

Title: Twistor-String Theory

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Abstract: I'll give an introduction to twistor-string theory, which is an attempt to reformulate supersymmetric gauge theory in four-dimensional space-time in terms of a certain generalisation of Gromov-Witten theory in twistor space. The resulting theory is closely related to the multi-dimensional residue calculus in $G(k,n)$ (introduced in Cachazo's talk).

Twistor String Theory appetizes

Twistor String Theory an appetizer



Twistor String Theory and appetizers



$\mathbb{R}^{2,1}$

M symplectic mfd.

$$S: M \rightarrow M$$

$$\bar{\Gamma}[\Phi]$$

$$\bar{\Phi}|_h = \phi$$

$$\bar{\Gamma}|_s = \phi_T$$

Twistor String Theory and appetizers



M symplectic manifold

$$S: M \rightarrow M$$

$$\mathbb{I}[\Phi]$$

$$\Phi|_a = \phi$$

$$\Phi|_b = \phi_T$$

Twistor String Theory as appetizer



M symplectic manifold

$$S: M \rightarrow M$$

$$\mathbb{I}[\Phi]$$

$$\Phi|_a = \phi$$

$$\Phi|_c = \phi_T$$

too hard \rightarrow work perturbatively



$H \cdot \oplus \text{Sym.}(F)$

S



$$H = \bigoplus \text{Sym.}(F)$$

$$S_{\dots p} \text{Sym.}(F) \otimes (\text{Sym}_p(F))^* \rightarrow \mathbb{C}$$

- Combinatorically very involved.



$$H = \bigoplus \text{Sym}_n(F)$$

$$S_{n,p} = \text{Sym}_n(F) \otimes (\text{Sym}_p(F))^* \rightarrow \mathbb{C}$$

- Combinatorially very involved.
- F.D. obscuring the str. of theory (e.g. gauge invariance off-shell)

ST.

Replace $\sum F.D.$ \Rightarrow



$$\int_{\mathcal{M}_{g,n}} \Omega$$

Normal ST. gives us constructions for Ω

ST.

Replace $\Sigma F.D.$ \Rightarrow



$$\int_{M_{2n}}$$

Normal ST. gives us constructions for Ω

- dimensional reduction.

- $\alpha' \rightarrow 0$

BCFW transition

$$\text{Diagram } A_{n,t} = \sum_{A \cup B \text{ eq. } k_0, k_1} \sum \text{Diagram } A_{n,t}$$

The diagram on the left is a circle labeled $A_{n,t}$ with n external lines. The diagram on the right is a circle labeled $A_{n,t}$ with a horizontal line connecting it to another circle labeled $A_{n,t}$, each with n external lines.



BCFW recursion

$$\begin{aligned}
 & \text{Diagram of } A_{n,t} \text{ (circle with } n \text{ external lines)} \\
 & = \sum_{\text{Aublag. } k, \text{ oberkol}} \sum_{\text{Klapp. } n_1, n_2} \text{Diagram of } A_{n_1,t} \text{ --- } A_{n_2,t} \text{ (two circles connected by a line)} \times \frac{1}{P_{k,t}}
 \end{aligned}$$

BCFW recursion

$$A_{n,t} = \sum_{A \cup B \text{ sep. } k, \ell, k+\ell=n} \sum_{\pi_1, \dots, \pi_2} A_{k,t} \cdot A_{\ell,t} \cdot \frac{1}{P_{A,B}}$$

stratified by "dual graph type"

each irreducible cusp $\Sigma_i \Leftrightarrow$ vertex v_i

If Σ_i, Σ_j share a node \Leftrightarrow draw line between v_i, v_j

- Marked point on $\Sigma_i \Leftrightarrow$ half-line ending in v_i

BCFW recursion

$$A_{n,t} = \sum_{A \cup B \text{ sep. } k, k', k''} \sum_{\pi_1, \dots, \pi_r} A_{n,t} \text{---} A_{j,t} \times \frac{1}{P_A}$$

