Title: Noncommutative algebras and (commutative) algebraic geometry

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Abstract: The study of D-branes at singular points of Calabi-Yau threefolds has revealed interesting connections between certain noncommutative algebras and singular algebraic varieties. In many respects, the choice of an appropriate noncommutative algebra is analogous to finding a resolution of singularities of the variety. We will explain this connection in detail, and outline a program for studying such "noncommutative resolutions" globally, for compact algebraic (Calabi--Yau) threefolds.

Noncommutative algebras and (commutative) algebraic geometry

David R. Morrison

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Connections in Geometry and Physics
Perimeter Institute
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The study of D-branes at singular points of Calabi-Yau threefolds has revealed interesting connections between certain noncommutative algebras and singular algebraic varieties. In many respects, the choice of an appropriate noncommutative algebra is analogous to finding a resolution of singularities of the variety. We will explain this connection in detail, and outline a program for studying such "noncommutative resolutions" globally, for compact algebraic (Calabi–Yau) threefolds.

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D-branes

D-branes were introduced into string theory in 1995. In the original formulation, D-branes are submanifolds on which strings end, and the submanifold must support a gauge field which provides a gauge degree of freedom at the endpoints of the string.

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In practice, when compactifying on a Calabi–Yau threefold X, the D-brane should be supported on a calibrated submanifold of X. For today's lecture, we set aside the interesting case of special Lagrangian submanifolds and focus on complex submanifolds. So the data for a D-brane in version one of our story is a complex submanifold and a holomorphic vector bundle on it.

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In version two of our D-brane story, one recognizes that a holomorphic vector bundle on a complex submanifold of X is a special case of a coherent sheaf on X, and that more general coherent sheaves can be used. So version two of D-branes is the category $Coh(X)$ of coherent sheaves on X.

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In topologically twisted string theory, Douglas showed that the natural structure is the (bouded) derived category $D^b(Coh(X))$. This is version three of D-branes, and was anticipated by Kontsevich's homological mirror symmetry conjecture, which was formulated before D-branes were even defined!

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Gauge theory on D-branes

There is a physical theory which exists directly on the branes: the gauge theory defined by the gauge fields we have specified. This theory is a Yang–Mills theory with maximal supersymmetry, whose gauge group is $U(N)$ when the corresponding vector bundle has rank N.

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An often-studied case is that of D3-branes located at a point of the associated Calabi–Yau manifold, and filling the effective $3+1$-dimensional spacetime. In this case, one gets the $\mathcal{N}=4$ super Yang–Mills theory in 4 dimensions.

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More generally, one can consider Calabi–Yau threefolds with singularities and place the D3-branes at the singularities. A field theory with less supersymmetry results, and this will be the topic of today's talk.

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In 1998, Maldacena conjectured that (conformal) field theories describing D3-branes have a well-behaved "large N limit" which is described by an anti-de Sitter vacuum of type IIB string theory. Even discussing a large N limit requires a certain uniformity for the D3-brane theories with $U(N)$ gauge group, independent of N, and this uniformity can be seen by writing the theory in $\mathcal{N} = 1$ terms.

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An $\mathcal{N} = 1$ gauge theory in 4 dimensions requires for its sepecification a (compact reductive) gauge group G, a "matter representation" ρ (which is a collection of adjoint and bifundamental fields for the local components G_i of the gauge group), and a superpotential $\mathcal{W}$ which is the trace of a polynomial W in the matter fields.

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For $\mathcal{N} = 4$ super Yang–Mills theory with gauge group $G = U(N)$, we have three matter fields X, Y, Z each taking values in the adjoint representation, and superpotential

$$\mathcal{W} = \mathrm{tr}(XYZ - XZY).$$

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The classical vacua for an $\mathcal{N} = 1$ gauge theory are given by specifying explicit "expectation values" for the matter fields X_j subject to the so-called F-term constraints $\mathrm{tr}(\partial W/\partial X_j) = 0$.

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$$\mathcal{A} = (\text{path algebra of quiver})/(\partial W/\partial X_\alpha).$$

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The representations of the D-brane algebra parameterize classical vacua for all of the D-brane theories simultaneously. In particular, when v_j is represented on a vector space of dimension N_j, the gauge group is $\prod U(N_j)$.

