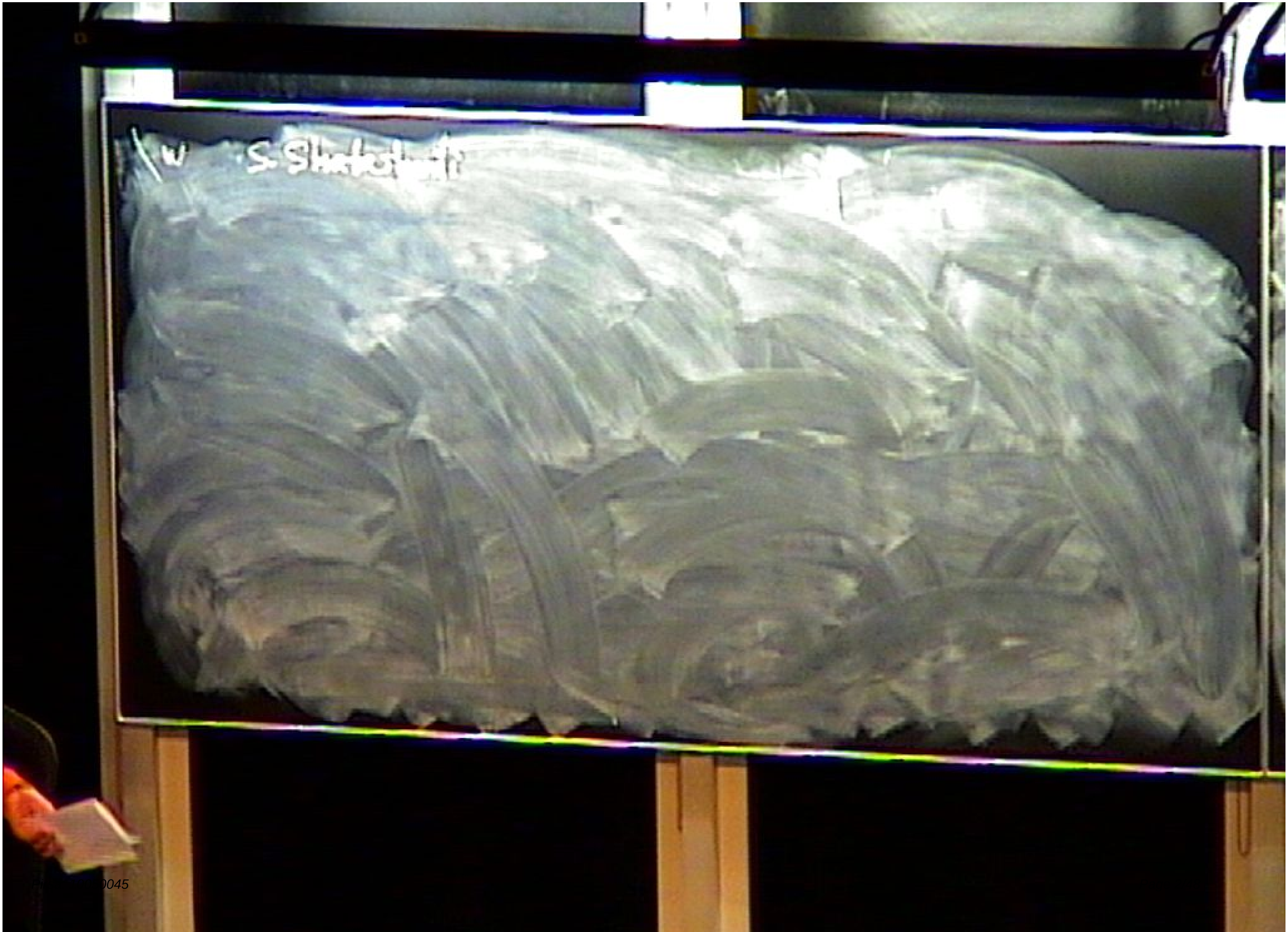
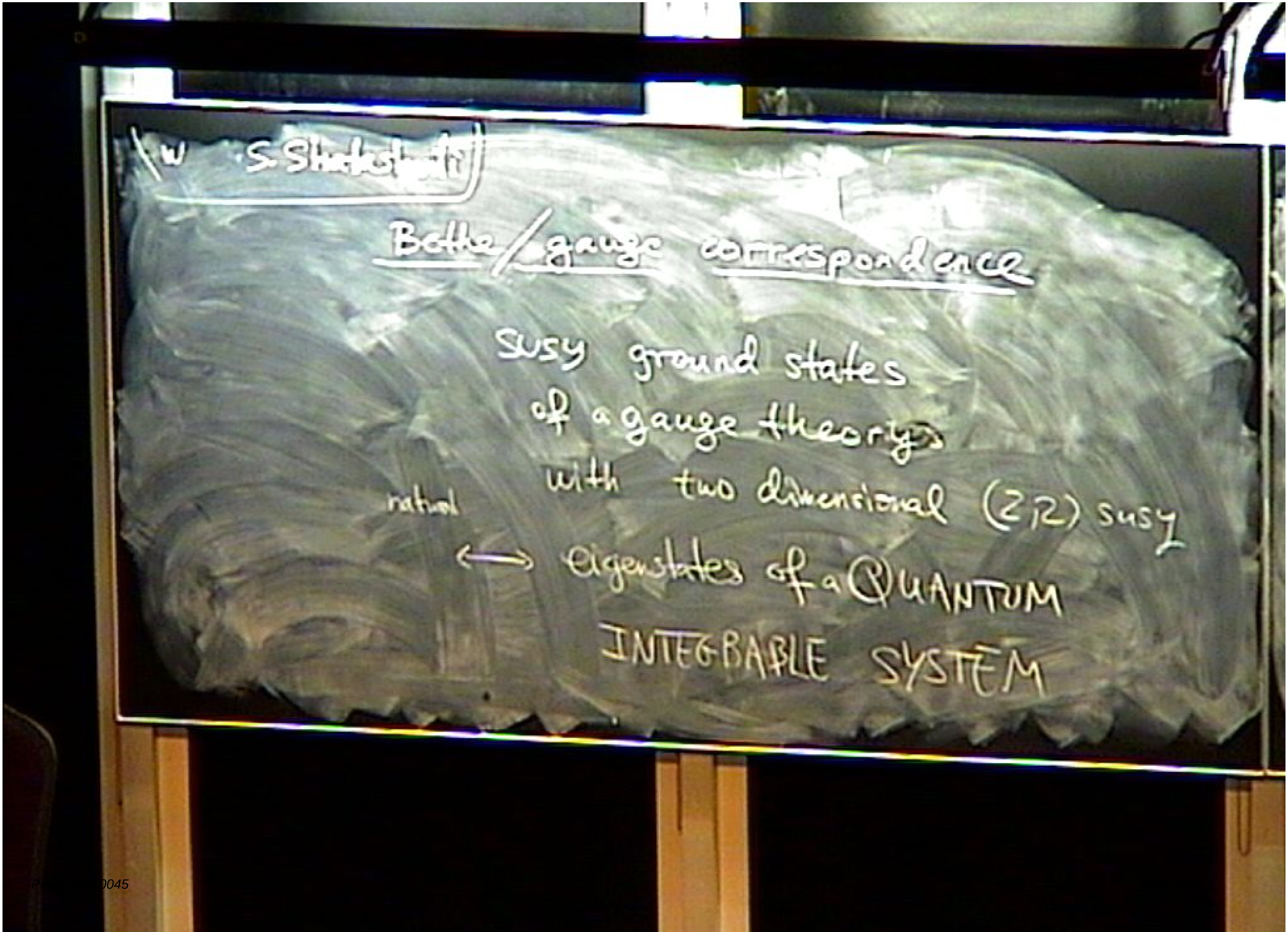
Title: Thermodynamic Bethe ansatz from instantons in super-Yang-Mills theory

Date: May 09, 2010 01:45 PM

URL: http://pirsa.org/10050045

Abstract: We show that the generating function of the equivariant (generalized) Donaldson invariants of ${\bf R}^2 X {\Sigma}$ is captured by the solution of a thermodynamic Bethe ansatz equation. Based on a joint work with S. Shatashvili.




Bethe/gauge correspondence
susy ground states
of a gauge theory
with two dimensional (2,2) susy
natural
⟷ eigenstates of a QUANTUM
INTEGRABLE SYSTEM

2d gauge $\mathcal{N}=(4,4)$

$U(N)$ gauge group $(\ \Phi\ )$ $\quad Q, \tilde{Q}$

L fundamental hypermultiplets (fund + antifund chiral)

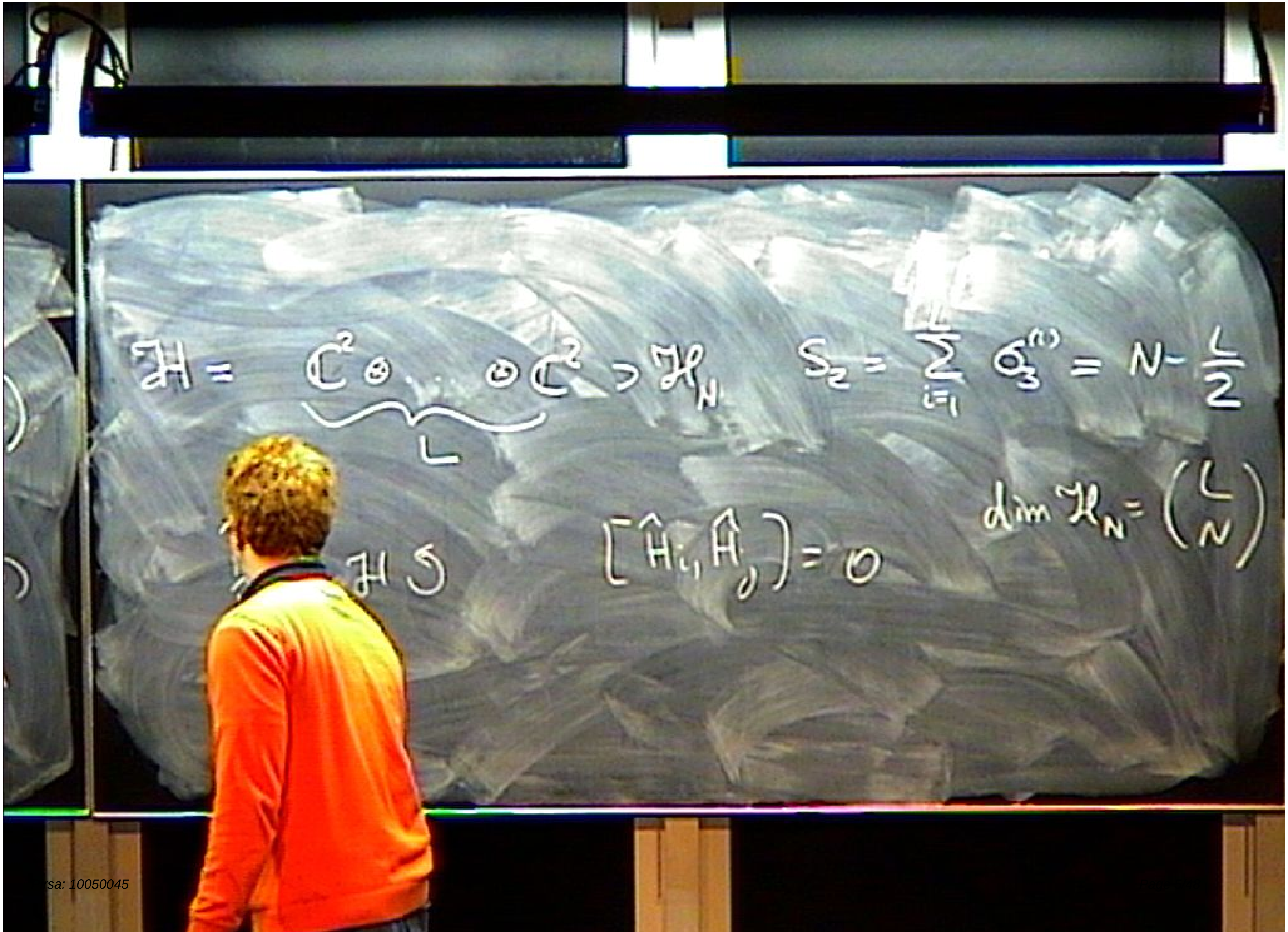
+ twisted masses compatible $\quad W = \mathrm{Tr}\, \tilde{Q} \Phi Q$

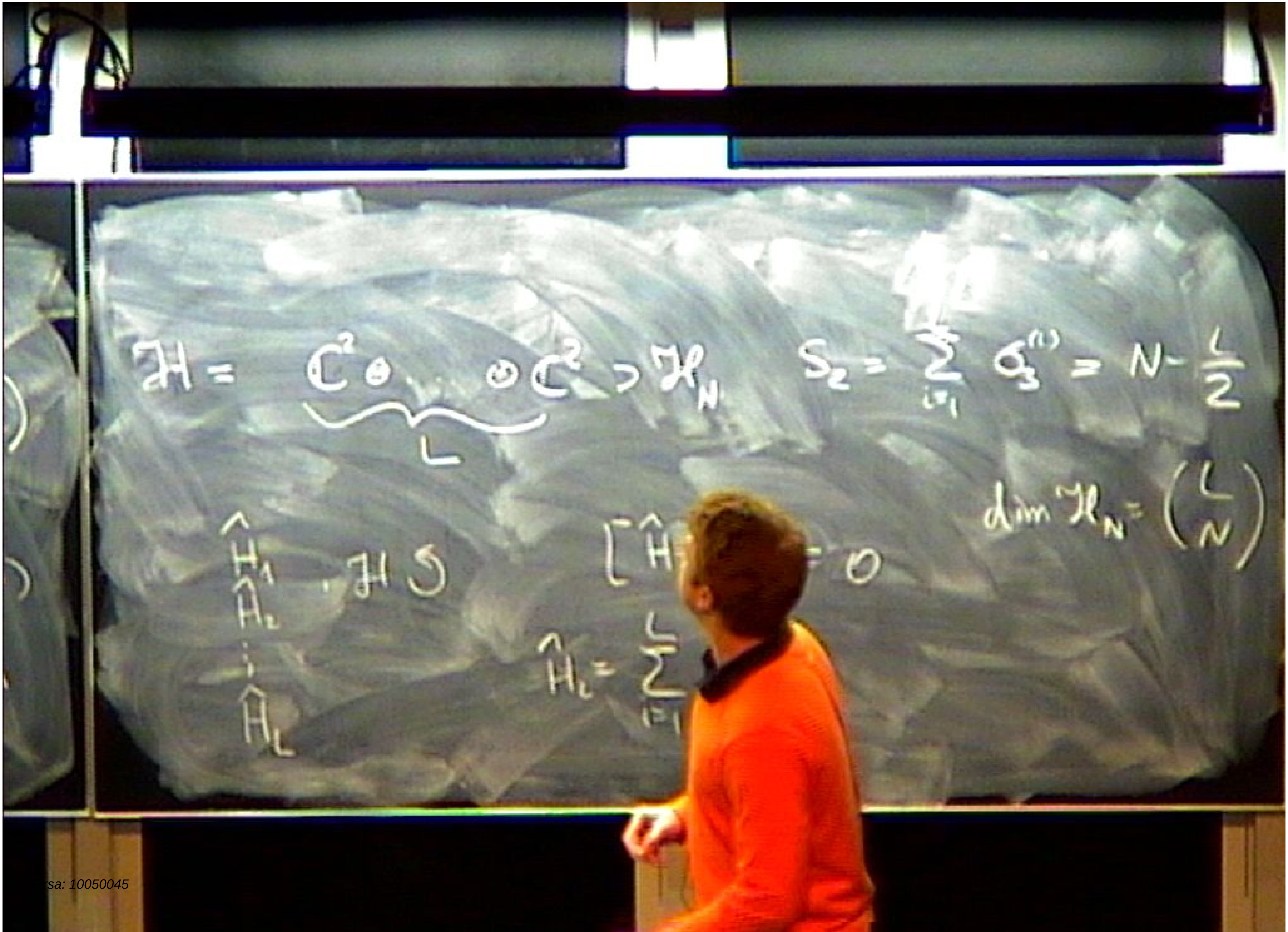
$\rightarrow \quad \uparrow \downarrow \uparrow \downarrow \uparrow$