$\overline{M}_{0,n}$ stratified by "dual graph type"

- to each irreducible cusp $\Sigma_i \iff$ vertex v_i
- if Σ_i, Σ_j share a node \iff draw line between v_i, v_j
- marked point on $\Sigma_i \iff$ half-line ending in v_i



P^2 is transition matrix of \mathbb{R}^n

Let $\{v_1, \dots, v_n\}$ be a basis of \mathbb{R}^n \Leftrightarrow a set of n linearly independent vectors in \mathbb{R}^n

\mathbb{P}^3 is twistor space of $\mathbb{R}^{3,1}$

holomorphic bundle $E \rightarrow \mathbb{P}^3 \iff$ self-dual solⁿ of YM eqⁿ on spacetime

$H^1(\mathbb{P}^3, \text{End } E) \iff$ -ve helicity gluons

$H^1(\mathbb{P}^3, \text{End } E \otimes \mathcal{O}(-4)) \iff$ +ve helicity gluons.

$\alpha \beta$

\mathbb{P}^3 is twistor space of $\mathbb{R}^{3,1}$

holomorphic bundle $E \rightarrow \mathbb{P}^3 \iff$ ASD solⁿ of YM eqⁿ on spacetime

$H^1(\mathbb{P}^3, \text{End } E) \iff$ -ve helicity gluons

$H^1(\mathbb{P}^3, \text{End } E \otimes \mathcal{O}(-4)) \iff$ +ve helicity gluons.

$$\int_{\mathbb{P}^3} (\alpha \cup \beta) = \langle + - \rangle$$

Scott's

$$H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}(1)) \xrightarrow{\cong} \mathbb{C}$$



Obvious generalisation $V = \mathcal{O}(1)^{\oplus 2}$

$$H^1(\mathbb{P}^1, \Lambda^1 V^{\vee}) \oplus H^1(\mathbb{P}^1, \Lambda^2 V^{\vee}) \oplus H^1(\mathbb{P}^1, \Lambda^3 V^{\vee}) \xrightarrow{\cong} \mathbb{C}$$

Scattering

$$H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}(4)) \xrightarrow{\cong} \mathbb{C}$$



Obvious generalization

$$V = \mathcal{O}(1)^{\oplus 4}$$

$$\Lambda^4 V \cong K_{\mathbb{P}^1}$$

$$H^0(\mathbb{P}^1, \Lambda^1 V) \oplus H^1(\mathbb{P}^1, \Lambda^1 V) \oplus H^1(\mathbb{P}^1, \Lambda^1 V) \xrightarrow{\cong} \mathbb{C}$$

$$\sum q_i = 4V$$



The ST

$$\bar{M}_n(\mathbb{P}^1, d) \quad \rho: \bar{M}_n(\mathbb{P}^1, d) \rightarrow \bar{M}_n$$

$$= \rho^{-1}((\epsilon, p_1, \dots, p_n))$$

$$\mathbb{C} \xrightarrow{\delta} \mathbb{P}^3$$

$$\begin{array}{c} \mathbb{C} \\ \downarrow \pi \\ M \end{array}$$

The ST

$$\bar{M}_{g,n}(\mathbb{P}^2, d) \quad \rho: \bar{M}_{g,n}(\mathbb{P}^2, d) \rightarrow \bar{M}_{g,n}$$

$$M = \rho^{-1}((\Sigma, p_1, \dots, p_n))$$

$$C \xrightarrow{\delta} \mathbb{P}^2$$

$$\pi \downarrow \\ M$$

$$W = \pi_* \Phi^* V$$

$$W|_f = H^0(\Sigma, f^* V)$$

$$h^1(\Sigma, f^* V) = 4(d+1)$$

$$c_1(W) = \pi_* (h(\Phi^* V) + Td(T_\Sigma)|_M)$$

The ST

$$\bar{M}_{g,n}(\mathbb{P}^1, d) \quad \rho: \bar{M}_{g,n}(\mathbb{P}^1, d) \rightarrow \bar{M}_{g,n}$$

