Title: What Do Grassmannians And Particle Colliders Have In Common?

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Abstract: In the past year, motivated by physics, a rich structure has emerged from studying certain contour integrals in Grassmannians. Physical considerations single out a natural meromorphic form in G(k,n) with a cyclic structure. The residues obtained from these contour integrals have been shown to be invariants of a Yangian algebra. These residues also control what happens deep inside collisions of protons taking place at colliders like the Large Hadron Collider or LHC at CERN. Applications of the Global Residue Theorem give rise to relations among residues which ensure important physical properties.

Perimeter Institute

Connections in Geometry and Physics 2010

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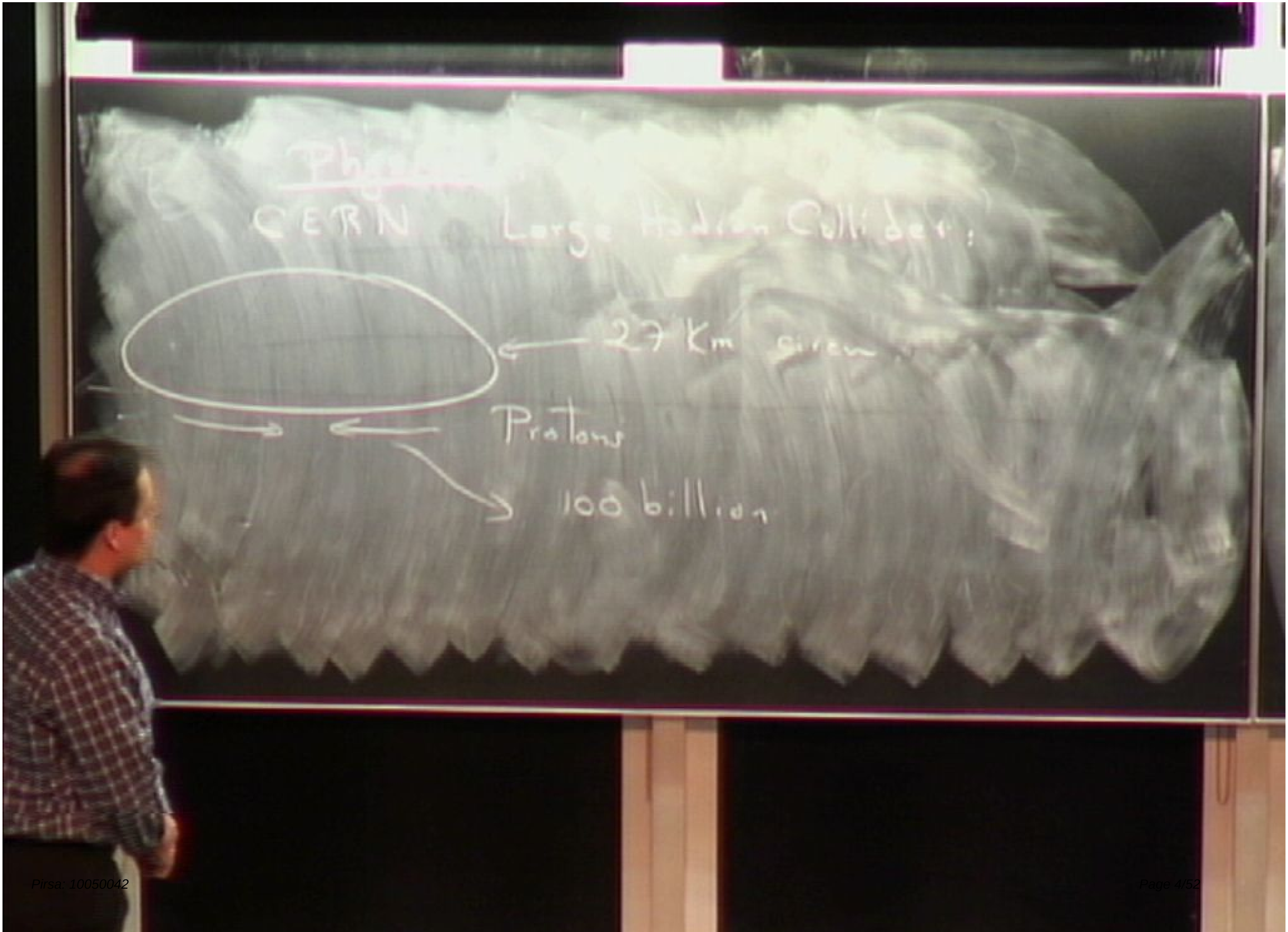
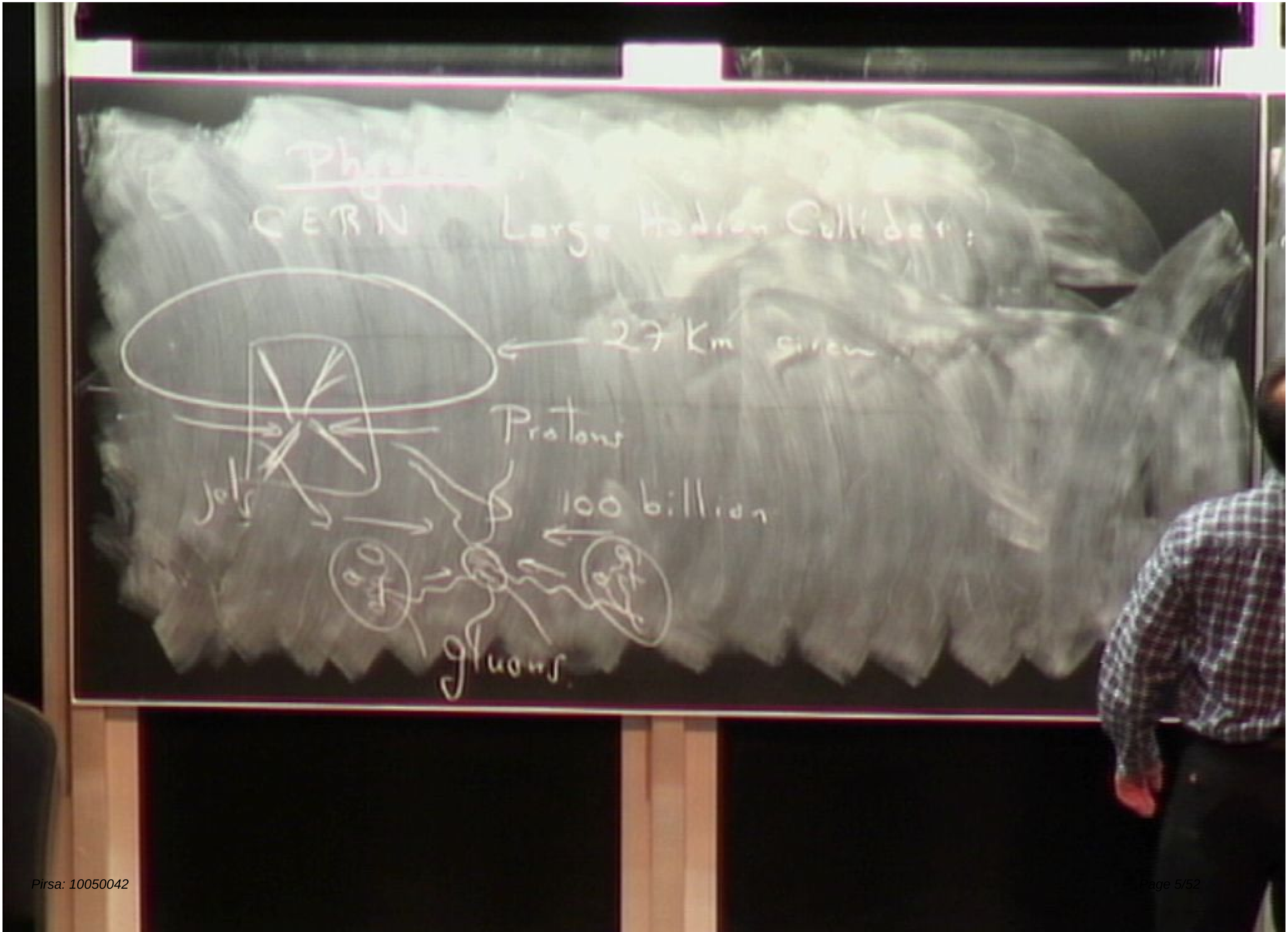
Based on N. Arkani-Hamed, F.C., J. Kaplan, C. Cheung,

N. Arkani-Hamed, F.C. Cheung,

N. Arkani-Hamed, J. Bourjaily, F.C. and J. Trnka.


CERN
Large Hadron Collider:
27 Km
Protons
100 billion


CERN
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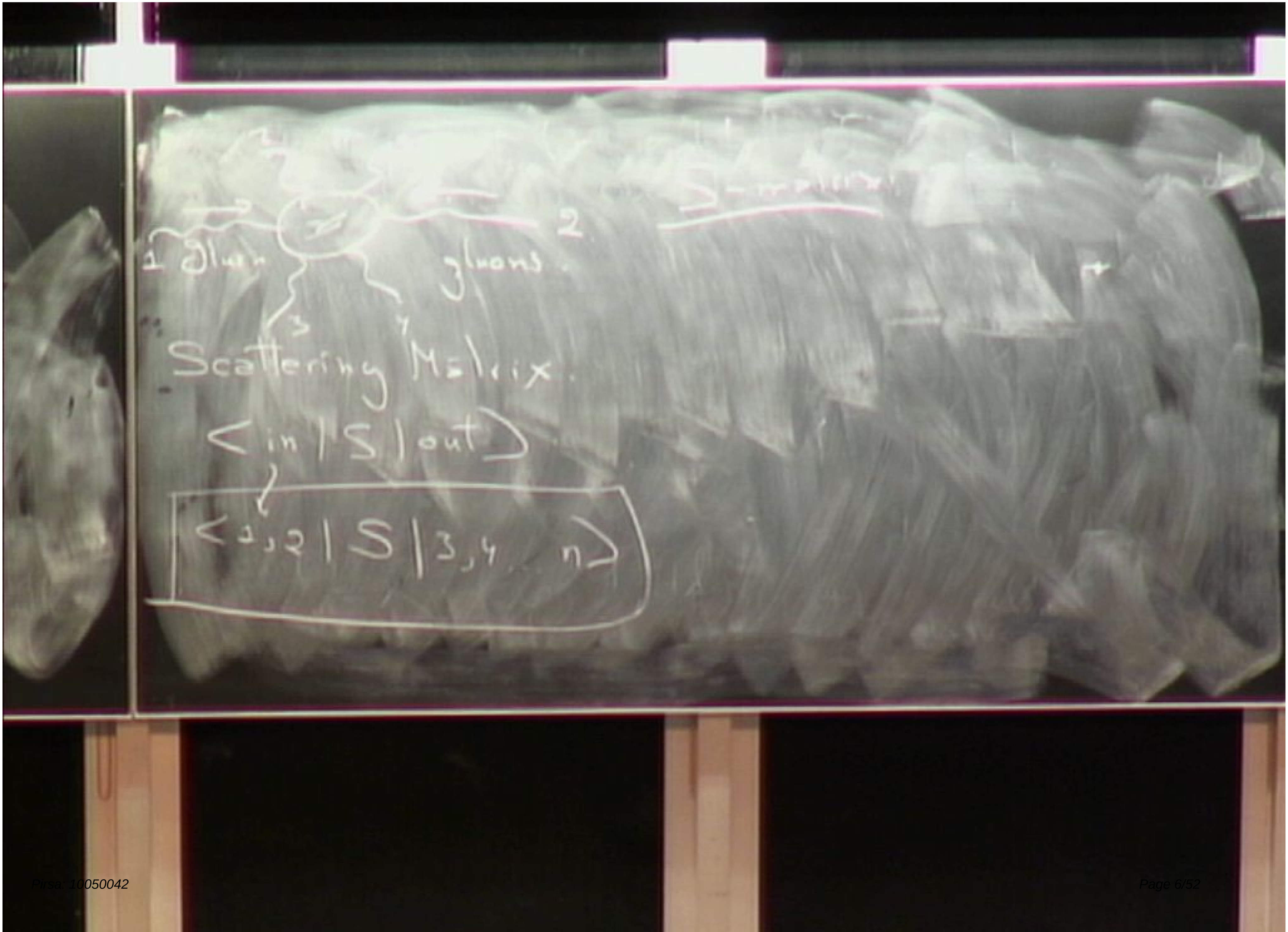
2
1 gluon
gluons
3
4
Scattering Matrix.
<in | S | out>
<1,2 | S | 3,4, n>

Space-Time:

Affine Space $\mathbb{R}^4$. Coordinates x^μ. Quadratic form: $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$.

