

Title: Twistor-String Theory

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Abstract: I'll give an introduction to twistor-string theory, which is an attempt to reformulate supersymmetric gauge theory in four-dimensional space-time in terms of a certain generalisation of Gromov-Witten theory in twistor space. The resulting theory is closely related to the multi-dimensional residue calculus in $G(k,n)$ (introduced in Cachazo's talk).

Twistor String Theory is appetizing

Twistor String Theory an appetizer



Twistor String Theory an appetizer



M symplectic mfd.

$$S: M \rightarrow M$$

$$\bar{\Gamma}[\bar{\Phi}]$$

$$\bar{\Phi}|_A = \phi$$

$$\bar{\Phi}|_B = \phi_T$$

Twistor String Theory an appetizer



M symplectic mfld.

$$S: M \rightarrow M$$

$$\mathbb{I}[\Phi]$$

$$\Phi|_A = \phi$$

$$\Phi|_B = \phi_T$$

Twistor String Theory an appetizer



M symplectic manifold

$$S: M \rightarrow M$$

$$\mathbb{I}[\bar{\Phi}]$$

$$\bar{\Phi}|_A = \phi$$

$$\bar{\Phi}|_B = \phi_T$$

too hard \rightarrow work perturbatively



$H \cdot \oplus \text{Sym.}(F)$

S



$$H = \bigoplus \text{Sym}_n(F)$$

$$S_{n,p} \text{Sym}_n(F) \otimes (\text{Sym}_p(F))^* \rightarrow \mathbb{C}$$

- Combinatorically very involved.



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$$S_{n,p} \text{Sym}_n(F) \otimes (\text{Sym}_p(F))^* \rightarrow \mathbb{C}$$

- Combinatorially very involved.
- F.D. obscuring the str. of theory (e.g. gauge invariance off-shell)

ST.

Replace $\sum F.D.$ \Rightarrow



$$\int_{\mathcal{M}_{2,n}}$$

Normal ST. gives us constructions for Ω

ST.

Replace Σ F.D. \Rightarrow



$$\int_{M_{2,n}}$$

Normal ST. gives us constructions for Ω

- dimensional reduction.

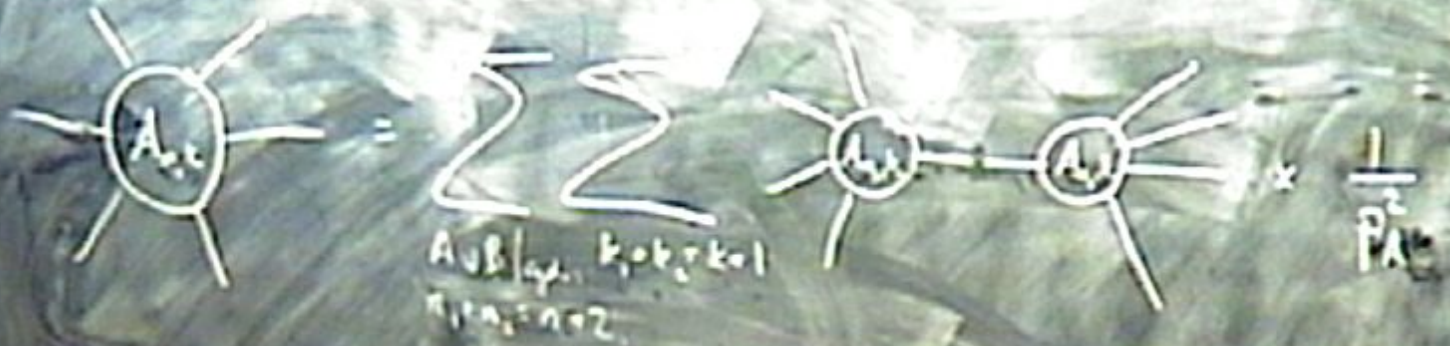
- $\alpha' \rightarrow 0$

BCFW recursion

$$A_{n,t} = \sum_{A \cup B \text{ sep. } k, k' \text{ col}} \sum A_{n_A, t_A} A_{n_B, t_B}$$



