$\longleftrightarrow \quad q + q^{-1}$$

Enhanced States

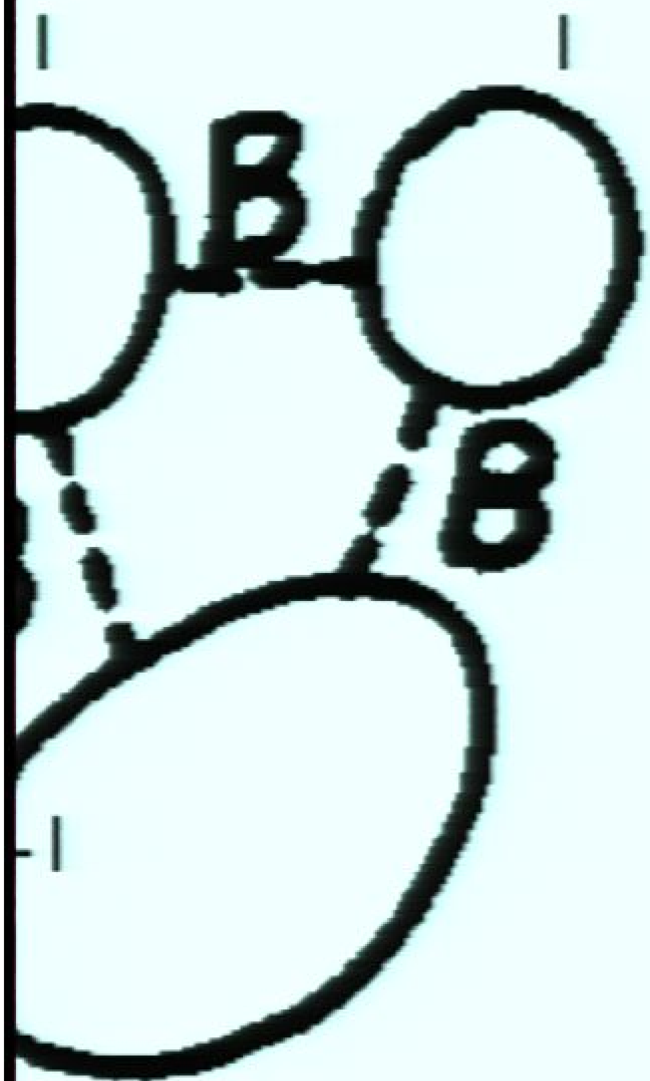
$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.



An enhanced state
that contributes

[(q)(q)(1/q)] [(-q) (-q) (-q)]
1 1 -1 **B B B**

to the revised
bracket state sum.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)}(-1)^{i(s)}$$

A Quantum Statistical Model for the Bracket Polynonmial.

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_{s} |s\rangle$$

na. The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle .$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

$$= \sum_s q^{j(s)}(-1)^{i(s)}$$

$$(-1)^{i(s)} q^{j(s)} |s\rangle = \sum_{s'} \sum_s (-1)^{i(s)} q^{j(s)} \langle s' | s \rangle$$

$$\sum_s (-1)^{i(s)} q^{j(s)} = \langle K \rangle.$$

$$\rangle = \delta(s, s').$$

$$= \langle \psi | U | \psi \rangle.$$

Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. … of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are … meanings to the term categorify, but here the quest is to find a way to express the … polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some hom… theory associated with $\langle K \rangle$.

We will formulate Khovanov Homology in the context of our quantum statistical model for the bracket polynomial.

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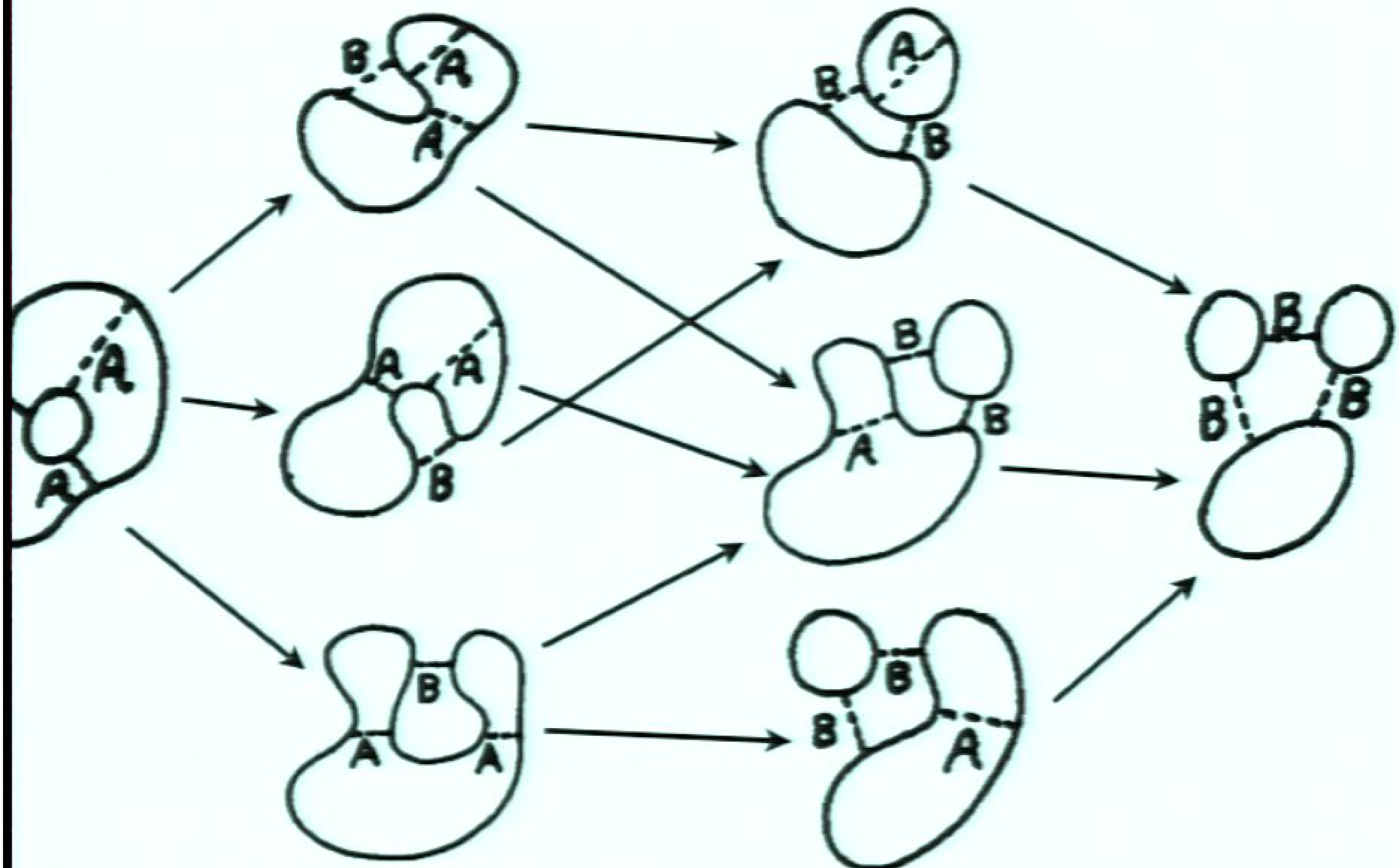
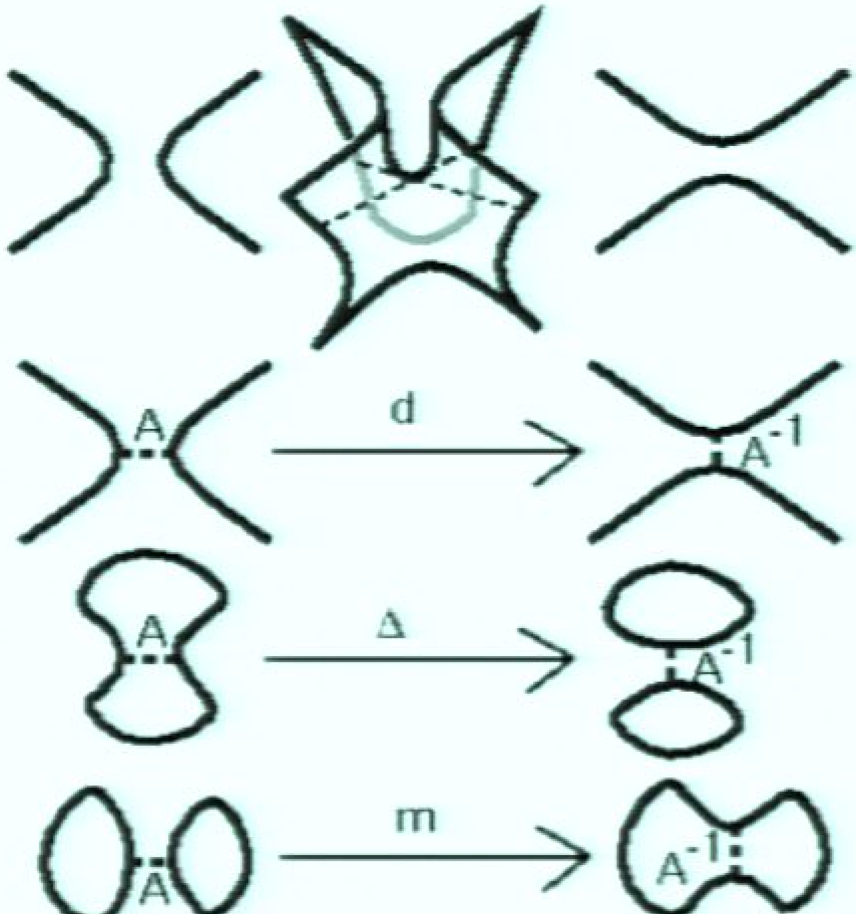
The Khovanov Complex



$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

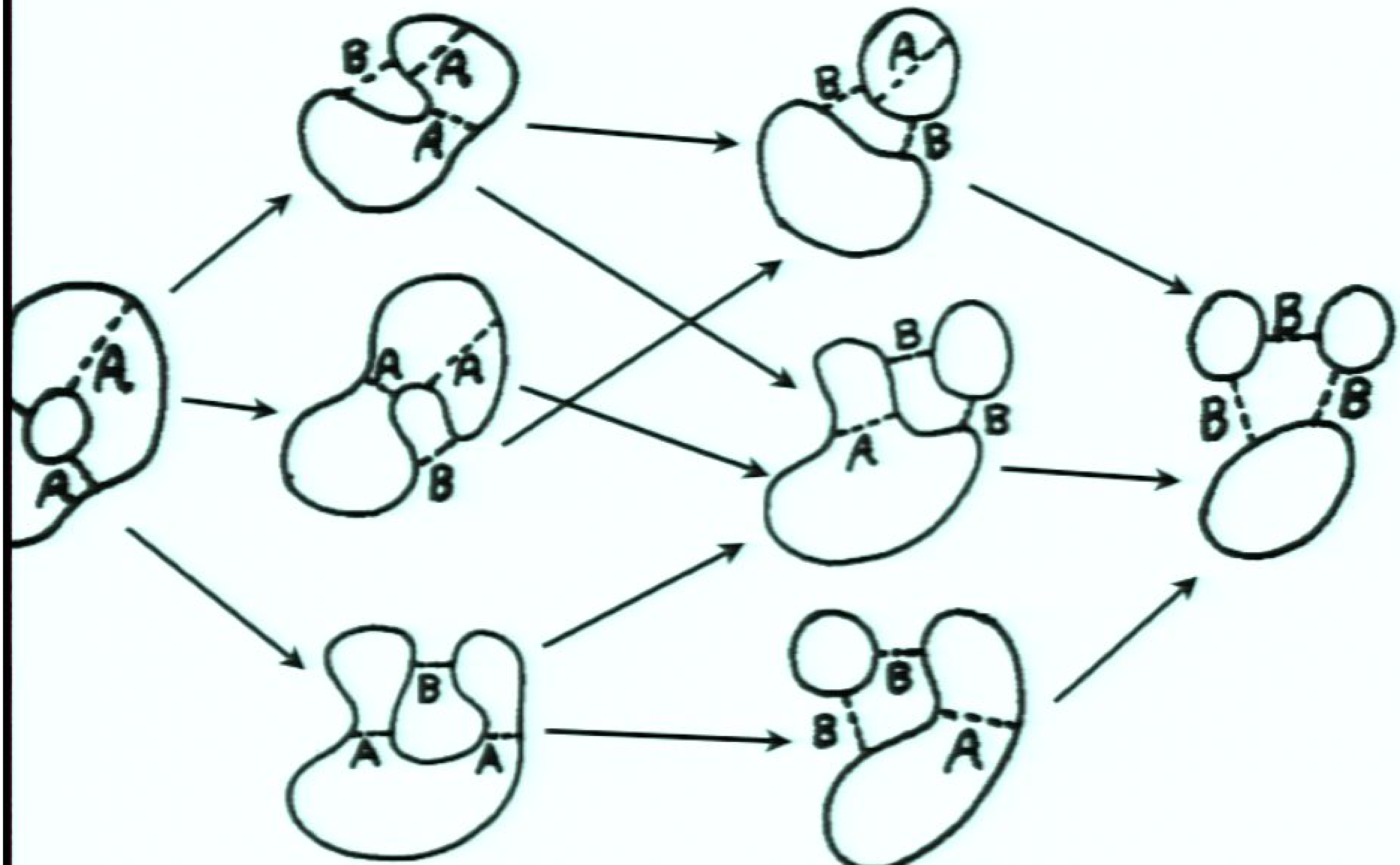
The boundary is a sum of partial differentials corresponding to resmoothings on the states.



Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.

The Khovanov Complex



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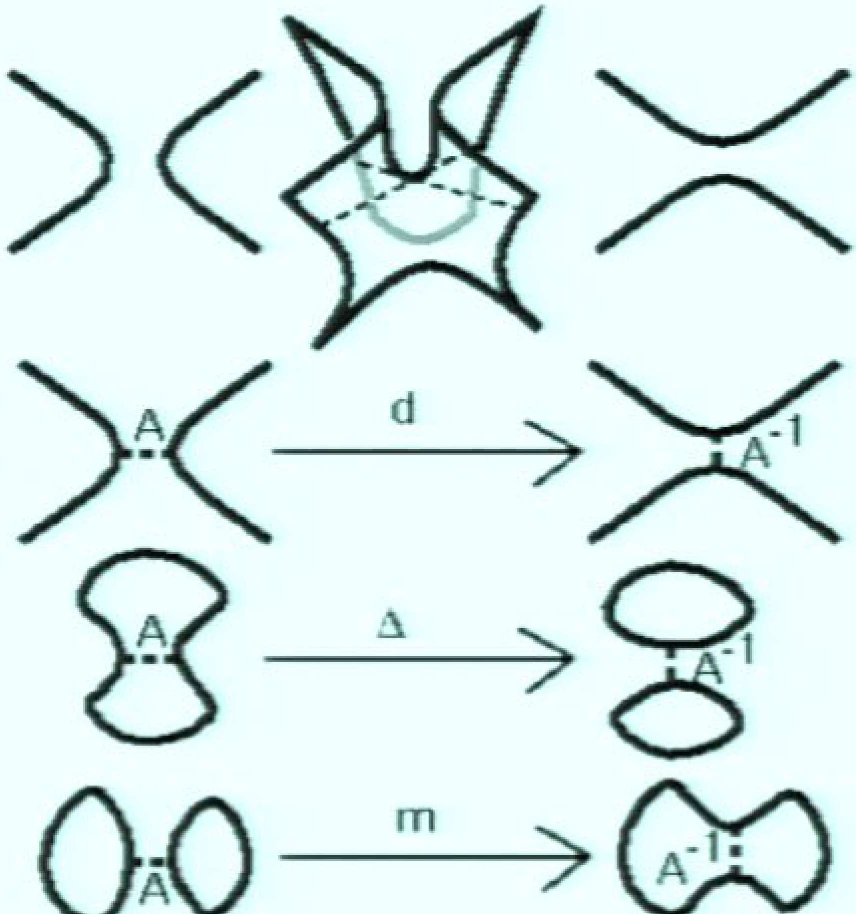

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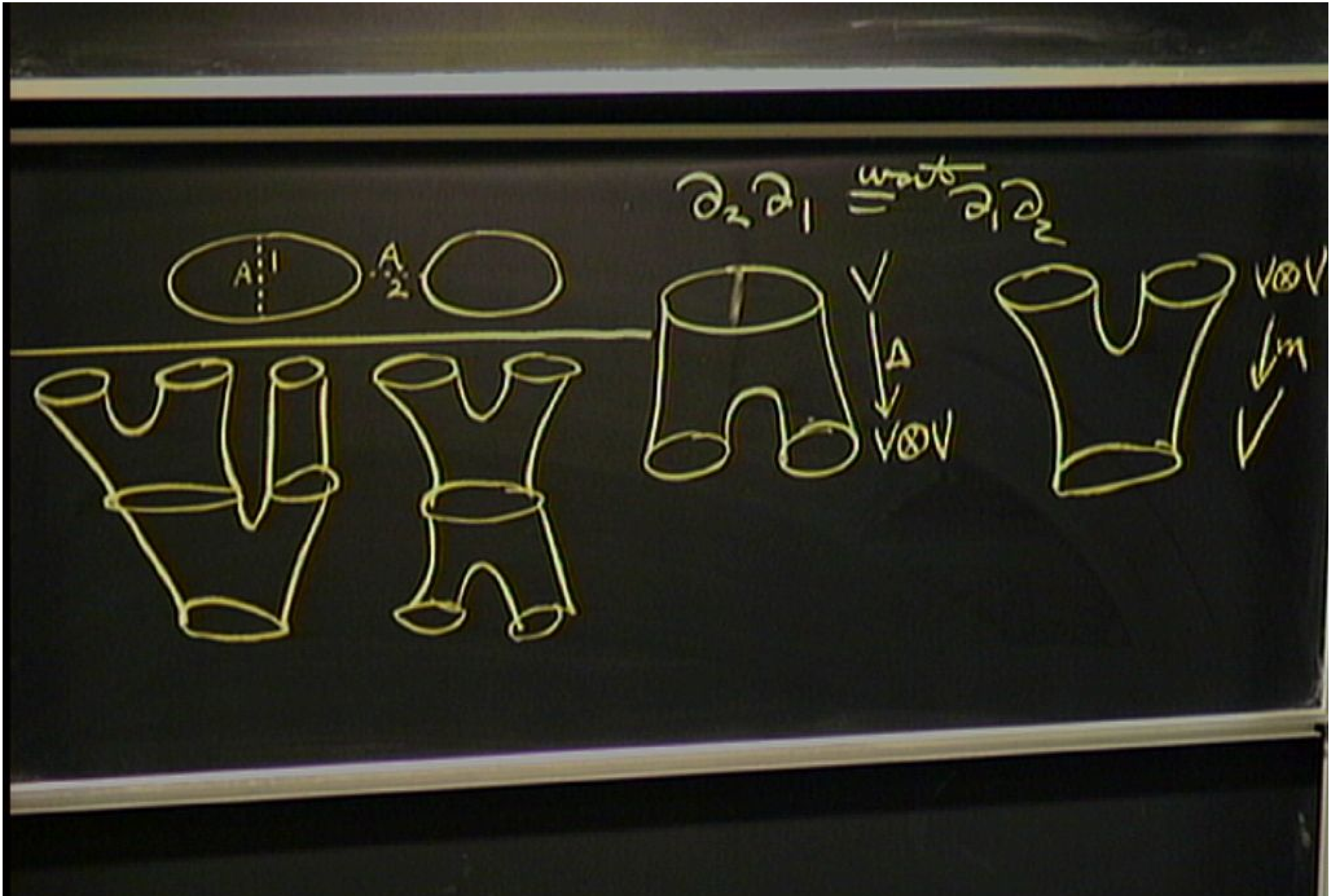
$\partial_2\partial_1 \overset{\text{wait}}{=} \partial_1\partial_2$
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V⊗V
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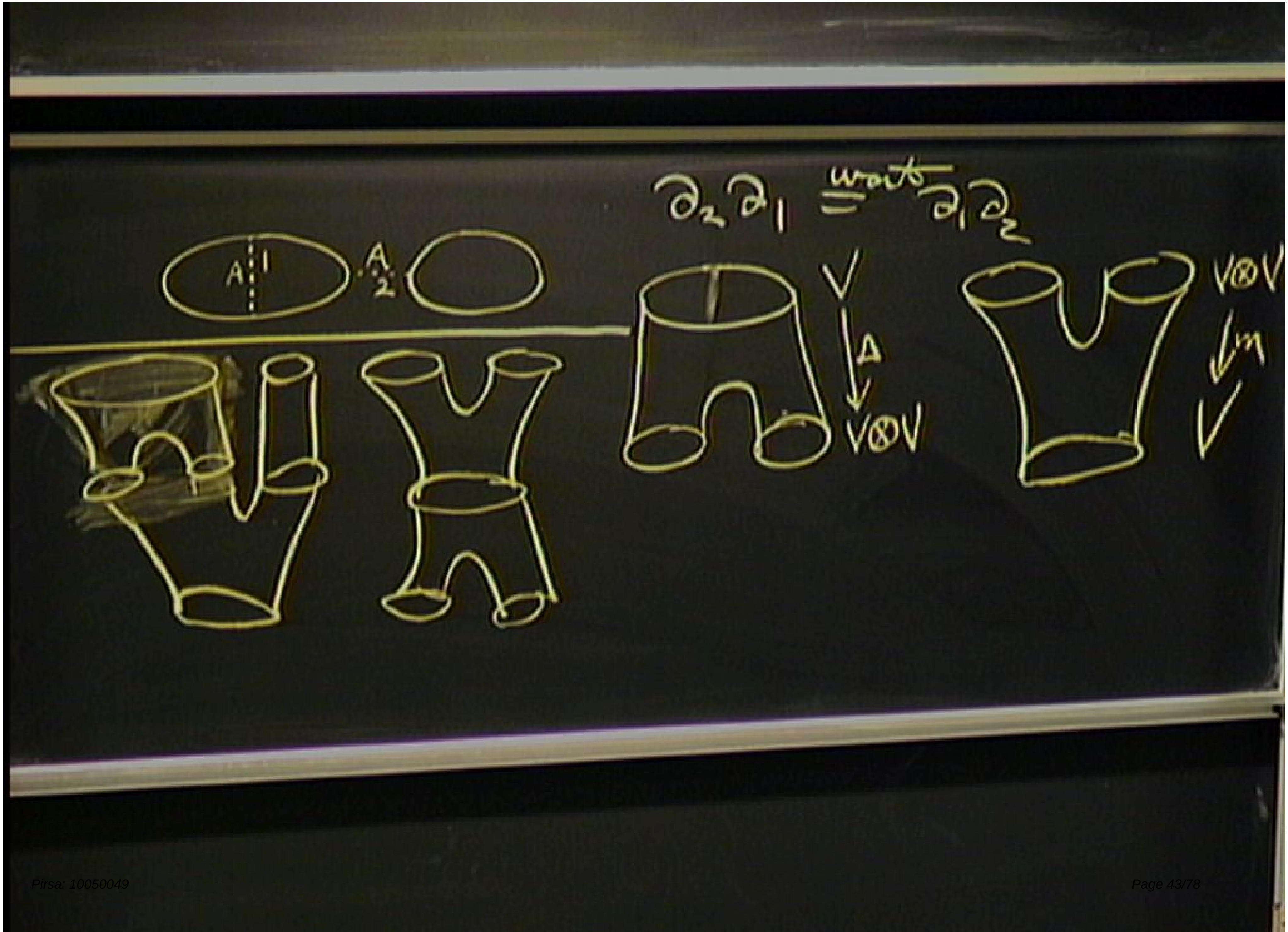