$$M = \rho^{-1}((\varepsilon, p_1, \dots, p_n))$$

$$C \xrightarrow{f} \mathbb{P}^1$$

$$\pi \downarrow \\ M$$

$$W = \pi_* \Phi^* V$$

$$W|_f = H^0(\varepsilon, f^* V)$$

$$h^0(\varepsilon, f^* V) = 4(d+1)$$

$$c_1(W) = \pi_* (ch(\Phi^* V) \cdot Td(T_\varepsilon)|_M) = c_1(T_M)$$

$$\Lambda^{top} W^v = K_M$$

The ST

$$\bar{M}_{g,n}(\mathbb{P}^d, d)$$

$$\rho: \bar{M}_{g,n}(\mathbb{P}^d, d) \rightarrow \bar{M}_{g,n}$$

$$M = \rho^{-1}((\varepsilon, f, \dots, p_n))$$

$$C \xrightarrow{\delta} \mathbb{P}^3$$

$$\pi \downarrow$$

$$M$$

$$W = \pi_* \Phi^* V$$

$$W|_f = H^1(\Sigma, f^* V)$$

$$h^1(\Sigma, f^* V) = 4(d+1)$$

$$c_1(W) = \pi_* (ch(\Phi^* V) \cdot Td(T_\Sigma)|_M) = c_1(T_M)$$

$$\Lambda^{top} W^v = K_M$$

BCFW transition



$$= \sum_{\text{AUS/leg. Klobzkel}} \sum_{\text{PINS/NOZ}}$$

AUS/leg. Klobzkel
PINS/NOZ



$$\times \frac{1}{P_{ij}^2}$$



BCFW transition

$$\begin{aligned}
 & \text{Diagram: A circle with } d \text{ external lines} \\
 & = \sum_{A \cup B = \{1, \dots, d\}} \sum_{\substack{k_1, k_2 \in A \\ n_1, n_2 \in B}} \text{Diagram: Two circles connected by a line} \\
 & \times \frac{1}{P_A}
 \end{aligned}$$

$k = \text{del}$

$$\begin{aligned}
 & \text{Diagram: A wavy line} \\
 & = \sum_{A \cup B = \{1, \dots, d\}} \sum_{\substack{d_1, d_2 = d}} \text{Diagram: A wavy line with a vertex labeled A and B}
 \end{aligned}$$

Claim (Witten points)

$$A(g, n) = \int_{\mathcal{M}_{g,n}} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i \partial u_i)}{(u_i - u_{i+1})} \wedge \text{tr}(\omega^k \bar{\Phi}_1 \dots \wedge \omega^p \bar{\Phi}_n)$$

$\mathcal{M}_{g,n}(P^1, A)$



Class (Hodge) (integrals)

$$A(g_1, \dots, g_n) = \int_{\mathcal{P} \cdot \mathcal{P}'_{g_1}(\mathbb{P}^1, d)} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i \bar{v}_i)}{(u_i - v_i)} \wedge \text{tr}(\omega_1^* \Phi_1 \wedge \dots \wedge \omega_n^* \Phi_n)$$

$u \in \mathbb{P}^1$

Claim (with points)

$$A(l, \dots, n) = \int_{\Gamma \backslash \mathbb{H}_n} \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H}_n)} \prod_{i=1}^n \frac{(u_i \phi_i)}{(u_i - u_{i+1})} \wedge \text{tr}(\alpha_1^* \Phi_1 \wedge \dots \wedge \alpha_n^* \Phi_n)$$

$u \in \mathbb{P}^1$

- No general proof, - but holds in many cases.
- Better formulation?

Claim (Witten's conjecture)

$$A(g, n) = \int_{\mathcal{M}_{g,n}} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, \bar{u}_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_1^* \bar{\Phi}_1 \alpha_2 \dots \wedge \alpha_n^* \bar{\Phi}_n)$$

$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(\mathbb{P}^1, d)$

$u^{\pm} = \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \in \mathbb{P}^1$

$(u_i, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$

- No general proof, but holds in many cases.
- Better formulation?