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The representations of the D-brane algebra parameterize classical vacua for all of the D-brane theories simultaneously. In particular, when v_j is represented on a vector space of dimension N_j, the gauge group is $\prod U(N_j)$. A large N limit can be studied with $N_j = N$ for all j.

One remarkable property which these D-brane algebras $\mathcal{A}$ appear to satisfy is that the center $\mathcal{Z}(\mathcal{A})$ is the coordinate ring of the associated Calabi–Yau threefold. In particular, the algebraic geometry should be recoverable from the field theory.

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The key thing to study will be the category *Mod*–$\mathcal{A}$ of representations of this algebra.

The McKay correspondence

An important example of the types of algebras we will encounter is the "twisted group algebra". Let G be a finite subgroup of SU(2) so that G acts on the polynomial algebra $\mathbb{C}[x,y]$. The *twisted group algebra* $\mathbb{C}[x,y] \star G$ consists of pairs $(f(x), g)$ and a multiplication

$$(f(x), g) \cdot (\phi(x), \gamma) = (f(x) \cdot g(\phi(x)), g \circ \gamma).$$

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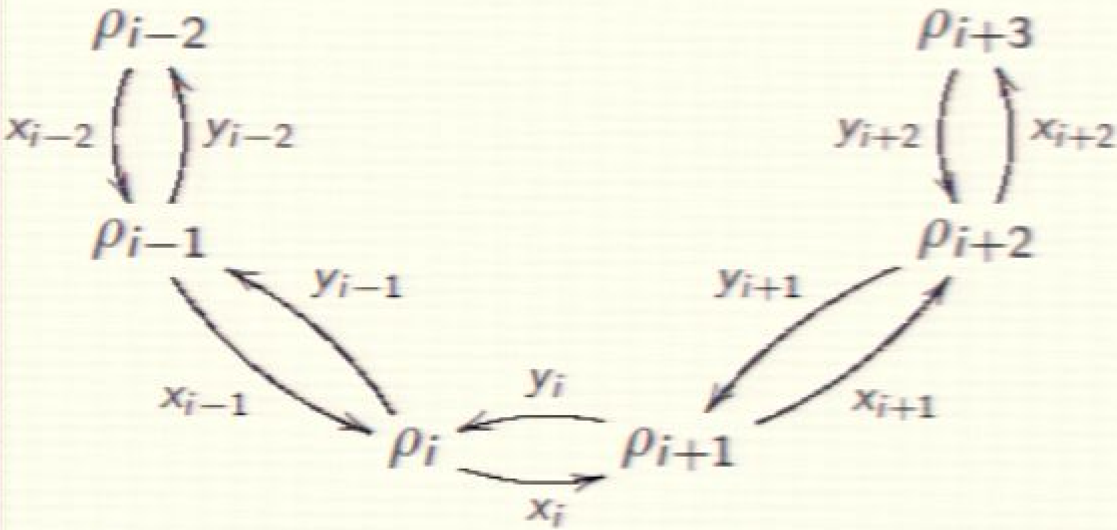
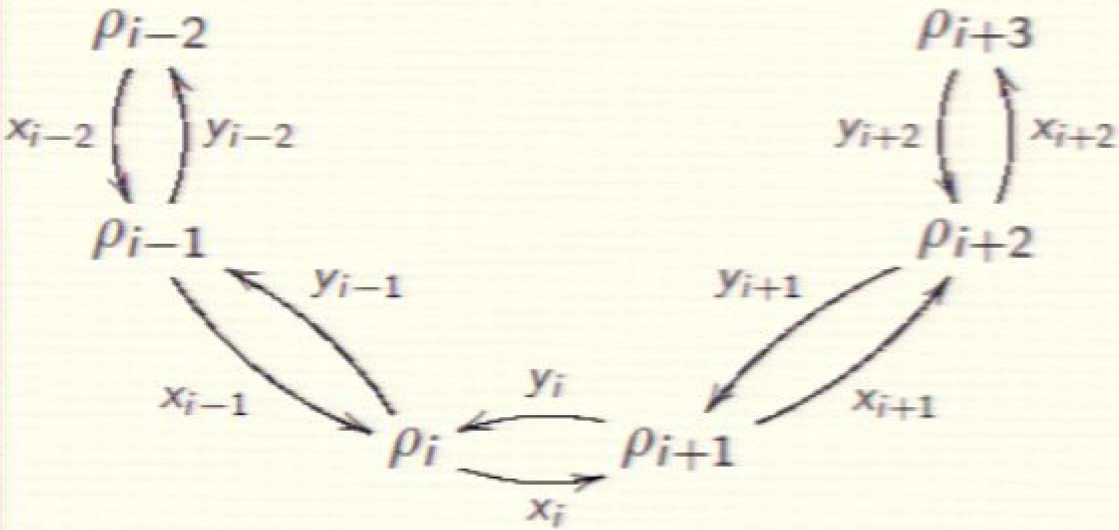
Kapranov and Vasserot interpreted the McKay correspondence as a statement about the structure of this algebra: we can write