$SU(2)$ $XXX_{\frac{1}{2}}$ Heisenberg spin chain

$L = \#$ of spins

10050045


$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} \supset \mathcal{H}_N$
$S_z = \sum_{i=1}^{L} \sigma_3^{(i)} = N - \frac{L}{2}$
$\dim \mathcal{H}_N = \binom{L}{N}$
$\mathcal{H} \circlearrowleft$
$[\hat{H}_i, \hat{H}_j] = 0$
rsa: 10050045


$\mathcal{H} = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 \supset \mathcal{H}_N$
L
$S_z = \sum_{i=1}^{L} \sigma_z^{(i)} = N - \frac{L}{2}$
$\dim \mathcal{H}_N = \binom{L}{N}$
$\hat{H}_1$
$\hat{H}_2$
$\hat{H}_L$
$\mathcal{H}$
$\hat{H}_L = \sum_{i=1}^{L}$
rsa: 10050045

quantum cohomology of $T^*Gr(N,L)$

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} \supset \mathcal{H}_N \qquad S_z = \sum_{i=1}^{L} \sigma_3^{(i)} = N - \frac{L}{2}$$

$$\dim \mathcal{H}_N = \binom{L}{N}$$

$$\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_L \quad \mathcal{H} \circlearrowleft \qquad [\hat{H}_i, \hat{H}_j] = 0$$

$$\hat{H}_2 = \sum_{i=1}^{L} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$$

rsa: 10050045

quantum cohomology of $T^*G_r(N,L)$

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} \supset \mathcal{H}_N \qquad S_z = \sum_{i=1}^{L} \sigma_3^{(i)} = N - \frac{L}{2}$$

$$\dim \mathcal{H}_N = \binom{L}{N}$$

$$\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_L \;\; \mathcal{H} \circlearrowright \qquad [\hat{H}_i, \hat{H}_j] = 0$$

$$\hat{H}_2 = \sum_{i=1}^{L} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$$

sa: 10050045

quantum cohomology of $T^*Gr(N,L)$

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} \supset \mathcal{H}_N \qquad S_z = \sum_{i=1}^{L} \sigma_3^{(i)} = N - \frac{L}{2}$$

$$\dim \mathcal{H}_N = \binom{L}{N}$$

$\hat{H}_1$
$\hat{H}_2$, $\mathcal{H} \circlearrowleft$
$\vdots$
$\hat{H}_L$

$$[\hat{H}_i, \hat{H}_j] = 0$$

$$\hat{H}_2 = \sum_{i=1}^{L} \sigma_i^1 \otimes \sigma_{i+1}^1 + \sigma_i^2 \otimes \sigma_{i+1}^2 + \Delta\, \sigma_i^3 \otimes \sigma_{i+1}^3$$

anisotropy

Intersection theory on the moduli space of solutions to certain PDE's (generalized vortex eq's)

$G \to P$ --- principal G-bundle

$\downarrow$

Σ

A - connection on P

Σ $\quad X \in \Gamma(R \times_G P)$

R - representation of G

Intersection theory on the moduli space of solutions to certain PDE's (generalized vortex eq's)

$G \to P$ --- principal G-bundle

$\downarrow$

Σ

A - connection on P

$\Gamma \rightsquigarrow \Sigma$

$X \in \Gamma\left(R_i \times_G P \otimes \mathcal{E}_i \right)^{\oplus_i K_\Sigma^{r_i}}$

$\oplus \mathbb{C}^{m_i} \otimes R_i = R$ - representation of G

sa: 10050045

$W : R_c \to \mathbb{C}$ (homogeneous

$$\nabla_{\bar A} X = h \, \overline{\frac{\partial W}{\partial X}}$$

hermitian metric on X

$$F_A + \mu(X, \bar X) = 0$$

$\mu : R_c \to \mathfrak{g}^*$ moment map

$W : R_c \to \mathbb{C}$ (homogeneous

$$\begin{cases} \nabla_{\bar{A}} X = h \, \overline{\dfrac{\partial W}{\partial X}} \\ F_A + \mu(X, \bar{X}) = 0 \end{cases}$$

hermitian metric on X

$\mu : R_c \to \mathfrak{g}^*$ moment map

$Q \in \mathbb{C}^N \otimes \mathbb{C}^L$

$\tilde{Q} \in (\mathbb{C}^N)^* \otimes \mathbb{C}^L$

$\Phi \in K^{1/2} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^L)$

$W : R_0 \to \mathbb{C}$ (homogeneous ...)

$$\begin{cases} \nabla_{\bar{A}} X = h \dfrac{\partial \overline{W}}{\partial \bar{X}} \\[2ex] F_A + \mu(X, \bar{X}) = 0 \end{cases}$$

hermitian metric on X

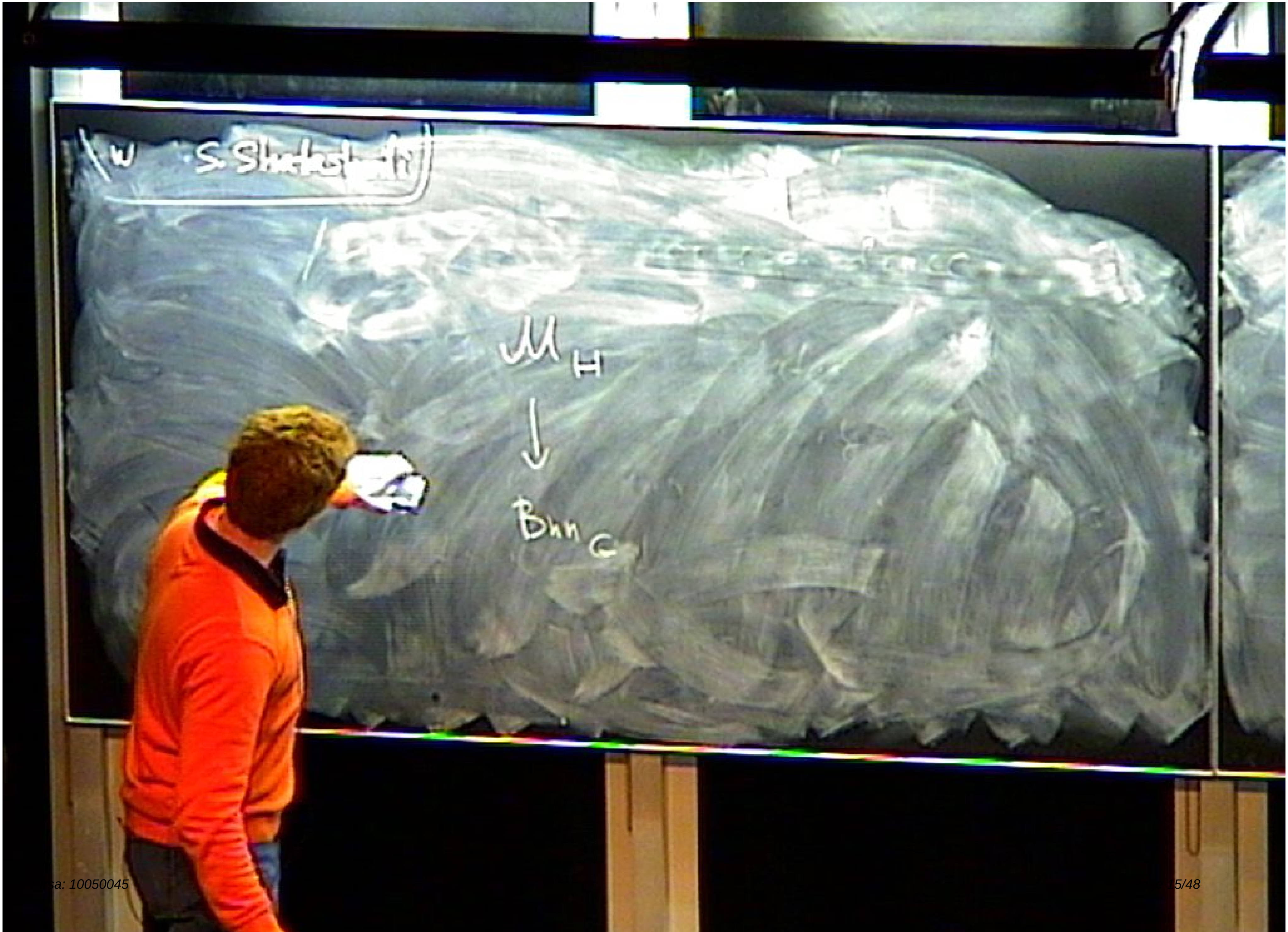
$\mu : R_0 \to \mathfrak{g}^*$ moment map

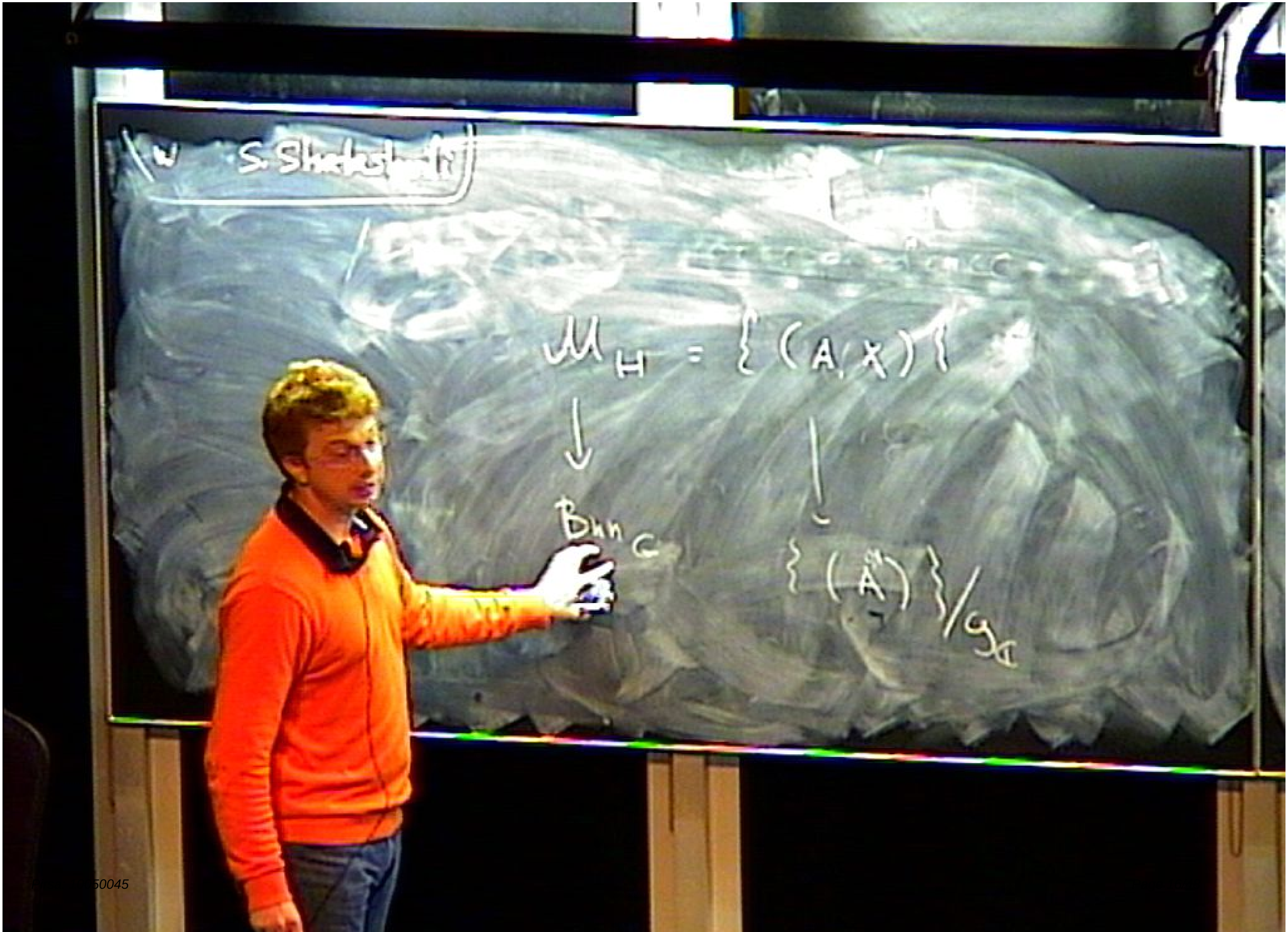
$Q \in \mathbb{C}^N \otimes \mathbb{C}^L$

$\tilde{Q} \in (\mathbb{C}^N)^* \otimes \mathbb{C}^L$

$\Phi \in (\oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^L))$

$\mathrm{Tr}\, \tilde{Q} \Phi Q = W$


$\mathcal{M}_H$
Bun_G



50045

w/ S. Shatashvili

$$\int^{\text{equiv}}_{\mathcal{M}_H} 1 = \int_{\mathrm{Bun}_G} \pi_* 1$$

$$\begin{array}{ccc} \mathcal{M}_H = \{ (A, X) \} & & \\ \downarrow \pi & & \downarrow \\ \mathrm{Bun}_G & & \{ (A) \} / \mathcal{G}_{\mathbb{C}} \end{array}$$