Lorentz group: $SO(3, 1)$

Poincare Group: $A \times' H$ with $A = \mathbb{R}^4$ and $H = SL(2, \mathbb{C})$.

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Particles:

Particles are described and classified as irreducible representations of the Poincare group (Wigner 1939).

$\hat{A}$: Group of characters of A: $\mathbb{R}^4$ with coordinates p_μ.

$$p = (p_\mu) \to \chi_p \qquad \chi_p(x) = e^{i\langle x,p\rangle}$$

This is what we call momentum space.

Orbits of $\hat{A}$ under H:

$$X_m = \{p^2 = m^2\} \quad m > 0$$

m is called the **mass** of the particle and the stabilizer of a point $p \in X_m$ is called the little group G.

Irreps of $A \times' H$ are classified by the irreps of the little group G.

For $m > 0$: $G = SU(2)$ and irreps are $j \in \frac{1}{2}\mathbb{Z}^+$ called **spin**.

For $m = 0$: G is the group of motions in $\mathbb{R}^2$ and irreps are $h \in \frac{1}{2}\mathbb{Z}$ called **helicity**.

In our case:

Gluons: Massless particles of helicity $|h| = 1$.

How do we construct objects with the correct properties?

Building blocks: For each particle (i)

$$\begin{pmatrix} \lambda_1^{(i)} \\ \lambda_2^{(i)} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\lambda}_1^{(i)} \\ \tilde{\lambda}_2^{(i)} \end{pmatrix}.$$

How do we construct objects with the correct properties?

Building blocks: For each particle (i)

$$\begin{pmatrix} \lambda_1^{(i)} \\ \lambda_2^{(i)} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\lambda}_1^{(i)} \\ \tilde{\lambda}_2^{(i)} \end{pmatrix}.$$

Properties: $F(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}\})$

- It only depends on $\epsilon^{ab}\lambda_a^{(i)}\lambda_b^{(j)} \equiv \langle i, j\rangle$ and $\epsilon^{ab}\tilde{\lambda}_a^{(i)}\tilde{\lambda}_b^{(j)} \equiv [i, j]$
- It is homogeneous

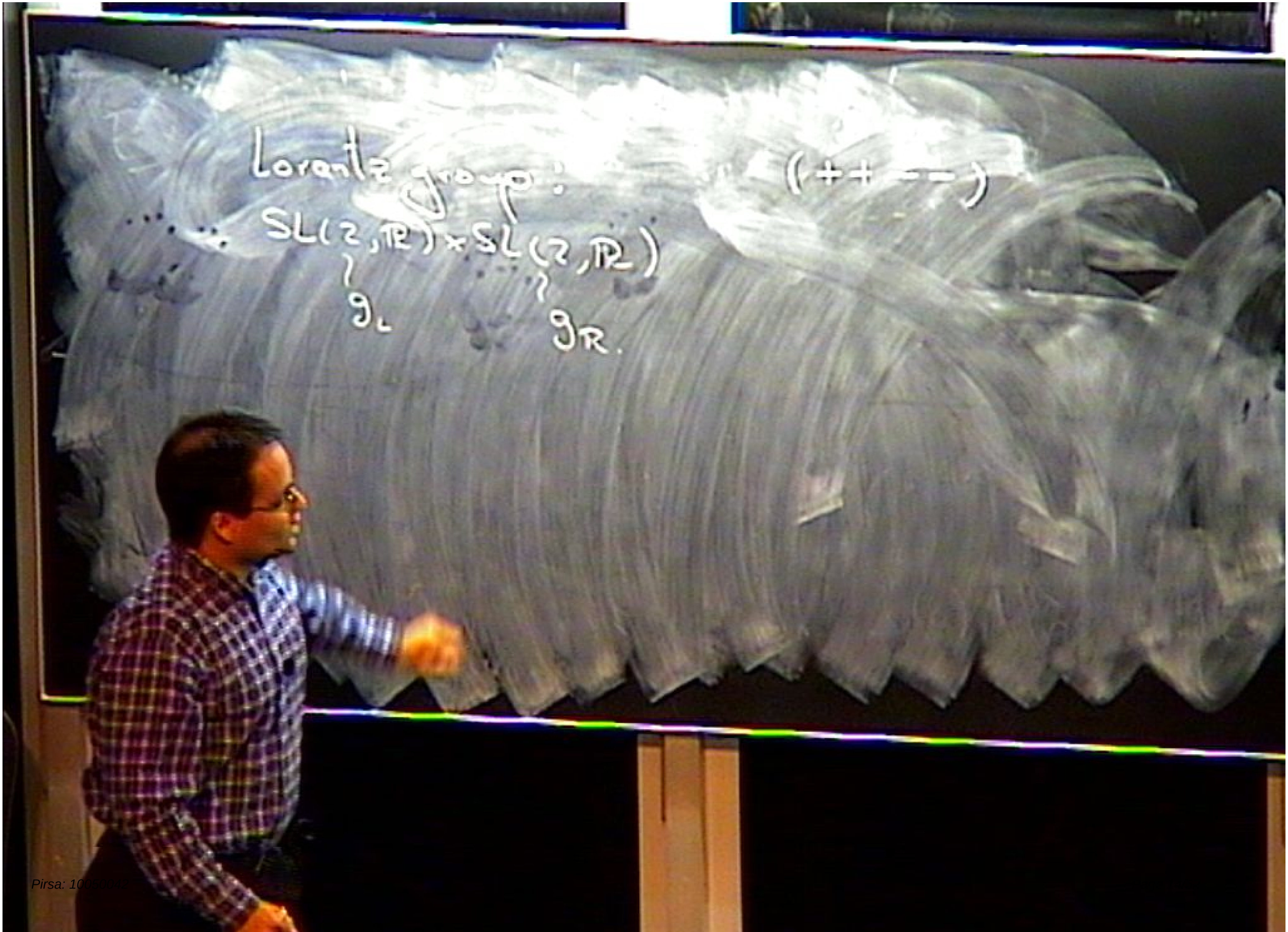
$$\left(\lambda_a^{(i)}\frac{\partial}{\partial\lambda_a^{(i)}} - \tilde{\lambda}_a^{(i)}\frac{\partial}{\partial\tilde{\lambda}_a^{(i)}}\right) F = -2h_i F$$

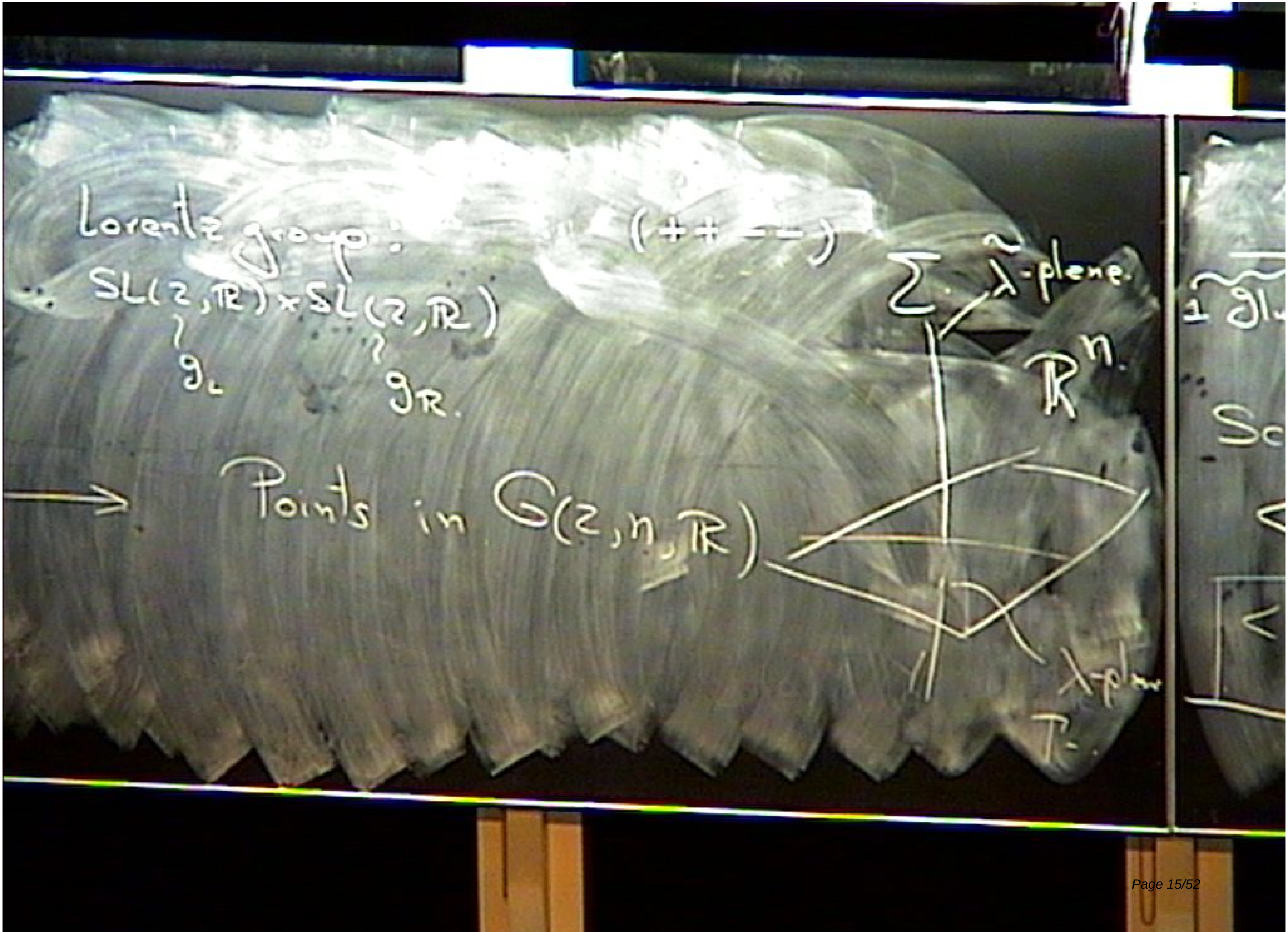
Hints of a Grassmannian

Physical Scattering Data:

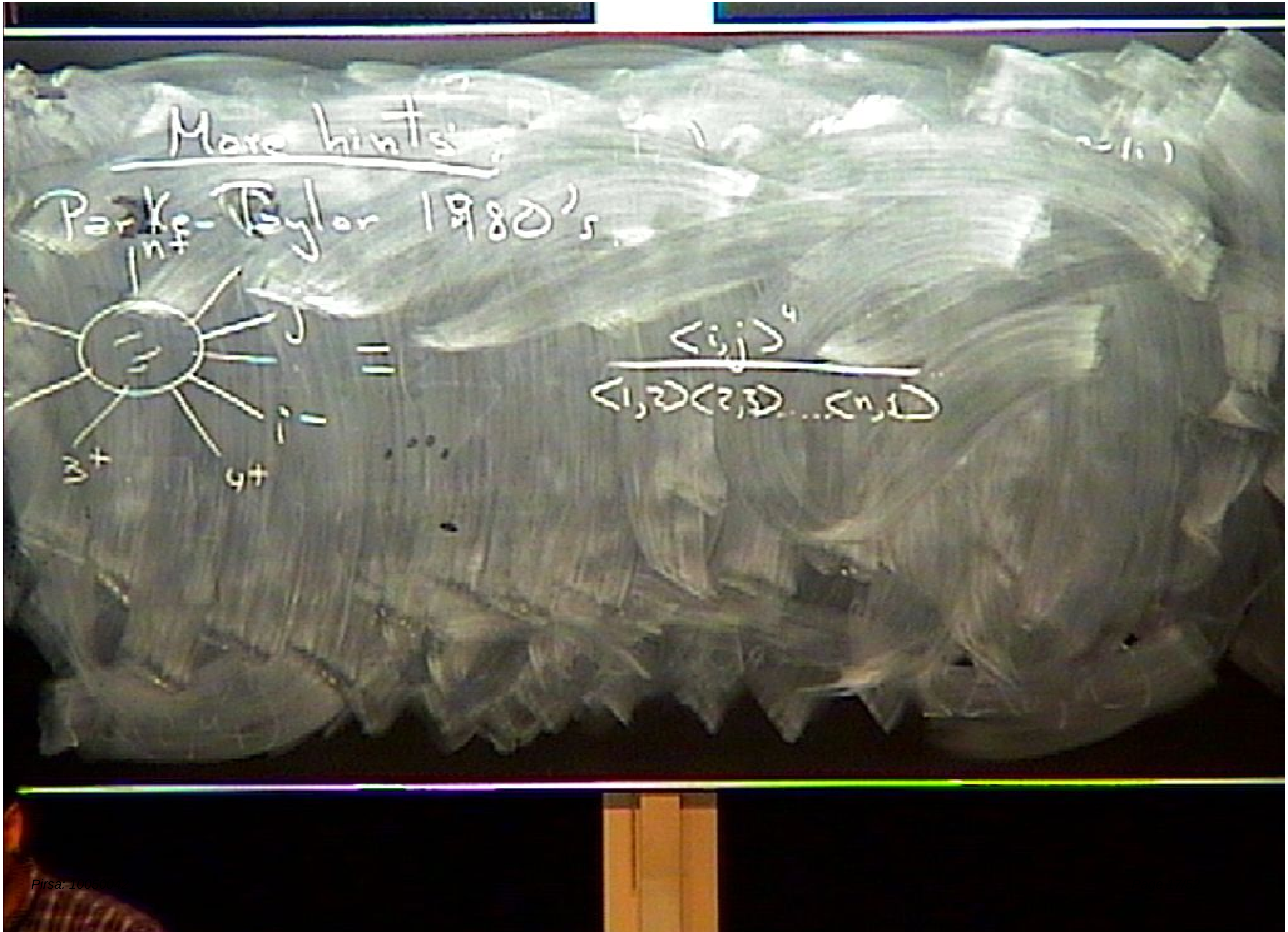
$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \lambda_1^{(3)} & \dots & \lambda_1^{(k)} & \lambda_1^{(k+1)} & \dots & \lambda_1^{(n-1)} & \lambda_1^{(n)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \lambda_2^{(3)} & \dots & \lambda_2^{(k)} & \lambda_2^{(k+1)} & \dots & \lambda_2^{(n-1)} & \lambda_2^{(n)} \end{pmatrix}$$