$$\mathbb{C}[x,y] \star G = \bigoplus_{\rho \in \mathrm{Irrep}(G)} M_\rho \otimes \rho$$

and describe the algebra in terms of the modules M_ρ. Its structure is determined by the graph of representations which McKay used:



In the A_{n-1} case, for example, the graph forms a cycle and one has relations $x_i y_i = y_{i-1} x_{i-1}$.



In the A_{n-1} case, for example, the graph forms a cycle and one has relations $x_i y_i = y_{i-1} x_{i-1}$.
Its not too hard to determine the center of the algebra from this description: it is generated by

$$X = x_0 \cdots x_{n-1} + \text{ cyclic permutations}$$
$$Y = y_0 \cdots y_{n-1} + \text{ cyclic permutations}$$
$$Z = x_i y_i + \text{ cyclic permutations}$$

subject to the relation $Z^n = XY$. Of course, the center can also be identified with $\mathbb{C}[x, y]^G$, so this was not unexpected.

Example 1

We have already described the D-brane algebra of a smooth point of a Calabi–Yau threefold: there is one vertex v and three matter fields X, Y, Z with superpotential

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$$\mathcal{W} = \mathrm{tr}\,(X(YZ - ZY))\,.$$

The F-term constraint in this case tells us $YZ = YZ$, $XZ = ZX$ and $XY = YX$. Thus, we find

$$\mathcal{A} = \mathbb{C}[X, Y, Z],$$

the (commutative) polynomial algebra in three variables.

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The associated category is the category of $\mathbb{C}[X, Y, Z]$-modules; when finite, these are given by triples of commuting $N \times N$ matrices.

xample 2

The McKay quiver algebra $\widehat{A_{n-1}}$ is not a D-brane algebra, but a closely related algebra is: add loops at each of the vertices, represented by fields ϕ_i; then the superpotential

$$W = \operatorname{tr}(\phi_i(x_i y_i - y_{i-1} x_{i-1}))$$

gives the same relations as before. (This is known to be the appropriate field theory description for D3-branes at a point P on a curve of A_{n-1} singularities on a Calabi–Yau threefold.)

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$$\mathcal{Z}(\mathcal{A}) = \mathbb{C}[x,y]^G \otimes \mathbb{C}[\Phi],$$

where $\Phi = \sum \phi_i$.

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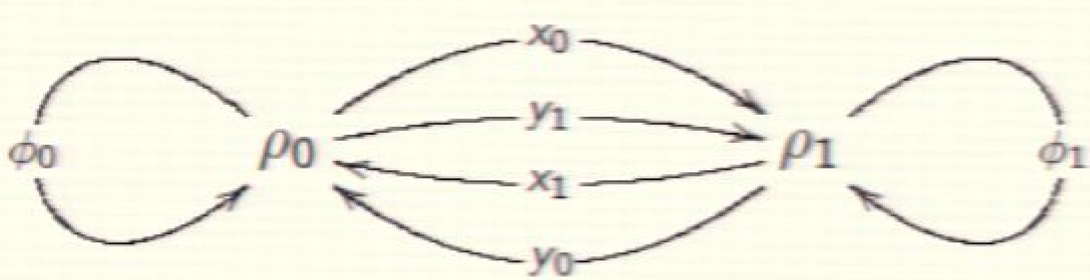
Example 3 (Cachazo–Katz–Vafa; Aspinwall–Katz)

Let us modify the theory for $\widehat{A_1}$ by adding another term to the superpotential. That is, we have fields x_i, y_i, ϕ_i for $i = 0, 1$ with superpotential

$$W = \mathrm{tr}\left(\phi_0(x_0 y_0 - y_1 x_1) + \phi_1(x_1 y_1 - y_0 x_0) + P(\phi_0) + P(\phi_1)\right)$$

where P is some fixed polynomial.