Claim (Witten's conjecture)

$$A(g, n) = \int_{\mathcal{M}_{g,n}} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, \bar{u}_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\omega_1^* \bar{\Phi}_1 \dots \wedge \omega_n^* \bar{\Phi}_n)$$

$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(P^1, d)$ $u^i = \begin{pmatrix} u_i \\ \bar{u}_i \end{pmatrix} \in \mathbb{P}^1$ $(u_i, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$

- No general proof, but holds in many cases.
- Better formulation?

$$G_f(K, \alpha)$$

$$S = \int_{\Sigma} (Y, \bar{\partial} z)$$



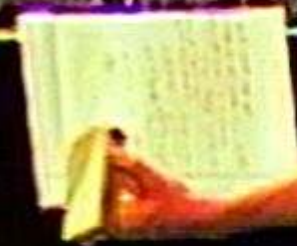
$$z: \Sigma \rightarrow \mathbb{P}^1$$

$$z^x \in \mathbb{C}^k \subset H^0(\Sigma, L)$$

pick a frame $(\{s_i\})$

$$\bigoplus_{i=1}^k (z_j^i, \{s_i\}) \in \mathbb{C}^n$$

$$H^0(\Sigma, L) \simeq \mathbb{C}^k$$



$G_1(K, \mathbb{C})$

$$S = \int_{\Sigma} (Y, \bar{\partial} z)$$



$$z: \Sigma \rightarrow \mathbb{P}^1$$

$$z^* \in \mathcal{O}^* \subset H^0(\Sigma, \mathcal{L})$$

pick a frame $(\{s_i\})$

$$\bigoplus_{i=1}^k (z|_{\sigma_i} \cdot s_i) \in \mathbb{C}^k$$

$$H^0(\Sigma, \mathcal{L}) \simeq \mathbb{C}^k$$

For each $(\Sigma, p_1, \dots, p_r) \in \tilde{M}_{g,n}$ (together with frame)
get $C \in G_1(K, \mathbb{C})$

Chern-Simons (fermions)

$$A(g, \mathbb{R}, n) = \int_{\mathcal{P}\text{-}\bar{M}_{g,n}(\mathbb{R}, n)} \frac{1}{|\det SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\omega_1^T \Phi_1 \wedge \dots \wedge \omega_n^T \Phi_n)$$

$\mathcal{P}(\text{framings} \rightarrow \bar{M}_{g,n})$

$$u^{\lambda} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) = \epsilon_{AB} u_i^{\lambda} u_{i+1}^{\bar{\lambda}}$$

Chern-Simons (fermions)

$$A(u_1, \dots, u_n) = \int_{\Gamma \cdot \bar{M}_{0,n}(\mathbb{P}^1, d)} \frac{1}{|\mathrm{Aut}(\mathbb{P}^1, d)|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i, u_{i+1})} \wedge \mathrm{tr}(\omega_1^d \bar{\Phi}_1 \wedge \dots \wedge \omega_n^d \bar{\Phi}_n)$$

$P(\text{framings} \rightarrow \bar{M}_{0,n})$

$$u^a = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$$

$$n-3 + n-1 = 2n-4$$

Chern-Simons (knots)

$$A(u, n) = \int_{\Gamma \cdot \bar{M}_{0,n}(\mathbb{R}, \epsilon)} \frac{1}{|\mathrm{SL}(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i - u_{i+1})} \wedge \mathrm{tr}(\omega_0^2 \Phi_0 \wedge \dots \wedge \omega_n^2 \Phi_n)$$

$P(\text{framings} \rightarrow \bar{M}_{0,n})$

$$u^{\lambda} = \begin{pmatrix} u_{\lambda} \\ u'_{\lambda} \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) \equiv \epsilon_{\alpha\beta} u_i^{\alpha} u_{i+1}^{\beta}$$

$$n-3 + n-1 = 2n-4$$