Title: Chern-Simons-Rozansky-Witten topological field theory

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Abstract: I will discuss a hybrid between Chern-Simons and Rozansky-Witten models. In particular, Wilson loops in this topological field theory are objects of a quantum deformation of the equivariant derived category of coherent sheaves.

Motivation

① Rozansky-Witten model

- topological $D=3$ σ-model

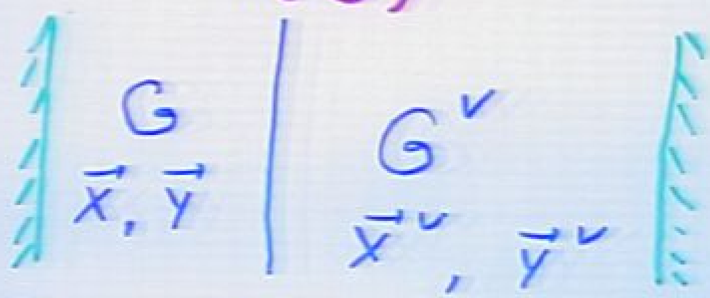
G - global symmetry

Can one gauge G?

② Topological twist

of $N=4$, $D=3$ Gaiotto - Witten theory

$T(G)$



$y=0$ $\qquad$ $y=L$

S-duality wall

$$\partial_y \vec{Y}(0)=0$$
$$\vec{X}(0)=0$$
$$CS_k$$

$$\partial_y \vec{Y}^\vee(L)=0$$
$$\vec{X}^\vee(L)=\xi$$

$\mathcal{N}=4$, $D=4$ SYM on $R^3 \times I$

Plan

1. Construction of CSRW model
2. Flat $X \to$ CS for supergroup
3. $X = T^*(G_{\mathbb{C}}/B)$
4. Wilson loops

hyper-kähler

$\boxed{G \text{ acts on } X}$

$\dim_{\mathbb{C}} X = 2n$

$I, \bar{I} = 1, \dots, 2n$

$$V_a = V_a^I \partial_I + V_a^{\bar{I}} \partial_{\bar{I}}$$

$a = 1, \dots, \dim G$

$$[V_a, V_b] = f_{ab}^c V_c$$

f_{ab}^c - structure constants of $\mathfrak{g}$

$\mathfrak{g}$ - Lie algebra of G

G is a global symmetry of RW model iff

1). $V_a^{\bar{I}}$ - holomorphic

2). $V_a^{\bar{I}}$ preserve symplectic structure Ω_{IJ}

We further assume: 3).

$$\boxed{\text{G-action preserves } J}$$

Kähler form on X

$\exists$ moment maps $\mu_+, \mu_-, \mu_3 : X \to g^*$

$$\boxed{\begin{aligned} d\mu_{+a} &= -i_{V_a}(\Omega) \\ d\mu_{-a} &= -i_{V_a}(\bar{\Omega}) \\ d\mu_{3a} &= i_{V_a}(J) \end{aligned}}$$

μ_+ - holomorphic, $\mu_- = \bar{\mu}_+$

$$\{\mu_{+a}, \mu_{+b}\} = -f_{ab}^{c}\mu_{+c}$$

$\varkappa_{ab}$ - G-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$

$$\boxed{\varkappa_{ad} f_{bc}^{\;d} + \varkappa_{bd} f_{ac}^{\;d} = 0}$$

It is crucial to require:

$$\boxed{\mu_+ \cdot \mu_+ = 0}$$

$$\mu_+ \cdot \mu_+ = \varkappa^{ab} \mu_{+a} \mu_{+b}$$

$Q_{BRST}^2 = 0$ on gauge-invariant observables

Bosonic fields in CSRW

$\boxed{A^a_\mu dx^\mu \text{ - a connection on a principle } G\text{-bundle } \mathcal{E} \text{ over } M_3}$

$X_\mathcal{E}$ - a fiber bundle over M_3 associated with $\mathcal{E}$ and typical fiber X.

