Title: Motivic degree zero Donaldson - Thomas invariants

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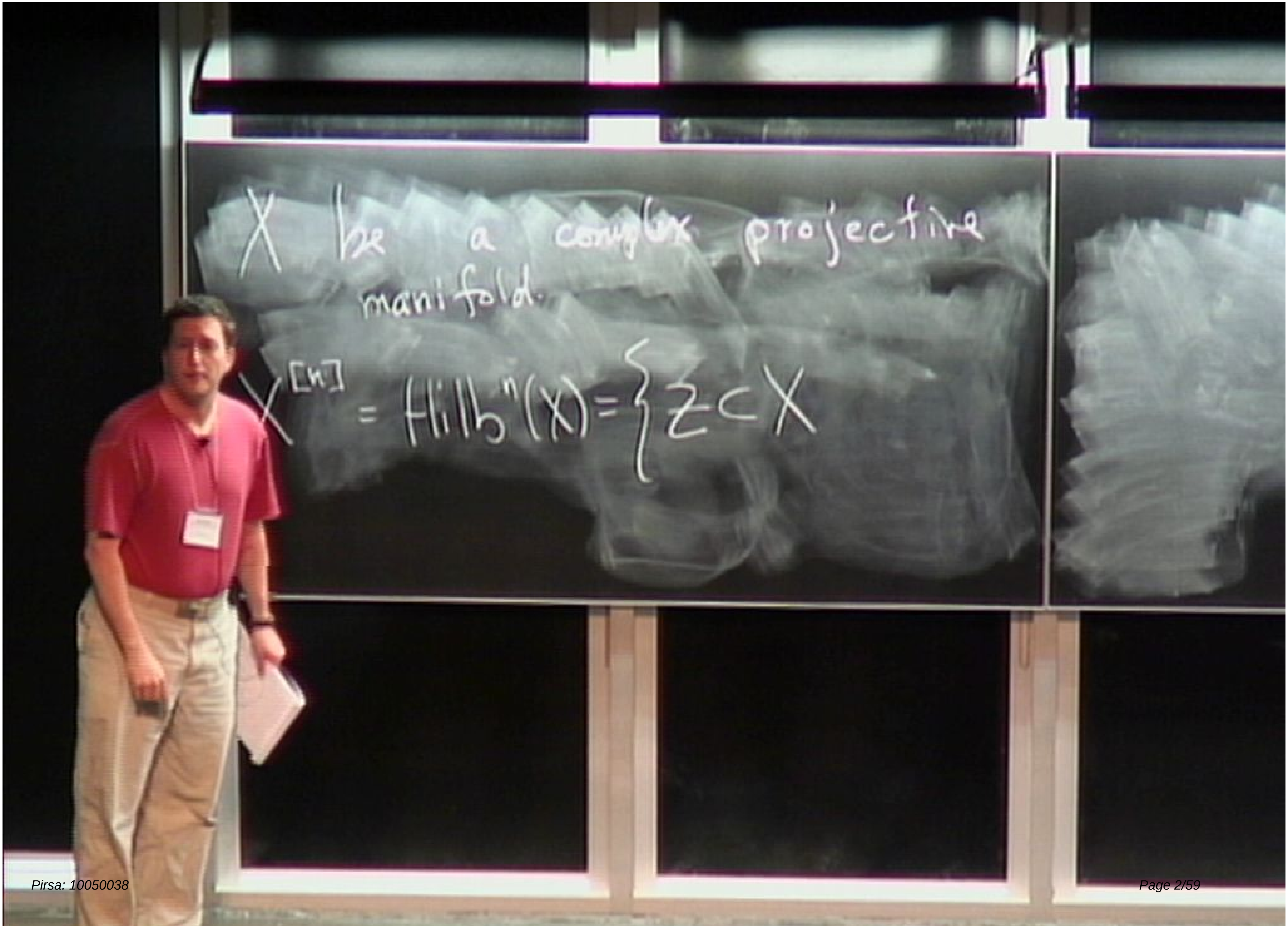
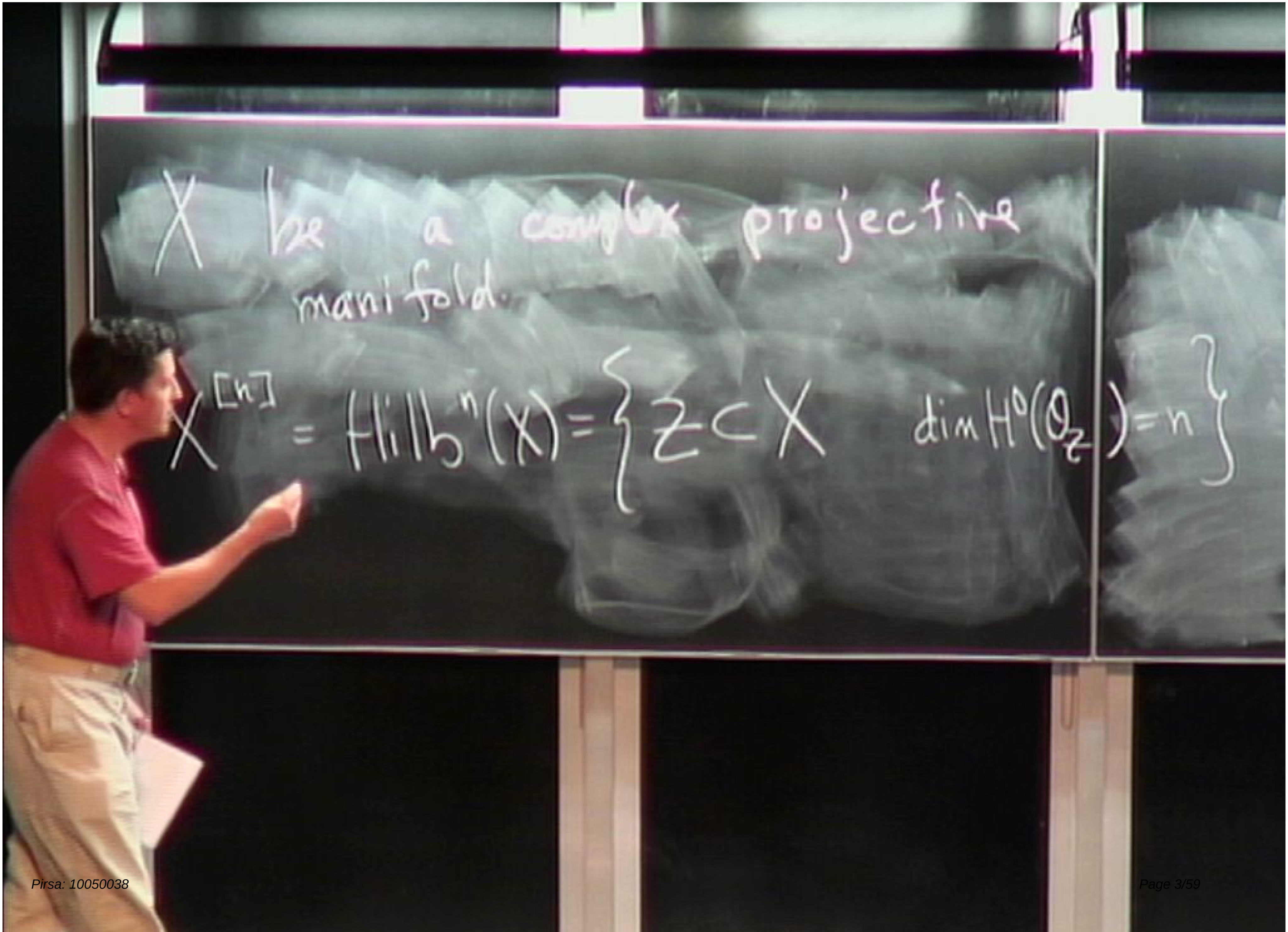
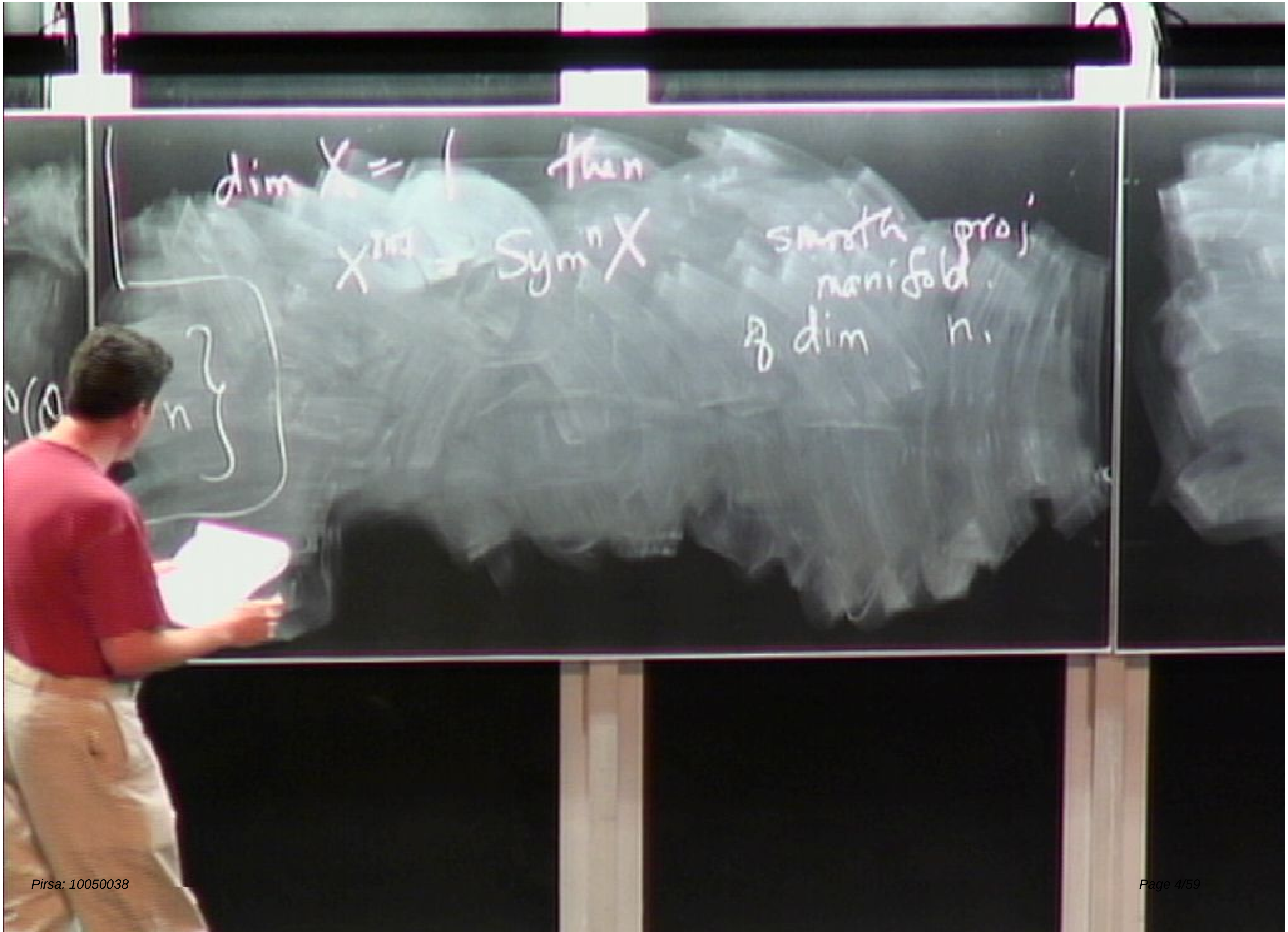
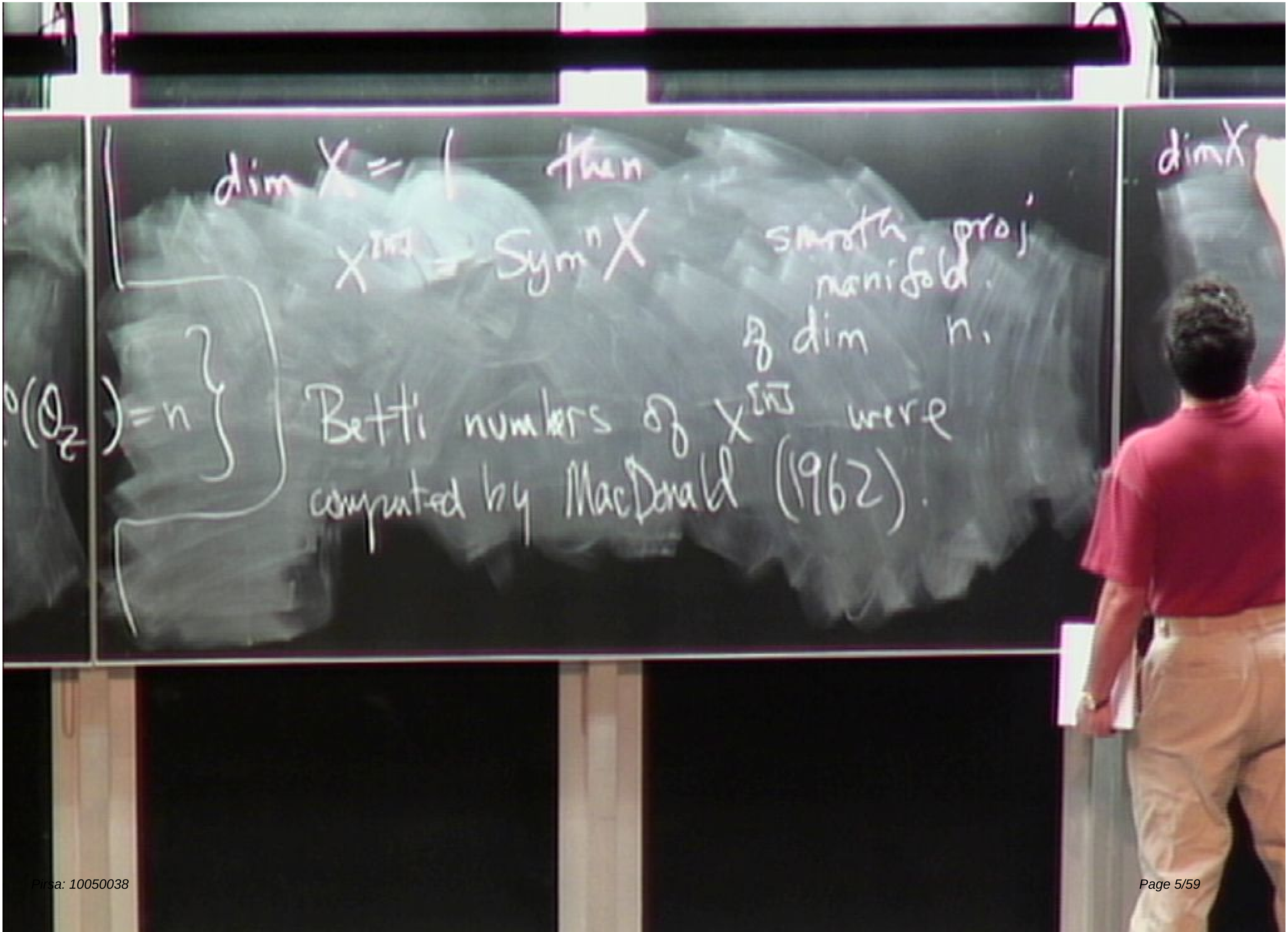
Abstract: The Hilbert scheme X[n] of n points on variety X parameterizes length n, zero dimensional subschemes of X. When X is a smooth surface, X[n] is also smooth and a beautiful formula for its motive was determined by Gottsche. When X is a threefold, X[n] is in general singular, of the wrong dimension, and reducible. However if X is a smooth Calabi-Yau threefold, X[n] has a canonical virtual motive --- a motification of the degree zero Donaldson-Thomas invariants. We give a formula analogous to Gottsche's for the virtual motive of X[n]. The key computation gives a q-refinement of the classical formula of MacMahon which counts 3D partitions.