BCFW recursion

$$A_{n,t} = \sum_{\text{Aubsp. } k, \text{ kob } k_1} \sum_{\text{Aubsp. } k_2, \text{ kob } k_2} A_{n_1,t_1} A_{n_2,t_2} \times \frac{1}{P_{k_1 k_2}}$$


BCFW recursion

$$A_{n+2} = \sum_{A \cup B \text{ plan. } k, k' \text{ col.}} \sum_{n_1 + n_2 = n} A_{n_1} A_{n_2} \times \frac{1}{P_{AB}}$$

stratified by "dual graph type"

each irreducible cusp $\Sigma_i \Leftrightarrow$ vertex v_i

If Σ_i, Σ_j share a node \Leftrightarrow draw line between v_i, v_j

- Marked point on $\Sigma_i \Leftrightarrow$ half-line ending in v_i

BCFW recursion

$$A_{n,t} = \sum_{\substack{A \cup B \text{ top } k, k \neq k_0 \\ \pi_1 \cap \pi_2 = \emptyset}} \sum_{\substack{A' \cup B' \text{ top } k', k' \neq k'_0 \\ \pi'_1 \cap \pi'_2 = \emptyset}} A_{n,t} \text{---} A_{n',t'} \times \frac{1}{P_{A \cup B}}$$

$\overline{M}_{0,n}$ stratified by "dual graph type"

- to each irreducible cusp $\Sigma_i \Leftrightarrow$ vertex v_i
- if Σ_i, Σ_j share a node \Leftrightarrow draw line between v_i, v_j
- marked point on $\Sigma_i \Leftrightarrow$ half-line ending in v_i



\mathbb{P}^3 is a vector space of \mathbb{R}^4

Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $\mathbb{R}^4 \iff$ a set of 4M of \mathbb{R}^4 on space time

\mathbb{P}^3 is twistor space of $\mathbb{R}^{3,1}$

holomorphic bundle $E \rightarrow \mathbb{P}^3 \iff$ self-dual solⁿ of YM eqⁿ on spacetime

$H^1(\mathbb{P}^3, \text{End } E) \iff$ -ve helicity gluons

$H^1(\mathbb{P}^3, \text{End } E \otimes \mathcal{O}(-4)) \iff$ +ve helicity gluons.

$\alpha \beta$

\mathbb{P}^3 is twistor space of $\mathbb{R}^{3,1}$

holomorphic bundle $E \rightarrow \mathbb{P}^3 \iff$ ASD solⁿ of YM eqⁿ on spacetime

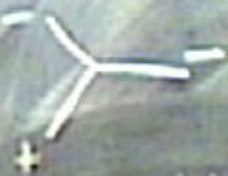
$H^1(\mathbb{P}^3, \text{End } E) \iff$ -ve helicity gluons

$H^1(\mathbb{P}^3, \text{End } E \otimes \mathcal{O}(4)) \iff$ +ve helicity gluons.

$$\int_{\mathbb{P}^3} (\alpha \cup \beta) = \langle + - \rangle$$

Scott's

$$H^0(\mathbb{P}^1, \mathcal{O}) \otimes H^0(\mathbb{P}^1, \mathcal{O}) \otimes H^0(\mathbb{P}^1, \mathcal{O}(2)) \xrightarrow{\cong} \mathbb{C}$$

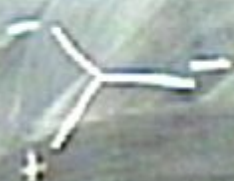


Obvious generalisation $V = \mathcal{O}(1)^{\oplus 2}$

$$H^0(\mathbb{P}^1, \Lambda^1 V^{\otimes 2}) \otimes H^0(\mathbb{P}^1, \Lambda^1 V^{\otimes 2}) \otimes H^0(\mathbb{P}^1, \Lambda^1 V^{\otimes 2}) \xrightarrow{\cong} \mathbb{C}$$

Scattering

$$H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, \mathcal{O}(4)) \xrightarrow{\cong} \mathbb{C}$$