∂₂∂₁ =wait ∂₁∂₂
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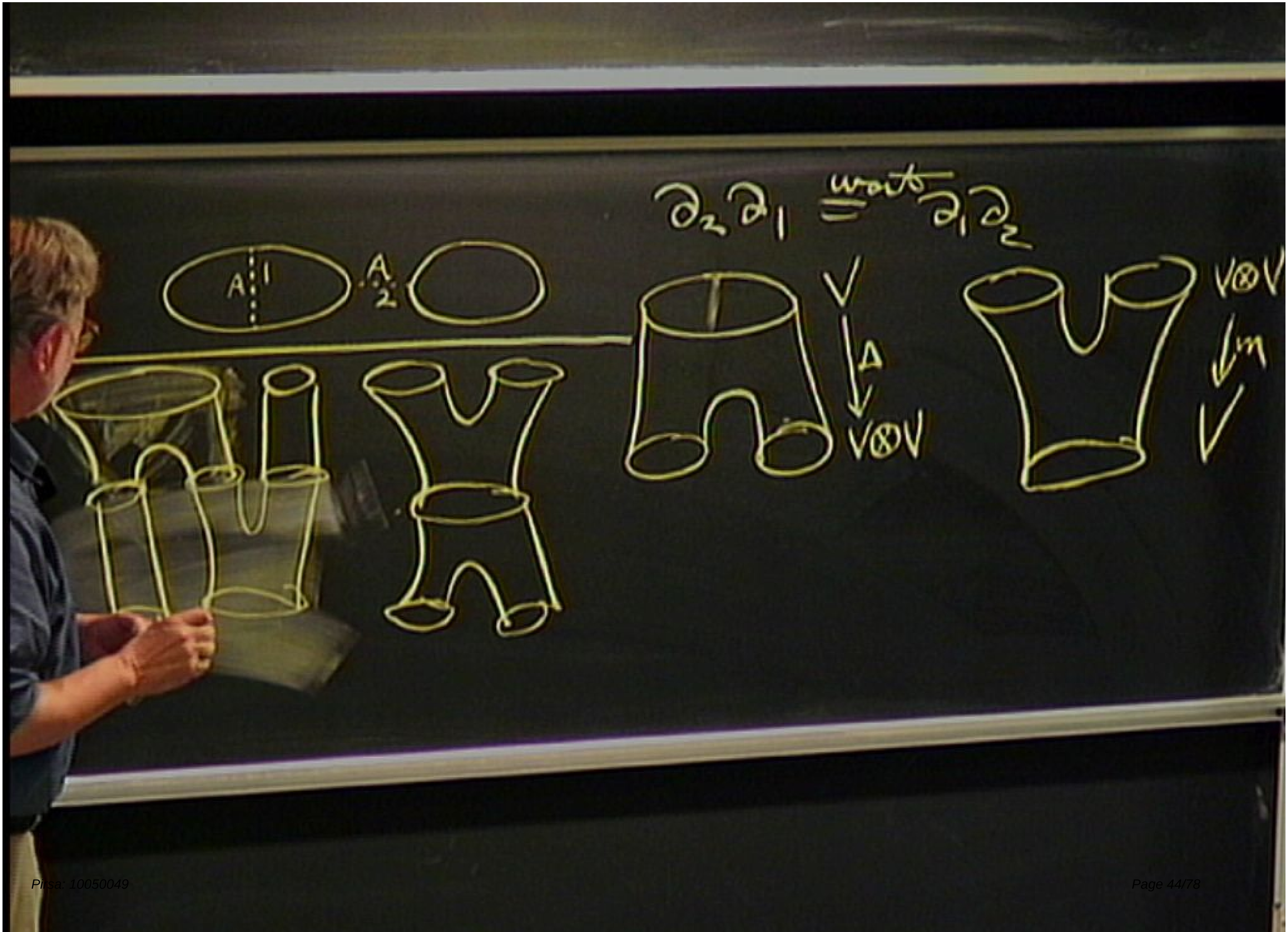
$\partial_2\partial_1 \overset{\text{wait}}{=} \partial_1\partial_2$
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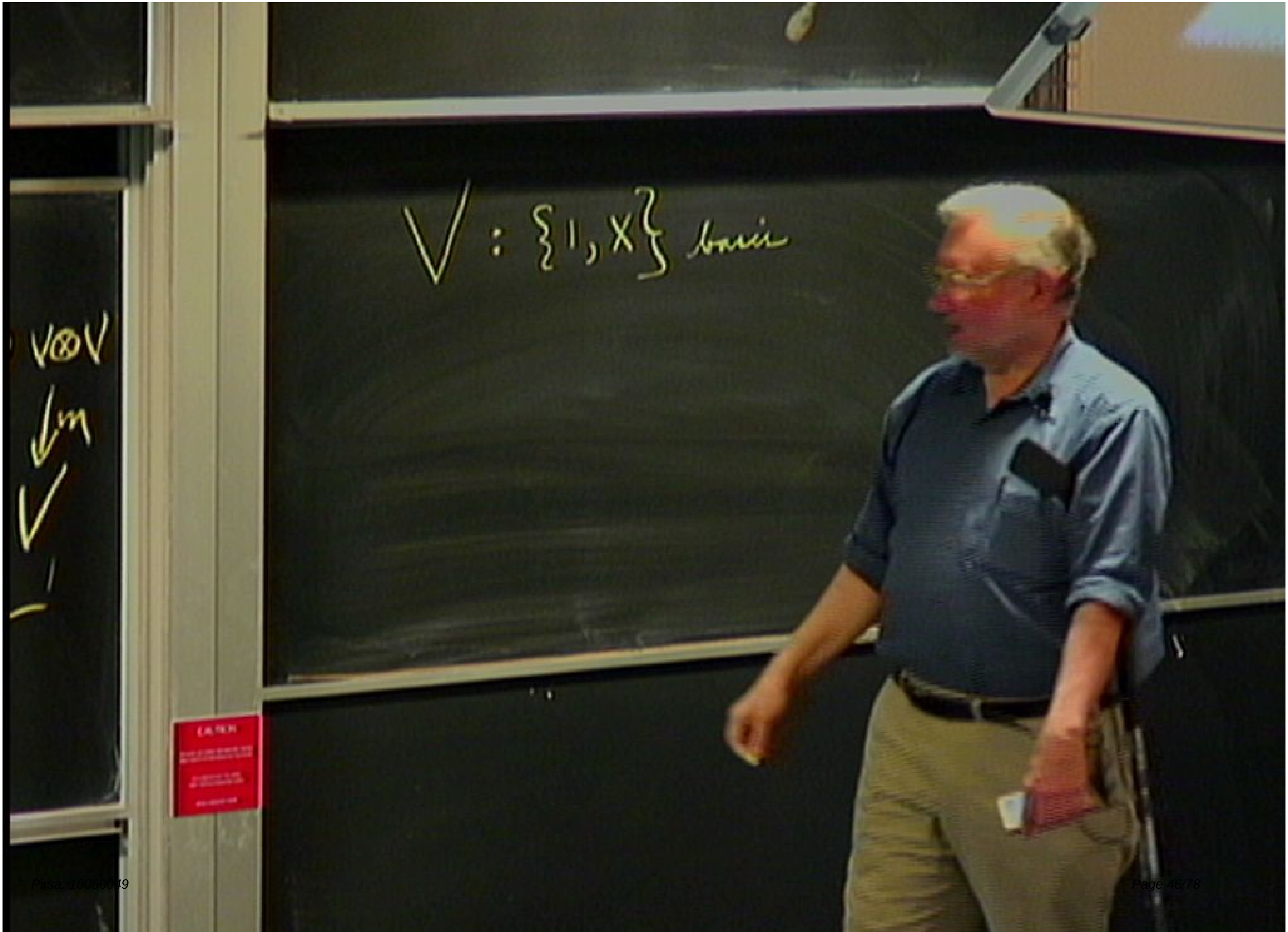
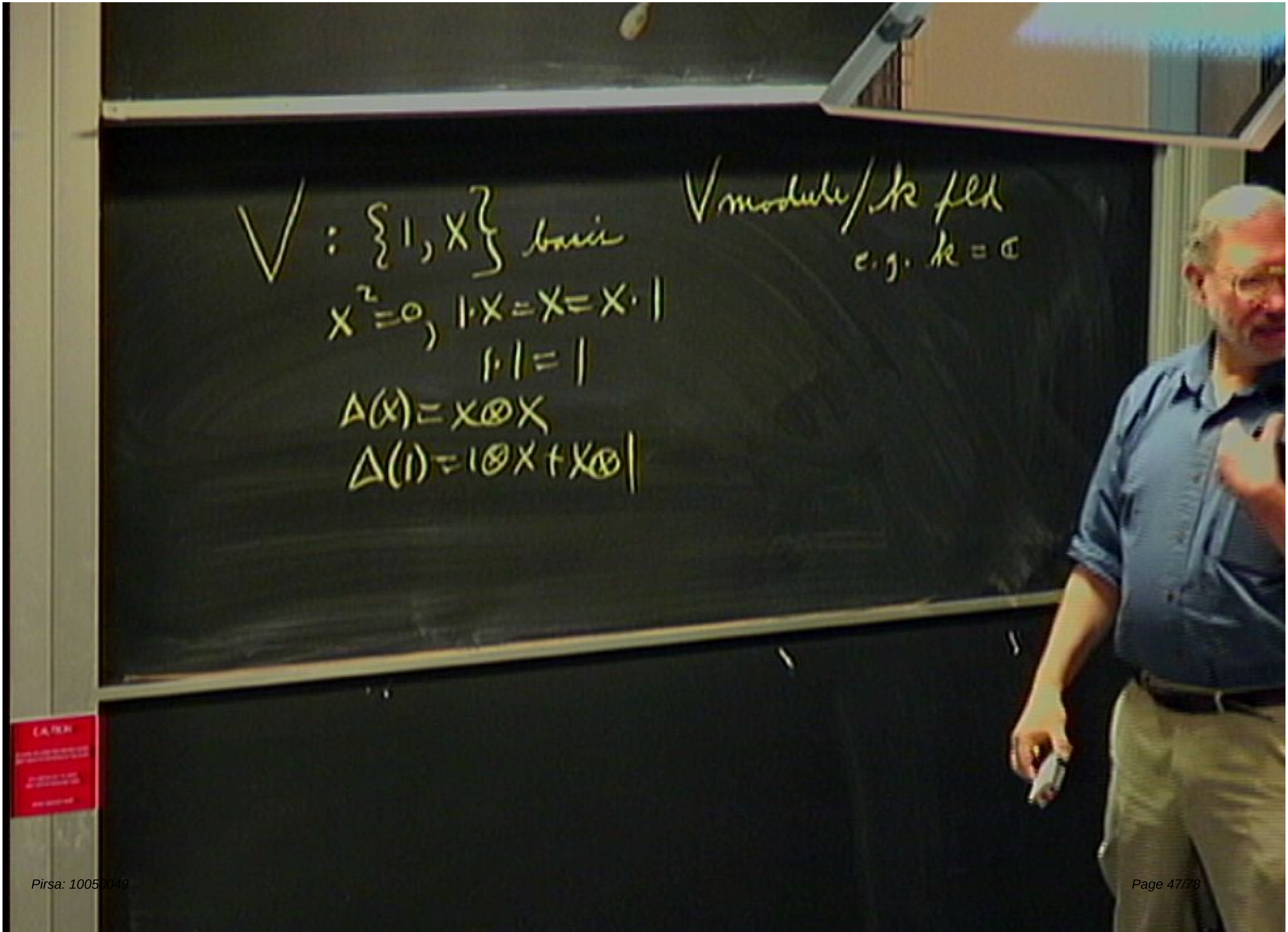
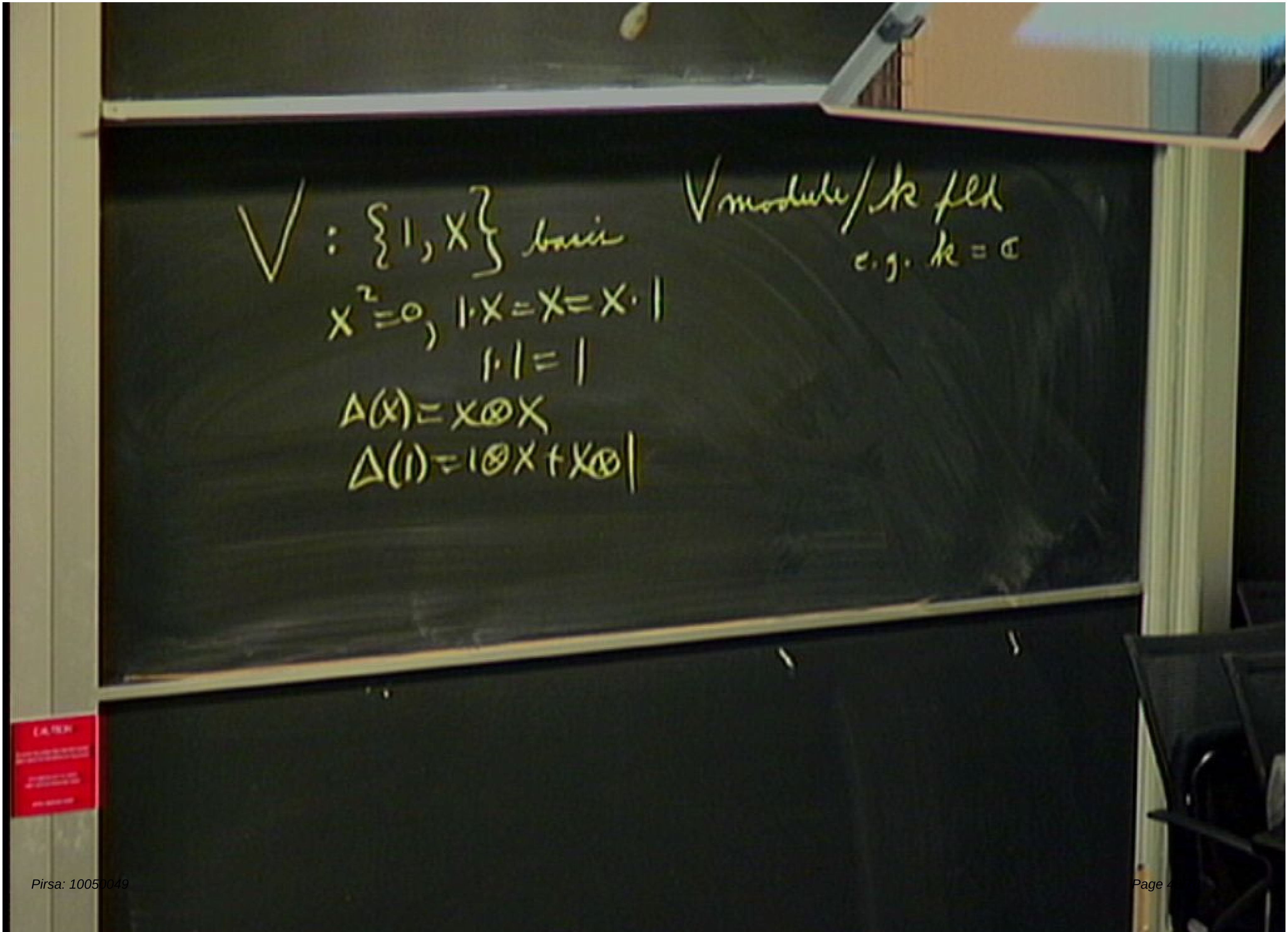