X be a complex projective manifold.
$X^{[n]} = \mathrm{Hilb}^n(X) = \{ Z \subset X$


X be a complex projective manifold.
$X^{[n]} = \mathrm{Hilb}^n(X) = \{ Z \subset X \quad \dim H^0(\mathcal{O}_Z) = n \}$


dim X = 1 then
X^[n] = Sym^n X smooth proj manifold.
of dim n.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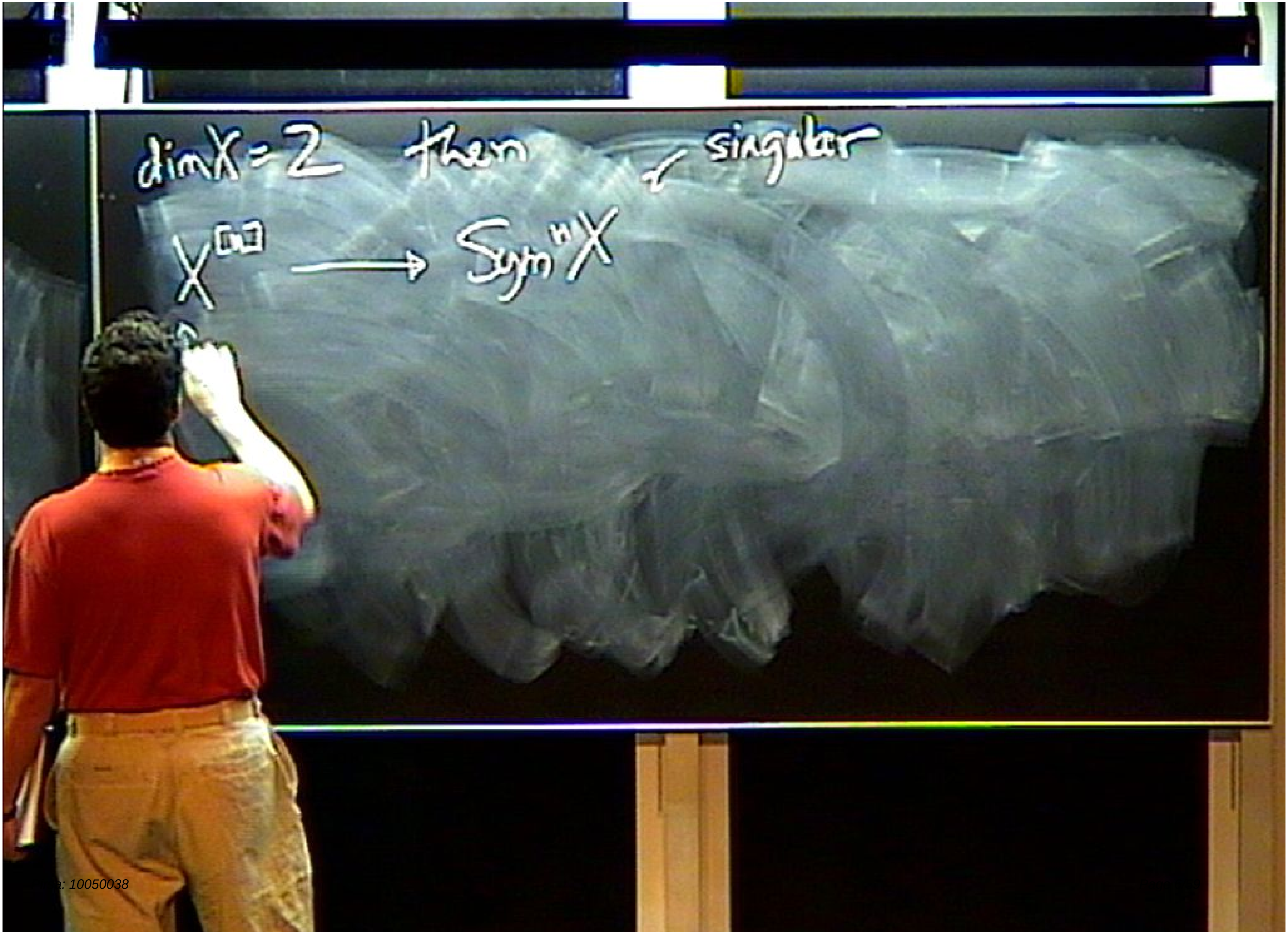
X^[n] = Sym^n X smooth proj manifold of dim n.
Betti numbers of X^[n] were computed by MacDonald (1962).
)=n
dimX


dim X = 2 then singular
X^[n] → Sym^n X
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$\dim X = 2$ then

singular

$$X^{[n]} \longrightarrow \mathrm{Sym}^n X$$

smooth it is a res. of singularities $\mathrm{Sym}^n X$

$\dim = 2n$

Betti numbers were computed by Göttsche (1990).

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dim $X = 3$ then

$X^{[n]}$ is singular

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$X^{[n]}$ is singular, not irreducible, not of the expected dimension, ...

But if X is a Calabi-Yau 3-fold, then $X^{[n]}$ is <u>virtually smooth</u>.

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But if X is a Calabi-Yau 3-fold, then $X^{[n]}$ is virtually smooth.

$X^{[n]}$ is a moduli space of sheaves (D-branes)

There are good notions of virtual Betti #s, virtual Euler char.