$\boxed{A^a V_a \text{ - connection on } X_\mathcal{E}}$

$$(D\varphi)^I = d\varphi^I + A^a V_a^I$$

$$(D\varphi)^{\bar{I}} = d\varphi^{\bar{I}} + A^a V_a^{\bar{I}}$$

$\boxed{\varphi^I(x^\mu),\ \varphi^{\bar{I}}(x^\mu) \text{ - section of } X_\mathcal{E}}$

$\boxed{x^\mu \quad \mu=1,2,3}$ local coordinates on M_3 (5)

fermionic fields in CSRW

$$\boxed{\chi^{I} \ - \text{ 1-form on } M_3 \text{ with values in } \varphi^*(T_{X_\varepsilon})}$$

$$\boxed{\eta^{\bar{I}} \ - \text{ 0-form on } M_3 \text{ with values in } \varphi^*(\overline{T}_{X_\varepsilon})}$$

T_{X_ε} – (1,0) part of the fiberwise-tangent bundle of X_ε

Covariant derivatives

$$\boxed{\begin{aligned} D\chi^{I} &= \nabla\chi^{I} + A\cdot\nabla_{K}(V^{I})\chi^{K} \\ D\eta^{\bar{I}} &= \nabla\eta^{\bar{I}} + A\cdot\nabla_{\bar{K}}(V^{\bar{I}})\eta^{\bar{K}} \end{aligned}}$$

∇ i...

BRST transformations

$$\delta_Q \varphi^{\bar I} = \eta^{\bar I} \qquad \delta_Q \varphi^{I} = 0$$

$$\delta_Q \eta^{\bar I} = -\bar\zeta^{\bar I} \qquad \delta_Q \chi^{I} = D\varphi^{I}$$

$$\delta_Q A^a = \varkappa^{ab} \chi^k \partial_k \mu_{+b}$$

$$\zeta^I = v^I \cdot \mu_- \,, \quad \bar\zeta^{\bar I} = v^{\bar I} \cdot \mu_+$$

δ_Q^2 – gauge transformation with $\epsilon^a = -\varkappa^{ab}\mu_{+b}$

$$\mu_+ \cdot \mu_+ = 0$$

Classical action

$$\boxed{S = \frac{1}{\hbar}\int_{M_3} \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_1 + \mathcal{L}_2}$$

$$\hbar = \frac{2\pi i}{k}, \quad k \in \mathbb{Z}$$

$$\boxed{\mathcal{L}_{CS} = \frac{1}{2}\varkappa_{ab}\left(A^a \wedge dA^b - \frac{1}{3} f^b_{cd} A^a \wedge A^c \wedge A^d\right)}$$

$$\mathcal{L}_1 = \delta_Q \left(g_{I\bar{K}}\, \chi^I \wedge * D\varphi^{\bar{K}} - \sqrt{\hbar}\, g_{I\bar{K}} \zeta^I \eta^{\bar{K}}\right)$$

$$= g_{I\bar{K}}\left(D\varphi^I \wedge * D\varphi^{\bar{K}} - \chi^I \wedge * D\eta^{\bar{K}}\right) +$$

$$+ \sqrt{\hbar}\left(g_{I\bar{K}} \zeta^I \zeta^{\bar{K}} + g_{I\bar{K}} (\partial_{\bar{P}} \zeta^I) \eta^{\bar{K}} \eta^{\bar{P}}\right)$$

$$\boxed{\mathcal{L}_2 = \frac{1}{2}\Omega_{IJ}\left(\chi^I \wedge D\chi^J + \frac{1}{3} R^J_{KL\bar{M}}\, \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}\right)}$$