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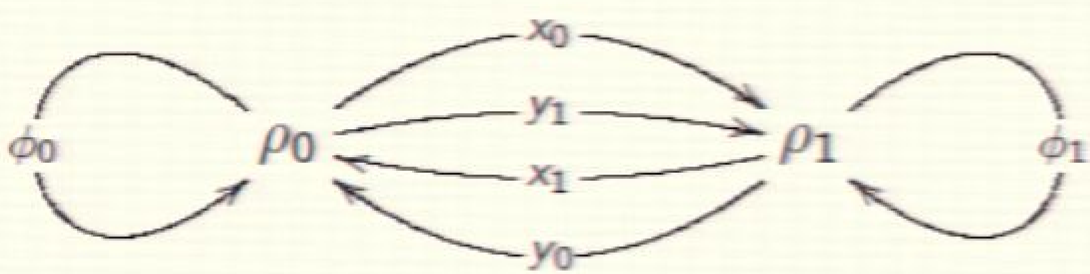

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The F-term relations are:

$$y_0\phi_0 = \phi_1 y_0$$
$$\phi_0 y_1 = y_1\phi_1$$
$$\phi_0 x_0 = x_0\phi_1$$
$$x_1\phi_0 = \phi_1 x_1$$
$$x_0 y_0 = y_1 x_1 - P'(\phi_0)$$
$$x_1 y_1 = y_0 x_0 - P'(\phi_1)$$

One can see that the following elements are central:

$$X = x_0 x_1 + x_1 x_0$$
$$Y = y_1 y_0 + y_0 y_1$$
$$Z = x_0 y_0 + x_1 y_1$$
$$\Phi = \phi_0 + \phi_1$$

and satisfy the relation

$$Z^2 = XY + ZP'(\Phi).$$

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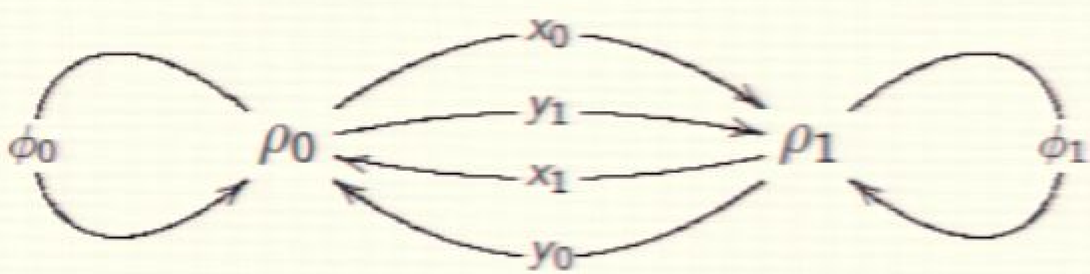

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and satisfy the relation

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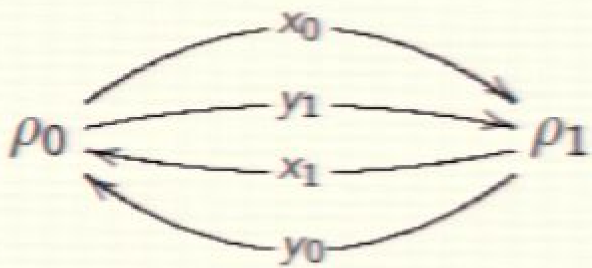
These singularities are closely related to flops: by an old result of Reid, the blowdown of a flop with normal bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ always has an equation of the above form, where P has a zero at $\Phi = 0$ of order at least 3.