vir Euler char = (deg 0) Donaldson-Thomas invs of X

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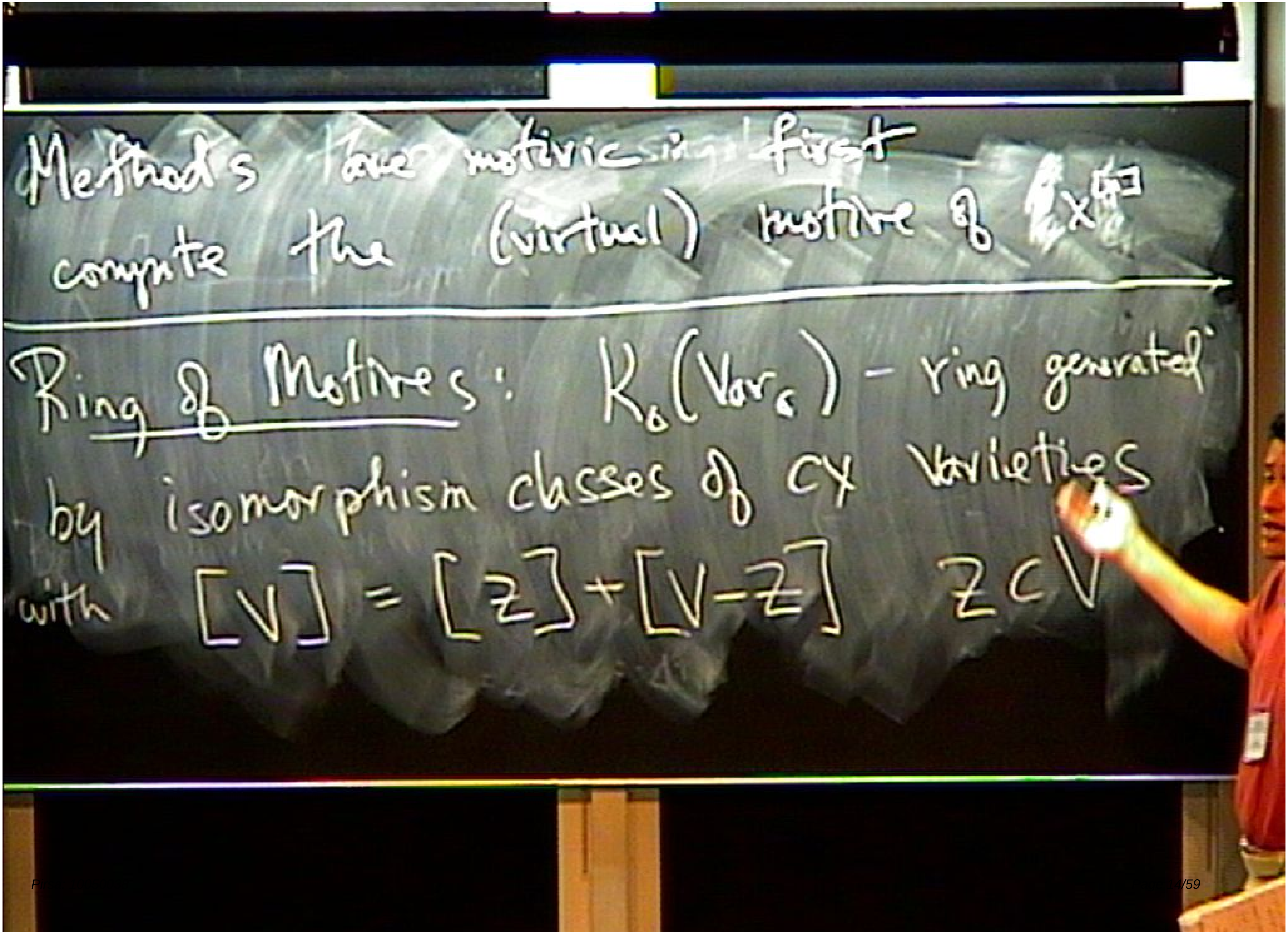
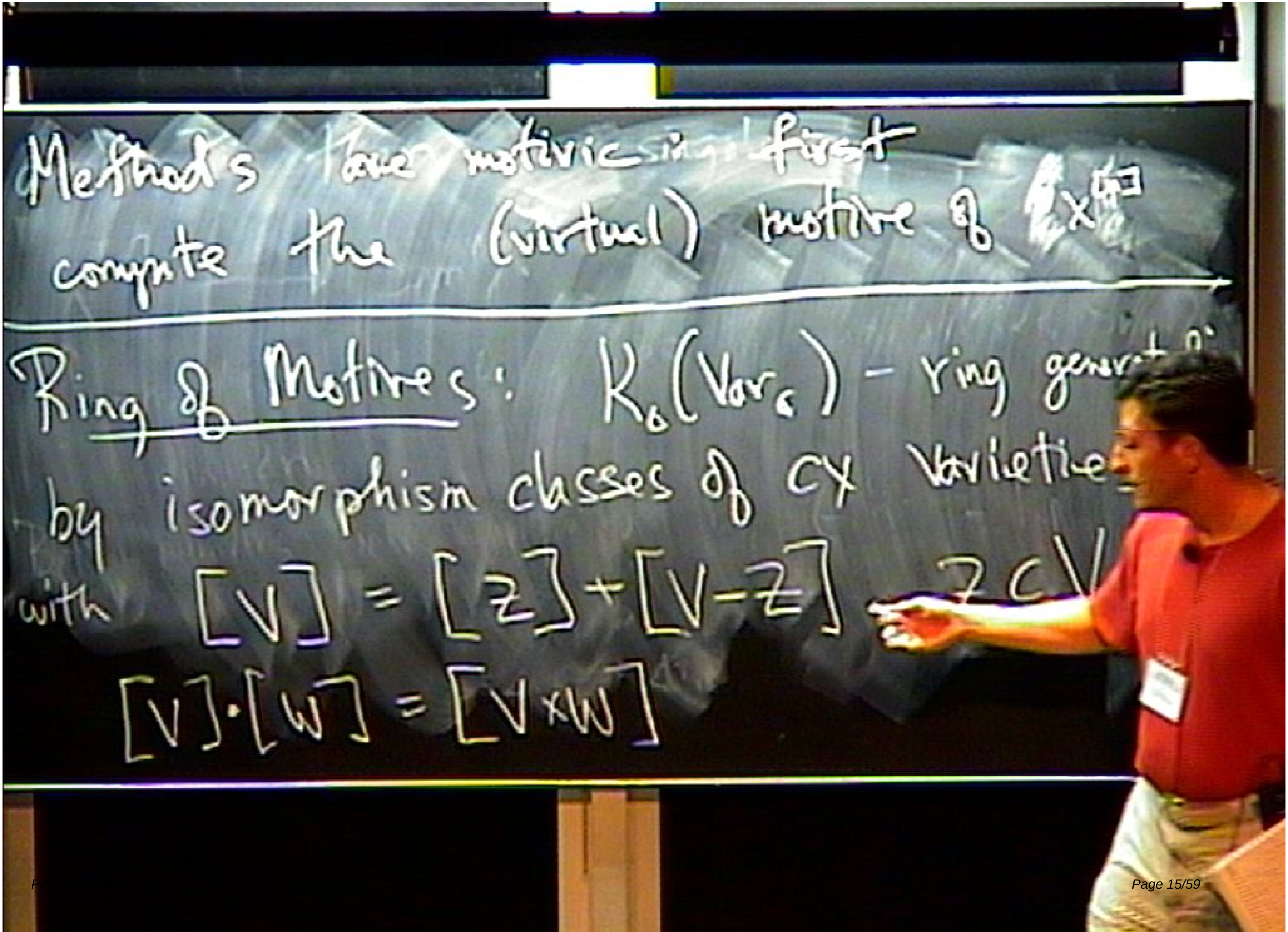

Methods motivic first
compute the (virtual) motive of X
Ring of Motives: K0(VarC) - ring generated
by isomorphism classes of cx varieties
with [V] = [Z] + [V-Z] Z ⊂ V


Methods
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Ring of Motives: $K_0(Var_{\mathbb{C}})$ - ring
by isomorphism classes of cx varieties
with $[V] = [Z] + [V-Z]$ $Z \subset V$
$[V] \cdot [W] = [V \times W]$

Let $\mathbb{L} = [\mathbb{C}]$

$$M_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{\pm 1/2}]$$

There is a ring homomorphism (use Deligne's mixed Hodge st

$$W : M_{\mathbb{C}} \longrightarrow \mathbb{Z}[q^{\pm 1/2}]$$

$$\mathbb{L} \longmapsto q^{1/2}$$

and if X a proj. manifold then

$W(X, q^{1/2})$ is the poincaré poly.