50045

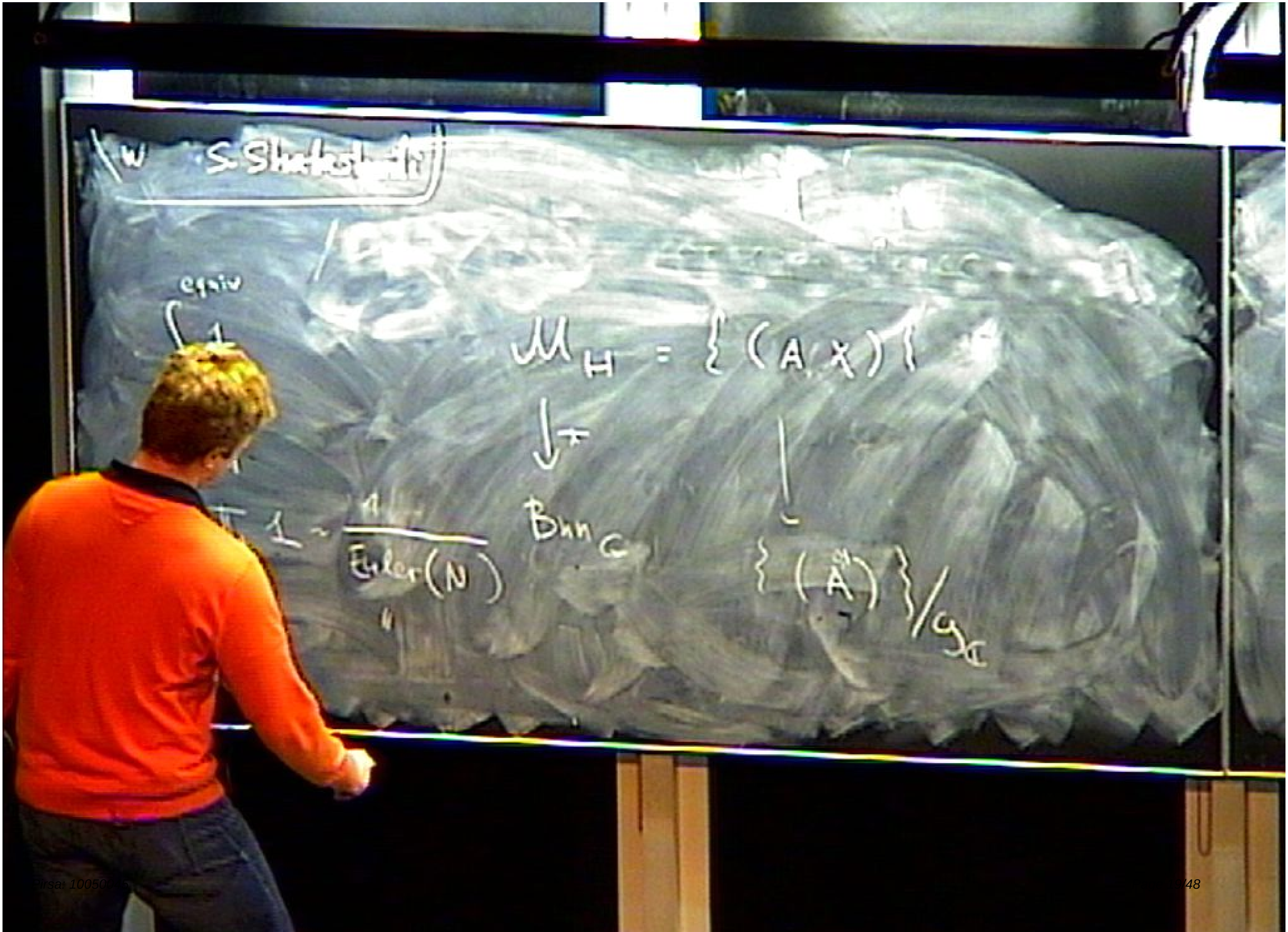
$$\tilde{V} \in \mathbb{C}[\mathfrak{g}]^G$$

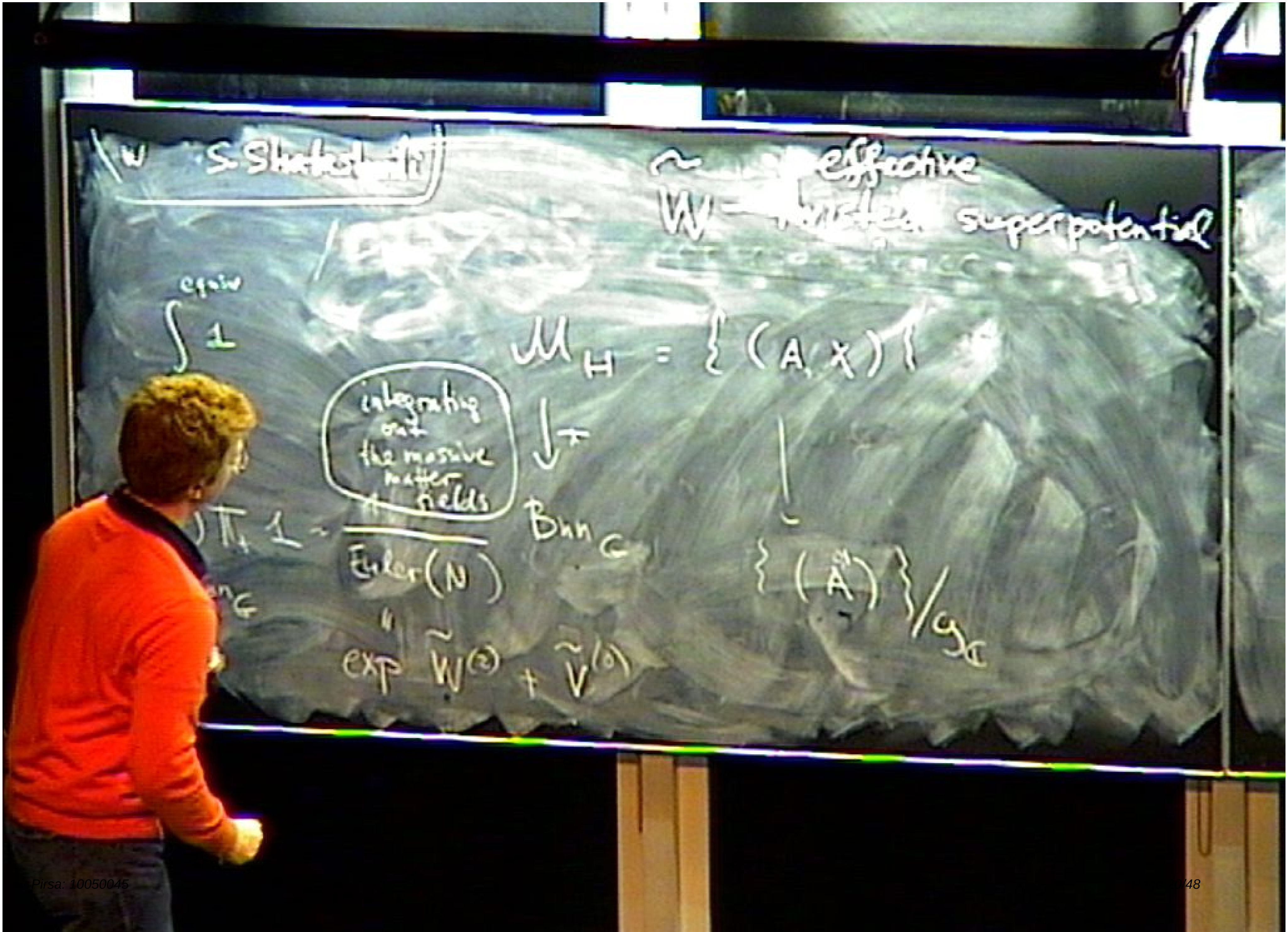
$$\tilde{W} \in \mathbb{C}[\mathfrak{g}]^G$$

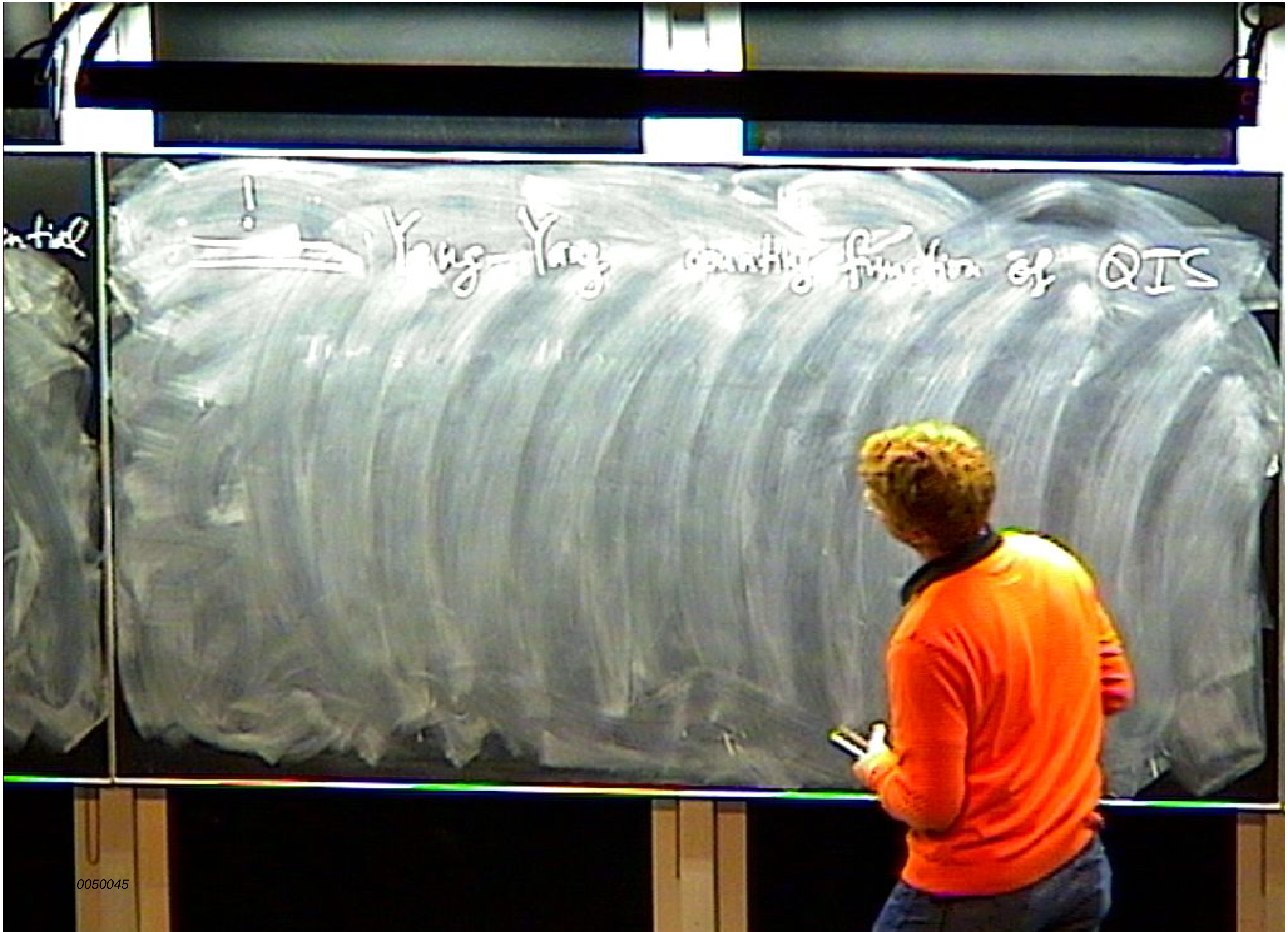
$$\tilde{W}^{(2)} = \int_\Sigma c_{\tilde{W}}(P)$$

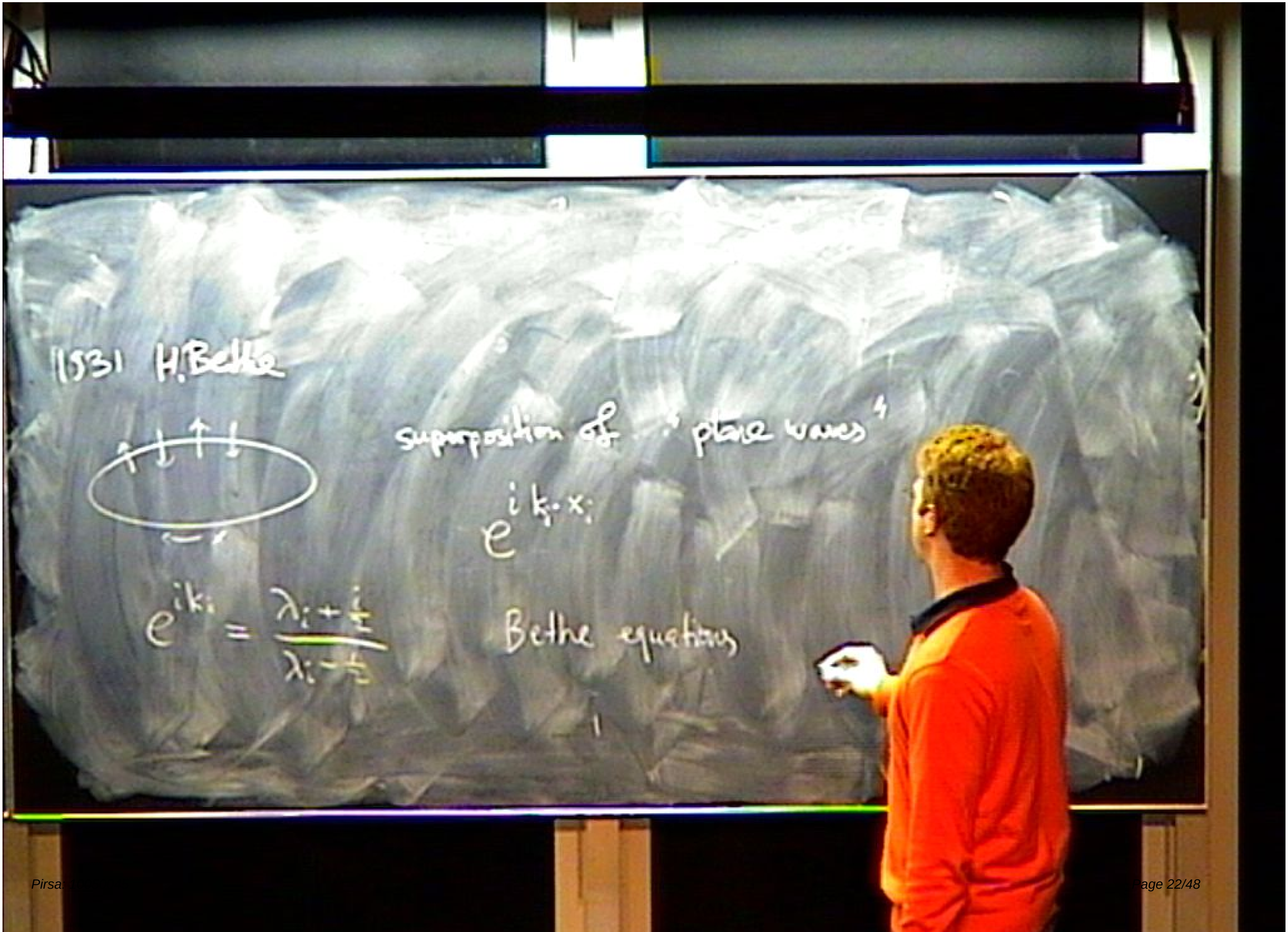
$$\begin{array}{c} P \\ \downarrow \\ \mathrm{Bun}_G \times \Sigma \longrightarrow \mathrm{Bun}_G \end{array}$$