$$\begin{pmatrix} \tilde\lambda_1^{(1)} & \tilde\lambda_1^{(2)} & \tilde\lambda_1^{(3)} & \dots & \tilde\lambda_1^{(k)} & \tilde\lambda_1^{(k+1)} & \dots & \tilde\lambda_1^{(n-1)} & \tilde\lambda_1^{(n)} \\ \tilde\lambda_2^{(1)} & \tilde\lambda_2^{(2)} & \tilde\lambda_2^{(3)} & \dots & \tilde\lambda_2^{(k)} & \tilde\lambda_2^{(k+1)} & \dots & \tilde\lambda_2^{(n-1)} & \tilde\lambda_2^{(n)} \end{pmatrix}$$


Lorentz group: (++ − −)
SL(2,ℝ) × SL(2,ℝ)
g_L
g_R


Lorentz group: (+ + − −)
SL(2, ℝ) × SL(2, ℝ)
g_L g_R
→ Points in G(2, n, ℝ)
Σ. λ̃-plane.
ℝ^n.
λ-plane


More hints:
Parke-Taylor 1980's
i−
3+
4+


More hints
Parke-Taylor 1980's
1+
j
i-
3+
4+
= <i,j>^4 / <1,2><2,3>...<n,1>

More hints:

Parke-Taylor 1980's

gluons come in "diff colors"

$$= \int \frac{\langle i,j\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle\cdots\langle n,1\rangle}\,\delta^4(p_1+\ldots+p_n)$$

More hints:

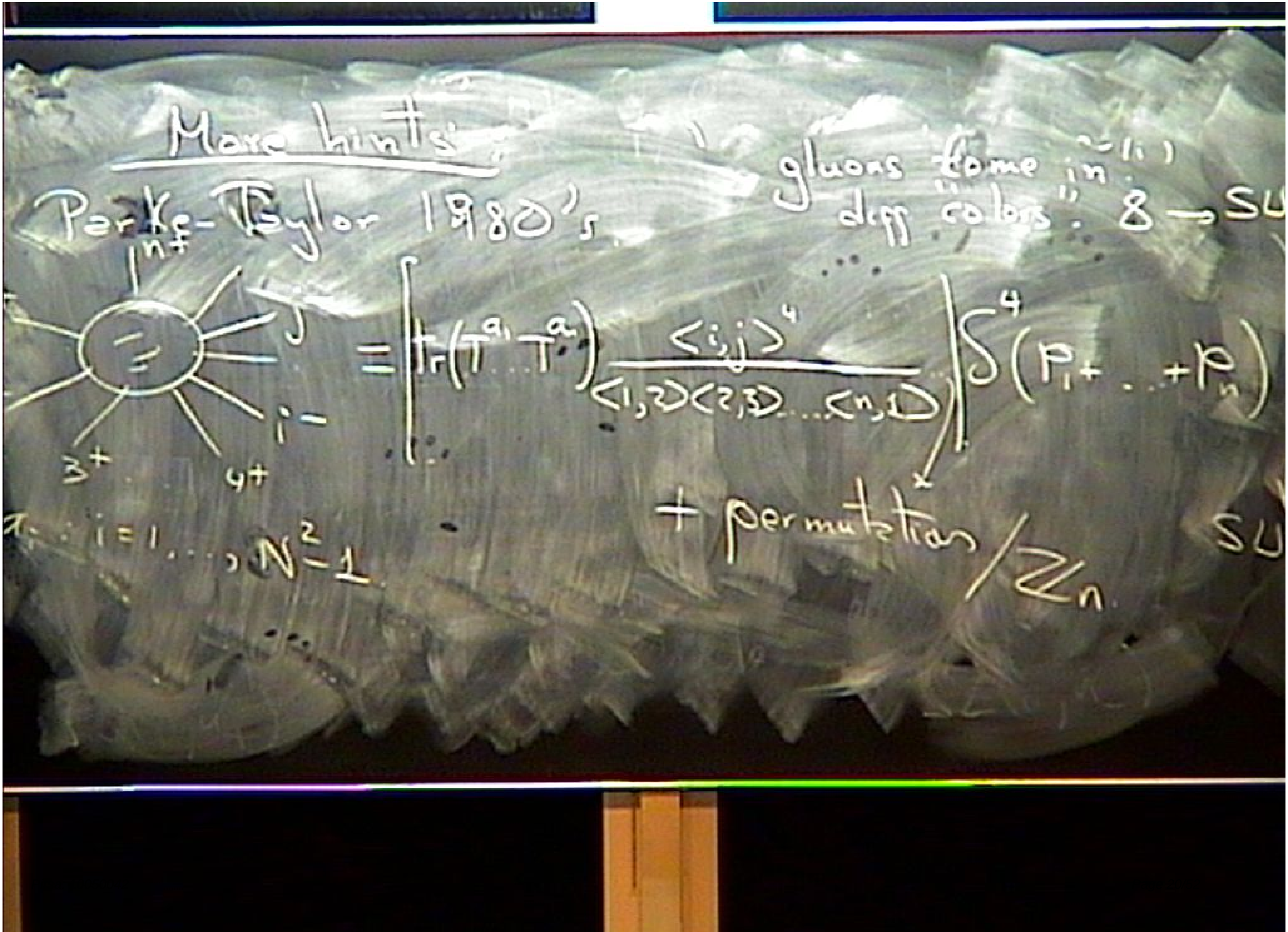
Parke-Taylor 1980's

gluons come in diff "colors": 8 → SU

$$= \int \mathrm{tr}\left(\cdots \frac{\langle i,j\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle \cdots \langle n,1\rangle} \right) \delta^4(p_1 + \ldots + p_n)$$

T_i : $i = 1, \ldots, N^2 - 1$

SU

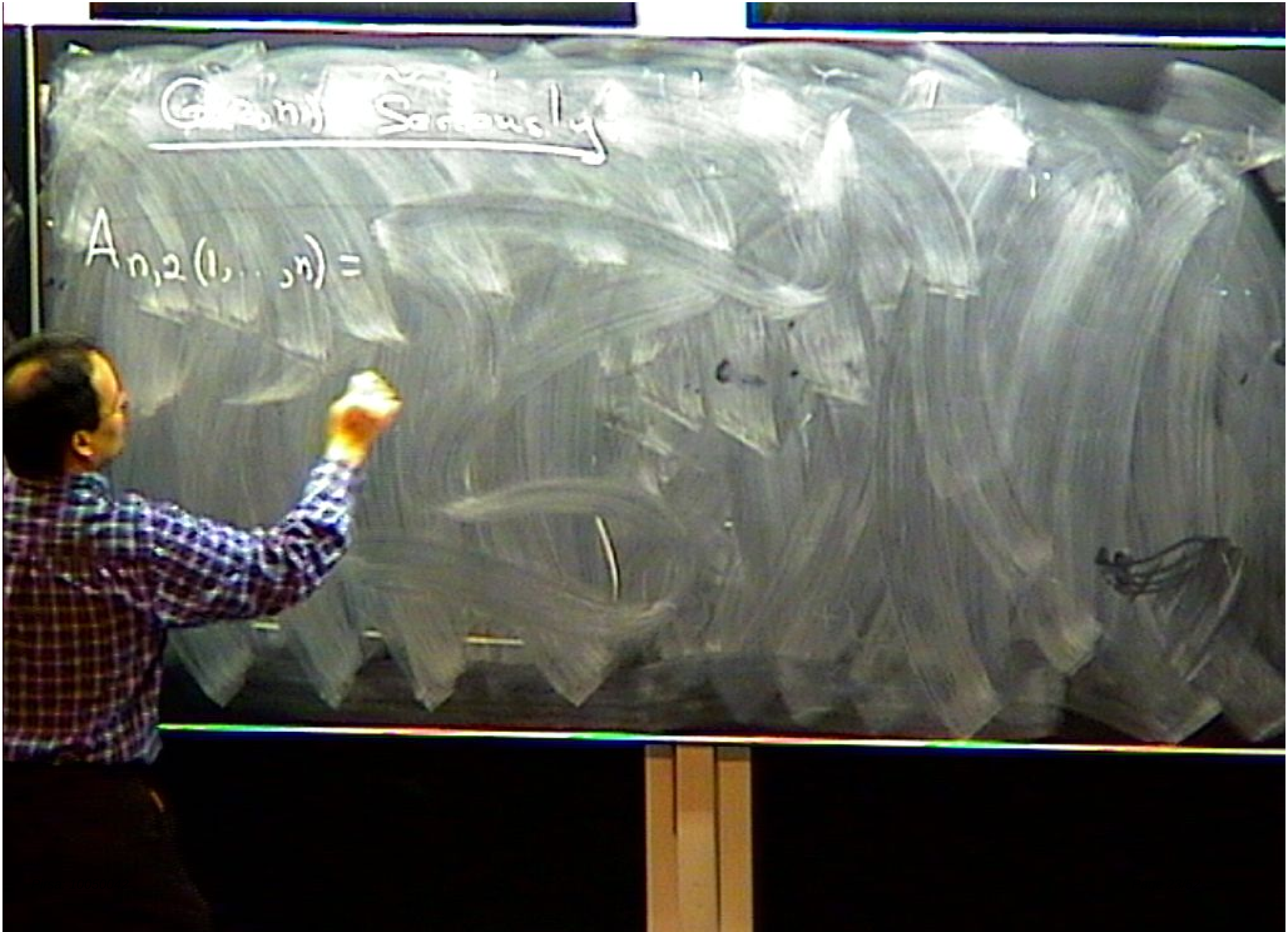

More hints:
Parke-Taylor 1980's
gluons come in
diff "colors": 8 → SU
j
i−
3+
4+
= ∫ tr(T^a1 ... T^an) <i,j>^4 / <1,2><2,3>...<n,1> δ^4(p1+ ... +pn)
+ permutation / Zn
i = 1, ..., N^2 − 1
SU