$$\chi : M_{\mathbb{C}} \longrightarrow \mathbb{Z}$$

$$V \longmapsto W(V, q^{1/2} = -1)$$

$$= \text{euler char}(V).$$

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Goal: write
in terms of L & [X]
P^n

Goal: write $[X^{(n)}]$ or $[\mathbb{P}^n_{\mathbb{C}}]$

in terms of $\mathbb{L}$ & $[X]$

$$[\mathbb{P}^n] = \left[\mathbb{C}^{n+1} - 0 \big/ \mathbb{C}^\times\right] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1} = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$$

&

Goal: write [illegible]

in terms of $\mathbb{L}$ & $[X]$

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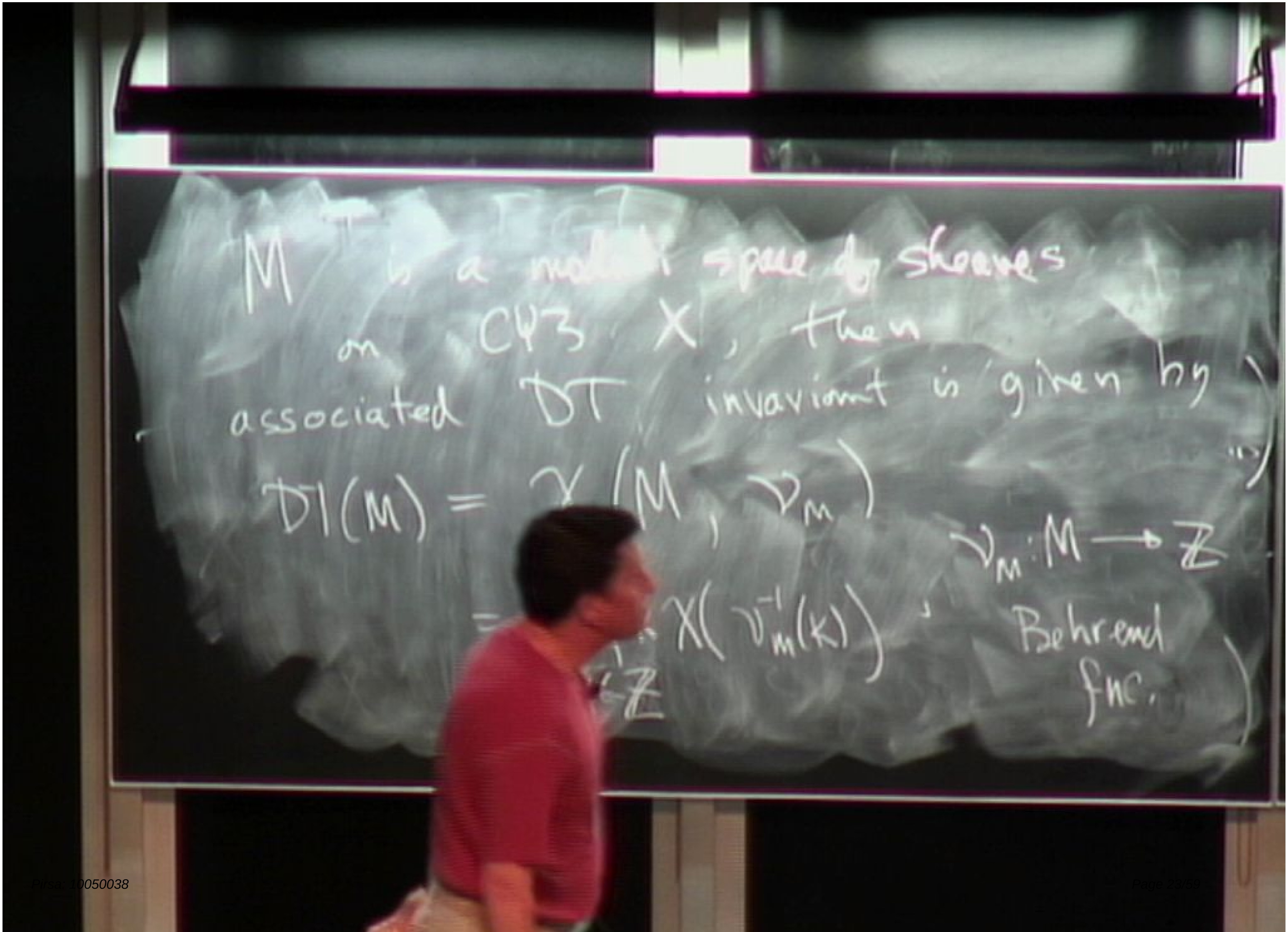
M is a moduli space of sheaves on CY3 X, then associated DT invariant is given by