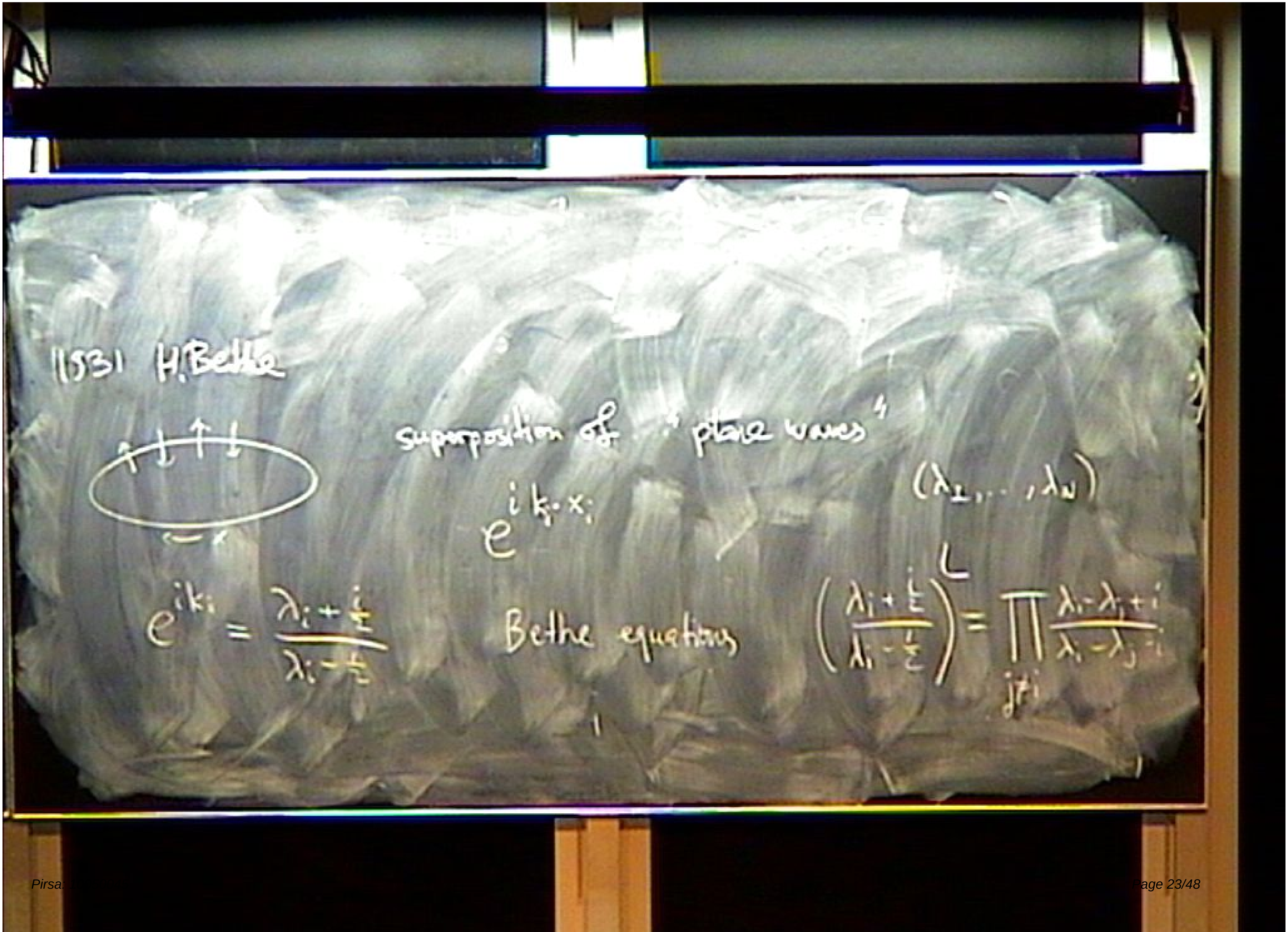
$$\tilde{V}^{(0)} = c_{\tilde{V}}(P)\Big|_{x \in \Sigma}$$


equiv
Euler(N)
Bun


effective
superpotential
equiv
integrating out the massive matter fields
Euler(N)
exp


Yang-Yang counting function of QIS
0050045


1931 H. Bethe
superposition of "plane waves"
$e^{i k_j \cdot x_j}$
$e^{ik_i} = \frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}$
Bethe equations


1931 H. Bethe
superposition of "plane waves"
$e^{i k_j x_j}$
$(\lambda_1, \dots, \lambda_N)$
x
$e^{ik_i} = \frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}$
Bethe equations
$\left(\frac{\lambda_i + \frac{i}{2}}{\lambda_i - \frac{i}{2}}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$

$\Longrightarrow$ Yang-Yang counting function of QIS

$$\tilde{W}(\lambda_1, \dots, \lambda_N) = L \sum_k (\lambda_k \pm \tfrac{i}{2}) \log (\lambda_k \pm \tfrac{i}{2}) +$$

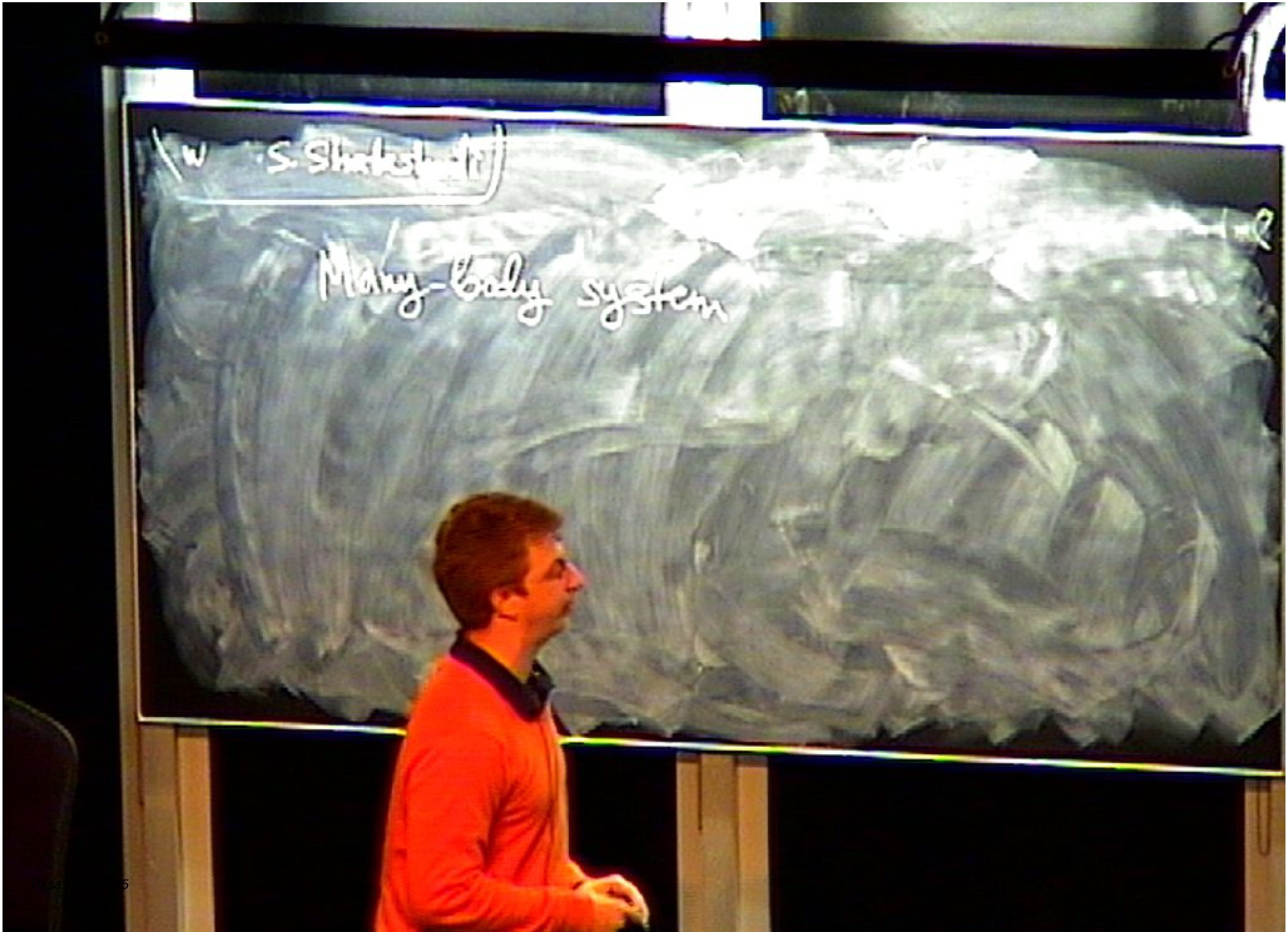
$$+ \sum_{j \neq k} (\lambda_k - \lambda_j + i) \log (\lambda$$

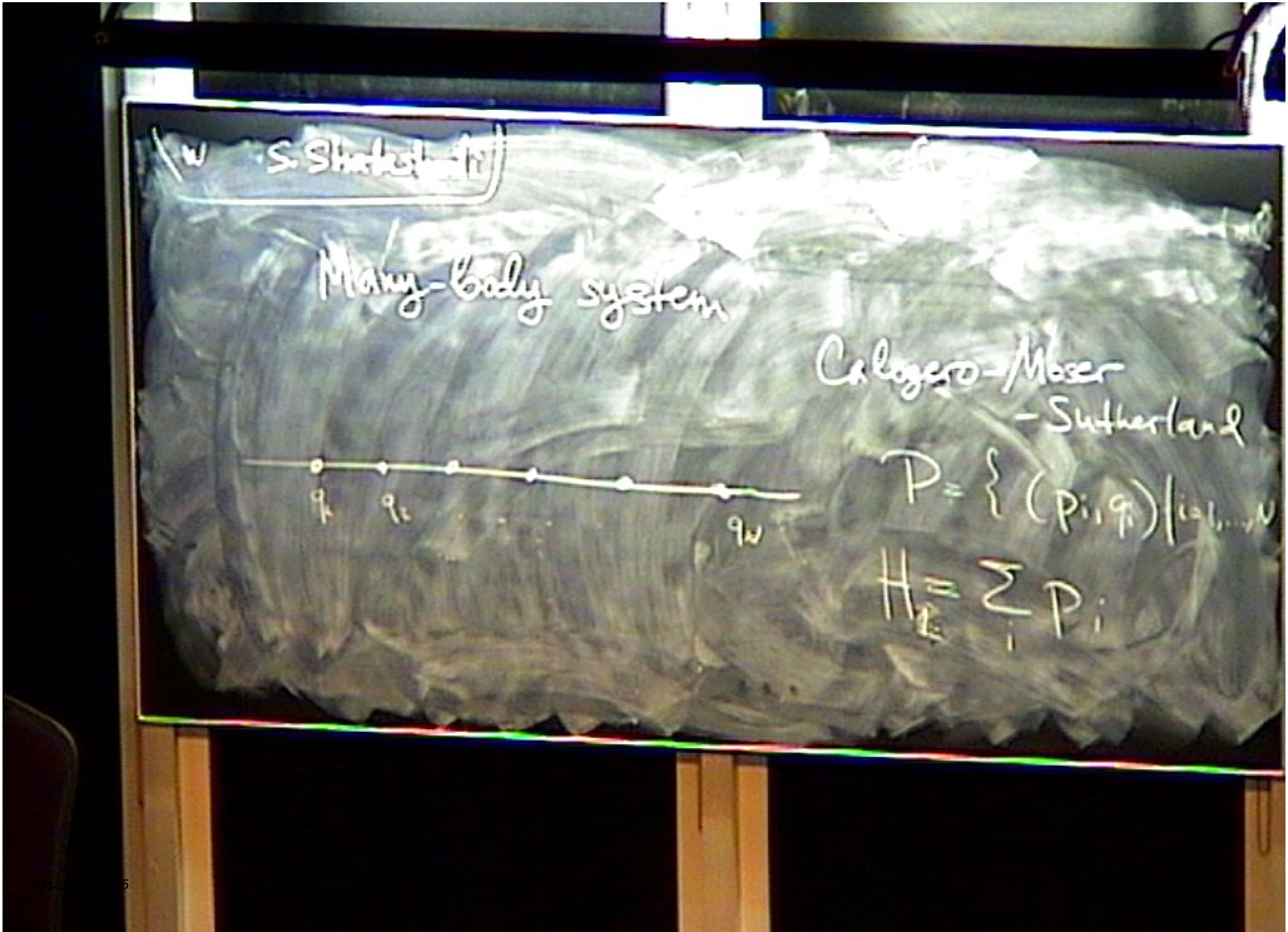
$$\exp \frac{\partial \tilde{W}}{\partial \lambda_i} = 1$$

$\Longrightarrow$ Yang–Yang counting function of QIS

$$\tilde{W}(\lambda_1,\dots,\lambda_N) = L\sum_k \left(\lambda_k \pm \frac{i}{2}\right)\log\left(\lambda_k \pm \frac{i}{2}\right) + \sum_{j\neq k}(\lambda_k-\lambda_j+i)\log(\lambda_k-\lambda_j$$

$$\boxed{\exp\frac{\partial \tilde{W}}{\partial \lambda_i} = 1}$$

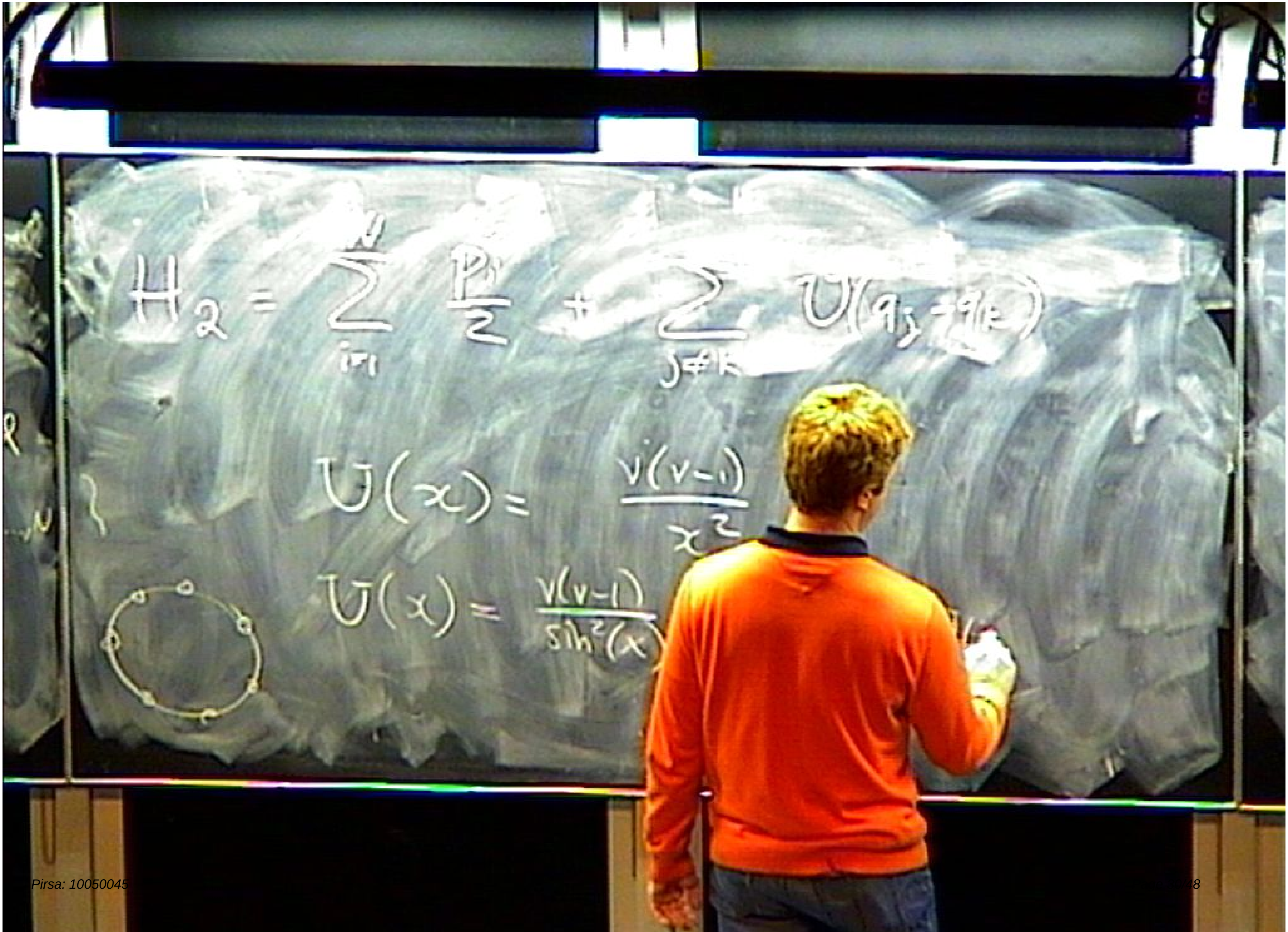

Many-body system


S. Shatashvili
Many-body system
Calogero-Moser
-Sutherland
q_1 q_2 q_N
$P = \{(p_i, q_i) \mid i = 1, \dots, N\}$
$H_2 = \sum_i p_i$