Gauge - ing

$c^a, \bar{c}_a$ - Faddeev-Popov ghosts and anti-ghosts

B_a - Lagrange multiplier for gauge-fixing condition

$$\delta_{\hat{Q}} = \delta_Q + \delta_{FP}$$

$$\delta_{\hat{Q}}^2 = 0$$

$$S_{g.f} = S_{cl} + \delta_{\hat{Q}} (\bar{c}_a f^a)$$

δ_{FP} acts in a standard way

$$\boxed{\delta_Q c^a = \varkappa^{ab}{}_{\nu + B}}, \quad \boxed{\delta_Q \bar{c}_a = \delta_Q B_a = 0}$$

$$\delta_{\hat{Q}} A_a = dc_a - f_{abd} A^b c^d + \chi^K \partial_K \mu_{+a}$$

$$\delta_{\hat{Q}} \varphi^I = -V^I \cdot c \qquad \delta_{\hat{Q}} \varphi^{\bar{I}} = \eta^{\bar{I}} - V^{\bar{I}} \cdot c$$

$$\delta_{\hat{Q}} \chi^I = D\varphi^I + (\partial_J V^{Ia}) \chi^J c_a$$

$$\delta_{\hat{Q}} \eta^{\bar{I}} = -\bar{\xi}^{\bar{I}} + (\partial_{\bar{J}} V^{\bar{I}a}) \eta^{\bar{J}} c_a$$

$$\delta_{\hat{Q}} c^a = \varkappa^{ab} \mu_{+b} - \frac{1}{2} f^a_{\ bd}\, c^b c^d$$

$$\delta_{\hat{Q}} \bar{c}_a = B_a, \qquad \delta_{\hat{Q}} B_a = 0$$

Flat target X → CS for supergroup

If X is a vector space with a linear action of G, then

$$\boxed{\begin{aligned} \mu_+ a &= \tfrac{1}{2}\, \varkappa_{ab}\, \tau^{b}_{IJ}\, \varphi^I \varphi^J \\ \mu_- a &= \tfrac{1}{2}\, \varkappa_{ab}\, \bar\tau^{b}_{\bar I \bar J}\, \varphi^{\bar I} \varphi^{\bar J} \\ \mu_3 a &= -i\, \varkappa_{ab}\, \varphi^I\, \tau^{b}_{IJ}\, \Omega^{JK}\, g_{K\bar M}\, \varphi^{\bar M} \end{aligned}}$$

Gaiotto-Witten introduced a Lie superalgebra $\mathfrak{g}$:

$$[M_a, M_b] = f_{ab}^{\ \ c} M_c \qquad [M_a, \Psi_I] = \varkappa_{ab}\, \tau^{b}_{IJ}\, \Omega^{JK}\, \Psi_K$$

$$\{\Psi_I, \Psi_J\} = \tau^{a}_{IJ}\, M_a$$

Super-Jacobi identities

$$\Rightarrow \boxed{\begin{array}{ll} \mu_+ \cdot \mu_+ = 0 & \mu_3 \cdot \mu_+ = 0 \\ \multicolumn{2}{c}{2\mu_3 \cdot \mu_3 - \mu_+ \cdot \mu_- = 0} \end{array}}$$

Constraints of $N=4, D=3$ superconformal symmetry

Why Lie superalgebra $\mathfrak{g}$ arises?

We found:

CSRW for flat X $\sim$ CS for supergroup associated to $\mathfrak{g}$ with gauge-fixed odd part of the gauge symmetry

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$$\boxed{\begin{aligned} &\mathcal{A} = \mathcal{A}_b + \mathcal{A}_f \\ &\mathcal{A}_b = A^a M_a \qquad \mathcal{A}_f = \chi^I \lambda_I \end{aligned}}$$

$$\boxed{\begin{aligned} &\bar{C} = \varphi^{\bar{I}} g_{\bar{I}K} \Omega^{KJ} \lambda_J \\ &C = \varphi^I \lambda_I \\ &B = \eta^{\bar{M}} g_{\bar{M}K} \Omega^{KJ} \lambda_J \end{aligned}}$$

$$\delta_Q \mathcal{A} = dC - [\mathcal{A}, C\}$$

$$\delta_Q \bar{C} = B \qquad \delta_Q C = 0$$

$$\delta_Q B = \frac{1}{2}[\bar{C}, [C, C\}\}$$

δ_Q^2 - gauge transformation with $\epsilon^a = -\chi^{ab} \ldots$

$$\mathcal{L}^{(\text{flat } X)}_{CSRW} = \frac{1}{2} \mathrm{STr}\left(A\, dA - \frac{1}{3} A [A, A]\right) - \delta_Q \Psi$$