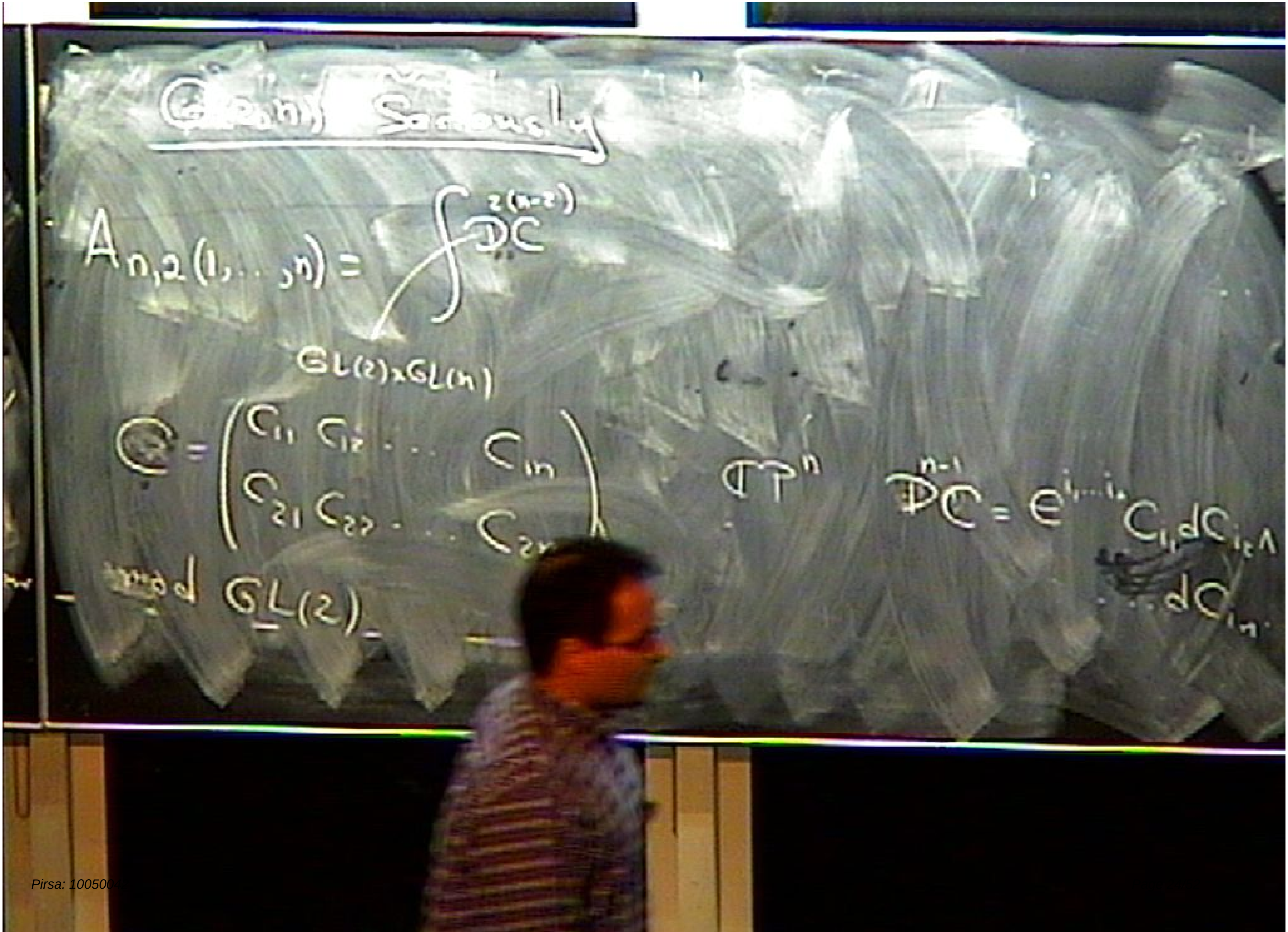
$$\Rightarrow \sum_{m=1}^{n} \lambda_a^{(m)} \tilde{\lambda}_{\dot{a}}^{(m)} = 0, \quad a = 1,2, \quad \dot{a} = 1,2.$$

a: 10050042

$$\Rightarrow \sum_{n=1}^{n} \lambda_{\alpha}^{(n)} \tilde{\lambda}_{\dot{\alpha}}^{(n)} = 0 \qquad \alpha = 1,2 \quad \dot{\alpha} = 1,2$$

$$\Gamma \perp \Sigma$$


$A_{n,2}(1,\ldots,n) =$



A_{n,2}(1,...,n) =
GL(2)xGL(n)
mod GL(2)
CP^n

G(2,n) Seriously

$$A_{n,2}(1,\ldots,n) = \int_{GL(2)\times GL(n)} \mathcal{D}C^{2(n-2)}$$

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \end{pmatrix} \bmod GL(2)$$

$\mathbb{CP}^n$ $\quad \mathcal{D}C^{n-1} = \epsilon^{i_1 \ldots i_n} C_{i_1} dC_{i_2} \wedge \ldots dC_{i_n}$

[illegible] Seriously

$$A_{n,2}(1,\ldots,n) = t^{2n}\int \frac{D^{2(n-2)}C}{(12)\cdots(n1)} \prod_{\alpha=1}^{2}\delta^{2}\Big(\sum C_{\alpha m}\tilde{\lambda}^{(m)}\Big)\int d^{2}\rho_{\alpha}\prod_{m=1}^{n}\delta^{2}(\rho_{\alpha}C_{\alpha m}=\lambda$$

$GL(2)\times GL(n)$

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \end{pmatrix} \quad \mathrm{mod}\ GL(2)$$

$\mathbb{CP}^{n}$

$$D^{n-1}C = \epsilon^{i_1\ldots i_n} C_{i_1} dC_{i_2}\wedge\cdots dC_{i_n}$$

$$C_{ij}\to tC_{ij}$$

Proposal: The Grassmannian Formula

The Grassmannian $G(k, n)$: Space of k-planes containing the origin in $\mathbb{C}^n$.

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1k} & c_{1k+1} & \cdots & c_{1n-1} & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2k} & c_{2k+1} & \cdots & c_{2n-1} & c_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ c_{k1} & c_{k2} & c_{k3} & \cdots & c_{kk} & c_{kk+1} & \cdots & c_{kn-1} & c_{kn} \end{pmatrix}$$

Modulo the action of $GL(k)$ on the right.

Proposal: The Grassmannian Formula

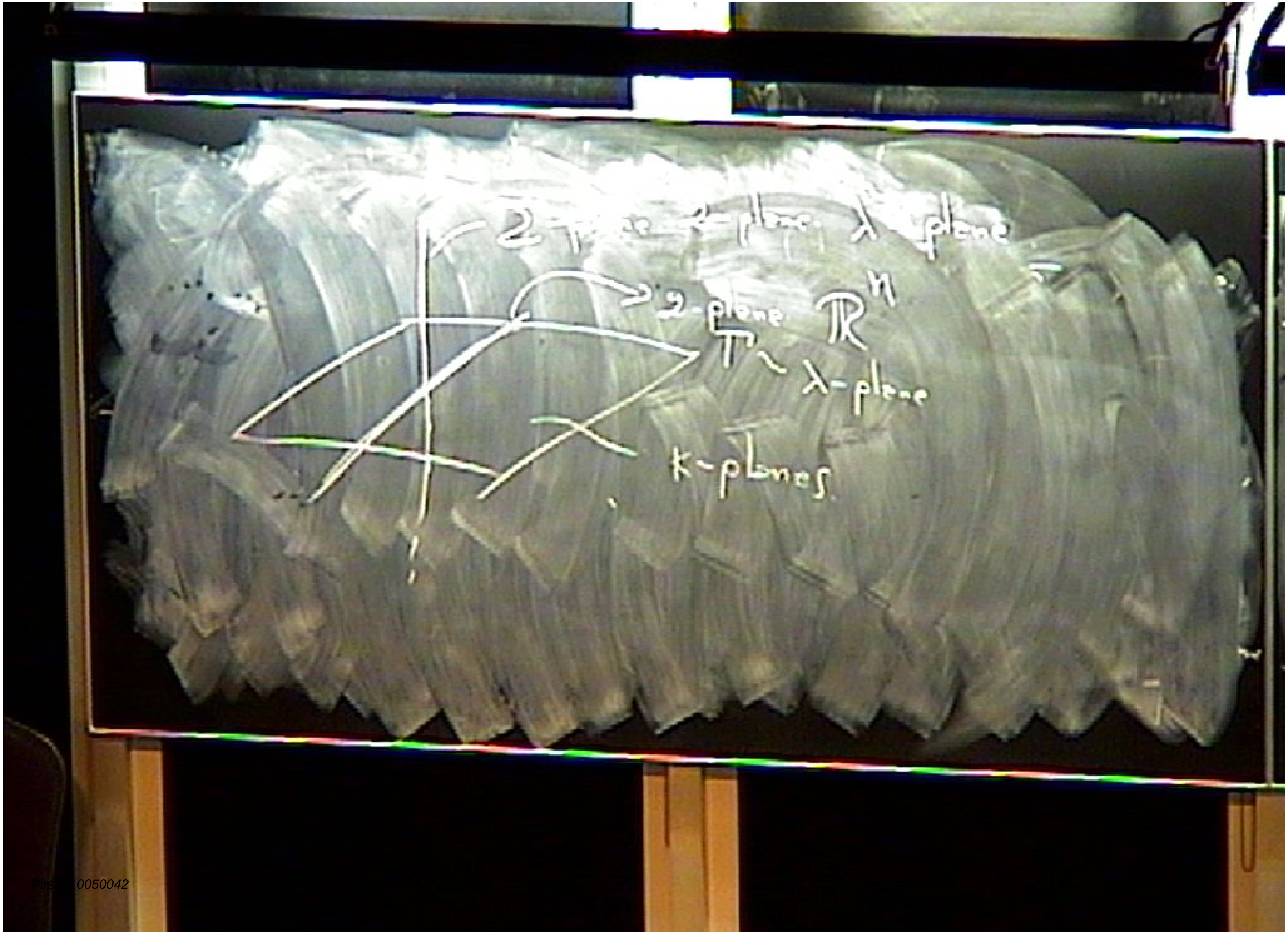
$$\mathcal{L}_{n,k} = \int \frac{D^{k(n-k)}c\ (i_1 i_2 \ldots i_k)^4\ \delta(C\cdot\Sigma)\delta(C^{\perp}\cdot\Gamma)}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots k-1)}$$