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} V(q_j - q_k)$$

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$$

$$U(x) = \frac{\nu(\nu-1)}{x^2}$$


$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} V(q_j - q_k)$
$V(x) = \frac{\nu(\nu-1)}{x^2}$
$V(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j\neq k} U(q_j - q_k)$$

$$U(x) = \frac{\nu(\nu-1)}{x^2}$$

$$U(x) = \frac{\nu(\nu-1)}{\sin^2(x)}$$

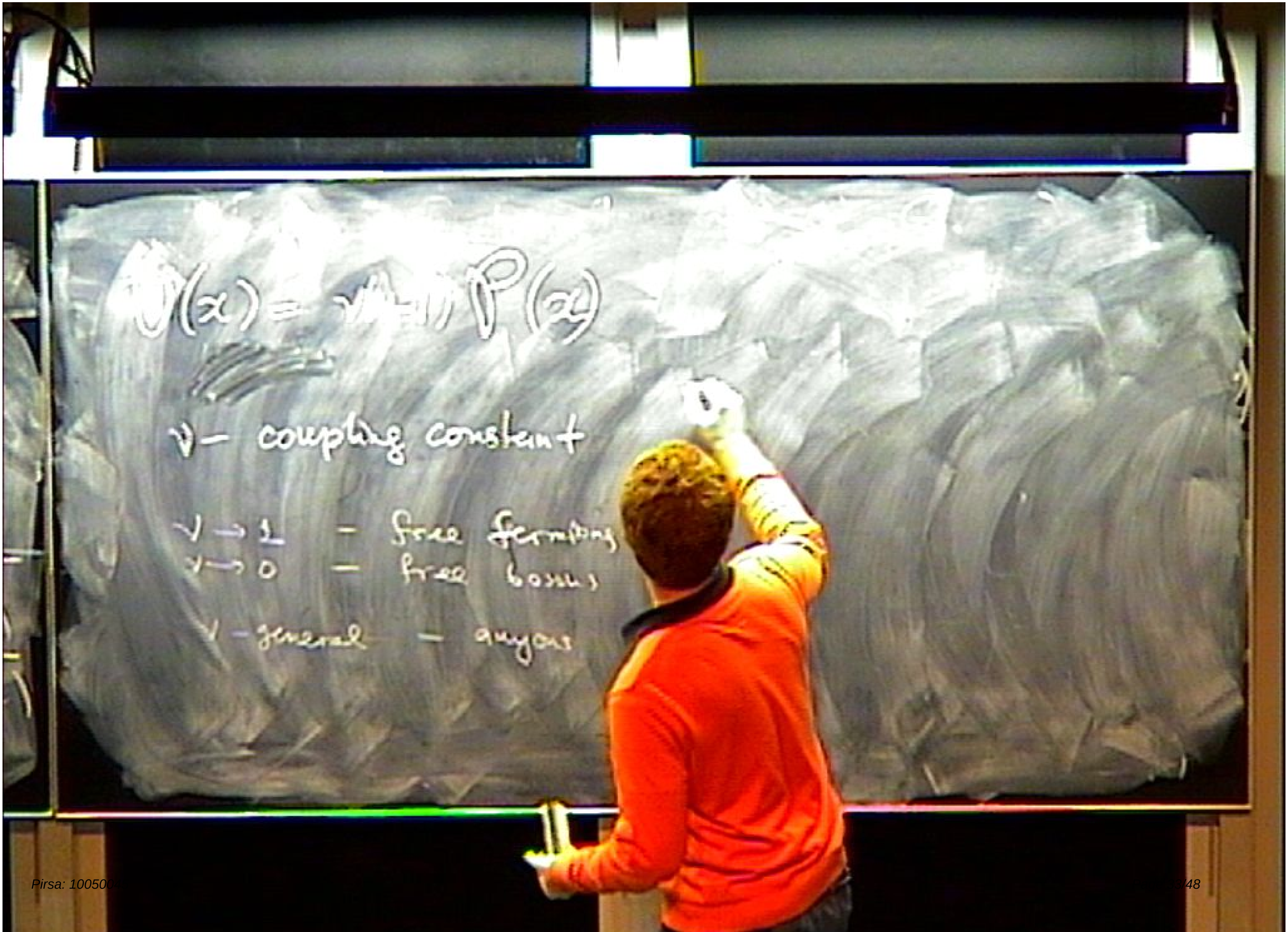
$$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$$

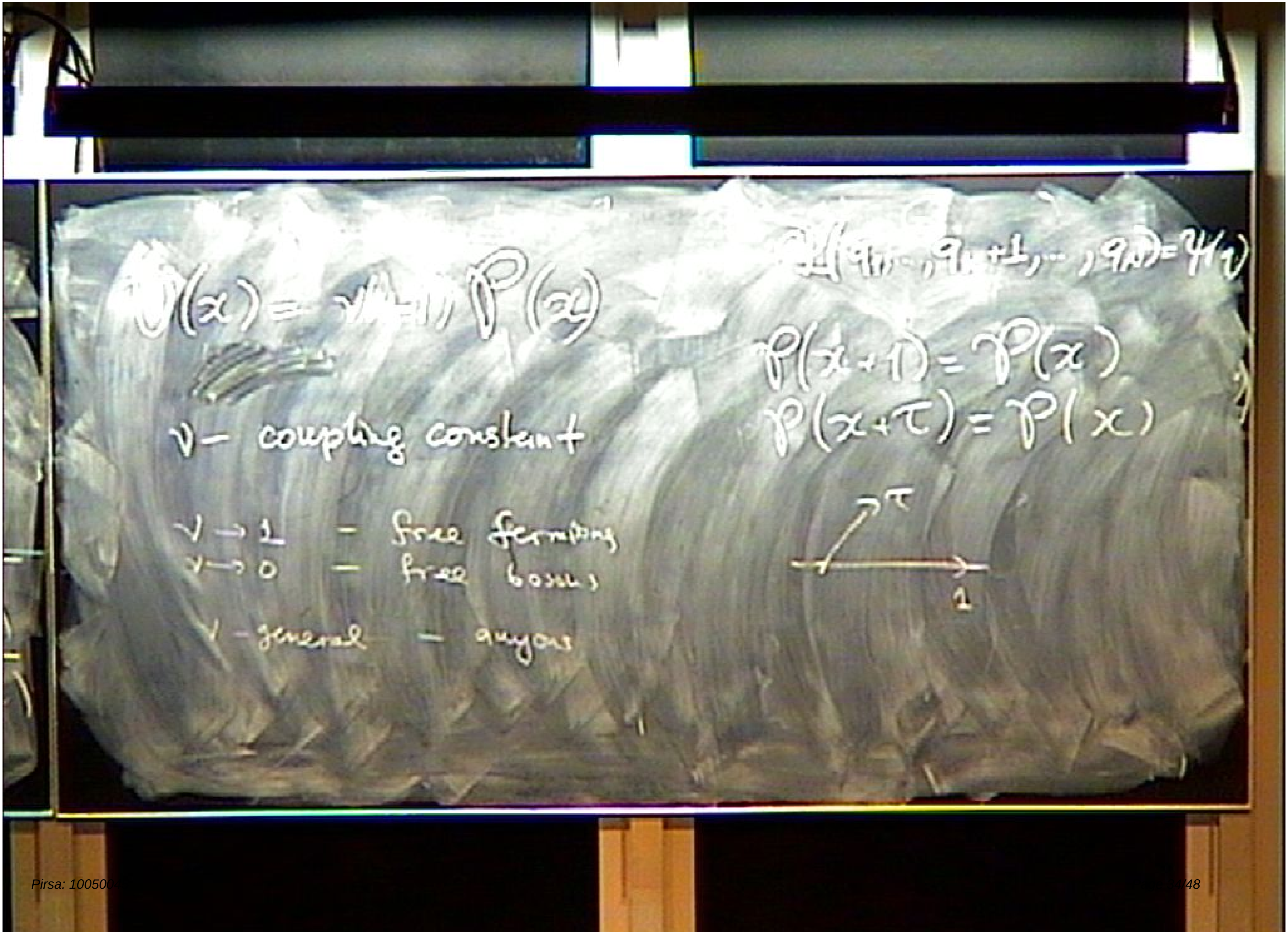
$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$$

$$U(x) = \frac{\nu(\nu-1)}{x^2}$$

$$U(x) = \frac{\nu(\nu-1)}{\sin^2(x)}$$

$$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$$


ν – coupling constant
ν → 1 – free fermions
ν → 0 – free bosons
ν – general – anyons


ρ(x+1) = ρ(x)
ρ(x+τ) = ρ(x)
ν − coupling constant
ν → 1 − free fermions
ν → 0 − free bosons
ν − general − anyons
τ
1