$$DT(M) = \chi(M, \nu_M)$$

$$= \sum_{k \in \mathbb{Z}} k \, \chi(\nu_M^{-1}(k))$$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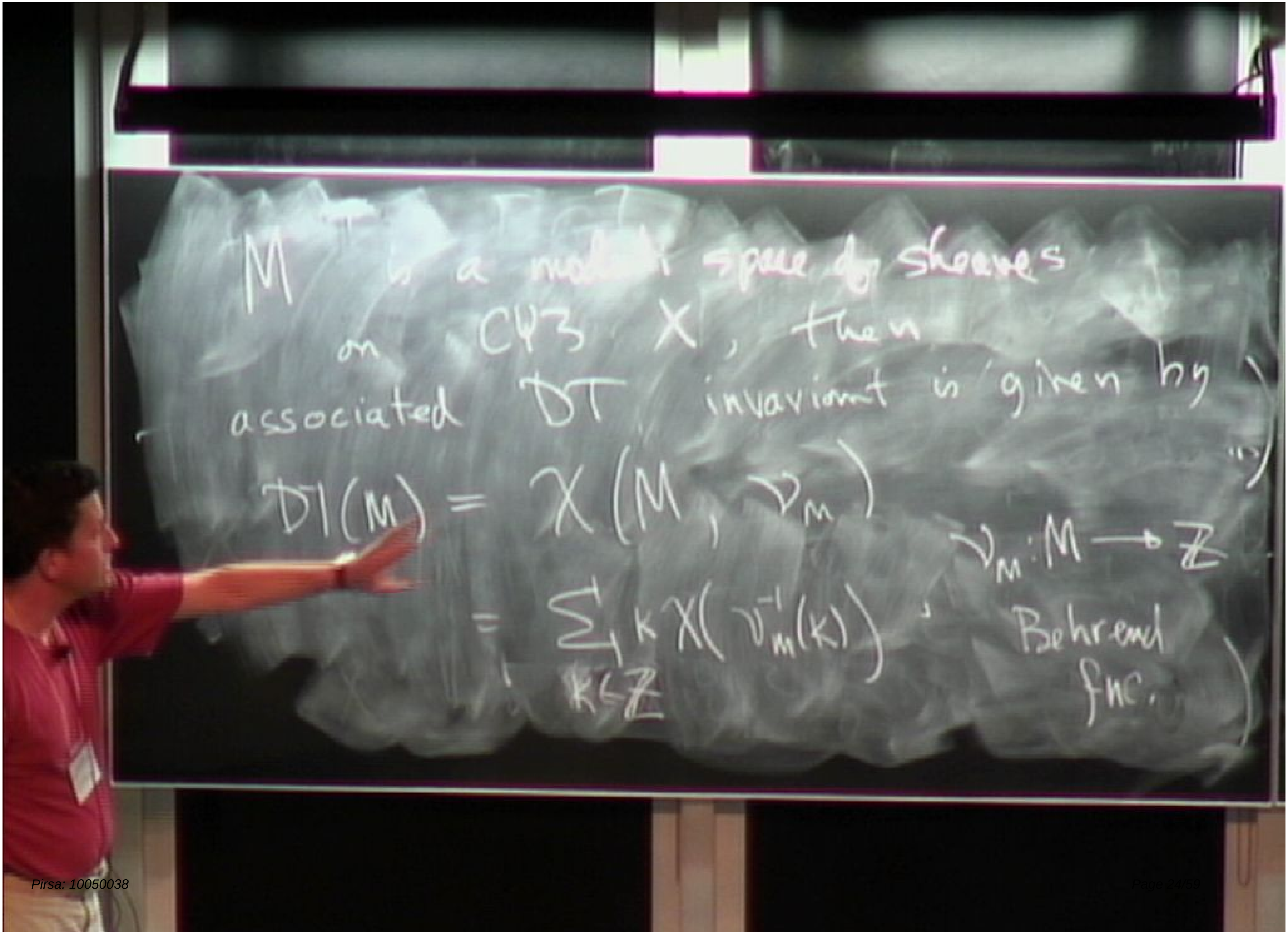
DT(M) = χ(M, ν_M)
ν_M : M → Z
= Σ k χ(ν_M^{-1}(k))
k∈Z
Behrend fnc.



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$\nu_M : M \to \mathbb{Z}$
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Problem: Associate a motive $[M]_{vir}$

such that $\chi([M]_{vir}) = DT(M)$

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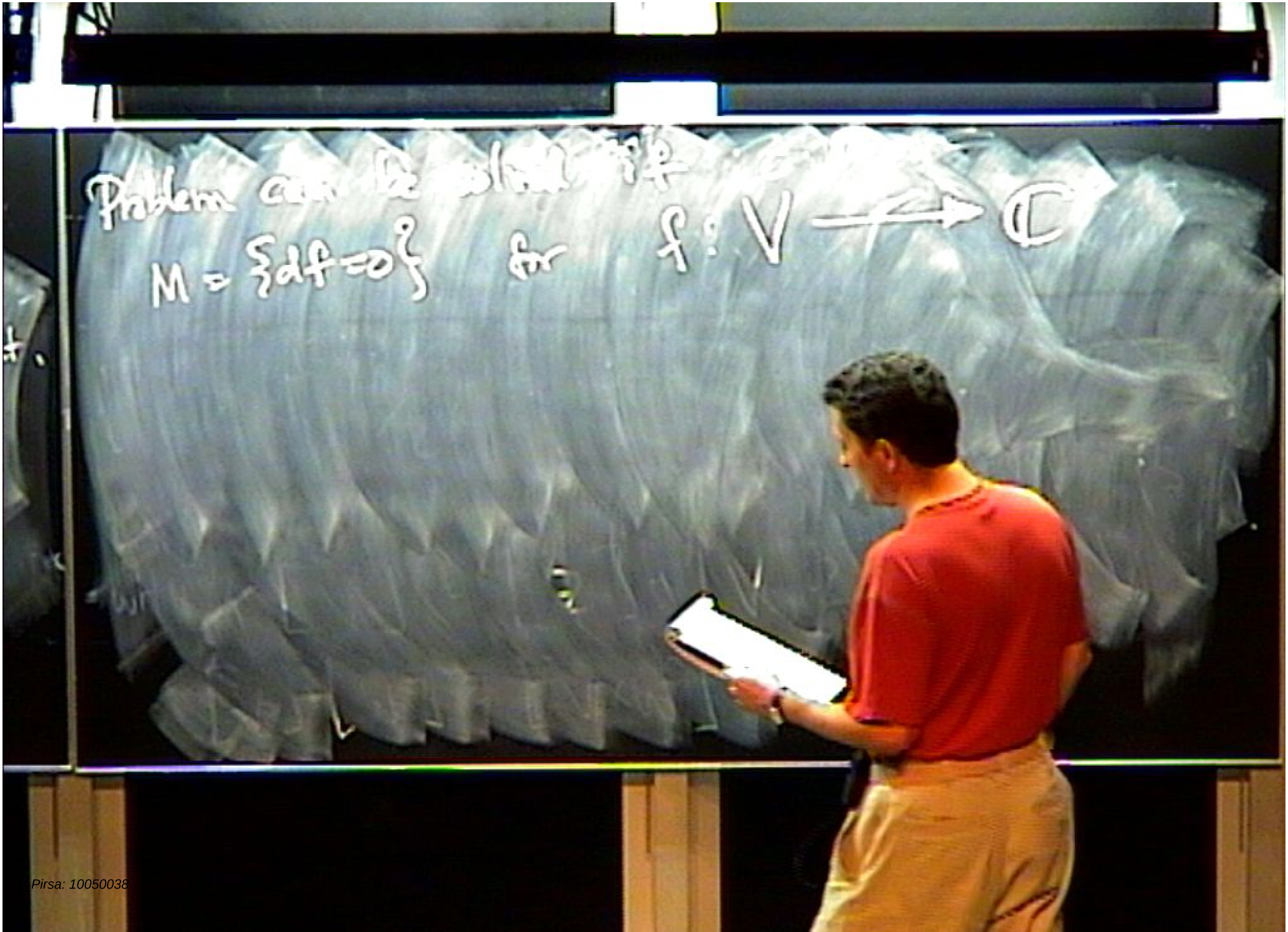
Can be done for $X^{[n]}$