V : {1, X} basis
V⊗V
m
V


V : {1, X} basis
V module / k field
e.g. k = ℂ
X² = 0, 1·X = X = X·1
1·1 = 1
Δ(X) = X⊗X
Δ(1) = 1⊗X + X⊗1


V : {1, X} basis
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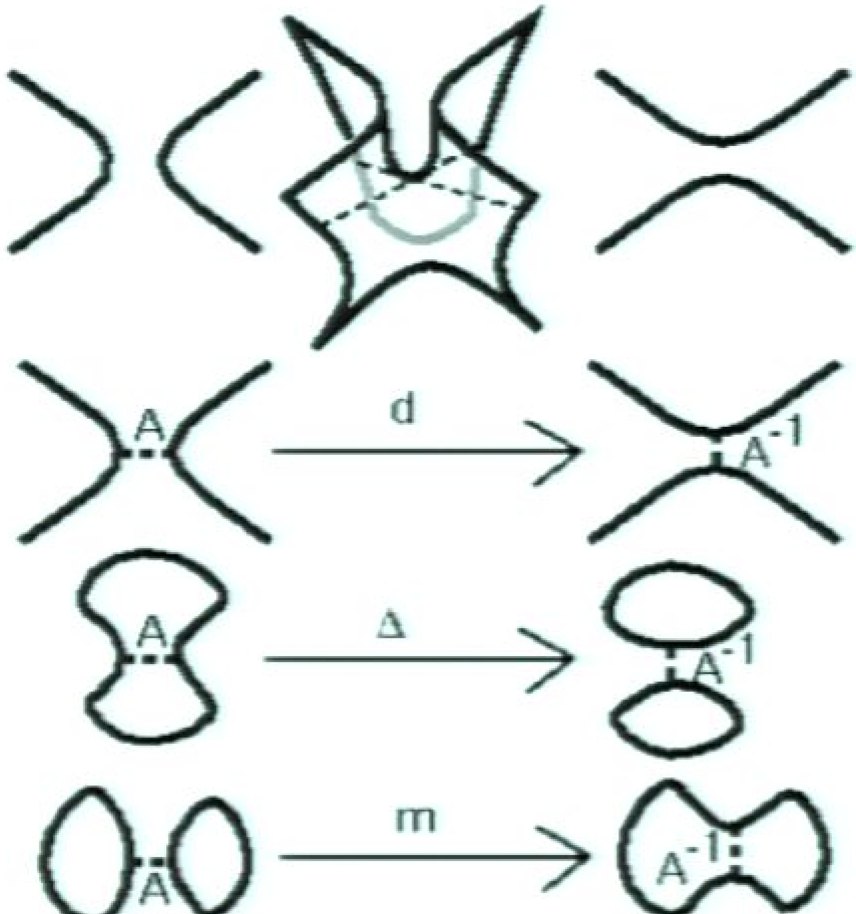

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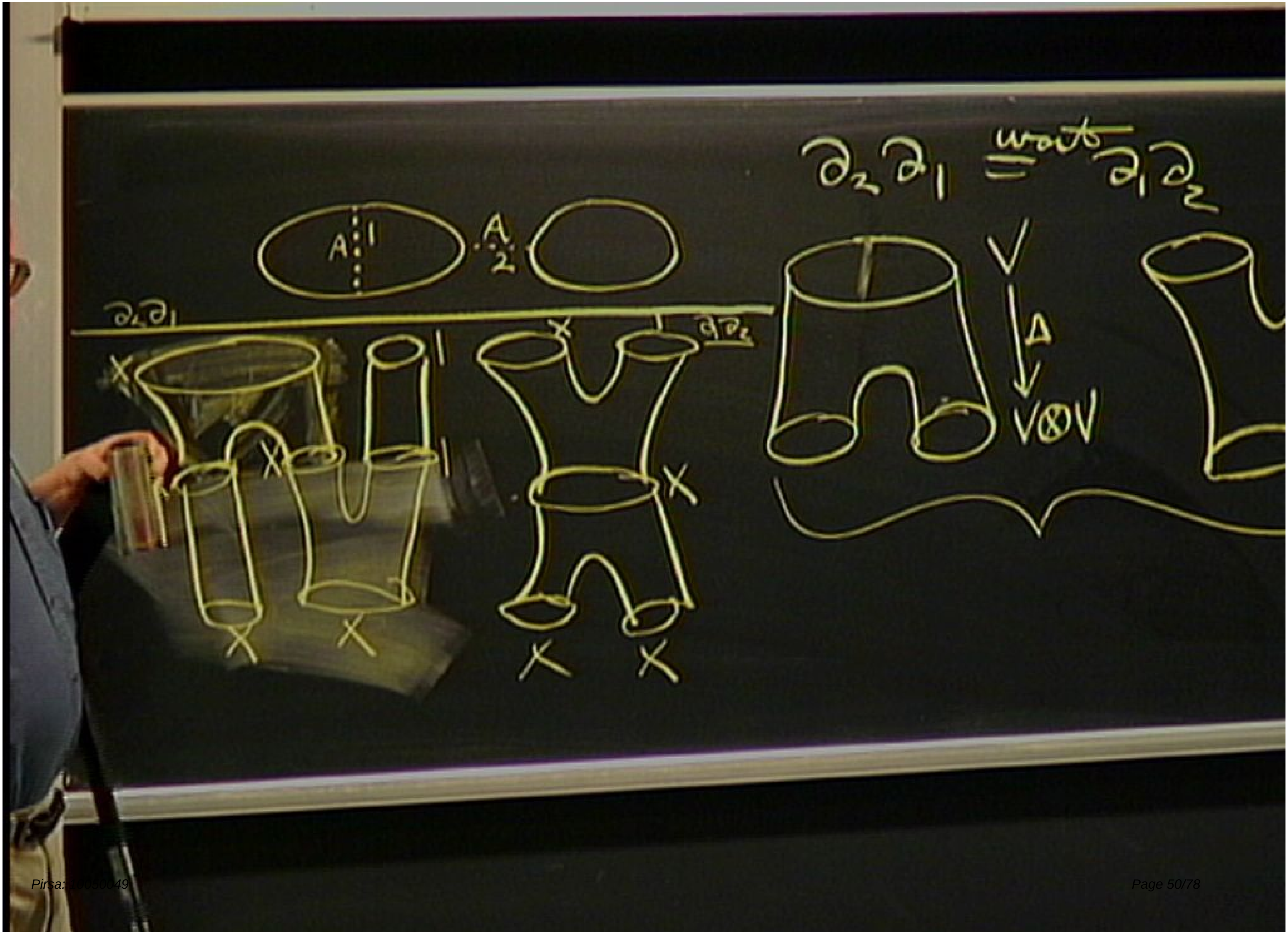
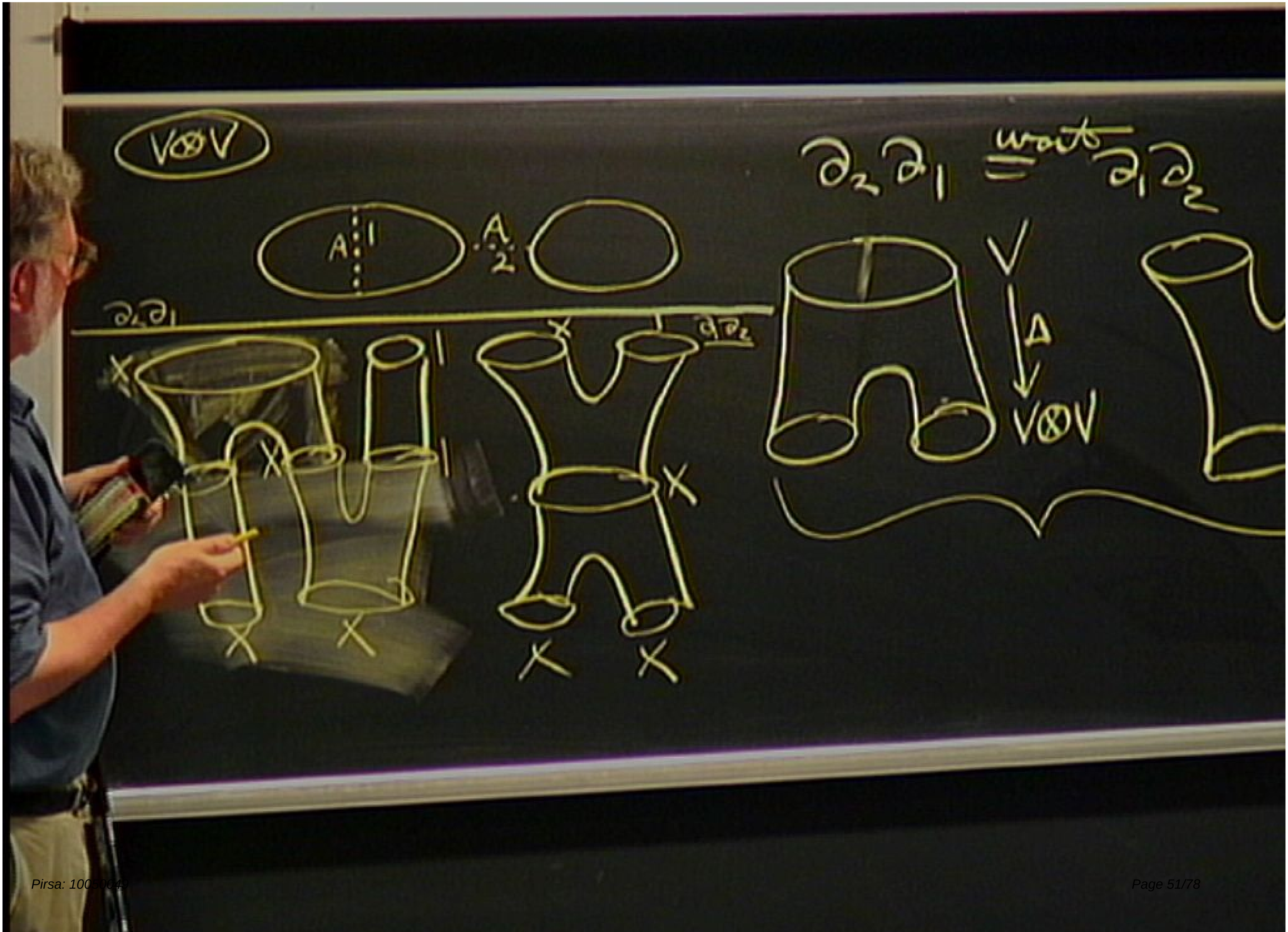
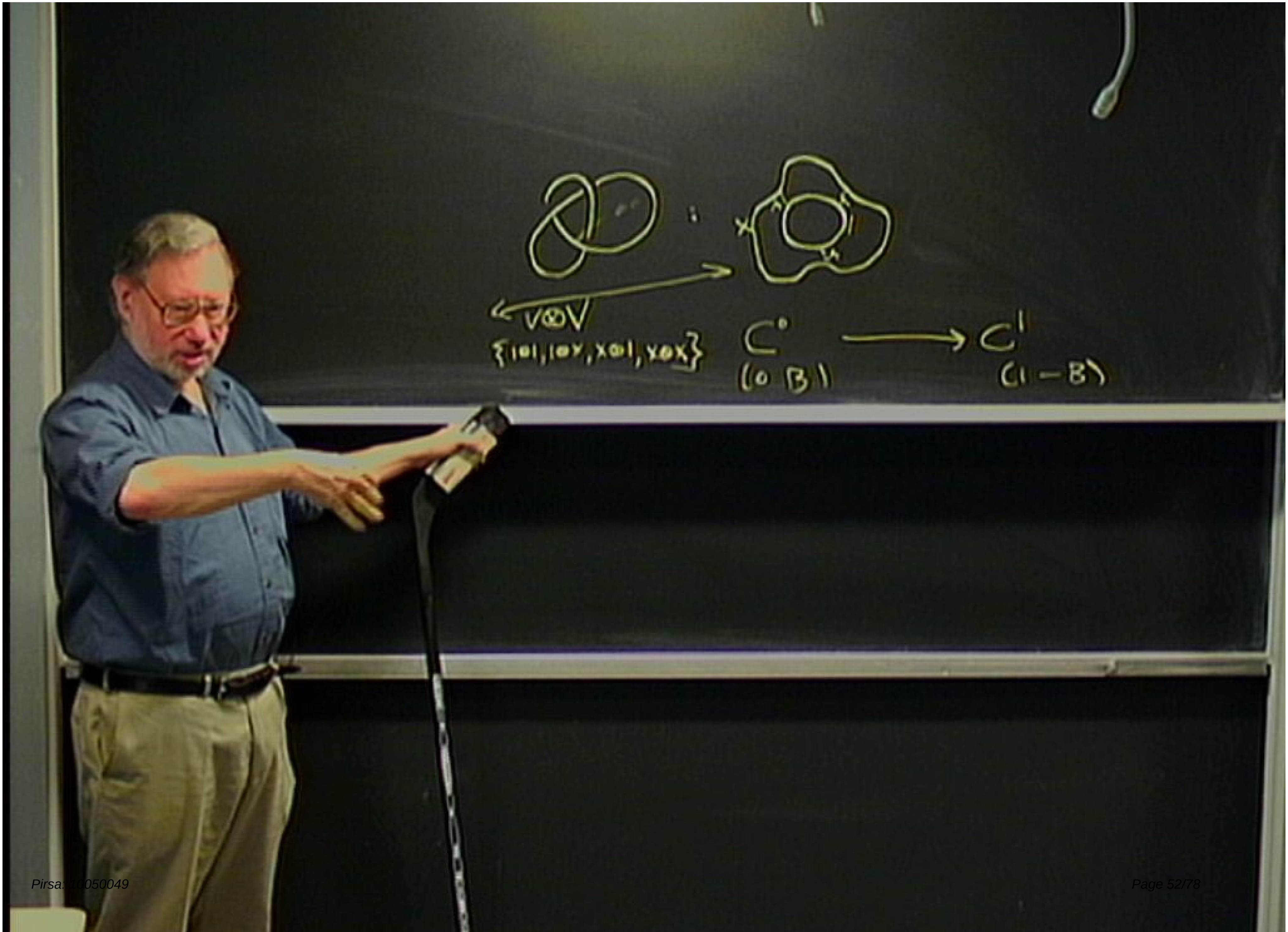
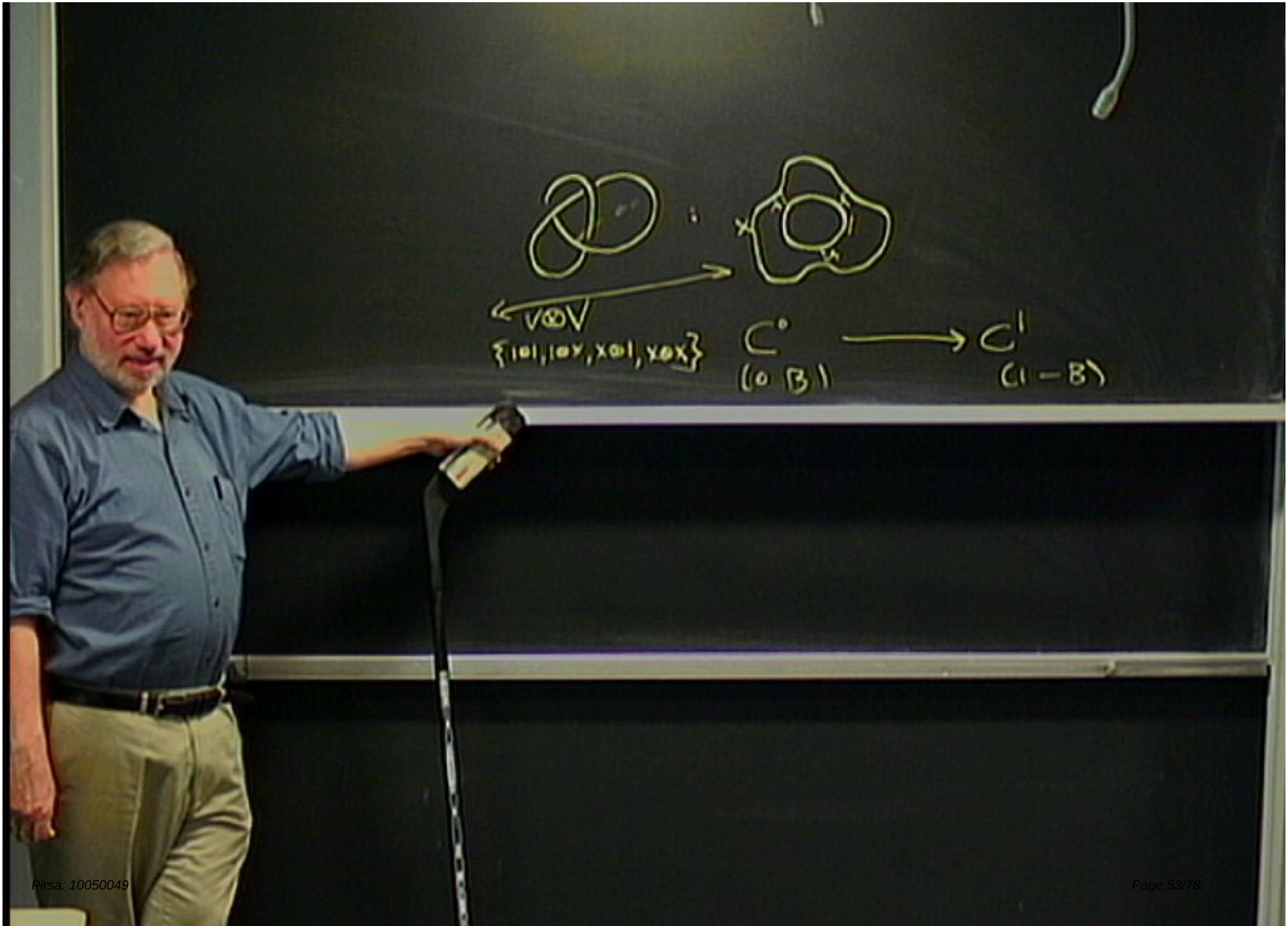