$$W(x) = \nu(\nu-1)\,\wp(x)$$

ν – coupling constant

$\nu \to 1$ – free fermions

$\nu \to 0$ – free bosons

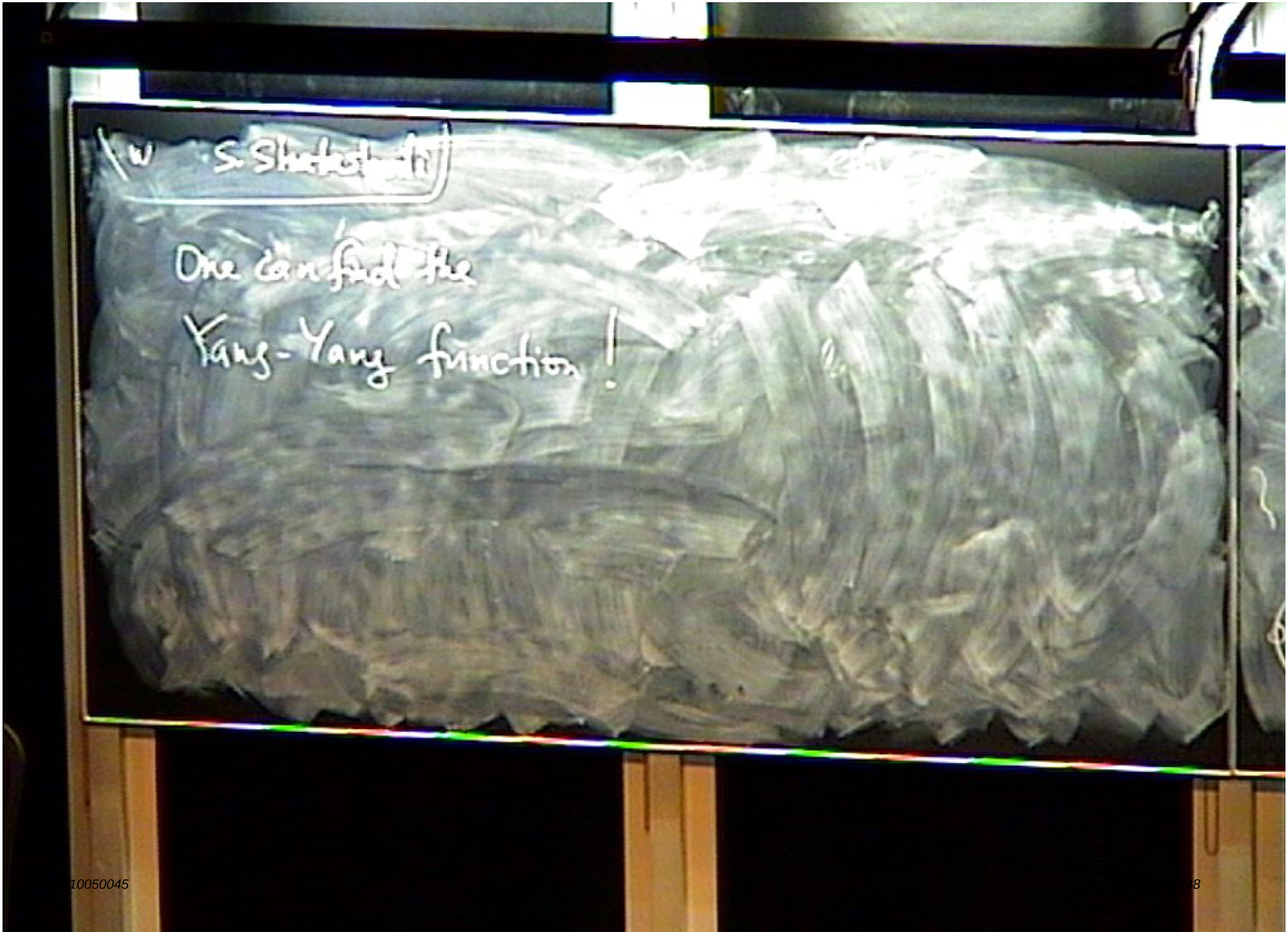
ν – general – anyons

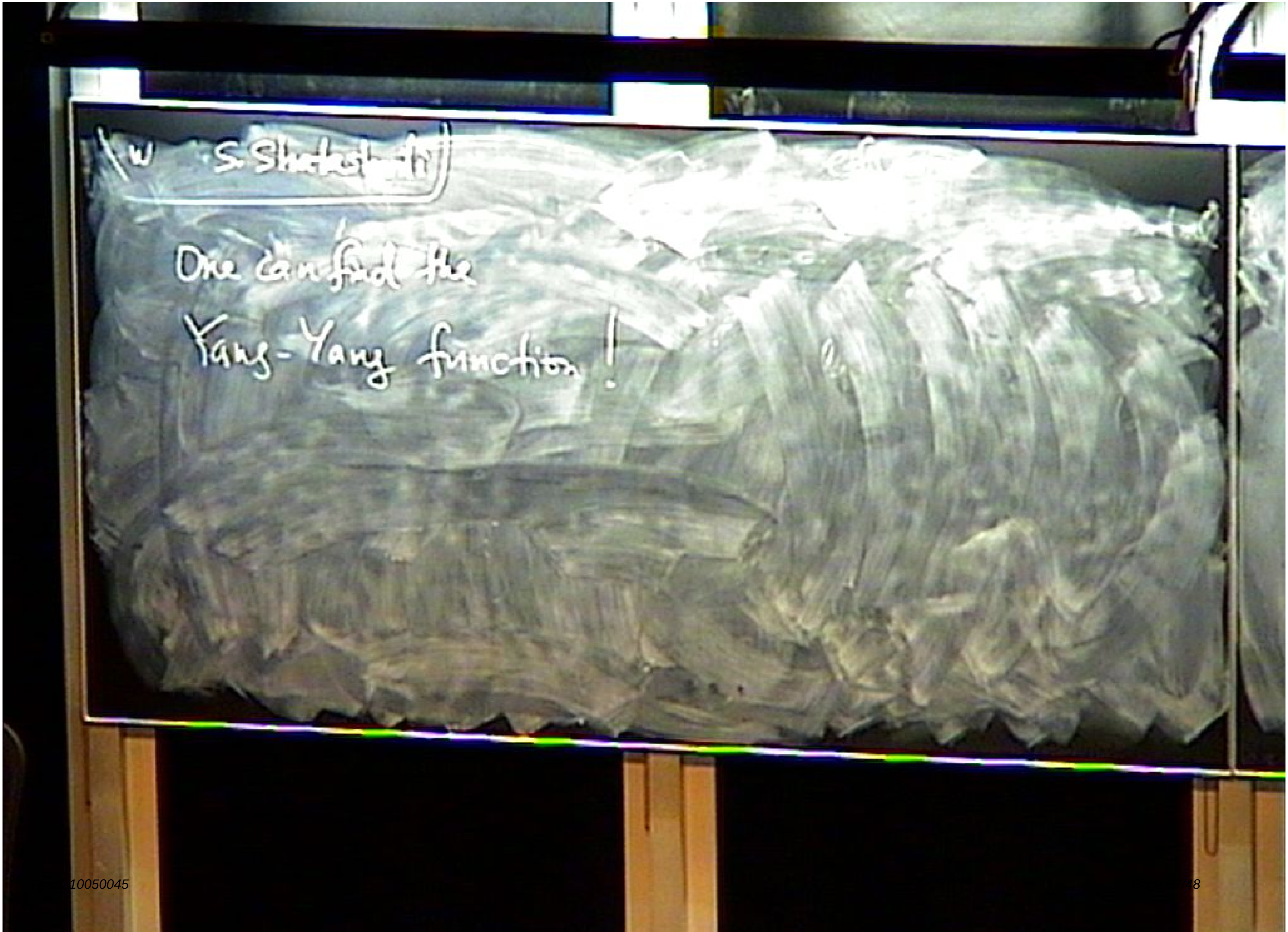
$$\Psi(q_1,\ldots,q_k+1,\ldots,q_N) = \Psi(q)$$

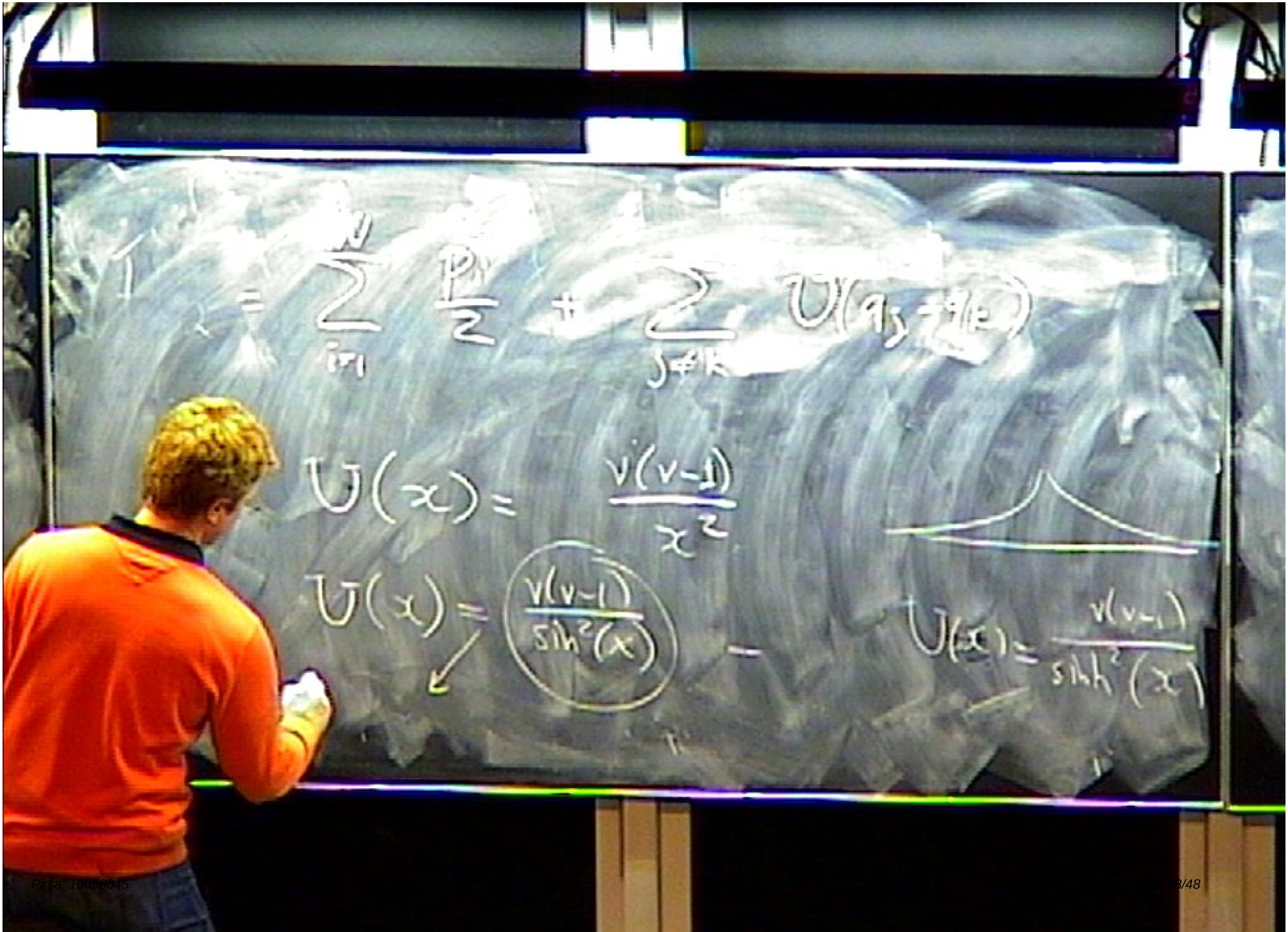
$$\wp(x+1) = \wp(x)$$

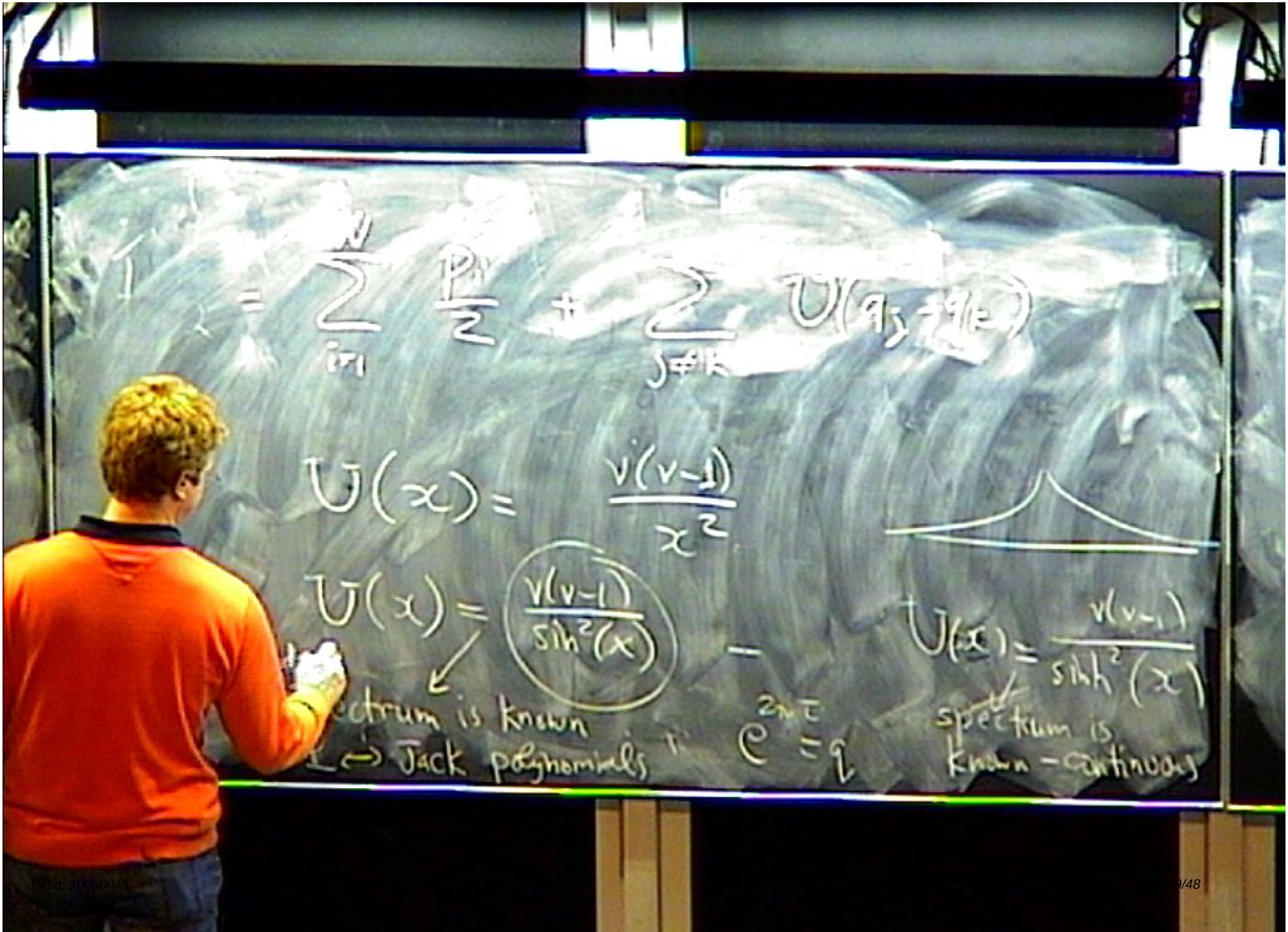
$$\wp(x+\tau) = \wp(x)$$

$$\hat{H}_k \Psi = E_k \Psi$$


One can find the
Yang-Yang function !


S. Shatashvili
One can find the
Yang-Yang function!


$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$
$U(x) = \frac{\nu(\nu-1)}{x^2}$
$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$
$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$



$= \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$
$U(x) = \frac{\nu(\nu-1)}{x^2}$
$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$
$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$
ectrum is known
$\longleftrightarrow$ Jack polynomials
$e^{2\pi i \tau} = q$
spectrum is known - continuous

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$$

$$\Psi_{(\lambda_1 \ldots \lambda_N)}(q) = \sum_{\sigma \in S_N} c(\lambda_\sigma)\, e^{i\ldots}$$