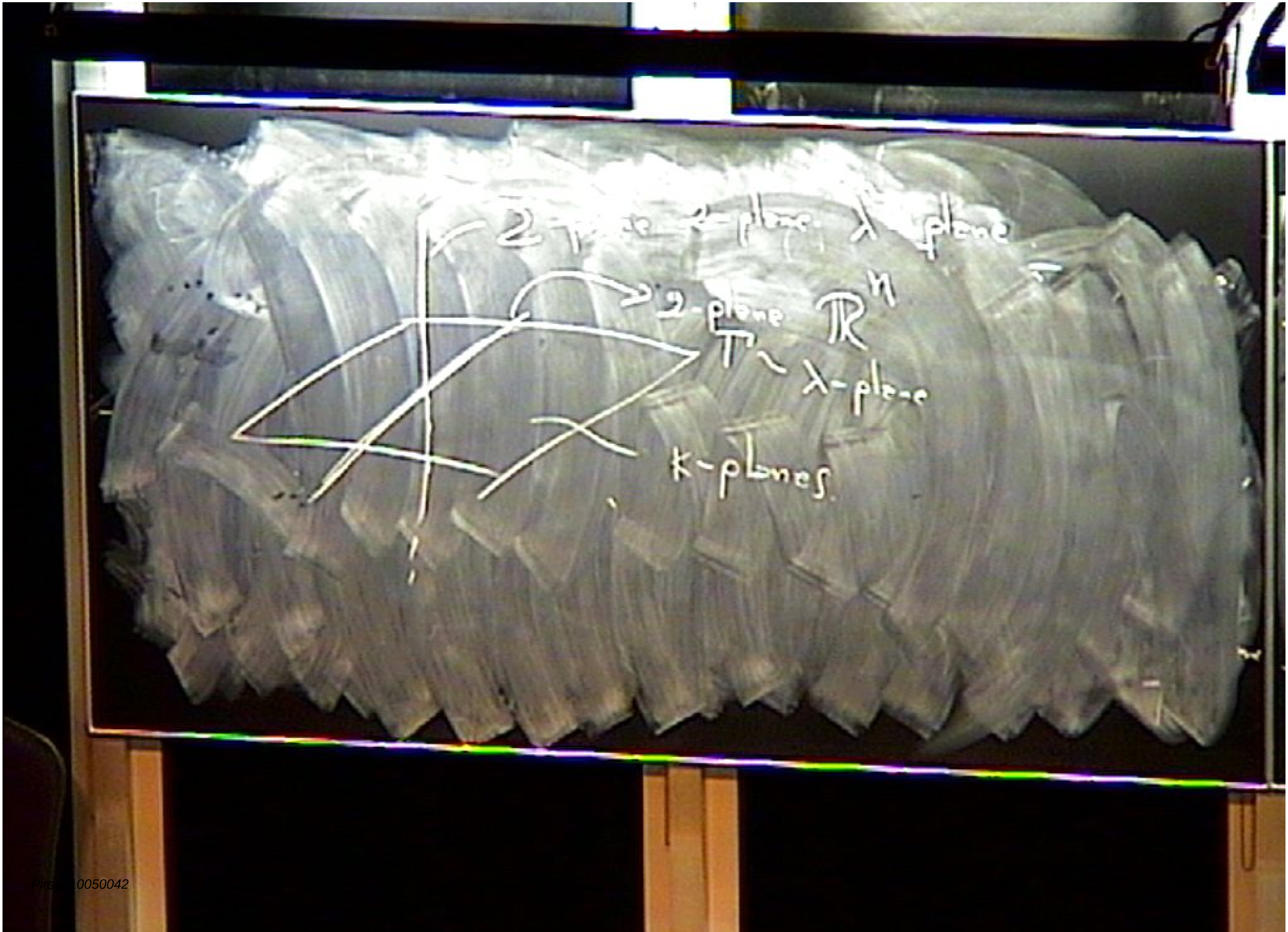
where

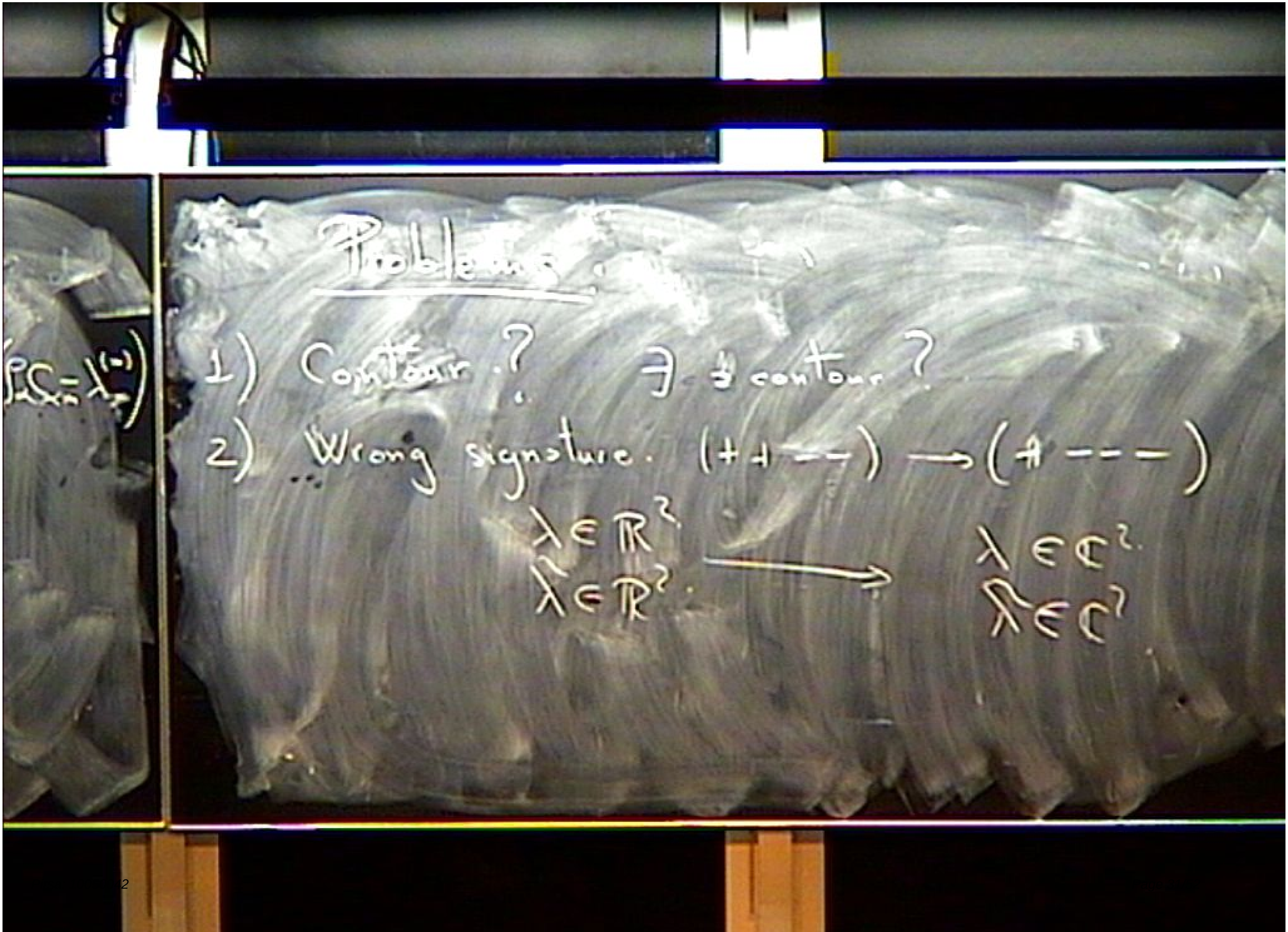
$$\delta(C\cdot\Sigma) = \prod_{\alpha=1}^{k} \delta^2\left(\sum_{m=1}^{n} c_{\alpha m}\tilde{\lambda}^{(m)}\right)$$

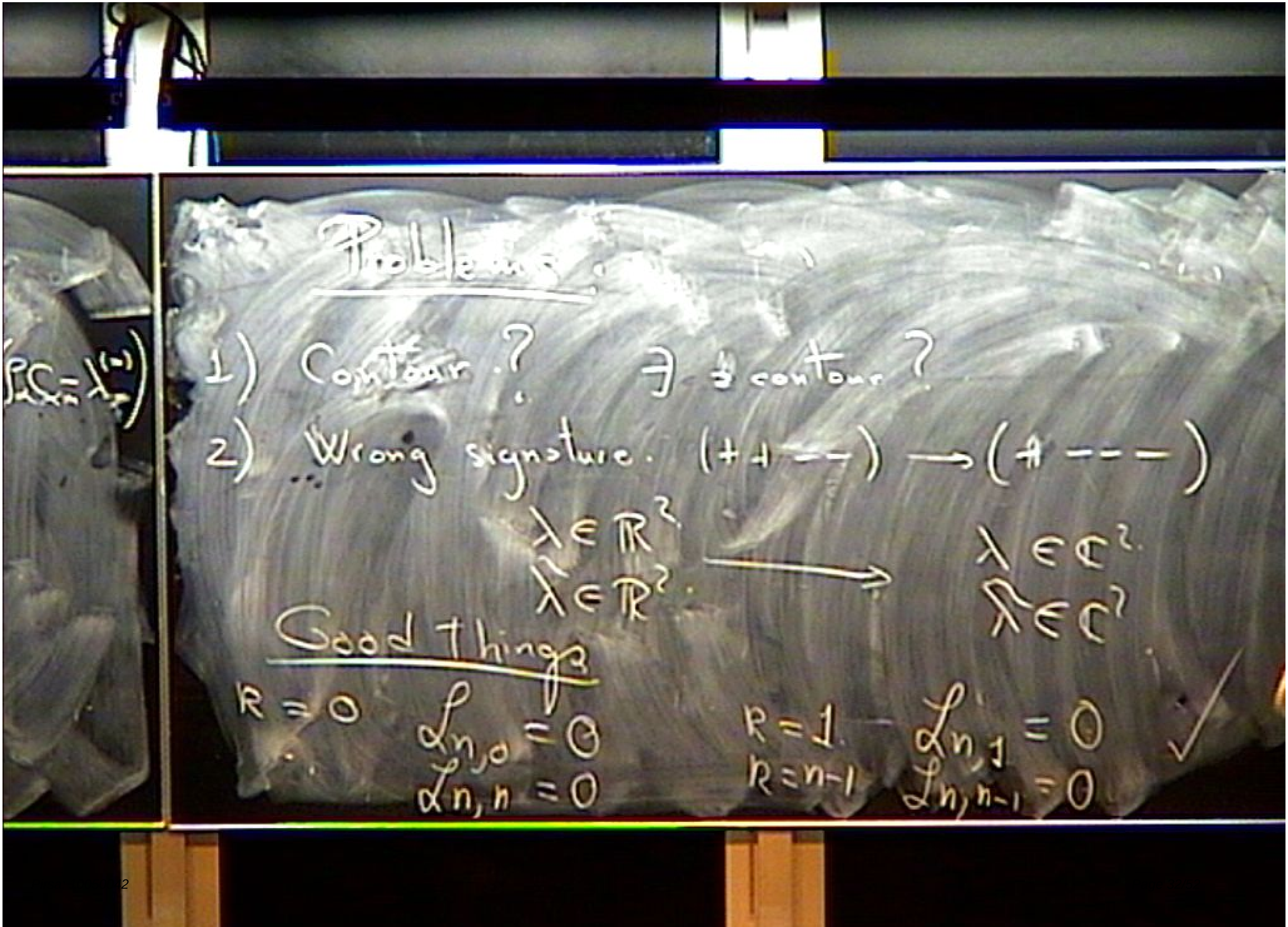
and

$$\delta(C^{\perp}\cdot\Gamma) = \prod_{\alpha=1}^{k}\int d^2\rho_\alpha \prod_{m=1}^{n}\delta^2\left(\rho_\alpha c_{\alpha m} - \lambda^{(m)}\right)$$


2-plane k-plane λ-plane
2-plane
$\mathbb{R}^n$
λ-plane
k-planes


k-plane λ-plane
2-plane
ℝⁿ
λ-plane
k-planes


Problems:
1) Contour? ∄ contour?
2) Wrong signature. (+ + − −) → (+ − − −)
λ ∈ ℝ² → λ ∈ ℂ²
λ̃ ∈ ℝ² → λ̃ ∈ ℂ²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1) Contour? ∃ ∄ contour?
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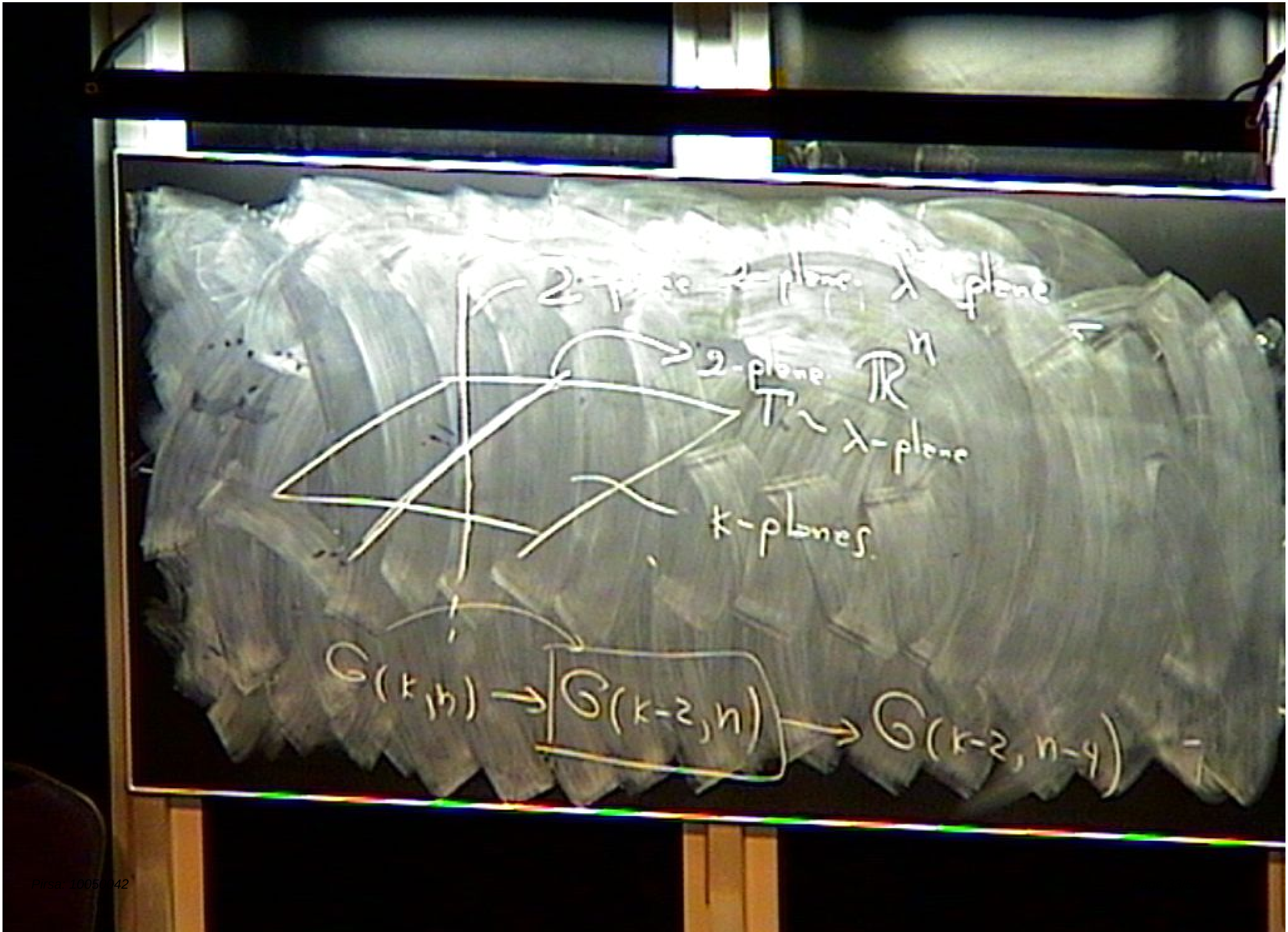
λ ∈ ℝ²
λ̃ ∈ ℝ² ⟹ λ ∈ ℂ²
λ̃ ∈ ℂ²
Good Things
R = 0 ℒn,0 = 0
ℒn,n = 0
R = 1 ℒn,1 = 0
R = n−1 ℒn,n−1 = 0