$$\Psi = \mathrm{STr}\left[A_f \wedge * \left(d\bar{C} - [A_B, \bar{C}\}\right)\right]$$

$$- 2\, \mathrm{vol}_M\ \mathrm{STr}\left[[\bar{C}, \bar{C}\}\, [C, B\}\right]$$

Gauge fixing both fermionic & bosonic gauge symmetries:

$$\mathbb{C} = c^a M_a + \varphi^I \perp_I$$

$$\bar{\mathbb{C}} = \bar{c}_a \varkappa^{ab} M_b + \bar{\varphi}^I g_{IK} \Omega^{KJ} \perp_J$$

$$\mathbb{B} = B_a \varkappa^{ab} M_b + (\eta^I - v^I \cdot c) g_{IK} \Omega^{KJ} \perp_J$$

$$\delta_{\hat{Q}} A = d\mathbb{C} - [A, \mathbb{C}]$$

$$\delta_{\hat{Q}} \mathbb{C} = -\frac{1}{2} [\mathbb{C}, \mathbb{C}]$$

$$\delta_{\hat{Q}} \bar{\mathbb{C}} = B \qquad \delta_{\hat{Q}} B = 0$$

Local observables in CSRW with flat X

use relation with CS for supergroup

$\Rightarrow$ candidate observables are polynomial functions of $\mathbf{c}$

$$\delta_{\hat{Q}} \mathbf{c} = -\tfrac{1}{2}[\mathbf{c}, \mathbf{c}\}$$

$\Rightarrow$ cohomology of $\mathfrak{g}$

For $\mathfrak{g} = gl(m|n)$ and $\mathfrak{g} = osp(m|n)$ cohomology is finite dimensional.

D. B. Fuks (1984)

For $gl(m|n)$ it is isomorphic to the cohomology of $gl(\max(m,n))$.

Monopole [illegible]rators

$X = \mathbb{C}^2, \quad G = U(1) \times U(1)$

A	1	-1
B	-1	1

$\mathfrak{g} = \mathfrak{gl}(1|1)$

$$\boxed{\begin{aligned} F_1 &= * d \frac{m_1}{2r} + \text{regular} \\ F_2 &= * d \frac{m_2}{2r} + \text{regular} \end{aligned}}$$

$$\delta_\epsilon S_{CS} = i k \,(m_1 \epsilon_1 - m_2 \epsilon_2)$$

$$\boxed{\delta_{\hat{Q}} M_{m_1, m_2} = - i k \,(m_1 c_1 - m_2 c_2)\, M_{m_1 m_2}}$$

CSRW model for $X = T^*\mathbb{P}^1$, $G = SU(2)$

Let z be coordinate on $\mathbb{P}^1$

b on fiber of $T^*\mathbb{P}^1$

$SU_{\mathbb{C}}(2) = SL(2,\mathbb{C})$ acts as

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta} \qquad b \to b(\gamma z + \delta)^2$$

$$J = t\,\hat{J}, \quad \hat{J} = i\,(f_1\, e_1 \wedge \bar{e}_1 + f_2\, e_2 \wedge \bar{e}_2)$$

$$f_1 = \frac{x^2}{\sqrt{1+x^2}} \qquad f_2 = \sqrt{1+x^2}$$

$$x^2 = |b|^2 (1+|z|^2)^2$$

$$\Omega = t\,\hat{\Omega}$$

$$\hat{\Omega} = db \wedge dz$$

$$e_1 = \frac{1}{2}\frac{db}{b} + \bar{z}\, e_2$$

$$e_2 = \frac{dz}{1+|z|^2}$$

Moment maps

$$\mu = t\,\hat{\mu}$$

$$\boxed{\begin{gathered}\hat{\mu}_+ = (-i)\,\beta \begin{pmatrix} -z & 1 \\ -z^2 & z \end{pmatrix} \qquad \hat{\mu}_- = i\bar{\beta} \begin{pmatrix} -\bar{z} & -\bar{z}^2 \\ 1 & \bar{z} \end{pmatrix} \\ \hat{\mu}_3 = \frac{f_2(x)}{1+|z|^2} \begin{pmatrix} \frac{1}{2}(1-|z|^2) & \bar{z} \\ z & -\frac{1}{2}(1-|z|^2) \end{pmatrix}\end{gathered}}$$