Example 4: the conifold

The previous example also describes the conifold, if we take $P(\Phi) = \frac{1}{2}\Phi^2$. The standard physics approach to this is to "integrate out" ϕ_i, using the F-term equations to solve $\phi_0 = y_1 x_1 - x_0 y_0$, $\phi_i = y_0 x_0 - x_1 y_1$. The new superpotential is

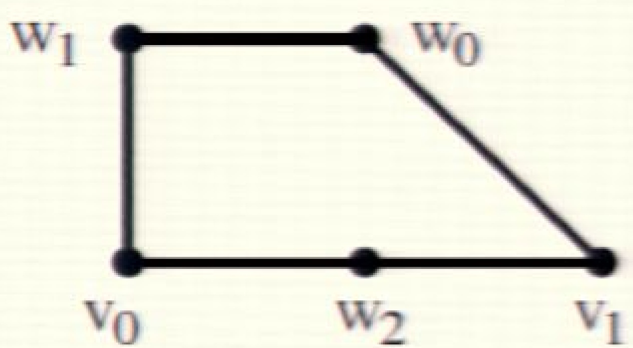
$$\mathcal{W} = \mathrm{tr}\left(x_0 x_1 y_1 y_0 - x_0 y_0 y_1 x_1\right).$$

This is the standard superpotential for the conifold, first found by Klebanov and Witten.

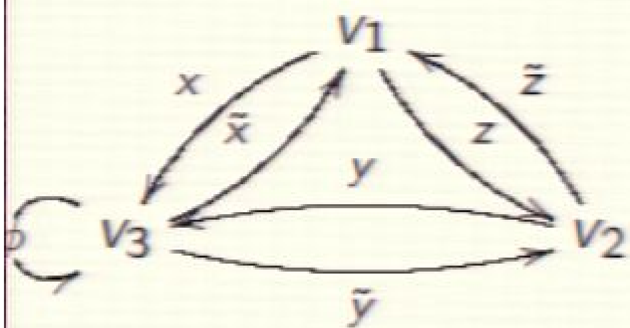


Example 5: the suspended pinch point

As our next example, we will compute the superpotential algebra and its center for the “suspended pinch point” (first considered in *hep-th/9810201*). The method which Plesser and I used to find the superpotential in this case was toric geometry. The suspended pinch point singularity can be described torically as the cone over the following lattice polyhedron:



Plesser and I calculated a field theory dual, which can be expressed in terms of the quiver



and superpotential

$$W = \mathrm{tr}\left(\phi(\tilde{Y}Y - \tilde{X}X) + \lambda(Z\tilde{Z}X\tilde{X} - \tilde{Z}Z\tilde{Y}Y)\right) .$$

Many other toric examples were subsequently computed, with a large amount of technology devoted to their efficient computation (work of Hanany and collaborators, among others).

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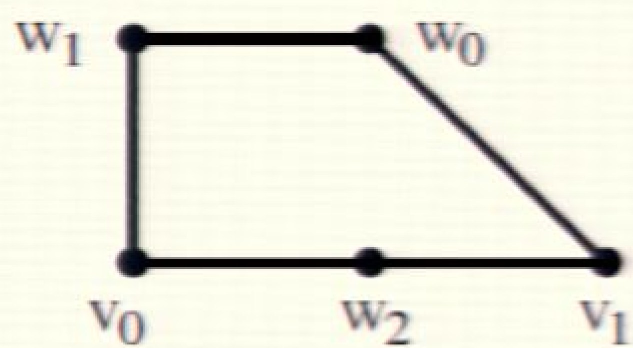

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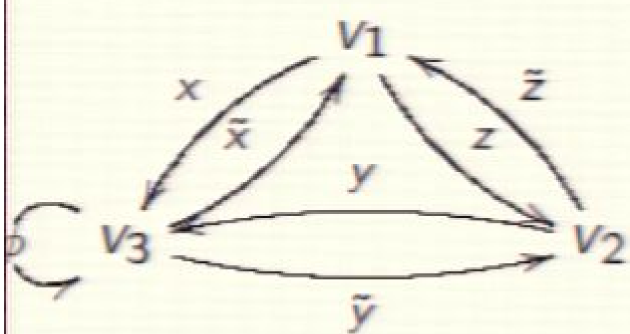

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In our example, the F-term constraints become:

$$\tilde{Y}Y = \tilde{X}X$$
$$\phi\tilde{X} = \lambda Z\tilde{Z}X$$
$$X\phi = \lambda X\tilde{Z}X$$
$$\phi\tilde{Y} = \lambda\tilde{Y}\tilde{Z}Z$$
$$Y\phi = \lambda\tilde{Z}ZY$$
$$\lambda\tilde{Z}X\tilde{Z} = \lambda Y\tilde{Y}\tilde{Z}$$
$$\lambda X\tilde{X}Z = \lambda ZY\tilde{Y}$$