$$U(x) = \frac{\nu(\nu-1)}{x^2}$$

$$U(x) = \frac{\nu(\nu-1)}{\sin^2(x)}$$

discrete

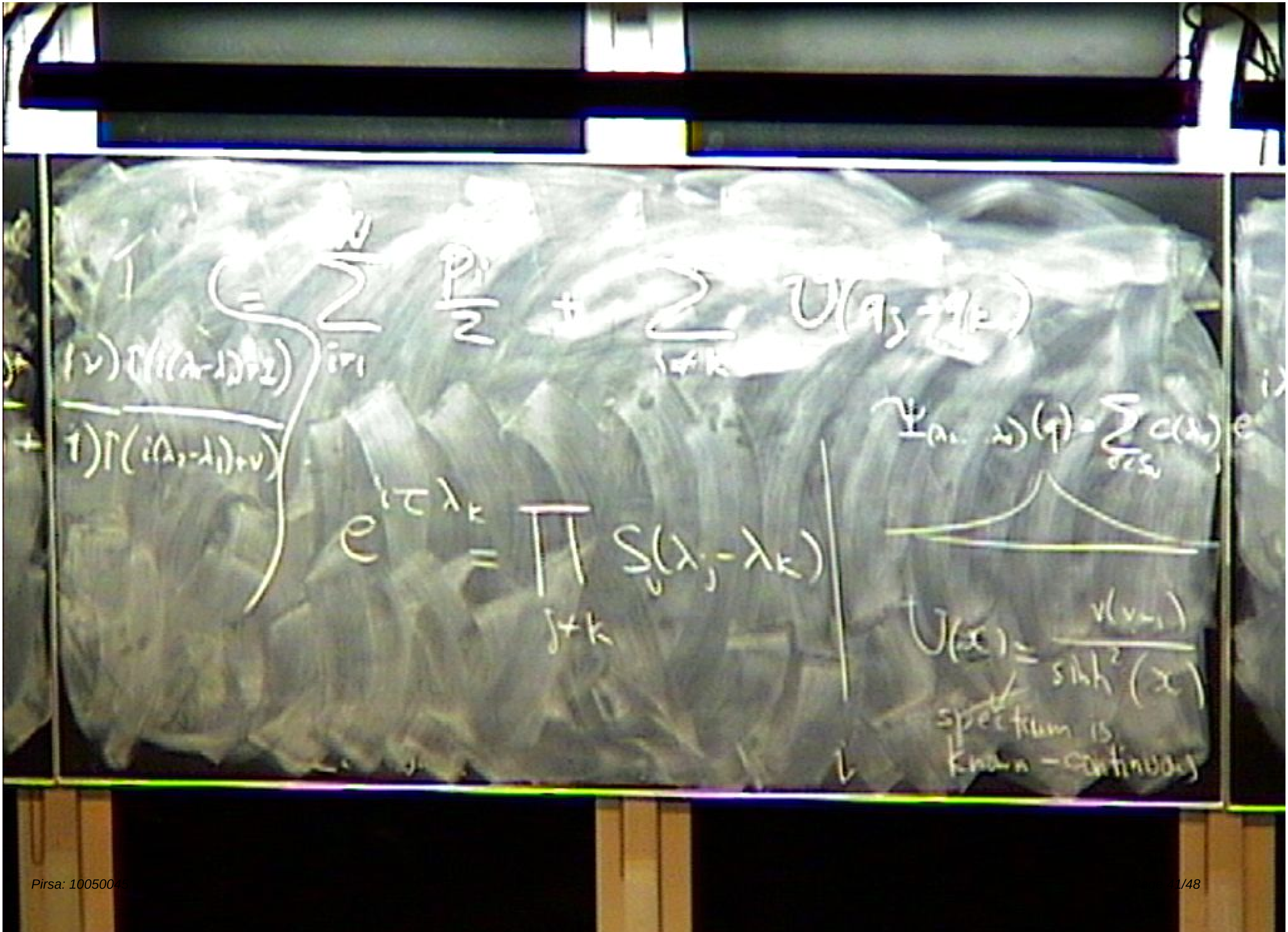
Spectrum is known

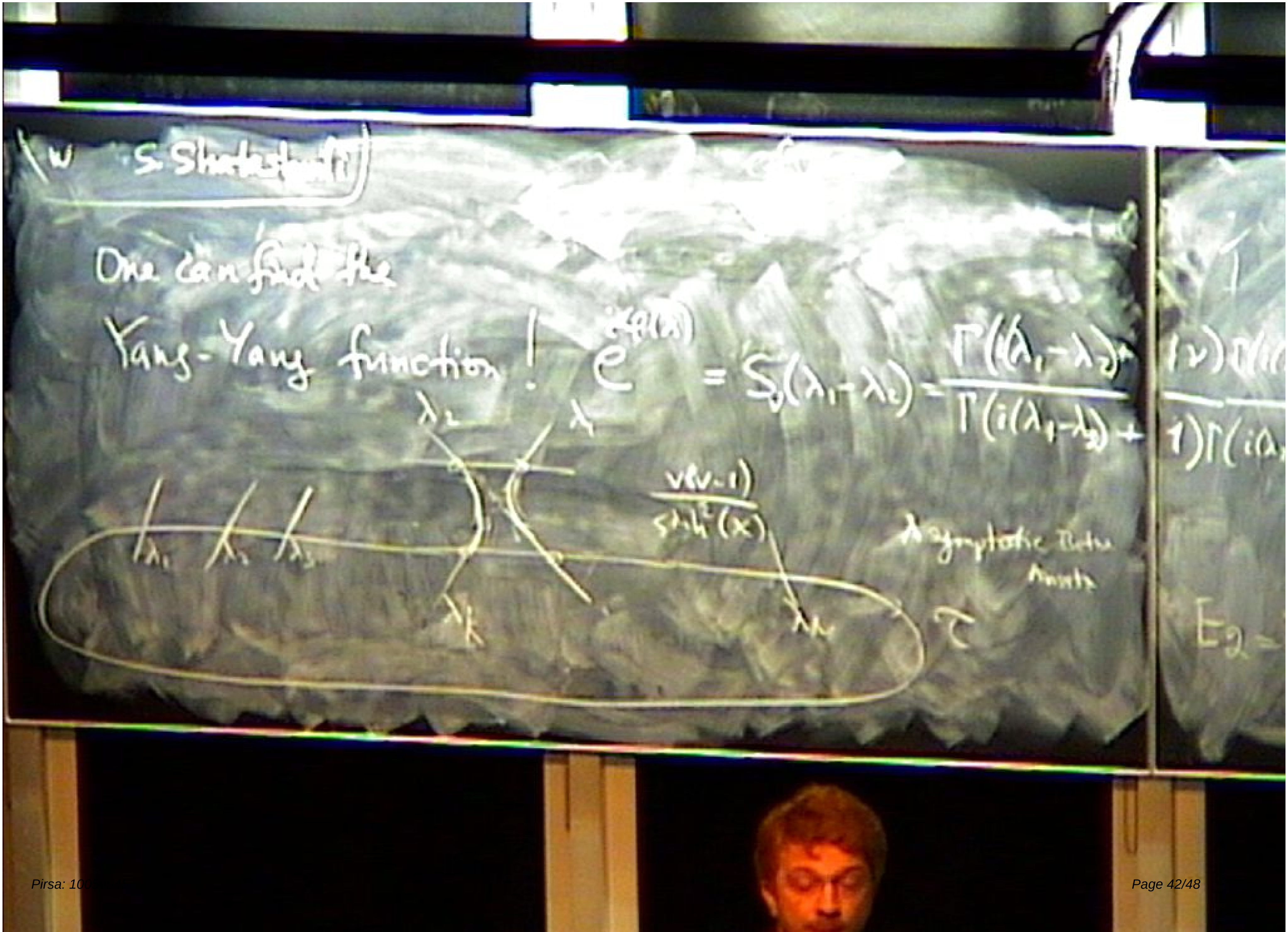
$\Psi \leftrightarrow$ Jack polynomials

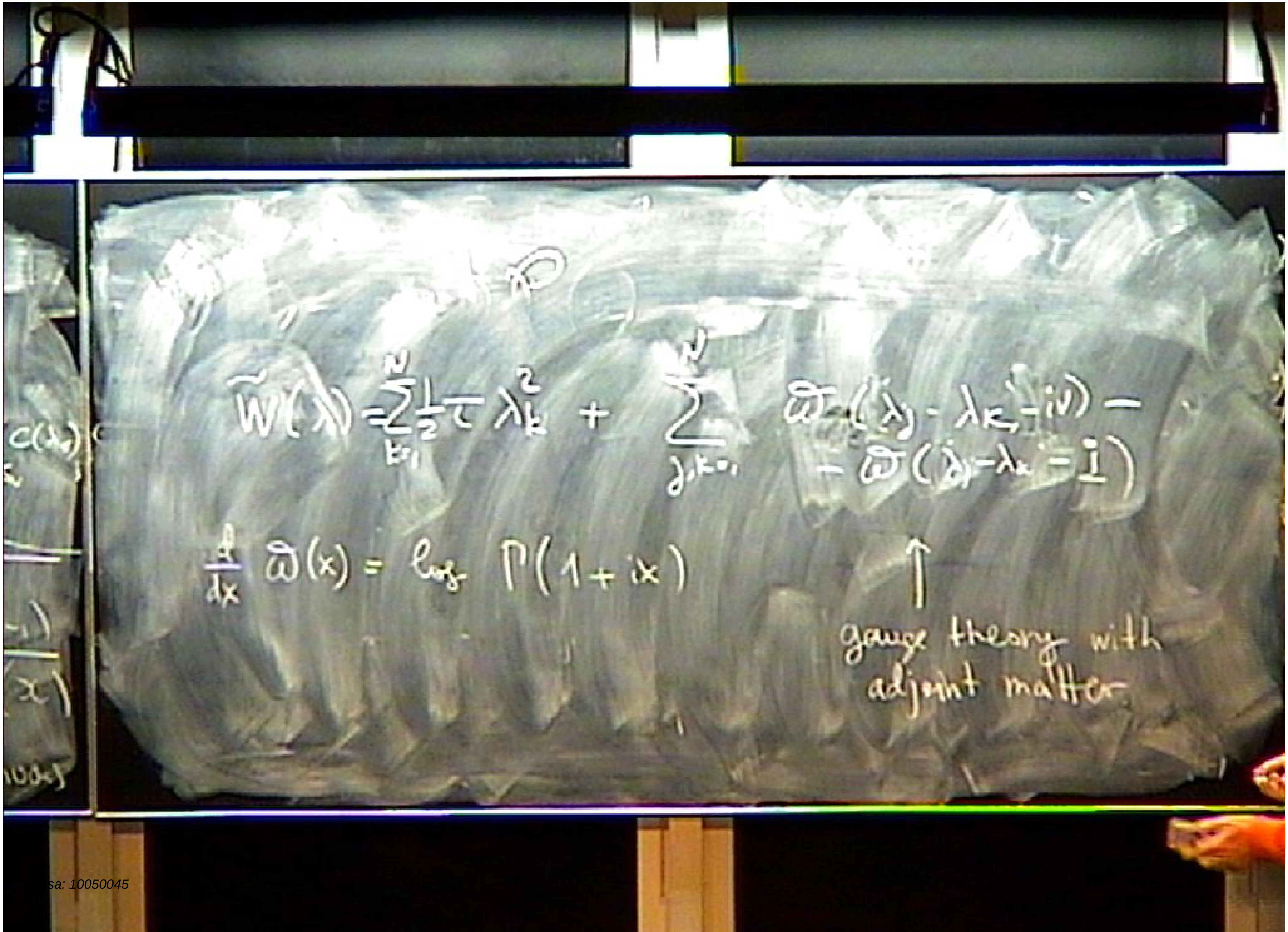
$$e^{2\pi i \tau} = q$$

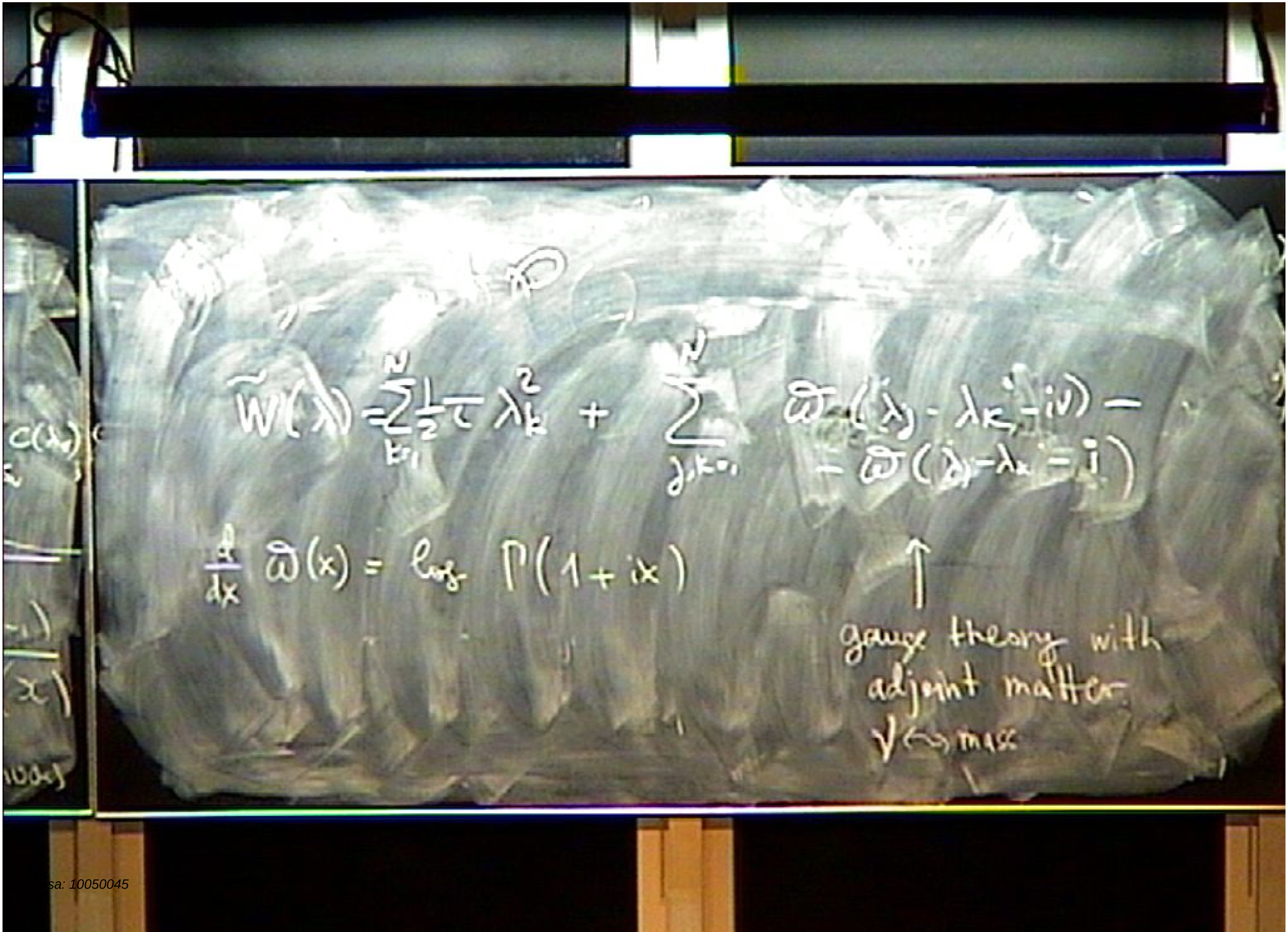
$$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$$