$$\boxed{\begin{gathered}\mathrm{Tr}(\mu_+^2) = 0 \qquad \mathrm{Tr}(\mu_+ \mu_3) = 0 \\ 2\,\mathrm{Tr}(\mu_3^2) - \mathrm{Tr}(\mu_+ \mu_-) = t^2\end{gathered}}$$

$$\boxed{V = g_{I\bar{J}}\, \zeta^I \bar{\zeta}^{\bar{J}} = t^3 x^2 \sqrt{1+x^2}}$$

$$x^2 = |\beta|^2 (1+|z|^2)^2$$

$$\Downarrow$$

$\boxed{\text{mass term for } \beta}$

$$g_{I\bar{K}}\, (\partial_{\bar{P}} \zeta^I)\, \eta^{\bar{K}} \eta^{\bar{P}} = t^2\, \eta^{\bar{z}} \eta^{\bar{\beta}}$$

mass for η's

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Algebra of local observables for $X = T^*(\mathbb{P}^1)$

$\mathbb{1}$ in $gh = 0$

$\frac{1}{3} f_{abd}\, c^a c^b c^d - \mu_{+a} c^a$ in $gh = 3$

No BRST-invariant monopole operators

$M_3 = \mathbb{R} \times S^2$

Flux $\int_{S^2} F$ in $U(1) \in SU(2)$ gives mass to z, so $\nexists$ zero modes which can be used to compensate for $U(1)$ charge of monopole

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Generalization for $X = T^*(G_{\mathbb{C}}/B)$

$\rho \in G/T$ near identity of G:

$$\boxed{\rho = e^{i\sum_{\alpha\in S}(w^\alpha E_\alpha + \bar{w}^{\bar\alpha} E_{\bar\alpha})} h, \quad h \in T}$$

S – set of positive roots

holomorphic Darboux coordinates on X: $\quad \varphi^I = (b_\alpha, z^\alpha) \quad \alpha \in S$

$$\boxed{\Omega = t \sum_{\alpha\in S} db_\alpha \wedge dz^\alpha}$$

$$z^\alpha = w^\alpha + O(|w|^3) \quad \text{for } |w| \ll 1$$

$$\boxed{\mu_+ = t\,\hat{\mu}_+, \quad \hat{\mu}_+ = \sum_{\alpha\in S} b_\alpha V^{z^\alpha}(z)}$$

$$\boxed{V^{z^\alpha} = E_\alpha + i z^\alpha \sum_{j=1}^{r} \alpha^j H_j + \sum_\delta{}' z^{\alpha-\delta} E_\delta + O(z^2)}$$

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$$\boxed{N^{\alpha\bar{\gamma}} = \mathrm{Tr}\left(V^{\bar{z}\bar{\gamma}}\, V^{z\alpha}\right)}$$

non-degenerate matrix

$\Rightarrow$ b_α, $\eta^{\bar{I}}$ - massive

$z_{0\alpha}$ - zero modes parametrizing $G_{\mathbb{C}}/B$

$\Rightarrow$ BRST cohomology can be computed in the space of holomorphic functions on $G_{\mathbb{C}}/B$ tensored with zero modes of FP ghosts.

$G_{\mathbb{C}}/B$ - compact $\Rightarrow$ BRST cohomology is isomorphic to the cohomology of Lie algebra $\mathfrak{g}$.