Obvious generalisation

$$V = \mathcal{O}(1)^{\oplus 4}$$

$$\Lambda^{1,1} V \cong K_{\mathbb{P}^1}$$

$$H^1(\mathbb{P}^1, \Lambda^1 V) \oplus H^1(\mathbb{P}^1, \Lambda^1 V) \oplus H^1(\mathbb{P}^1, \Lambda^1 V) \xrightarrow{\cong} \mathbb{C}$$

$$\sum q_i = h^1 V$$



The ST

$$\bar{M}_n(\mathbb{P}^1, d) \quad \rho: \bar{M}_{n,n}(\mathbb{P}^1, d) \rightarrow \bar{M}_n$$

$$= \rho^{-1}((\varepsilon, p_1, \dots, p_n))$$



The ST

$$\bar{M}_{g,n}(\mathbb{P}^2, d)$$

$$\rho: \bar{M}_{g,n}(\mathbb{P}^2, d) \rightarrow \bar{M}_{g,n}$$

$$M = \rho^{-1}((\Sigma, p_1, \dots, p_n))$$

$$C \xrightarrow{\delta} \mathbb{P}^2$$

$$\begin{array}{c} \pi \\ \downarrow \\ M \end{array}$$

$$W = \pi_* \Phi^* V$$

$$W|_f = H^0(\Sigma, f^* V)$$

$$h^1(\Sigma, f^* V) = 4(d+1)$$

$$c_1(W) = \pi_* (h(\Phi^* V) - Td(T_\Sigma)|_M)$$

The ST

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$$\Lambda^{top} W^v = K_M$$

The ST

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$$c_1(W) = \pi_* (ch(\Phi^* V) \cdot Td(T_\Sigma)) = c_1(T_M)$$

$$\Lambda^{top} W^v = K_M$$

BCFW transition



$$= \sum_{\text{AUS/ops. } n_1, n_2} \sum_{\text{Kontur}}$$

AUS/ops. Kontur
 n_1, n_2



$$\times \frac{1}{P_A^2}$$



BCFW transition

$$\begin{aligned}
 & \text{Diagram 1: A circle with } d \text{ external lines, labeled } A_{\text{ext}} \\
 & = \sum_{A \cup B | \text{ ext. } k_1, k_2, k_3} \sum_{n_1, n_2} \text{Diagram 2: Two circles connected by a line, labeled } A_{\text{int}} \text{ and } A_{\text{ext}} \\
 & \times \frac{1}{P_A}
 \end{aligned}$$

$k = d-1$

$$\begin{aligned}
 & \text{Diagram 3: A wavy line with } d \text{ external lines} \\
 & = \sum_{A \cup B | \text{ ext.}} \sum_{d_1, d_2 = d} \text{Diagram 4: A diagram with two vertices labeled } A \text{ and } B \text{ connected by a line}
 \end{aligned}$$

Claim (Witten's formula)

$$A(L, \mathfrak{g}, \mathfrak{h}) = \int_{\Gamma \backslash \mathbb{H}_g} \frac{1}{\text{vol SL}(2, \mathbb{C})} \prod_{i=1}^g \frac{(u_i \partial u_i)}{(u_i - u_{i+1})} \wedge \text{tr}(\omega^{\mathfrak{g}} \bar{\Phi}_1 \Phi_1 \dots \wedge \omega^{\mathfrak{h}} \bar{\Phi}_g \Phi_g)$$



Class (Witten Perkowski)

$$A(l_1, \dots, l_n) = \int_{\Gamma \backslash \mathbb{H}^n} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, \partial u_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_i^* \Phi_{i+1} \wedge \dots \wedge \alpha_i^* \Phi_n)$$

$u \in \mathbb{P}^1$

Claim (Witten, Kontsevich)

$$A(l_1, \dots, l_n) = \int_{\Gamma \backslash \mathbb{H}^d} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, \partial u_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_1^{\epsilon_1} \Phi_1 \alpha_2^{\epsilon_2} \dots \alpha_n^{\epsilon_n} \Phi_n)$$

$$u \in \mathbb{P}^1$$

- No general proof, but holds in many cases.
- Better formulation?

Claim (Witten, Perlmutter)

$$A(g_1, \dots, g_n) = \int_{\Gamma \backslash \mathbb{H}^n} \frac{1}{|SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, \bar{u}_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_1^* \bar{\Phi}_1 \wedge \dots \wedge \alpha_n^* \bar{\Phi}_n)$$

$\Gamma \backslash \mathbb{H}^n(\mathbb{R}^1, d)$
 $u^A = \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \in \mathbb{P}^1$
 $(u, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$

- No general proof, but holds in many cases.
- Better formulation?