$\partial_2 \partial_1 \stackrel{\text{wait}}{=} \partial_1 \partial_2$
A
1
A
2
$\partial_2\partial_1$
$\partial_1\partial_2$
V
Δ
V⊗V


V⊗V
∂₂∂₁ =wait= ∂₁∂₂
A
1
A
2
∂₂∂₁
∂₁∂₂
V
Δ
V⊗V


V⊗V
{1⊗1, 1⊗X, X⊗1, X⊗X}
C°
(0 B 1)
C¹
(1 − B)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C⁰
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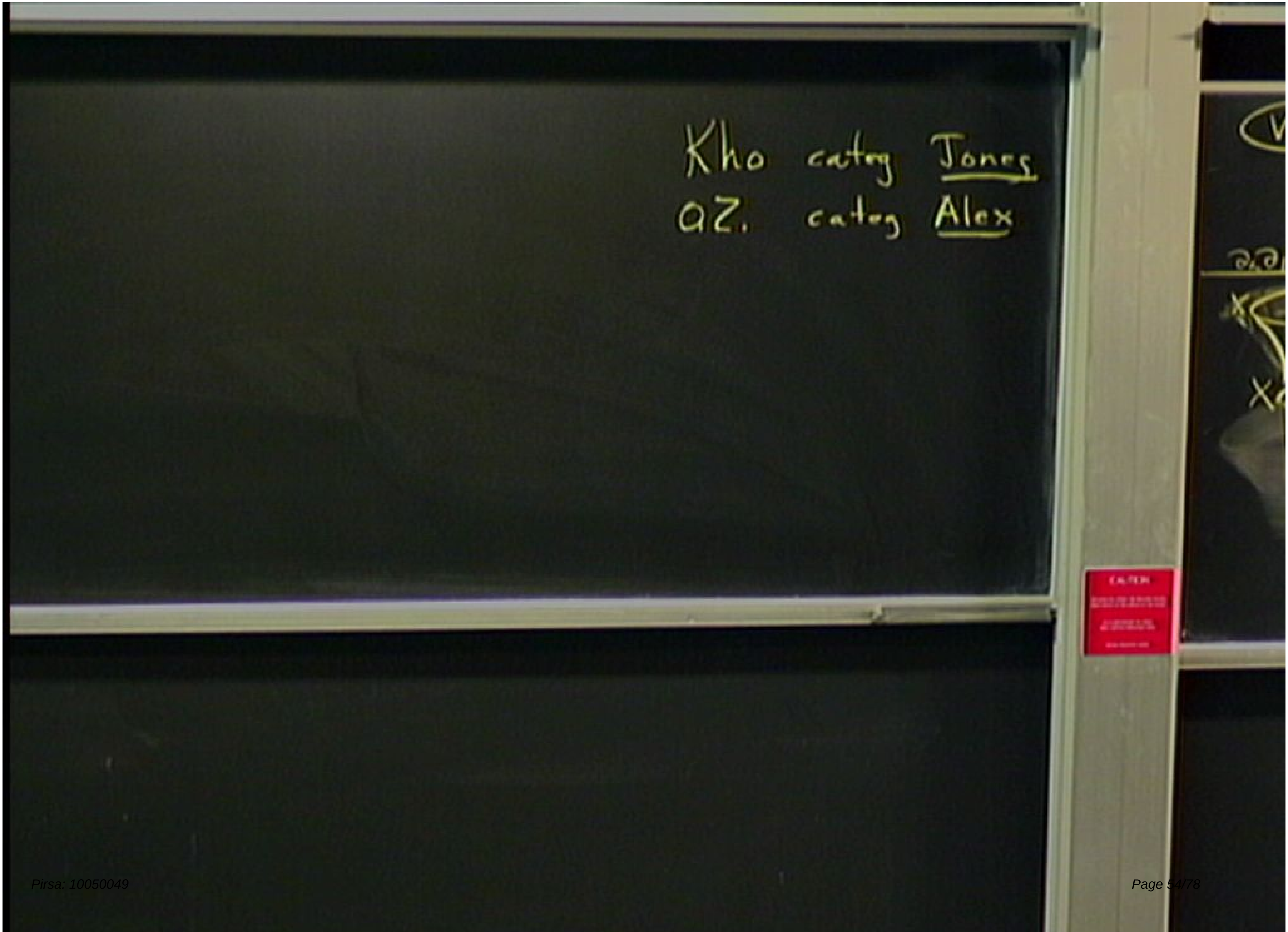
Kho categ Jones
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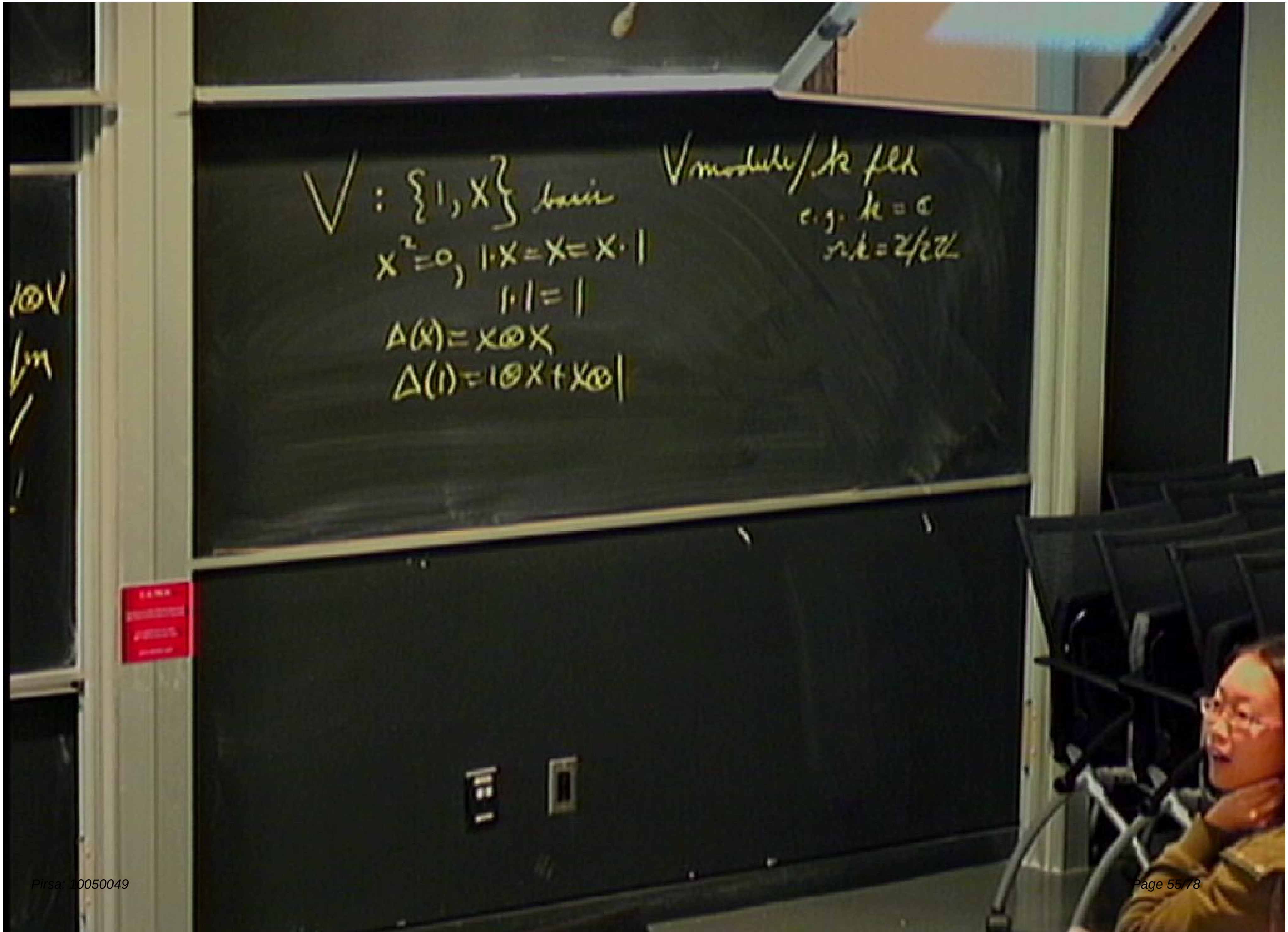
V : {1, X} basis
V module / k field
e.g. k = C
or k = Z/2Z
X^2 = 0, 1·X = X = X·1
1·1 = 1
Δ(X) = X⊗X
Δ(1) = 1⊗X + X⊗1