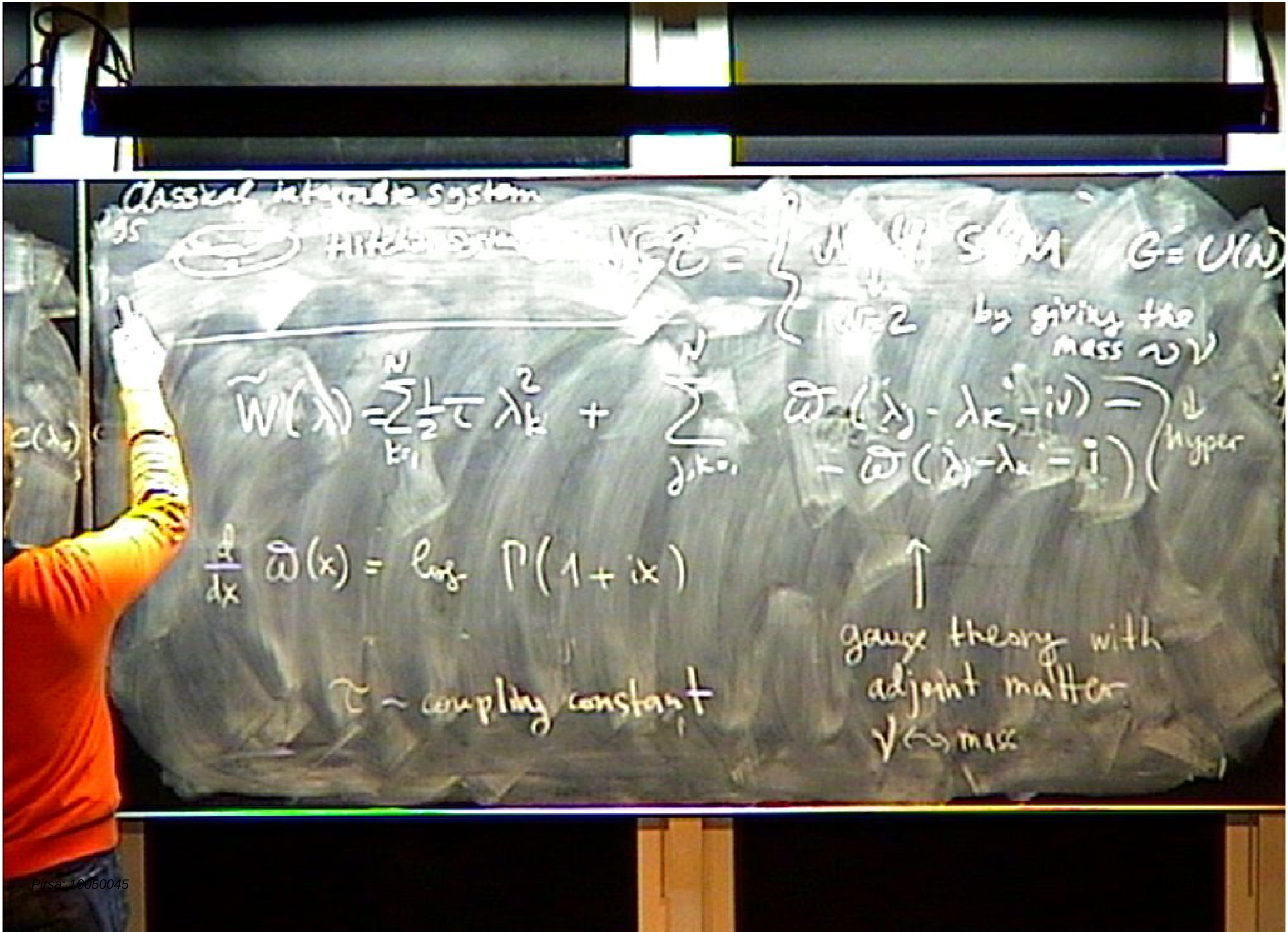
spectrum is known – continuous

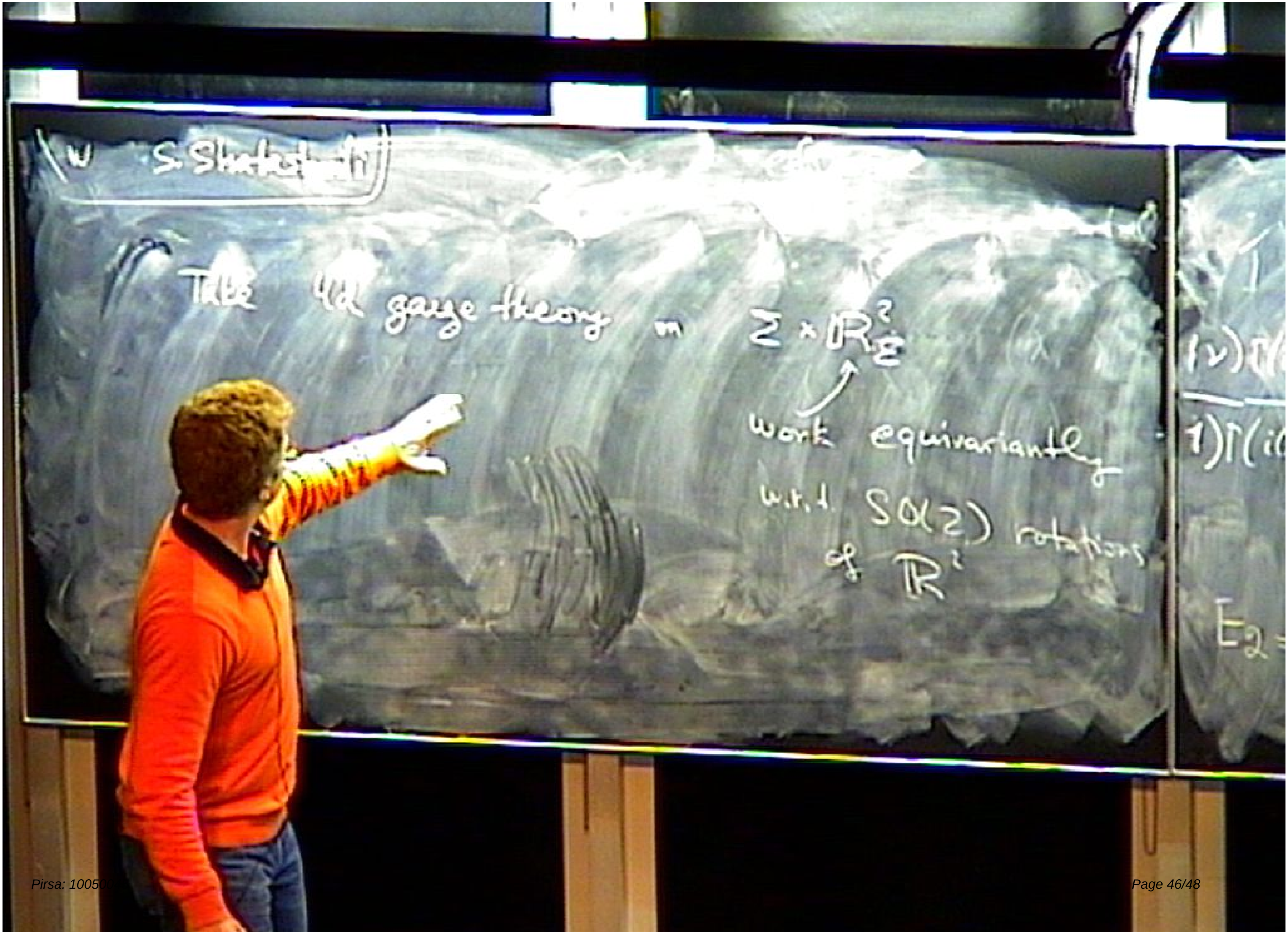

$\sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{j \neq k} U(q_j - q_k)$
$e^{i\tau\lambda_k} = \prod_{j \neq k} S_\nu(\lambda_j - \lambda_k)$
$\Psi_{(\lambda_1 \dots \lambda_N)}(q) = \sum_{\sigma \in S_N} c(\lambda_\sigma)\, e$
$U(x) = \frac{\nu(\nu-1)}{\sinh^2(x)}$
spectrum is
known - continuous

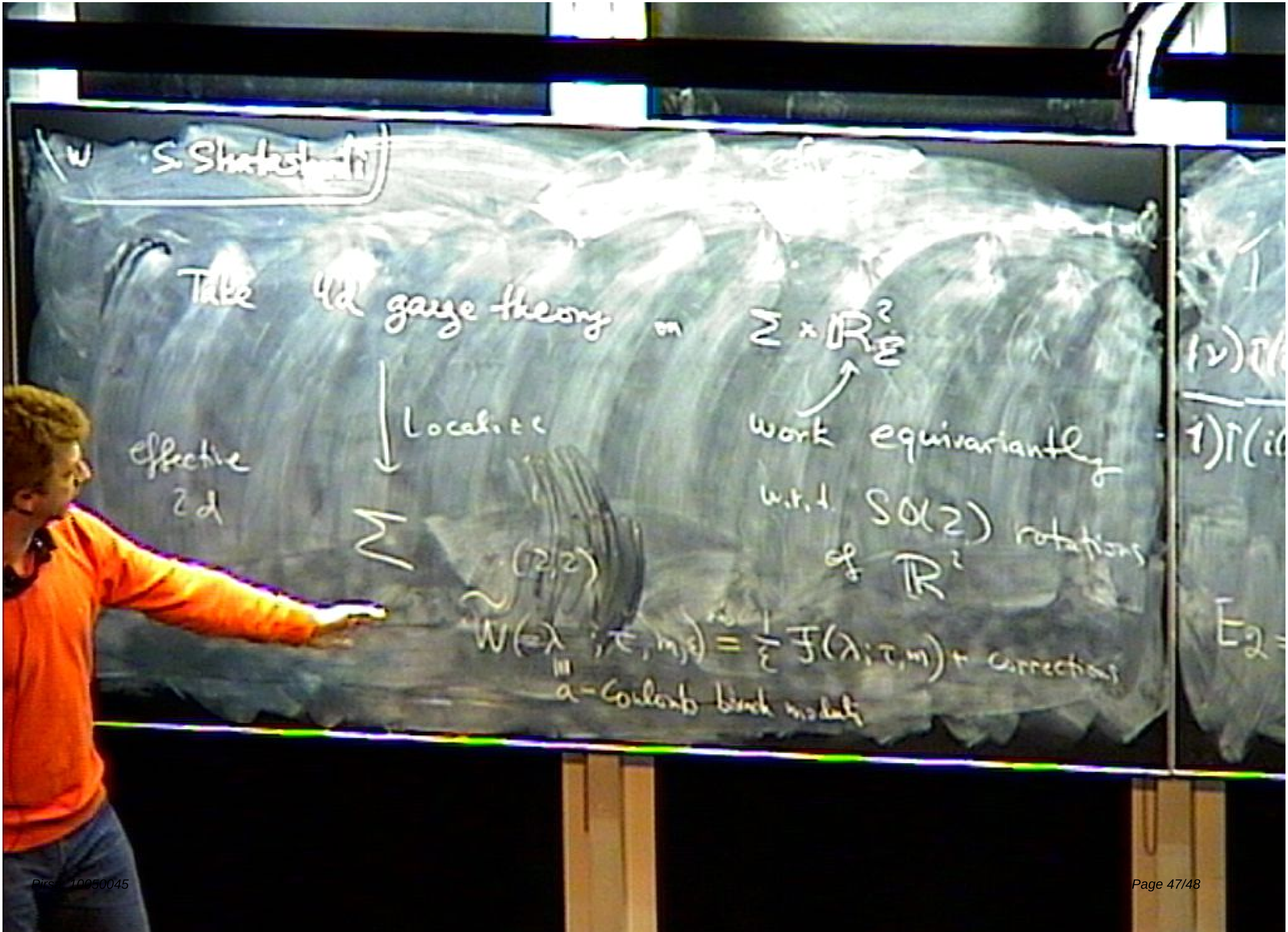

One can find the
Yang-Yang function!
Asymptotic Bethe Ansatz

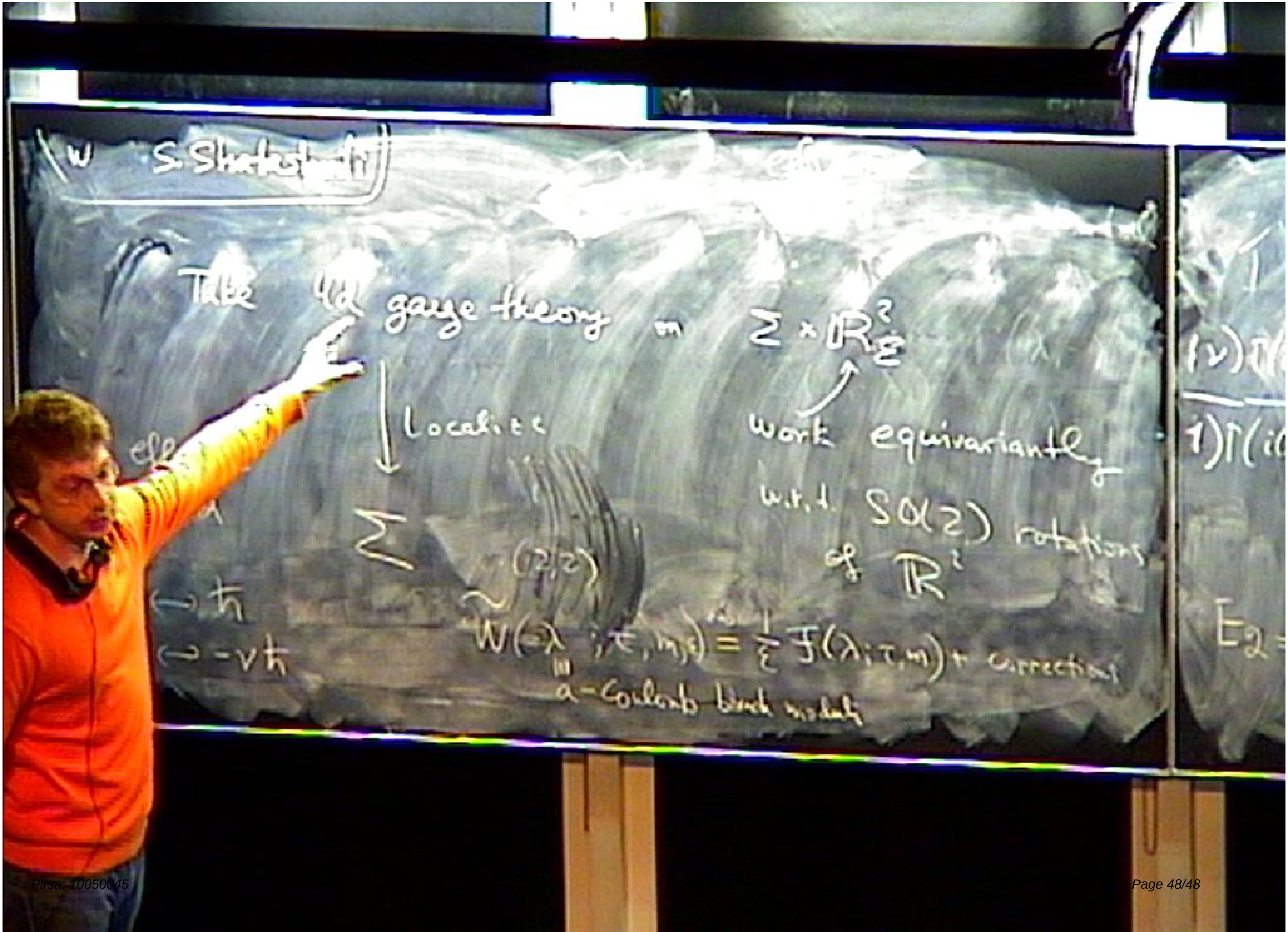

$\tilde{W}(\lambda) = \sum_{k=1}^{N} \frac{1}{2}\tau \lambda_k^2 + \sum_{j,k=1}^{N} \varpi(\lambda_j - \lambda_k - i\nu) - \varpi(\lambda_j - \lambda_k - i)$
$\frac{d}{dx}\varpi(x) = \log \Gamma(1 + ix)$
gauge theory with adjoint matter
sa: 10050045


$\tilde{W}(\lambda) = \sum_{k=1}^{N} \frac{1}{2}\tau \lambda_k^2 + \sum_{j,k=1}^{N} \varpi(\lambda_j - \lambda_k - i\nu) - \varpi(\lambda_j - \lambda_k - i)$
$\frac{d}{dx}\varpi(x) = \log \Gamma(1 + ix)$
gauge theory with
adjoint matter
$\nu \leftrightarrow$ mass
sa: 10050045


Classical integrable system
$G = U(N)$
$N=2$
by giving the mass $\sim \nu$
$\tilde{W}(\lambda) = \sum_{k=1}^{N} \frac{1}{2}\tau \lambda_k^2 + \sum_{j,k=1}^{N} \varpi(\lambda_j - \lambda_k - i\nu) - \varpi(\lambda_j - \lambda_k - i)$
hyper
$\frac{d}{dx}\varpi(x) = \log \Gamma(1 + ix)$
$\tau \sim$ coupling constant
gauge theory with
adjoint matter
$\nu \leftrightarrow$ mass


Take 4d gauge theory on $\Sigma \times \mathbb{R}^2_\varepsilon$
work equivariantly
w.r.t. SO(2) rotations
of $\mathbb{R}^2$


w/ S. Shatashvili
Take 4d gauge theory on Σ × R²_ε
Localize
effective 2d
Σ
(2,2)
work equivariantly
w.r.t. SO(2) rotations of R²
W(λ; τ, m, ε) = (1/ε) F(λ; τ, m) + corrections
a – Coulomb branch moduli
E₂ =


w S. Shatashvili
Take 4d gauge theory on Σ × R²
localize
work equivariantly
w.r.t. SO(2) rotations of R²
Σ
corrections
a – Coulomb branch moduli