Claim (Witten Perkovits)

$$A(g, \dots, n) = \int_{\Gamma \cdot \mathcal{M}_{g,n}(\mathbb{R}^1, d)} \frac{1}{|\mathcal{M}_{g,n}(\mathbb{R}^1, d)|} \prod_{i=1}^n \frac{(u_i, \partial u_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_i^* \Phi_0 \wedge \dots \wedge \alpha_i^* \Phi_n)$$

$u_i^* = \begin{pmatrix} u_i \\ \bar{u}_i \end{pmatrix} \in \mathbb{P}^1 \quad (u_i, u_{i+1}) \equiv \epsilon_{AB} u_i^A u_{i+1}^B$

- No general proof, but holds in many cases.
- Better formulation?

$$G_f(K, \alpha)$$

$$S = \int_{\Sigma} (Y, \bar{\partial} z)$$



$$z: \Sigma \rightarrow \mathbb{P}^1$$

$$z^k \in \mathbb{C}^k \subset H^0(\Sigma, \mathcal{L})$$

pick a frame $(\{s_i\})$

$$\bigoplus_{i=1}^k (z_i | \{s_i\}) \in \mathbb{C}^k$$

$$H^0(\Sigma, \mathcal{L}) \simeq \mathbb{C}^k$$

$G(k, n)$

$$S = \int_{\Sigma} (Y, \bar{\partial} z)$$



$$z: \Sigma \rightarrow \mathbb{P}^1$$

$$z^x \in \mathcal{O}^x \subset H^0(\Sigma, \mathcal{L})$$

pick a frame $(\{s_i\})$

$$\bigoplus_{i=1}^n (\mathbb{C} |_{p_i} \cdot s_i) \subset \mathbb{C}^n$$

$$H^0(\Sigma, \mathcal{L}) \simeq \mathbb{C}^k$$

For each $(\Sigma, p_1, \dots, p_n) \in \tilde{M}_{g,n}$ (together with frame)
get $C \in G(k, n)$

Chern-Simons invariants

$$A(g, \dots, n) = \int_{\mathbb{R} \cdot \overline{M}_{0,n}(\mathbb{R}, \epsilon)} \frac{1}{|\det SL(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i, u_{i+1})} \wedge \text{tr}(\alpha_1^* \Phi_1 \wedge \dots \wedge \alpha_n^* \Phi_n)$$

$$P(\text{framings} \rightarrow \overline{M}_{0,n})$$

$$u^{\lambda} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$$

Chern-Simons (fermions)

$$A(u_1, \dots, u_n) = \int_{\mathcal{P} \cdot \bar{M}_{0,n}(\mathbb{P}^1, d)} \frac{1}{|\mathrm{SL}(2, \mathbb{C})|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i, u_{i+1})} \wedge \mathrm{tr}(\omega_1^a \Phi_1 \wedge \dots \wedge \omega_n^a \Phi_n)$$

$\mathcal{P}(\text{framings} \rightarrow \bar{M}_{0,n})$

$$u^a = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) = \epsilon_{AB} u_i^A u_{i+1}^B$$

$$n-3 + n-1 = 2n-4$$

Chern-Simons (fermions)

$$A(u_1, \dots, u_n) = \int_{\Gamma \cdot \bar{M}_{0,n}(\mathbb{P}^1, d)} \frac{1}{|\mathrm{Aut}(\mathbb{P}^1, \sigma)|} \prod_{i=1}^n \frac{(u_i, du_i)}{(u_i - u_{i+1})} \wedge \mathrm{tr}(\omega_1^a \Phi_1 \wedge \dots \wedge \omega_n^a \Phi_n)$$

$P(\text{framings} \rightarrow \bar{M}_{0,n})$

$$u^a = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{P}^1$$

$$(u_i, u_{i+1}) \equiv \epsilon_{ab} u_i^a u_{i+1}^b$$

$$n-3 + n-1 = 2n-4$$