2-plane λ-plane
2-plane $\mathbb{R}^n$
λ-plane
k-planes
$G(k,n) \to G(k-2,n) \to G(k-2,n-4)$

Making The proposal Well-Defined

New variables:

$$\begin{pmatrix} \lambda_1^{(i)} \\ \lambda_2^{(i)} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\lambda}_1^{(i)} \\ \tilde{\lambda}_2^{(i)} \end{pmatrix} \longrightarrow \quad Z^{(i)} = \begin{pmatrix} \lambda_1^{(i)} \\ \lambda_2^{(i)} \\ \mu_1^{(i)} \\ \mu_2^{(i)} \end{pmatrix}.$$

where

$$\tilde{\lambda}^{(i)} = \frac{\langle i+1, i\rangle \mu^{(i-1)} + \langle i-1, i\rangle \mu^{(i+1)} + \langle i, i+1\rangle \mu^{(i-1)}}{\langle i-1, i\rangle \langle i, i+1\rangle}$$

(Hodges 2009, Skinner, Mason 2009, Arkani-Hamed,F.C.,Cheung 2009)

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(Hodges 2009, Skinner, Mason 2009, Arkani-Hamed,F.C.,Cheung 2009)

Making The proposal Well-Defined

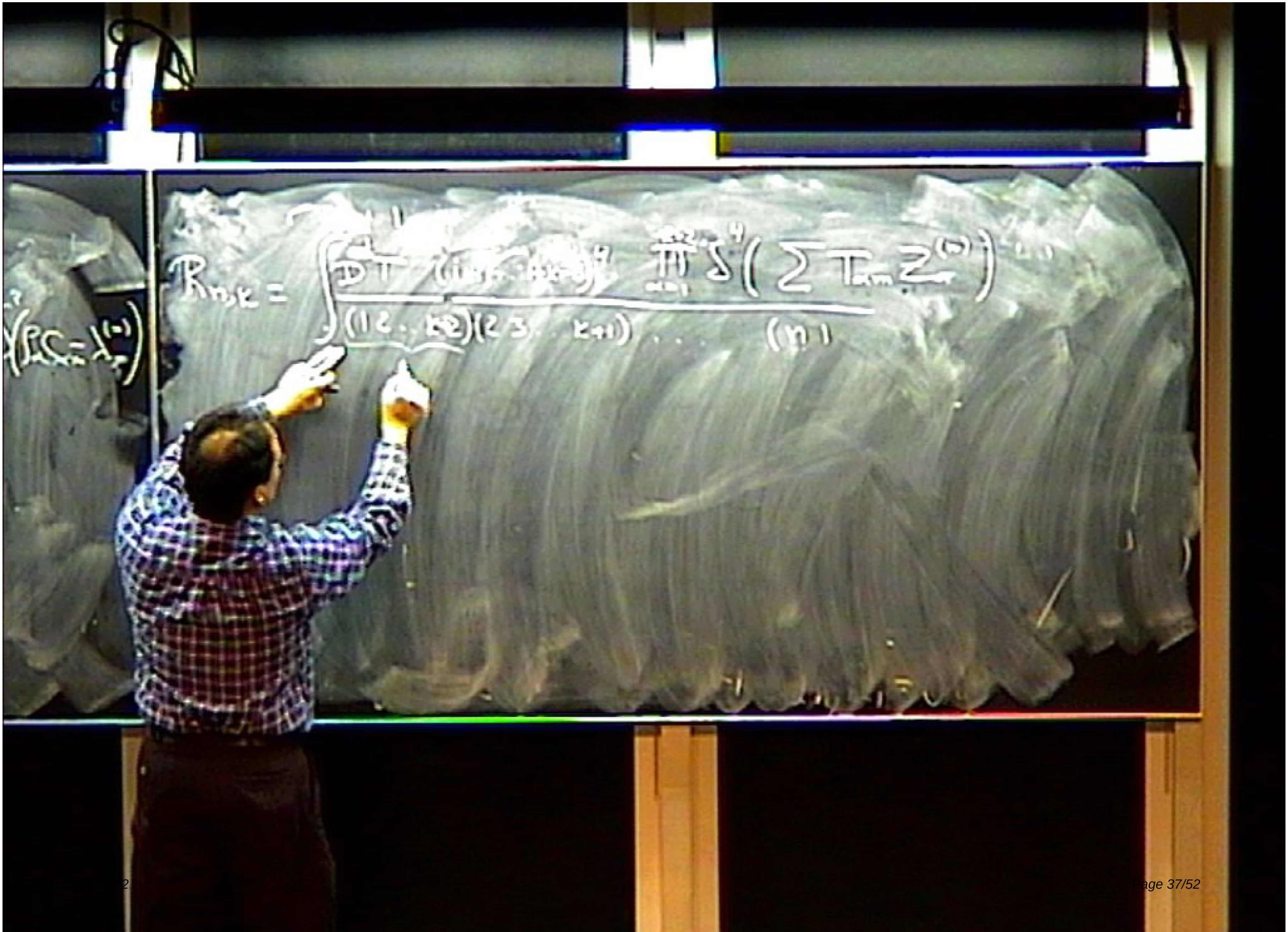
$$\mathcal{L}_{n,k} = \mathcal{L}_{n,2} \times R_{n,k}$$

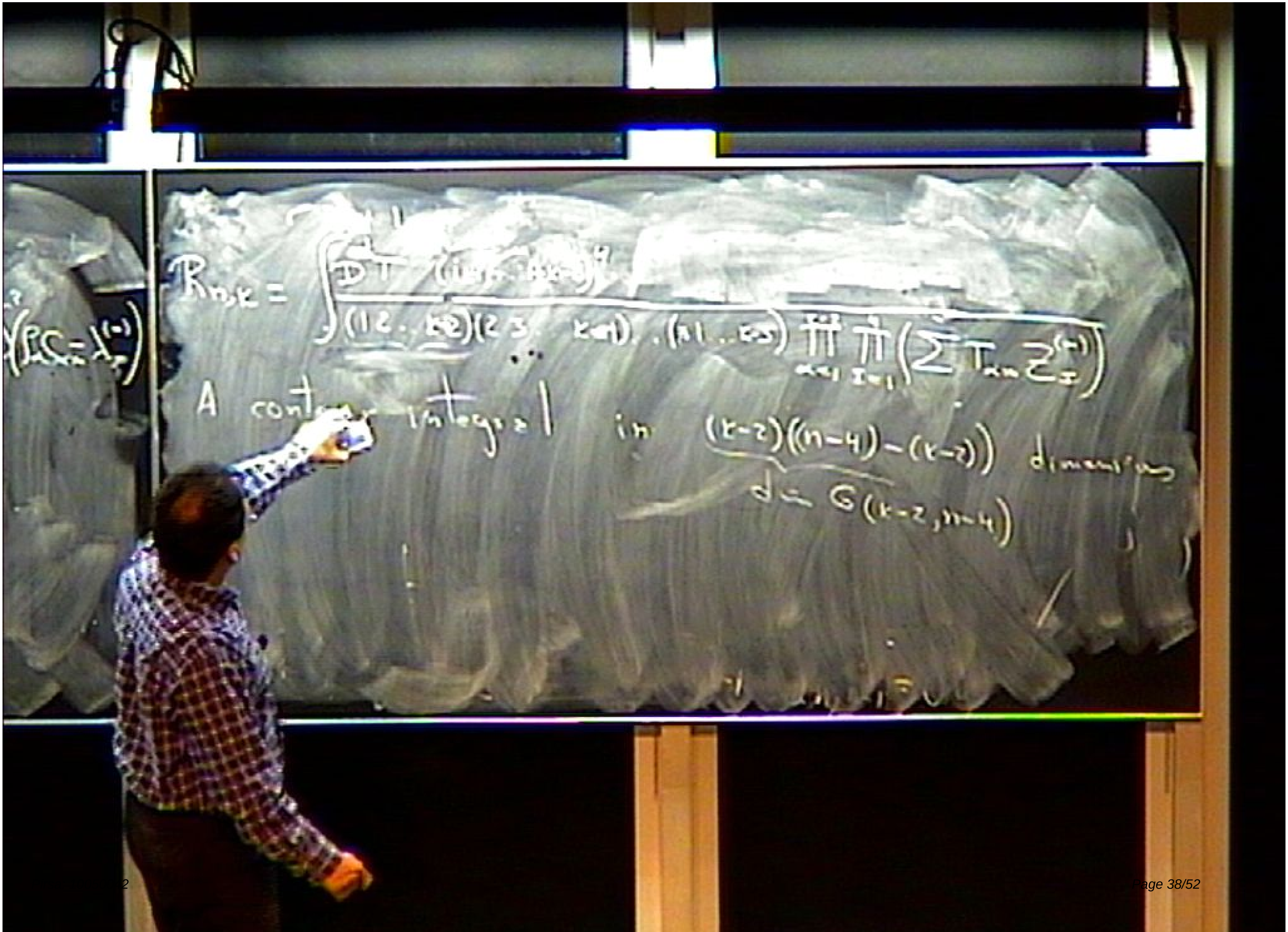
where

$$R_{n,k} = \int \frac{D^{(k-2)(n-k+2)} T\ (i+1,\ldots,i+k-2)^4\ \prod_{\alpha=1}^{k-2} \delta^4 \left(\sum_{m=1}^{n} T \right.}{(12\ldots k-2)(23\ldots k-1)\ldots(n1\ldots k-3)}$$

Obs: $\mathcal{L}_{n,2}$ contains the delta function imposing momentum conservation. This means that the left over piece is not a distribution and our task of defining it has simplified.

The new object $R_{n,k}$ is defined over $G(k-2,n)$.




A contour integral in (k-2)((n-4)-(k-2)) dimensions
dim G(k-2,n-4)

General definition of residues:

Given a holomorphic map $f : \mathbb{C}^m \rightarrow \mathbb{C}^m$ one defines a functional called the local residue $\mathrm{res}_{f,p}[\,]$ at $p \in f^{-1}(0)$ which is a assumed to be isolated. One gets a number if one plugs a function h, holomorphic at p,

$$\mathrm{res}_{f,p}[h] = \int_{T^m} d^m\tau \frac{h(\tau)}{f_1(\tau) f_2(\tau) \ldots f_m(\tau)}$$

where T^m is defined by $|f_i| = \epsilon_i$ with orientation $d(\arg f_1) \wedge \ldots d(\arg f_m) \geq 0$.

A consequence of this is that residues are antisymmetric under the exchange of f_i with f_{i+1}. In other words, if $f = (f_1, f_2)$ and $g = (f_2, f_1)$, then their local residues are the same up a sign.