There are central elements $A = \phi + \lambda Z\tilde{Z} + \lambda\tilde{Z}Z$, $B = \tilde{X}X + X\tilde{X} + Y\tilde{Y}$, $C = \tilde{Y}\tilde{Z}X + \tilde{Z}X\tilde{Y} + X\tilde{Y}\tilde{Z}$, $D = \tilde{X}ZY + ZY\tilde{X} + Y\tilde{X}Z$.
The relation is:

$$AB^2 = \lambda CD.$$

Example 6: A flop of length 2

(Worked out by Aspinwall and myself in *arXiv:1005.1042*.) Consider the hypersurface singularity with defining polynomial

$$F(x, y, z, w) = x^2 + y^3 + wz^2 + w^3 y - \lambda w y^2 - \lambda w^4$$

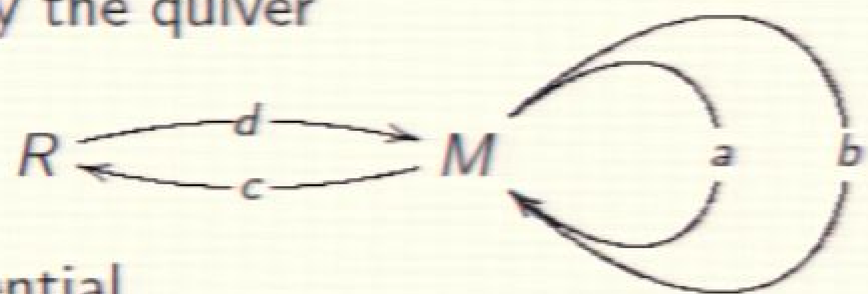
There is a matrix factorization for this polynomial of the form

$$(xI_4 + \Xi)(xI_4 - \Xi) = FI_4,$$

where

$$\Xi := \begin{bmatrix} 0 & y & z & -w \\ -(y-\lambda w)y & 0 & (y-\lambda w)w & z \\ -wz & -w^2 & 0 & -y \\ (y-\lambda w)w^2 & -wz & (y-\lambda w)y & 0 \end{bmatrix}$$

The cokernel of $xI_4 + \Xi$ defines a module M over $R = \mathbb{C}[x,y,z,w]/(F)$ and the ring $\mathcal{A} = \mathrm{End}(R \oplus M)$ is a D-brane algebra for this singularity. It turns out that $\mathcal{A}$ can be described by the quiver



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$$\mathcal{W} = \mathrm{tr}(b^2dc + \tfrac{1}{2}dcdc + a^2b + \tfrac{1}{3}\lambda b^3 + \tfrac{1}{4}b^4).$$

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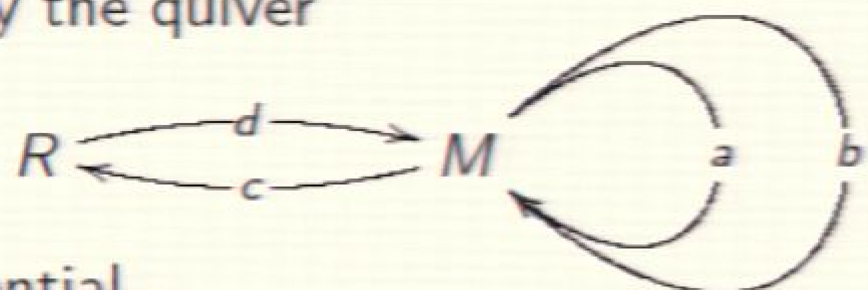

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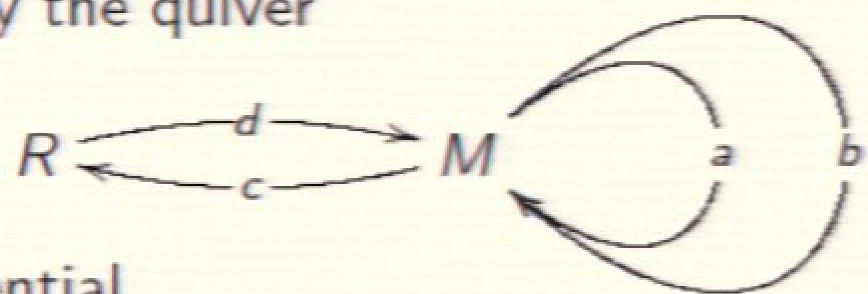

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Remarks

1. The algebra $\mathcal{A}$ is not uniquely determined by the D-brane configuration (this is related to Seiberg duality and has been studied extensively).