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)} (-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j dim(C^{ij})$$

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

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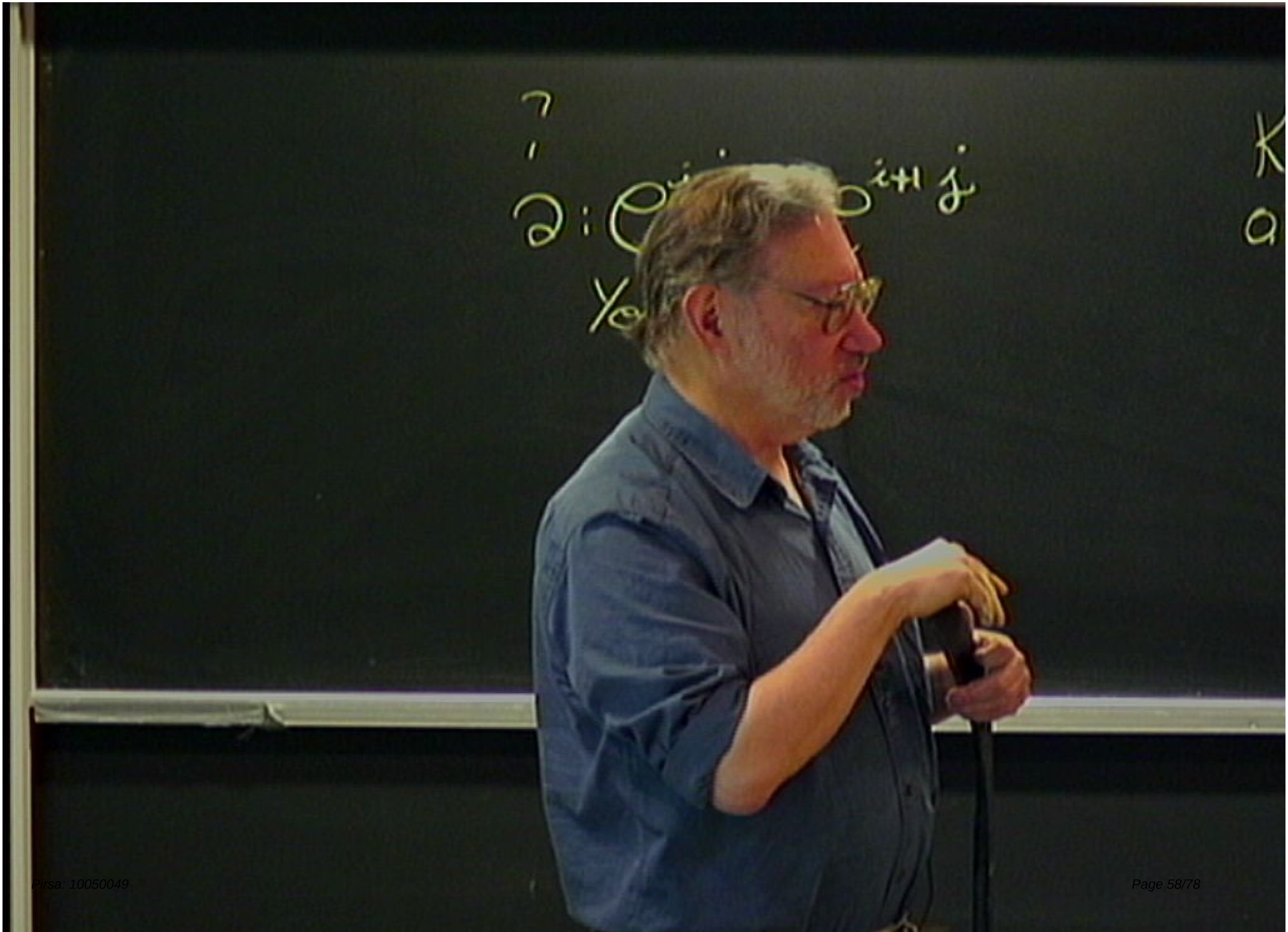


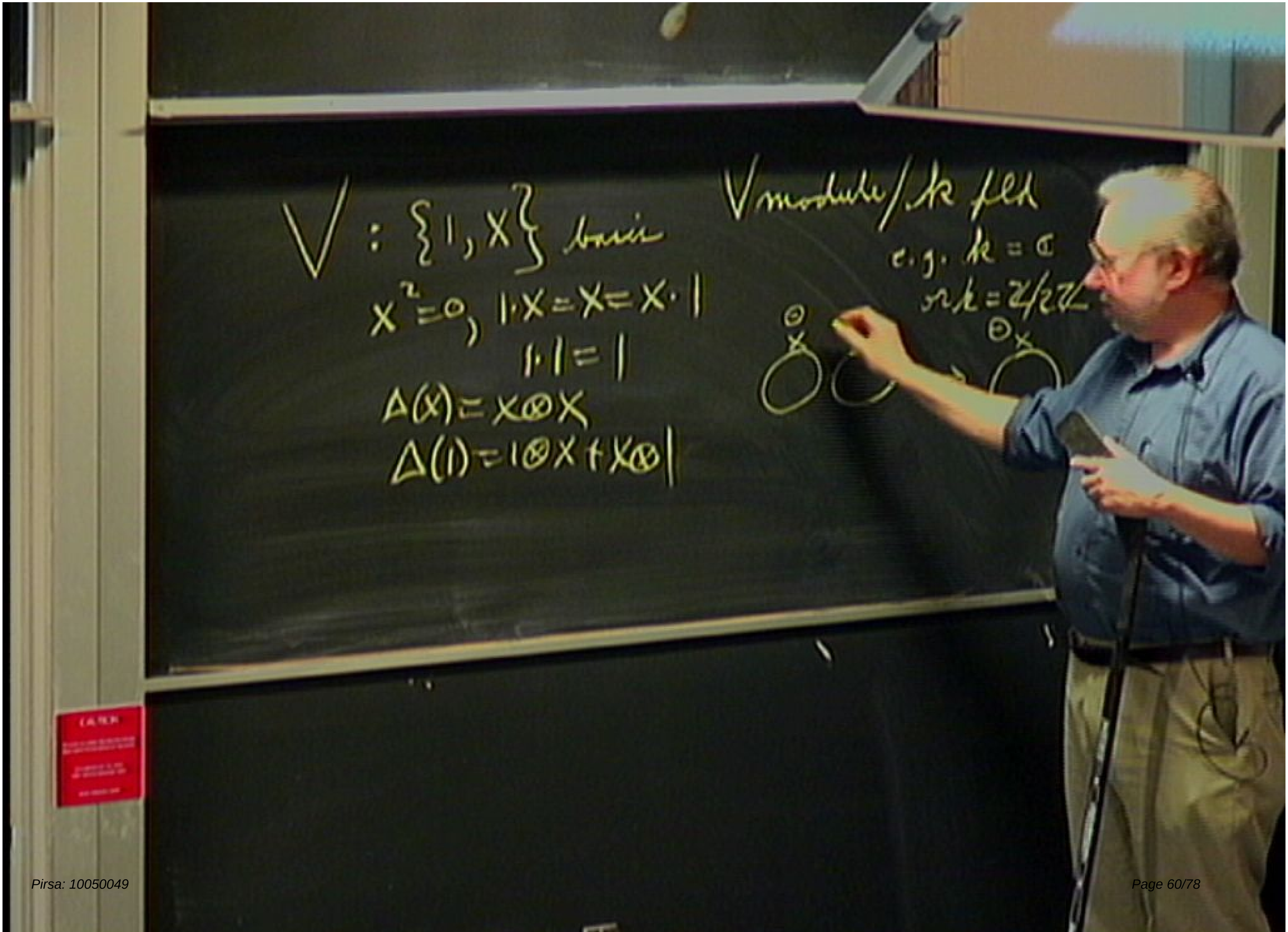
$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

hovanov constructs differential acting in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.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e.g. k = ℂ
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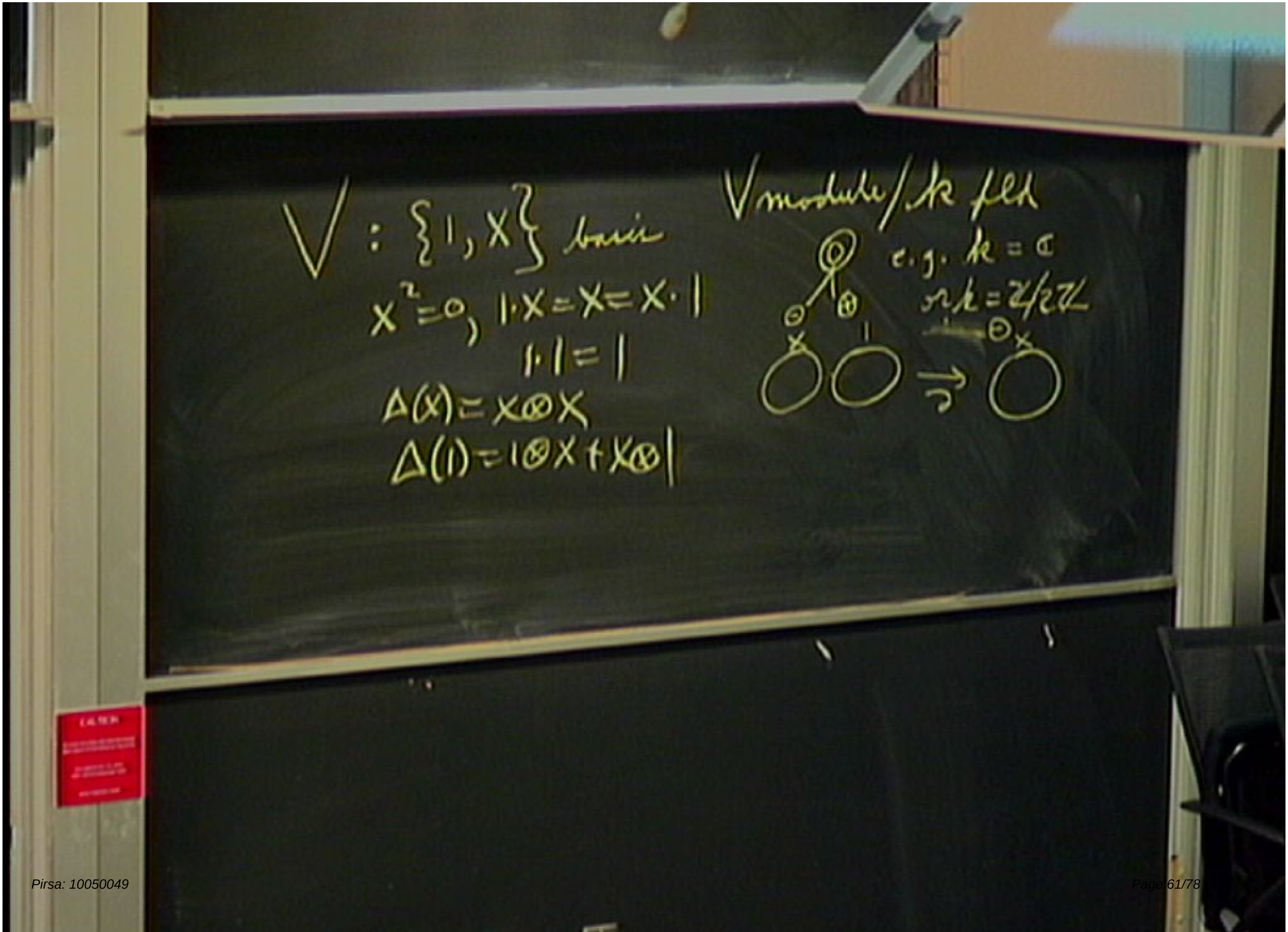
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For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.