Wilson loops in CSRW model

$\underline{E = E^+ + E^-}$ – $\mathbb{Z}_2$ graded vector bundle over X

$\underline{\text{Fiber of } E}$ over $p \in X$ – space of $\underline{\text{d.o.f.}}$ living on $\underline{\text{Wilson loop } \mathcal{W}}$

Observables on $\mathcal{W}$ – sections of $\mathrm{End}(E) \otimes \Omega^{0,\bullet}$

$\bar\partial$ - superconnection on $\underline{E}$:

$$D = \bar\partial + \mathcal{K} = \bar\partial + \begin{pmatrix} \omega^+_{\bar I} d\varphi^{\bar I} & T \\ S & \omega^-_{\bar I} d\varphi^{\bar I} \end{pmatrix}$$

∂ - connection on $\underline{E^\pm}$:

$$\partial^\pm = \partial + \omega^\pm_I d\varphi^I$$

$$\mathcal{F}^\pm = d\omega^\pm + \omega^\pm \wedge \omega^\pm$$

Action $\mathfrak{g}$ on $\Gamma(E)$

↖ smooth sections of E

$$\boxed{\begin{gathered} e_a : \; s \to \mathcal{V}_a(s) = V^{\hat\rho}\,\nabla_{\hat\rho}\, s + T_a s \\ e_a \in \mathfrak{g}, \; s \in \Gamma(E) \end{gathered}}$$

$$T_a = \begin{pmatrix} t_a^+ & 0 \\ 0 & t_a^- \end{pmatrix} \qquad \hat\rho = (\rho, \bar\rho)$$

← even section of End (E)

$\mathcal{V}_a$ maps covariantly constant sections to covariantly constant sections:

$$\nabla_{\hat\rho}\, t_a^{\pm} = V_a^{\hat\kappa}\, \mathcal{F}^{\pm}_{\hat\kappa\hat\rho}$$

$$\boxed{[\mathcal{V}_a, \mathcal{V}_b] = f_{ab}^{\ c}\, \mathcal{V}_c}$$

$$\boxed{[t_a^{\pm}, t_b^{\pm}] = f_{ab}^{\ c}\, t_c^{\pm} + V_a^{\hat\jmath}\, V_b^{\hat\kappa}\, \mathcal{F}^{\pm}_{\hat\jmath\hat\kappa}}$$

Impose conditions on $\mathbb{D}$:

$$\boxed{\begin{aligned} \mathbb{D}^2 &= \varkappa^{ab} \mathcal{N}_{+a} T_b \\ [\mathbb{D}, \nu_a] &= 0 \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\mathcal{F}^{\pm}_{IJ} = 0 \qquad \nabla_I S = 0 \qquad \nabla_I \tau = 0}$$

$$\boxed{\begin{aligned} S\tau &= \varkappa^{ab} \mathcal{N}_{+a} t^-_b \\ \tau S &= \varkappa^{ab} \mathcal{N}_{+a} t^+_b \end{aligned}} \qquad \varkappa^{ab} \sim \frac{1}{k}$$

$$\boxed{\begin{aligned} V^I_a \nabla_I \tau &= \tau t^-_a - t^+_a \tau \\ V^I_a \nabla_I S &= S t^+_a - t^-_a S \end{aligned}}$$

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γ - closed curve in M_3

$t \in [0,1)$

$$\mathcal{N} = \begin{pmatrix} A_t^c t_c^+ - \omega_{\bar{I}}^+ \partial_t \varphi^{\bar{I}} - \chi_t^N \eta^{\bar{M}} \mathcal{F}^+_{N\bar{M}} & -\chi_t^I \nabla_I \mathcal{T} \\ -\chi_t^I \nabla_I S & A_t^c t_c^- - \omega_{\bar{I}}^- \partial_t \varphi^{\bar{I}} - \chi_t^N \eta^{\bar{M}} \mathcal{F}^-_{N\bar{M}} \end{pmatrix} dt$$