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2. As stated, this provides a way of passing from a Lagrangian description of a family of field theories to the algebro-geometric structure near the corresponding singular point. It is also possible to go backwards (work of Aspinwall and collaborators): given $P \in X$, one studies the (derived) category of coherent sheaves on X supported at P, and determines a so-called tilting module for the category. That tilting module, and some further computations of *Ext* groups of the sheaves, gives a Lagrangian description for the family of D-branes (including a superpotential).

3. To the best of my knowledge, there is no known algorithm for computing the center of $\mathcal{A}$. There is also no known algorithm for producing a tilting module for the derived category of coherent sheaves supported at X. And of course, there is no proof at present that the center $\mathcal{Z}(\mathcal{A})$ is isomorphic to the the coordinate ring of X at P.

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4. Van den Bergh has conjectured that a canonical singularity $P \in X$ in dimension 3 admits an algebra $\mathcal{A}$ whose category of modules describe the coherent sheaves supported at P if and only if there is a resolution of singularities $\pi : Y \to X$ which is relatively Calabi–Yau (that is, no zeros are introduced into the holomorphic 3-form), and he has proven this conjecture in a number of cases. However, neither his conjecture nor his proof address the question of whether $\mathcal{A}$ can be written as a D-brane algebra, that is, as a quiver algebra modulo relations coming from a superpotential.

5. One of the key properties of the algebras which van den Bergh uses (and a property enjoyed by all of our examples) is that the ring $\mathcal{A}$ has finite global dimension. Generally, the coordinate ring of an affine algebraic variety only has finite global dimension when the variety is nonsingular. Here, we have a class of noncommutative algebras with a similar property, even for a class of singular varieties. This is one of the ways in which specifying a noncommutative algebra of the type we are considering may be seen as an analogue of resolving the singularities.

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$$\mathcal{A}_f := \mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}_f .$$

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McKay localized

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$$X = x_0 x_1 + x_1 x_0, \quad Y = y_0 y_1 + y_1 y_0, \quad Z = x_0 y_0 + x_1 y_1,$$

subject to $Z^2 = XY$.

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Now, to invert X, we must invert x_0 and x_1 and then it easily follows that $y_0 = (X^{-1}Z)x_1$ and $y_1 = (X^{-1}Z)x_0$. It turns out that the matrix algebra $M_2(\mathcal{Z}_X)$ is isomorphic to the localized algebra $\mathcal{A}_X$ via:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto x_0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto X^{-1} x_1.$$

Morita equivalence

One fundamental notion in the theory of noncommutative rings and algebras is the notion of *Morita equivalence*. Two rings $\mathcal{A}$ and $\mathcal{B}$ are said to be Morita equivalent if there is an equivalence of categories between the category of left $\mathcal{A}$-modules and the category of left $\mathcal{B}$-modules.

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A key fact is that if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then the centers of the rings $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{B})$ are isomorphic.

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The application in the present context is clear: once we localized away from the singularity, we obtained a Morita-equivalent ring.

Globalizing

The key observation to globalizing this setup is to notice that we don't really care about the noncommutative rings *per se*, we care about their representations, which will determine sheaves on the space.

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The details are not all worked out yet, but the construction we want to give of noncommutative spaces roughly goes like this.

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The details are not all worked out yet, but the construction we want to give of noncommutative spaces roughly goes like this. We work with spaces modeled on $\mathrm{Spec}_{NC}(\mathcal{A})$ (a noncommutative Spec which we are trying to define). Given a representation $M \in Mod\text{–}\mathcal{A}$, we define a sheaf $\widetilde{M}$ by sheafifying the presheaf

$$\widetilde{M}(\mathrm{Spec}(\mathcal{Z}(\mathcal{A})_f) := M \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{A})_f \in Mod\text{–}\mathcal{A}_f.$$

These are the quasi-coherent sheaves.