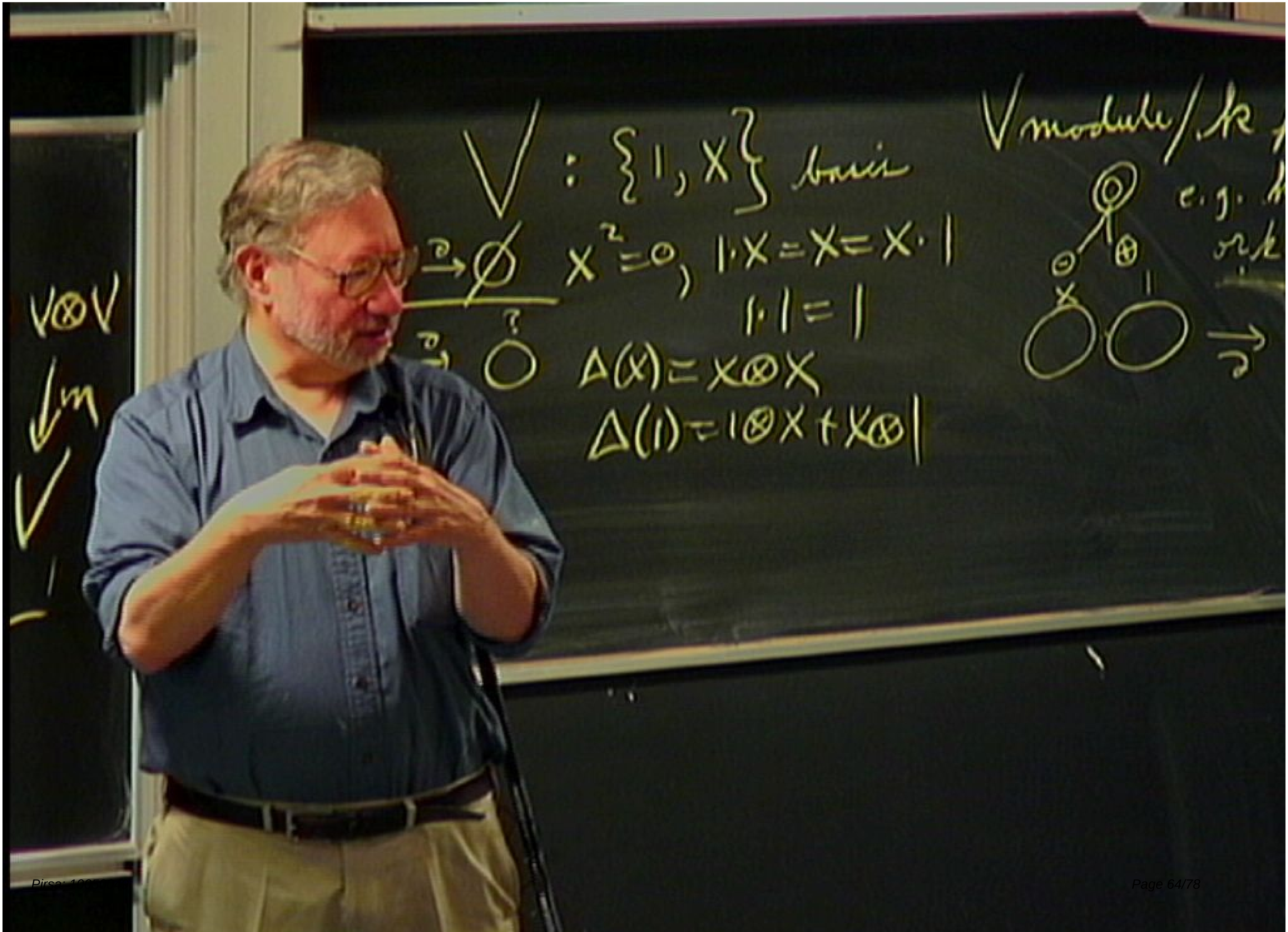
The differential increases the homological ading i by 1 and leaves fixed the quantum grading j.

Then

$$K\rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j})$$

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j})$$

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$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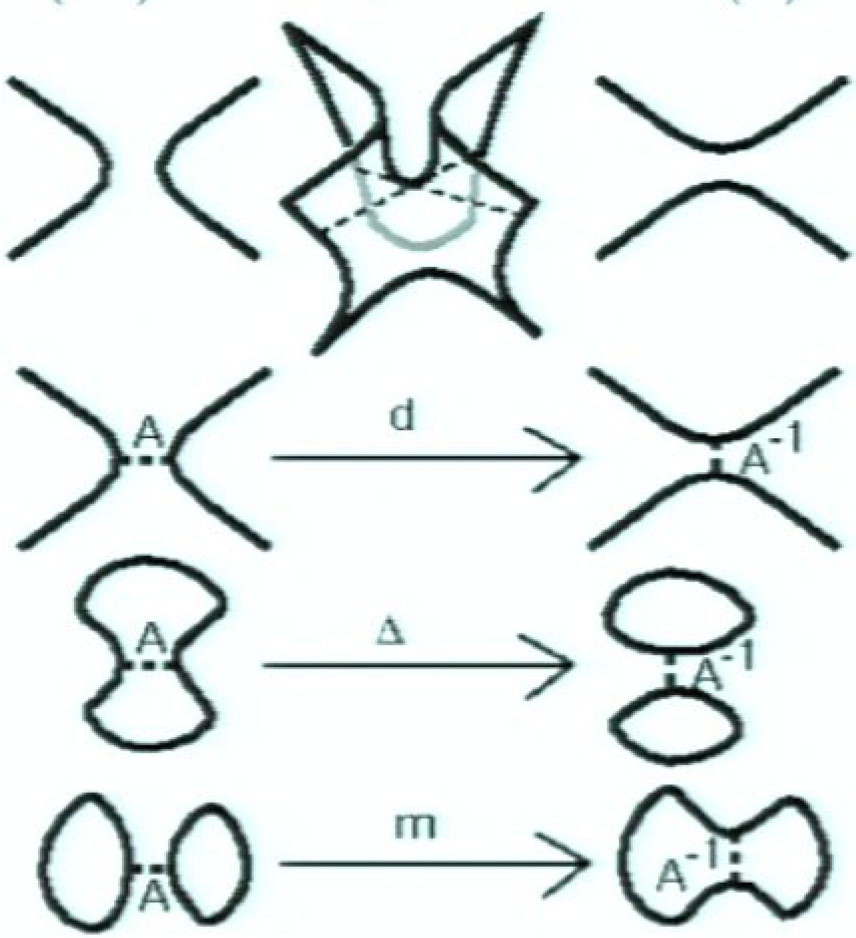
V module/k
e.g.
V⊗V
m
V

$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

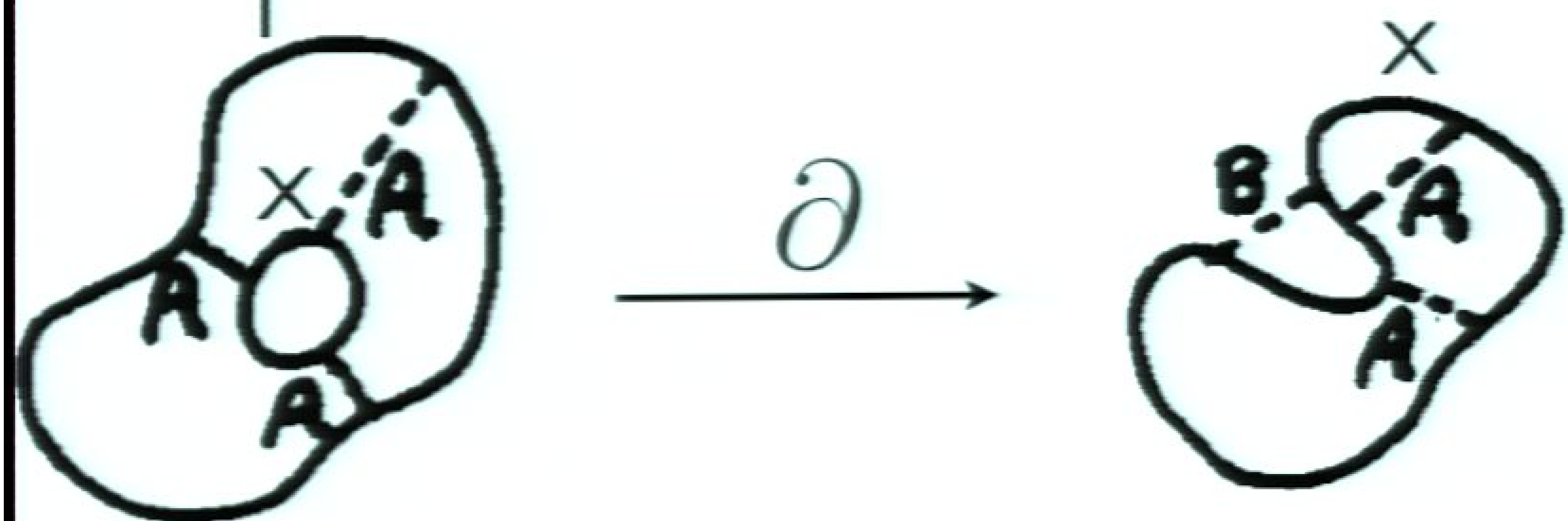
The boundary is a sum of partial differentials corresponding to resmoothings on the states.

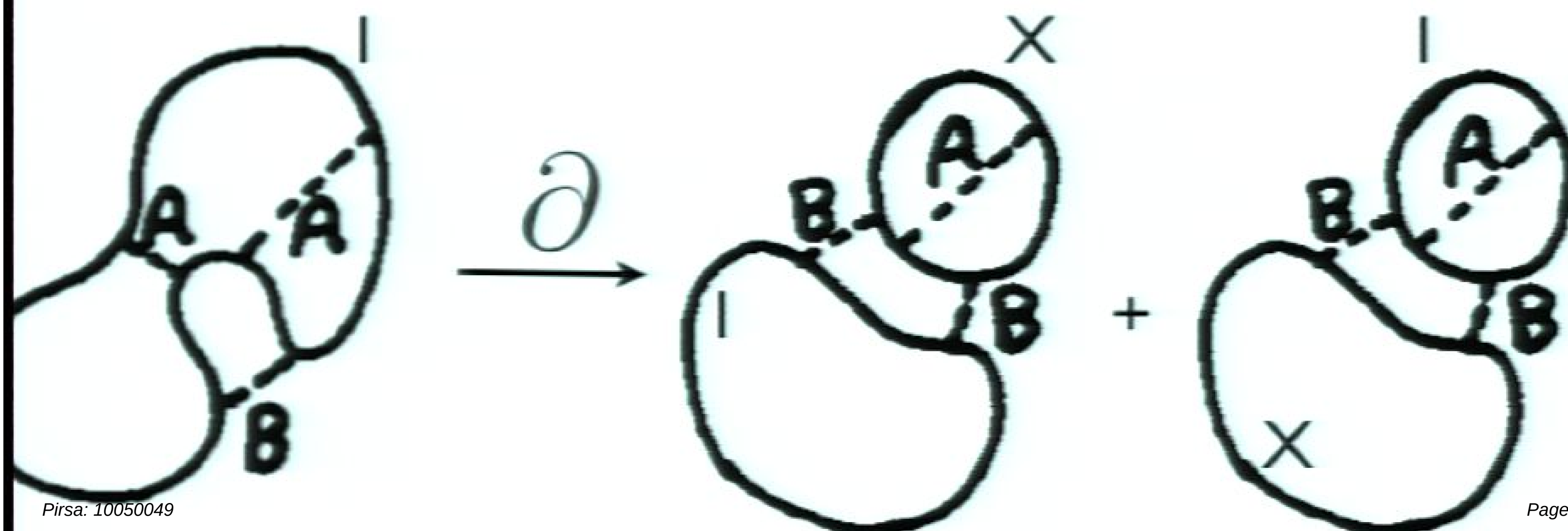
$\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1.$