$\mathcal{N}$ - connection on $\varphi^*(E)$

$\varphi = (\varphi, \eta) : \gamma \to \Pi \bar{T}_X$

supermanifold with odd coordinates $\eta^{\bar{I}}$

$$\delta_Q \mathcal{N} = -d_t(\mathcal{K}) + [\mathcal{N}, \mathcal{K}]$$

$$\boxed{\mathcal{W} = \mathrm{STr}\, \mathcal{U}(0,1)}$$

BRST invariant Wilson loop

$$\boxed{(d_t - \mathcal{N})\, \mathcal{U}(0,t) = 0 \quad \text{with } \mathcal{U}(0,0) = \mathbb{1}}$$

Impose conditions on $\mathbb{D}$:

$$\boxed{\begin{aligned} \mathbb{D}^2 &= \varkappa^{ab} \mathcal{N}_{+a} T_b \\ [\mathbb{D}, \nabla_a] &= 0 \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\mathcal{F}^{\pm}_{\bar{I}\bar{J}} = 0 \qquad \nabla_{\bar{I}} S = 0 \qquad \nabla_{I} \tau = 0}$$

$$\boxed{\begin{aligned} S\,\tau &= \varkappa^{ab} \mathcal{N}_{+a} t^-_b \\ \tau\, S &= \varkappa^{ab} \mathcal{N}_{+a} t^+_b \end{aligned}} \qquad \varkappa^{ab} \sim \frac{1}{k}$$

$$\boxed{\begin{aligned} V_a^I \nabla_I \tau &= \tau\, t_a^- - t_a^+ \tau \\ V_a^I \nabla_I S &= S\, t_a^+ - t_a^- S \end{aligned}}$$

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Examples of Wilson loops (30)

1). flat X

$$\boxed{W_R = \mathrm{STr}_R \, P \, e^{\oint A^a M_a^{(R)} + \chi^I \perp_I^{(R)}}}$$

R - finite dim. rep. of $\mathfrak{g}$

This is a special case of our general construction: E - trivial vector bundle with fiber R, $D = \bar{\partial} - \varphi^I \perp_I^{(R)}$

2). $X = T^* \mathbb{P}^1$

E - holomorphic line bundle over X with $SL(2, \mathbb{C})$ action.

$\tau = S = 0$, T_a must satisfy:

$$\boxed{\varkappa^{ab} \mu_{+a} T_b = 0} \qquad \boxed{\partial_{\hat{\rho}} T_a = V_a^{\hat{k}} F_{\hat{k}\hat{\rho}}}$$

$$\boxed{f_{ab}^c T_c + V_a^{\hat{j}} V_b^{\hat{k}} F_{\hat{j}\hat{k}} = 0}$$

Class of solu ns: $E = \mathcal{L}^n, n \in \mathbb{Z}$

$\mathcal{F} = (-i) n \hat{\mathcal{J}} \qquad T_a = -in \hat{\mu}_{3a}$

$\int_{\mathbb{P}^1} \hat{\mathcal{J}} = 2\pi \qquad \mathcal{J} = t \hat{\mathcal{J}}$

at $z = b = 0$

$T_1 = T_2 = 0 \qquad T_3 = (-i) n$

i.e. fiber of $\mathcal{L}^n$ over $z = b = 0$ transforms in rep. with charge n

$$\boxed{W_n \cdot W_m = W_{n+m}}$$ ← classically

no endomorphisms of W_n with $gh = 1$

$\Rightarrow$ no quantum corrections

Role of level k

1. Braiding phase of Wilson loops

γ_1, γ_2 - closed curves in R^3 with linking # = 1.

$$\langle W_{n_1} W_{n_2} \rangle \to e^{\frac{\pi i n_1 n_2}{k}}$$

2. $\boxed{n \sim n + 2k}$

Periodic identification of Wilson loops

This is due to screening by monopole operator

Open problems

1. Find more non-trivial examples of Wilson lines

2. CSRW model with $X = T^*(G_{\mathbb{C}}/B)$ has finite partition sum

$\Rightarrow$ new invariants of 3-manifolds

3. Find examples of CSRW with curved X other than $T^*(G_{\mathbb{C}}/B)$