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Now we patch together these local objects, but we don't require an equivalence of rings when matching affine open sets, we only require an equivalence of module-categories.

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That is, a gluing should be a ring isomorphism

$$\psi : \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{B})$$

together with a compatible equivalence of categories

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Andrei Caldararu has pointed out to me that this construction has many points of contact with algebraic stacks, at least if the rings $\mathcal{A}$ are suitably restricted. It is not clear yet if we are providing a new perspective on stacks, or a generalization (of at least some of them).

I hope I have convinced you that there is a natural class of noncommutative algebras which can be used to study singularities of algebraic varieties.

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Example 1

We have already described the D-brane algebra of a smooth point of a Calabi–Yau threefold: there is one vertex v and three matter fields X, Y, Z with superpotential

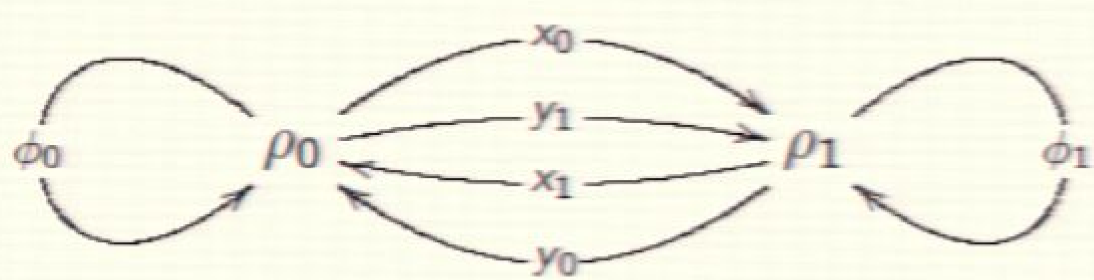
$$\mathcal{W} = \mathrm{tr}\left(X(YZ - ZY)\right).$$

xample 3 (Cachazo–Katz–Vafa; spinwall–Katz)

Let us modify the theory for $\widehat{A_1}$ by adding another term to the superpotential. That is, we have fields x_i, y_i, ϕ_i for $i = 0, 1$ with superpotential

$$W = \mathrm{tr}\left(\phi_0(x_0 y_0 - y_1 x_1) + \phi_1(x_1 y_1 - y_0 x_0) + P(\phi_0) + P(\phi_1)\right)$$

where P is some fixed polynomial.



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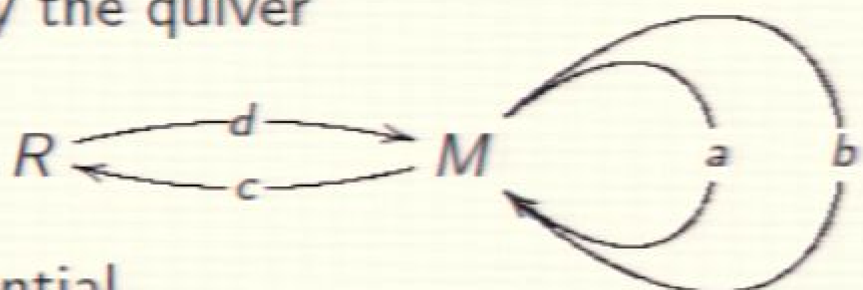

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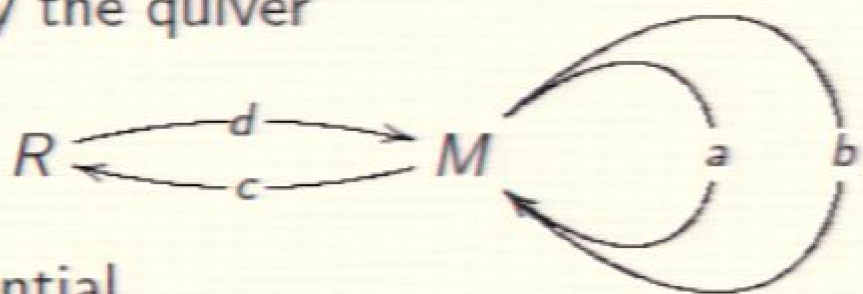

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