The **global residue theorem** states that the sum over all the local residues is zero. (Under some assumptions)

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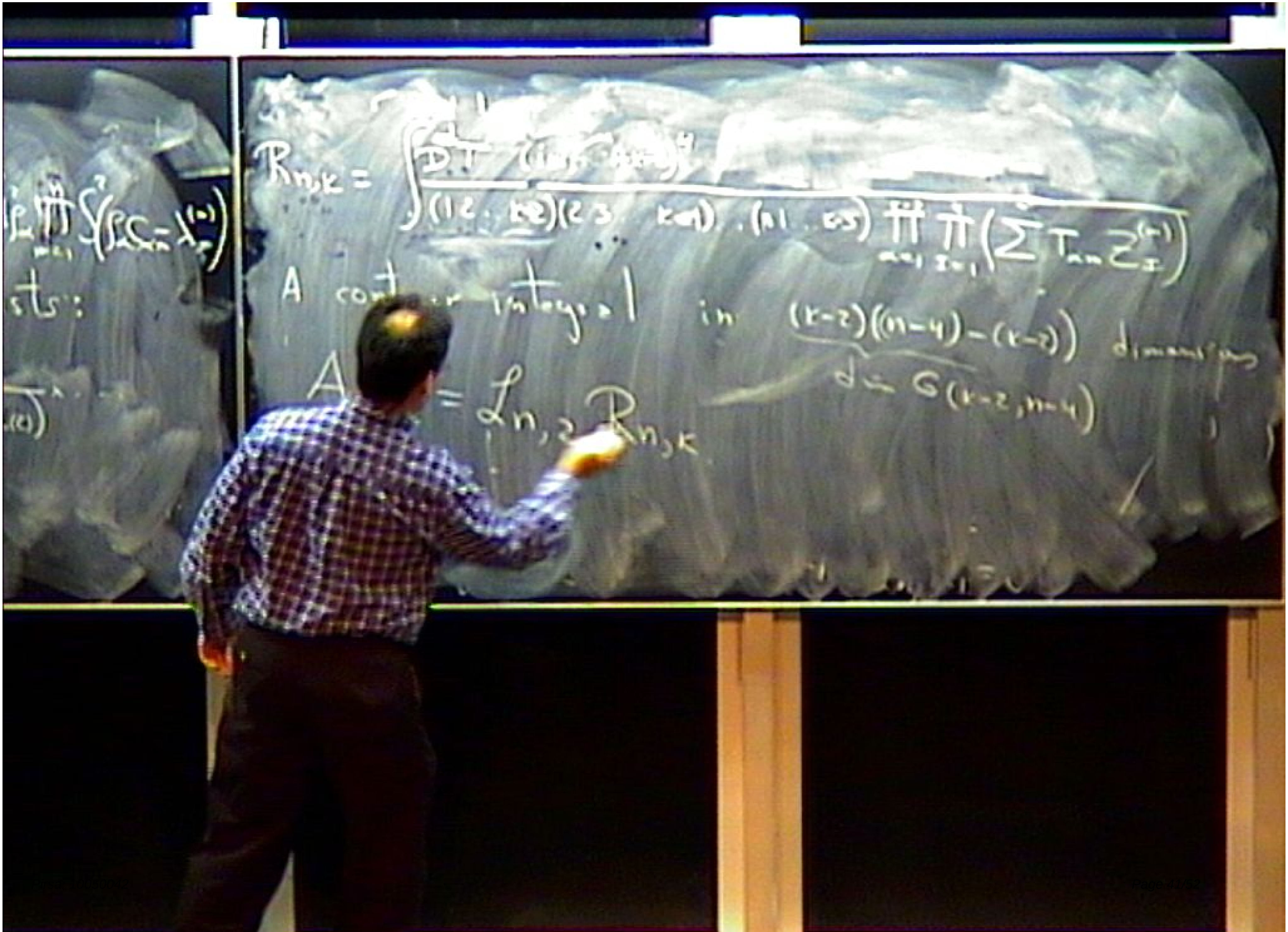
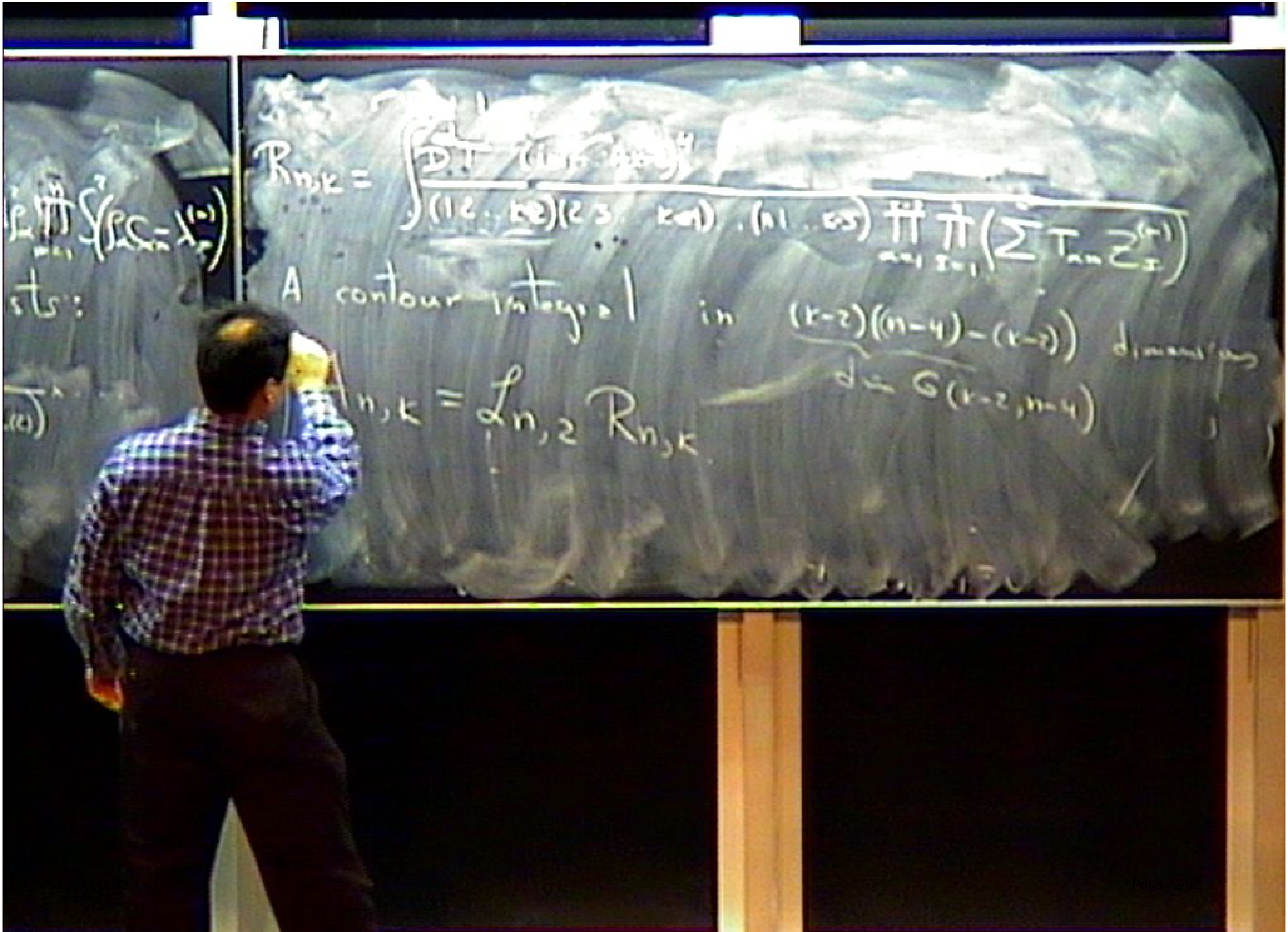


Rn,k =
A contour integral in (k-2)((n-4)-(k-2)) dimensions
d = G(k-2, n-4)
Ln,2 Rn,k


A contour integral in (k-2)((n-4)-(k-2)) dimensions
d = G(k-2, n-4)
Ln,k = Ln,2 Rn,k

$$R_{n,k} = \int \frac{D^{k-2}\tau \; (\ldots)^{4}}{(1\,2\ldots k-2)(2\,3\ldots k-1)\ldots(n\,1\ldots k-3)\prod_{a=1}^{n}\prod_{I=1}^{k-2}\left(\sum \tau_{a m} Z_I^{(m)}\right)}$$

A contour integral in $(k-2)((n-4)-(k-2))$ dimensions

$m = \dim G(k-2, n-4)$

$$\boxed{A_{n,k} = \mathcal{L}_{n,2}\, R_{n,k}}$$

sts:

Back to Physics:

It has been explicitly shown that:

All $k = 3$ and $k = 4$ scattering amplitudes are given as sums of local residues obtained from $R_{n,k}$.

Example:

$A_{6,3} = A + B = C + D + E$

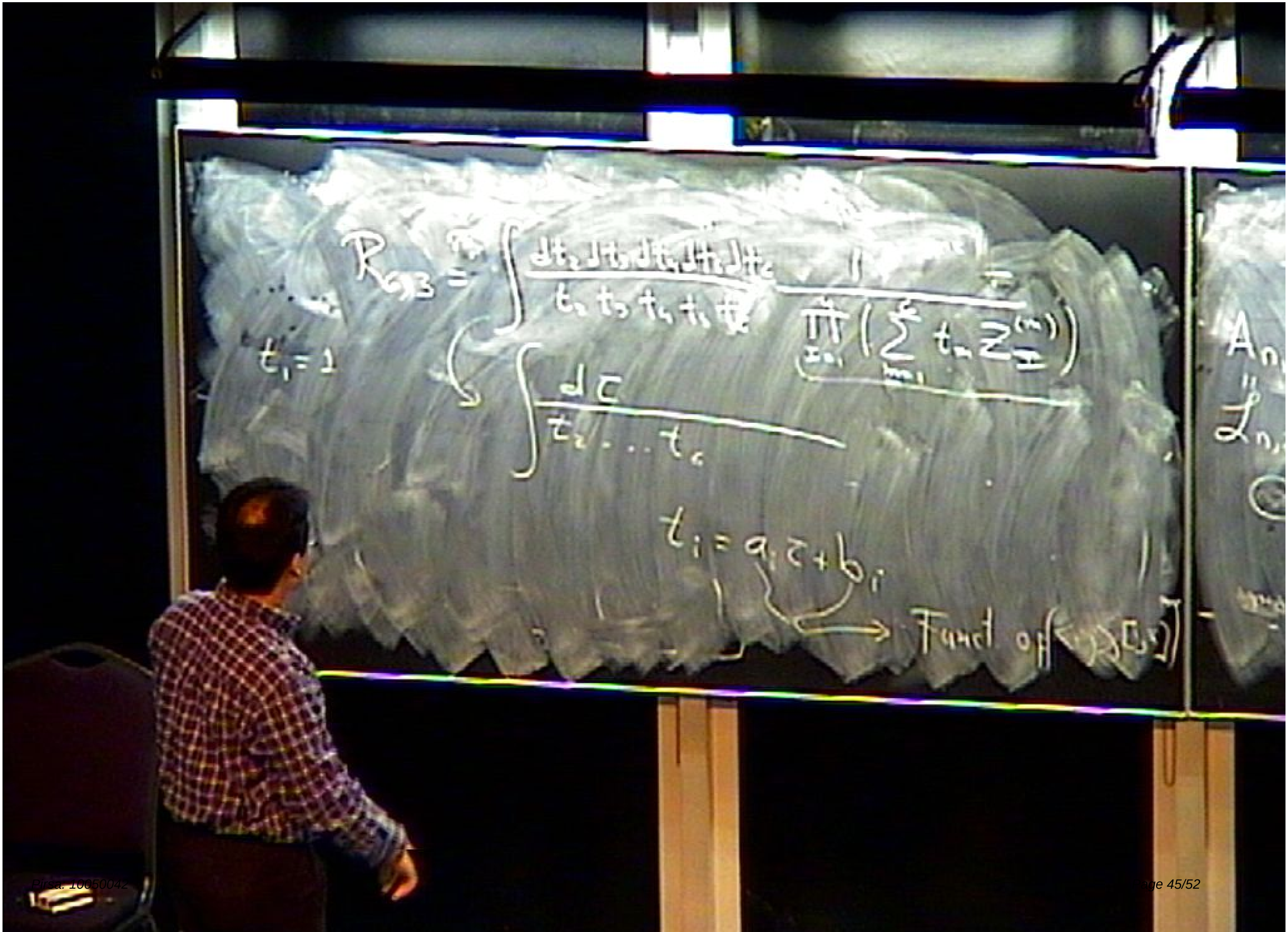
with

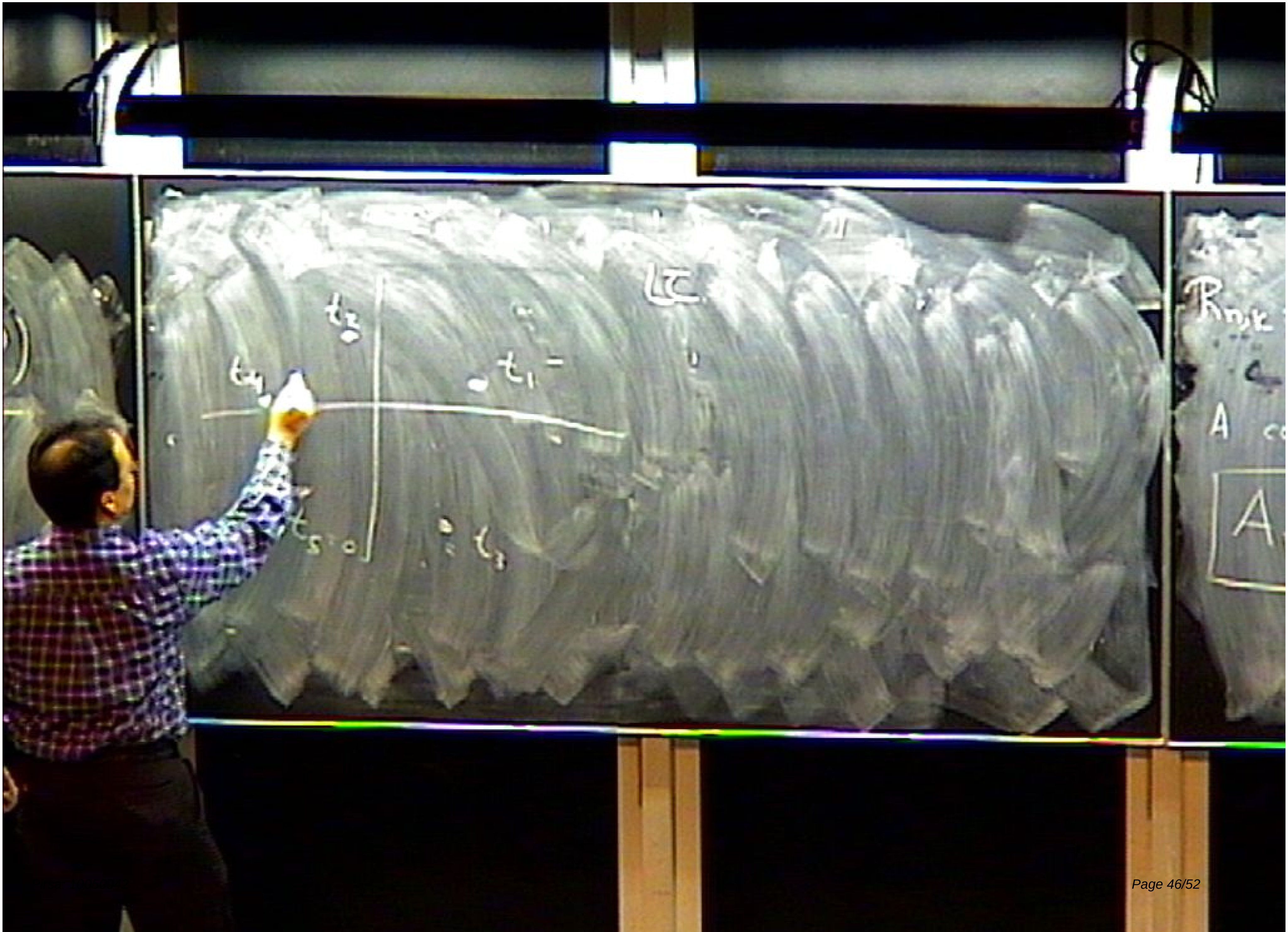
$$A = \frac{[4|5+6|1\rangle^3}{[3,4][2,3]\langle 5,6\rangle\langle 6,1\rangle[2|3+4|5\rangle(p_2+p_3+p_4)^2}$$

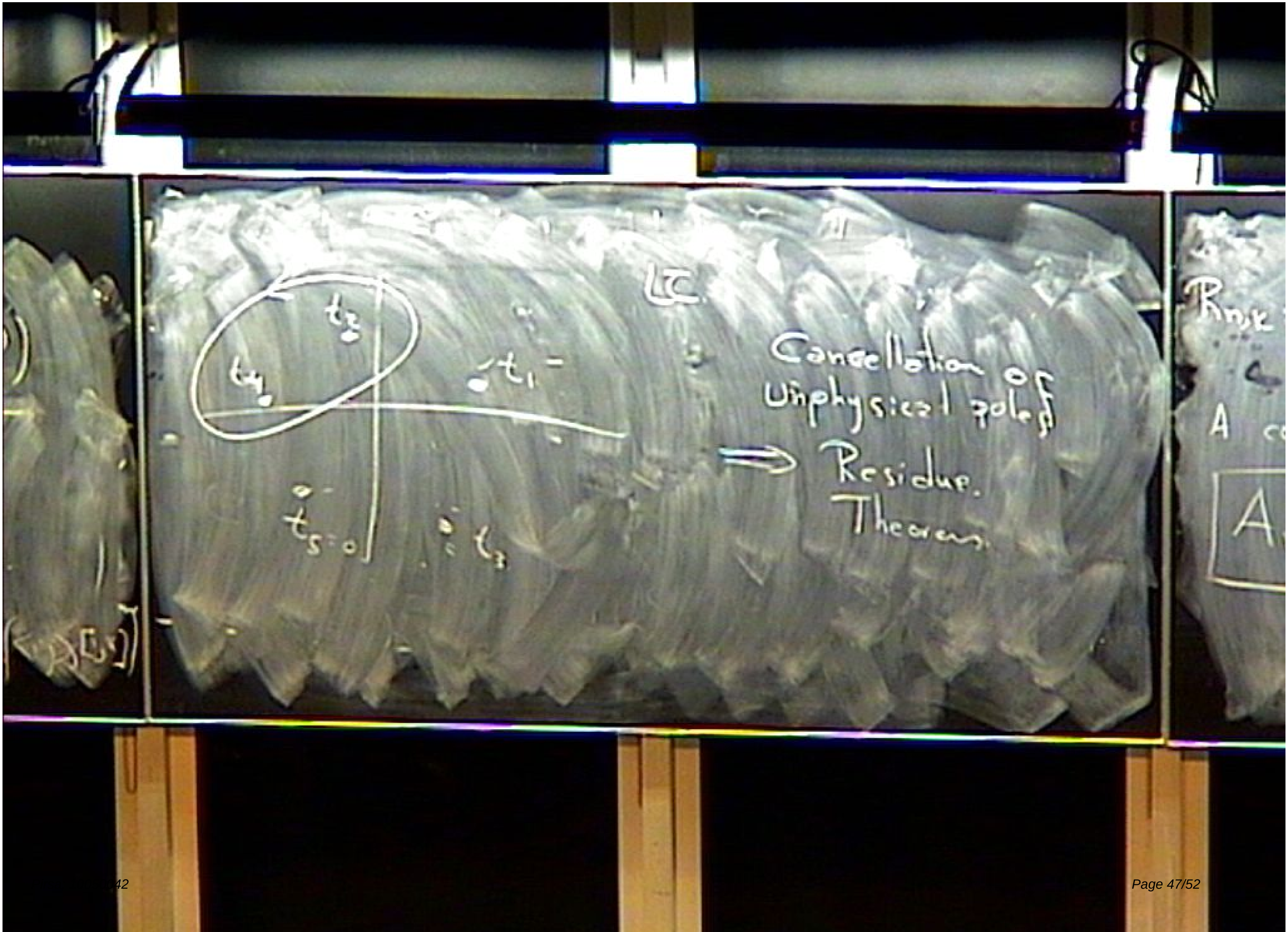
$$B = \frac{[6|1+2|3\rangle^3}{[6,1][1,2]\langle 3,4\rangle\langle 4,5\rangle[2|3+4|5\rangle(p_6+p_1+p_2)^2}$$

and

$$[a|b+c|d\rangle = [a,b]\langle b,d\rangle + [a,c]\langle c,d\rangle.$$




LC
t_2
t_4
t_1
t_5
t_3
A


LC
Cancellation of
unphysical poles
⟹ Residue
Theorem
t2
t4
t1
t5=0
t3

New Formulation Makes Hidden Symmetries Manifest

Small Improvement: Introduce 4 superdirections η^I and promote Z to be a superobject!

$$Z^{(i)} = \begin{pmatrix} \lambda_1^{(i)} \\ \lambda_2^{(i)} \\ \mu_1^{(i)} \\ \mu_2^{(i)} \end{pmatrix} \quad \to \quad \mathcal{Z}^{(i)} = \begin{pmatrix} Z^{(i)} \\ \eta^{(i)} \end{pmatrix}.$$

$$\mathcal{R}_{n,k} = \int \frac{D^{(k-2)(n-k+2)}T \; \prod_{\alpha=1}^{k-2} \delta^{4|4}\left(\sum_{m=1}^{n} T_{\alpha m} \mathcal{Z}^{(m)}\right)}{(12\ldots k-2)(23\ldots k-1)\ldots(n1\ldots k-3)}$$

Yangian

$$J_B^{(0)A} = \sum_i \mathcal{Z}_i^A \frac{\partial}{\partial \mathcal{Z}_i^B}$$

$$J_B^{(1)A} = \sum_{i<j} \left[\mathcal{Z}_i^A \frac{\partial}{\partial \mathcal{Z}_i^C} \mathcal{Z}_j^C \frac{\partial}{\partial \mathcal{Z}_j^B} - (i \leftrightarrow j) \right]$$

These are the level 0 and level 1 generators of the Yangian of the superconformal group $psu(2,2|4)$.

It has been shown that $R_{n,k}$ with contours that compute residues is annihilated by these operators.

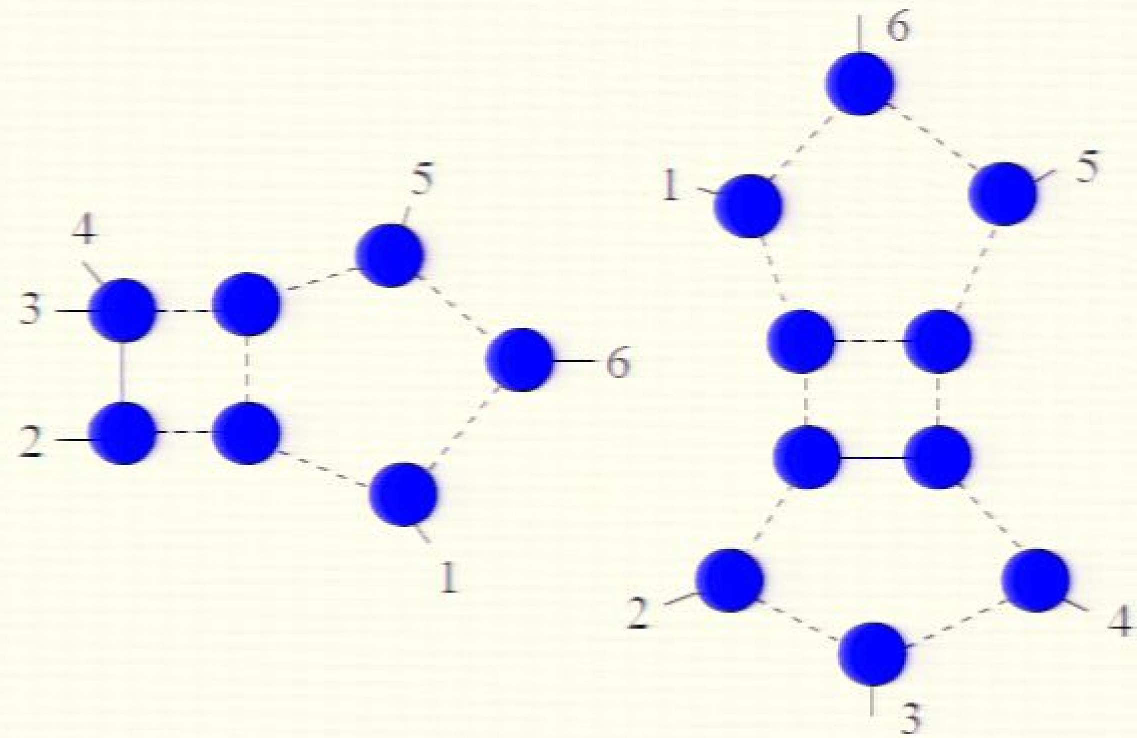
(Drummond, Ferro 2010)

Conjecture: The set of all local residues of $R_{n,k}$ provide all invariants of $Y(psu(2,2|4))$ and their relations in the form of global residue theorems.

Conjecture: The set of residues of $R_{n,k}$ and the set of leading singularities of scattering amplitudes (in $\mathcal{N}=4$) are the same.

Conjecture: The set of all local residues of $R_{n,k}$ provide all invariants of $Y(psu(2,2|4))$ and their relations in the form of global residue theorems.

Conjecture: The set of residues of $R_{n,k}$ and the set of leading singularities of scattering amplitudes (in $\mathcal{N}=4$) are the same.



Fascinating product structure!

Conclusion:

We have only started to understand the properties of $R_{n,k}$ and its connection with physics!