$x^2 = 0$



Partial Boundaries

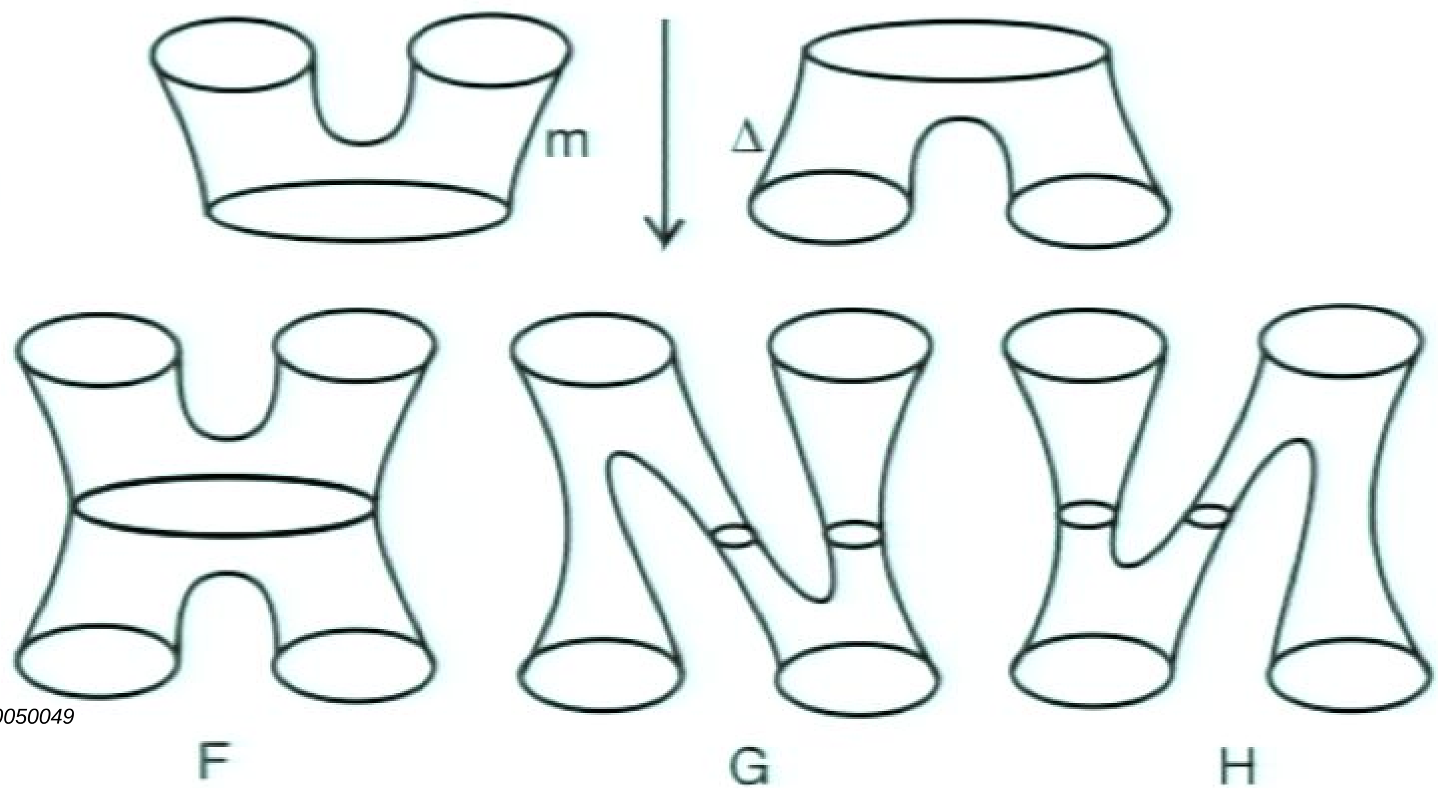




ition. The partial differentials $\partial_\tau(s)$ are uniquely determined by the condition $') = j(s)$ for all s' involved in the action of the partial differential on the en-state s. This unique form of the partial differential can be described by the fol-structures of multiplication and comultiplication on the algebra $A = k[X]/(X^2)$ $= Z/2Z$ for mod-2 coefficients, or $k = Z$ for integral coefficients.

he element 1 is a multiplicative unit and $X^2 = 0$.

$(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$.



Quantum Statistical Model for Khovanov Homolgy and the Bracket Polynonmial.

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_s |s\rangle$$

na. The evaluation of the bracket polynomial is given by the following formula

$$\langle K\rangle = \langle\psi|U|\psi\rangle.$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

With $U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle,$

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

$$U\partial + \partial U = 0.$$

This means that the unitary transformation U acts on the homology so that

U: H(C(K) -----> H(C(K))

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This means that the unitary transformation U acts on the homology so that

U: H(C(K) -----> H(C(K))

$$\mathcal{C}^{\bullet,j} = \bigoplus_i \mathcal{C}^{i,j}$$

$$\rangle = \sum_s q^{j(s)}(-1)^{i(s)} = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij})$$

$$= \sum_j q^j \chi(\mathcal{C}^{\bullet,j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet,j})).$$

This shows how <K> as a quantum amplitude contains information about the homology.

Eigenspace Picture

$$\mathcal{C}^0 = \oplus_\lambda \mathcal{C}^0_\lambda$$

$$\mathcal{C}^\bullet_\lambda : \mathcal{C}^0_\lambda \longrightarrow \mathcal{C}^1_{-\lambda} \longrightarrow \mathcal{C}^2_{+\lambda} \longrightarrow \cdots \mathcal{C}^n_{(-1)^n\lambda}$$

$$\mathcal{C} \; = \; \oplus_\lambda \mathcal{C}^\bullet_\lambda$$

$$\langle\psi|U|\psi\rangle = \sum_\lambda \lambda\chi(H(C^\bullet_\lambda))$$

SUMMARY

We have interpreted the bracket polynomial as a uantum amplitude by making a Hilbert space C(K) ose basis is the collection of enhanced states of the bracket.

This space C(K) is naturally intepreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

The homology and the unitary transformation U speak to one another via the formula

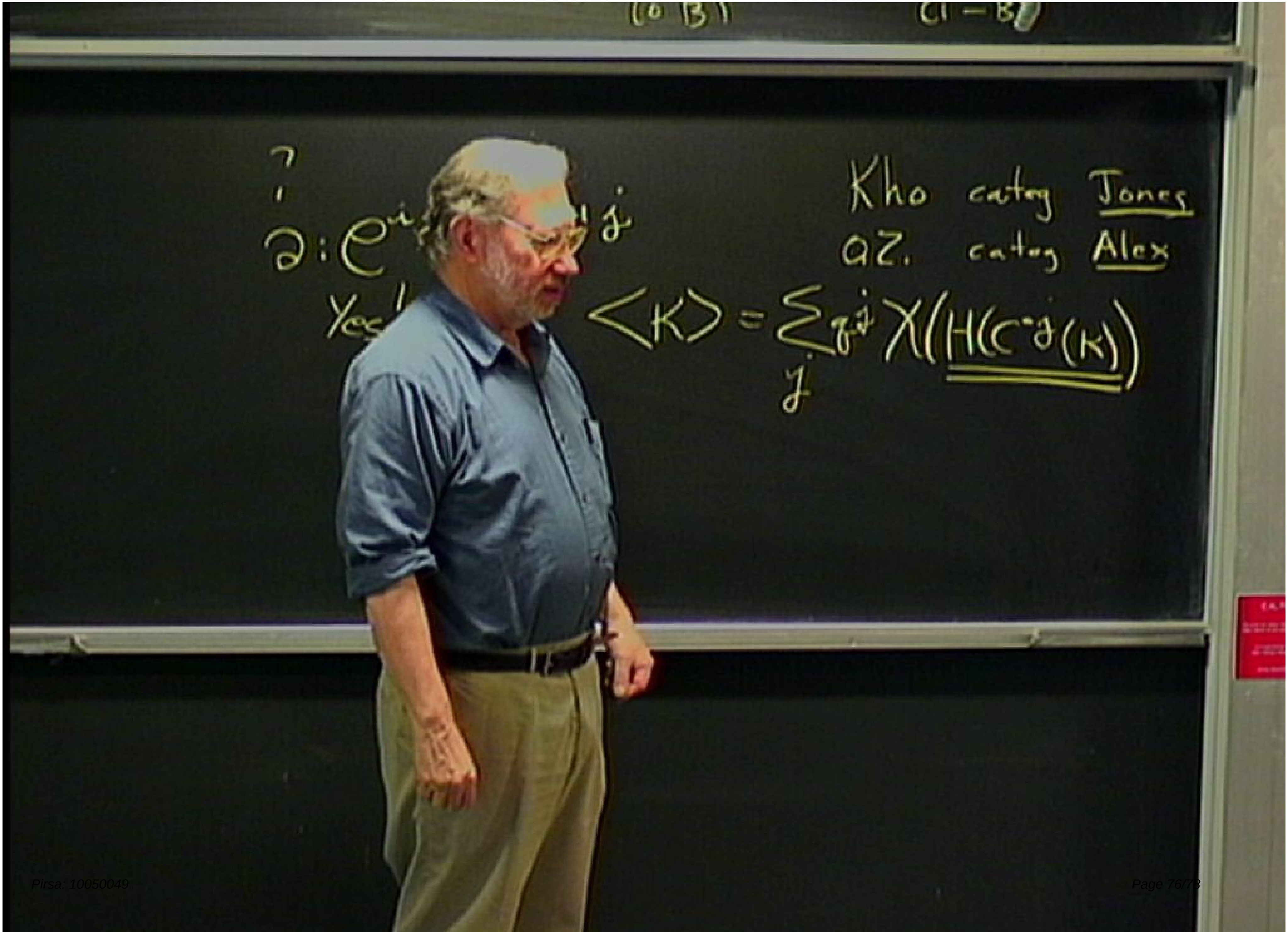
$$U\partial + \partial U = 0.$$

Questions

We have shown how Khovanov homology fits
nto the context of quantum information related to
the Jones polynomial and how the polynomial is
placed in this context by a unitary transformation U
he Hilbert space of the model. This transformation U
s on the homology, and its eigenspaces give a natural
ecomposition of the homology that is related to the
tum amplitude corresponding to the Jones polynomial.

The states of the model are intensely
combinatorial, related to the
representation of the knot or link.

How can this formulation be used in
quantum information theory and in
statistical mechanics?!


Kho categ Jones
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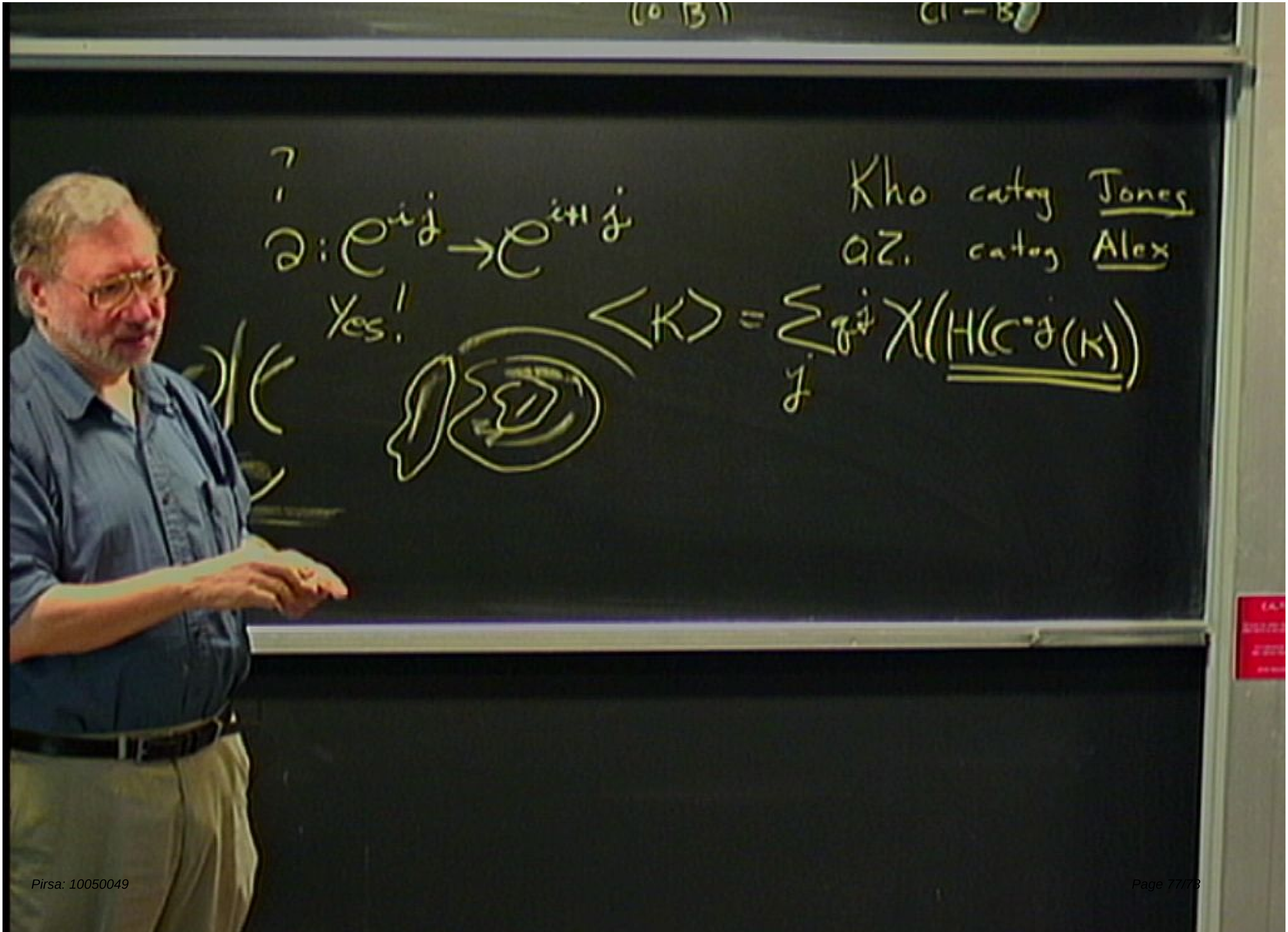
Yes!


Kho categ Jones
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Yes!

(0 B) (1 − B)

?

$\partial : C^{i,j} \to C^{i+1,j}$

Yes!

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$$\langle K \rangle = \sum_j q^j \chi\left(H(C^{*,j}(K))\right)$$