Problem can be solved if
M = {df = 0} for f : V → ℂ

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$M = \{df = 0\}$ for $f : V \to \mathbb{C}$

V smooth manifold

Problem can be solved if [illegible] or super potential.

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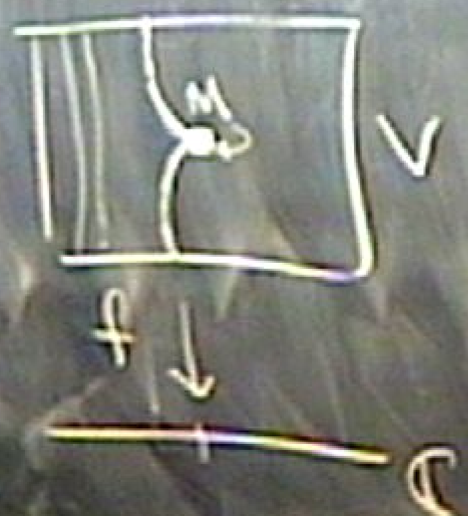
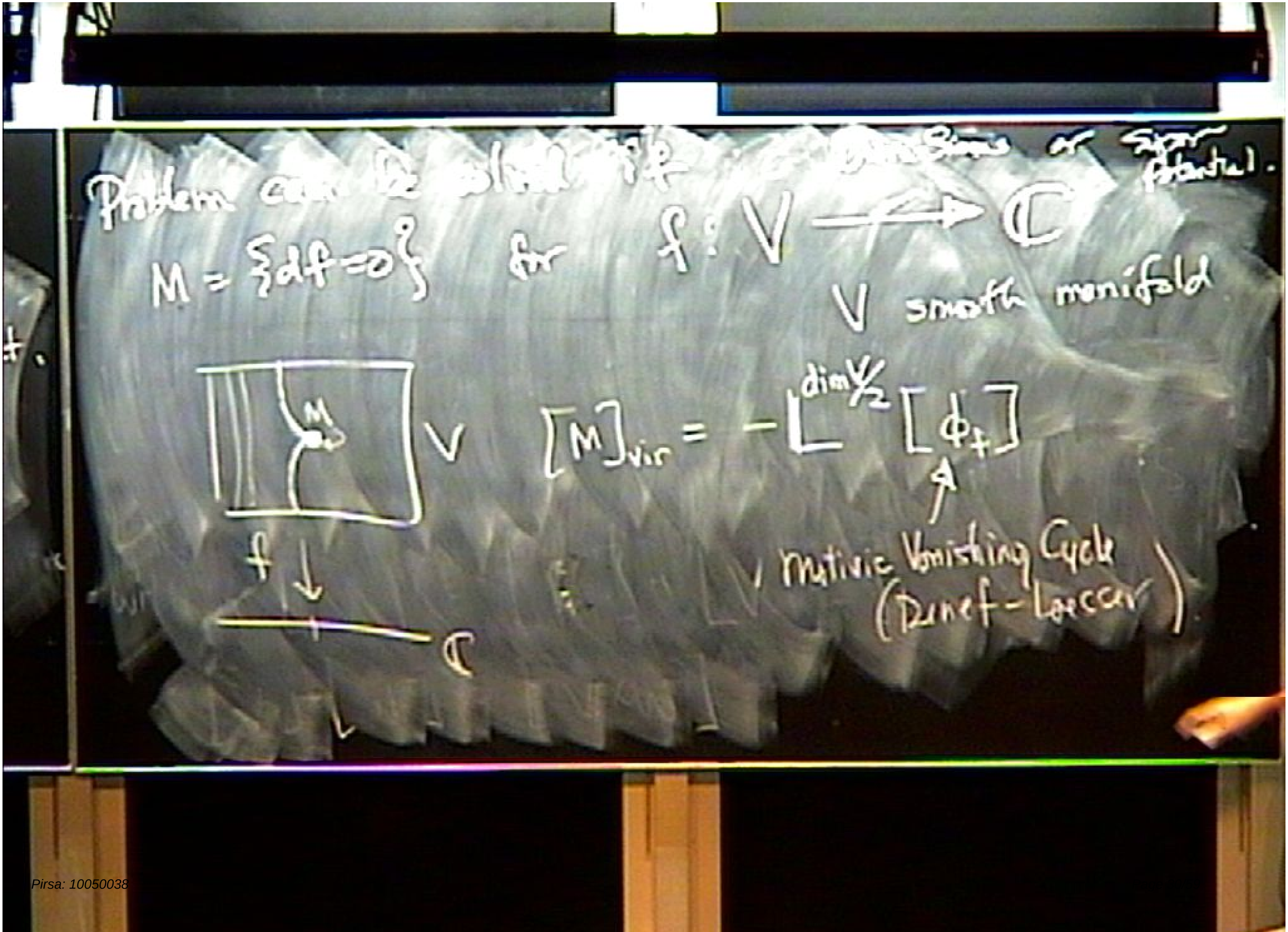
V smooth manifold



$$[M]_{vir} = -\mathbb{L}^{\dim V/2}[\phi_f]$$



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M = {df = 0} for f : V → ℂ
V smooth manifold
M
V
f ↓
ℂ
[M]vir = − L^(dim V/2) [φf]
motivic Vanishing Cycle
(Denef–Loeser)

e.g.

$$[M]_{vir} = [\quad [M]$$

e.g.

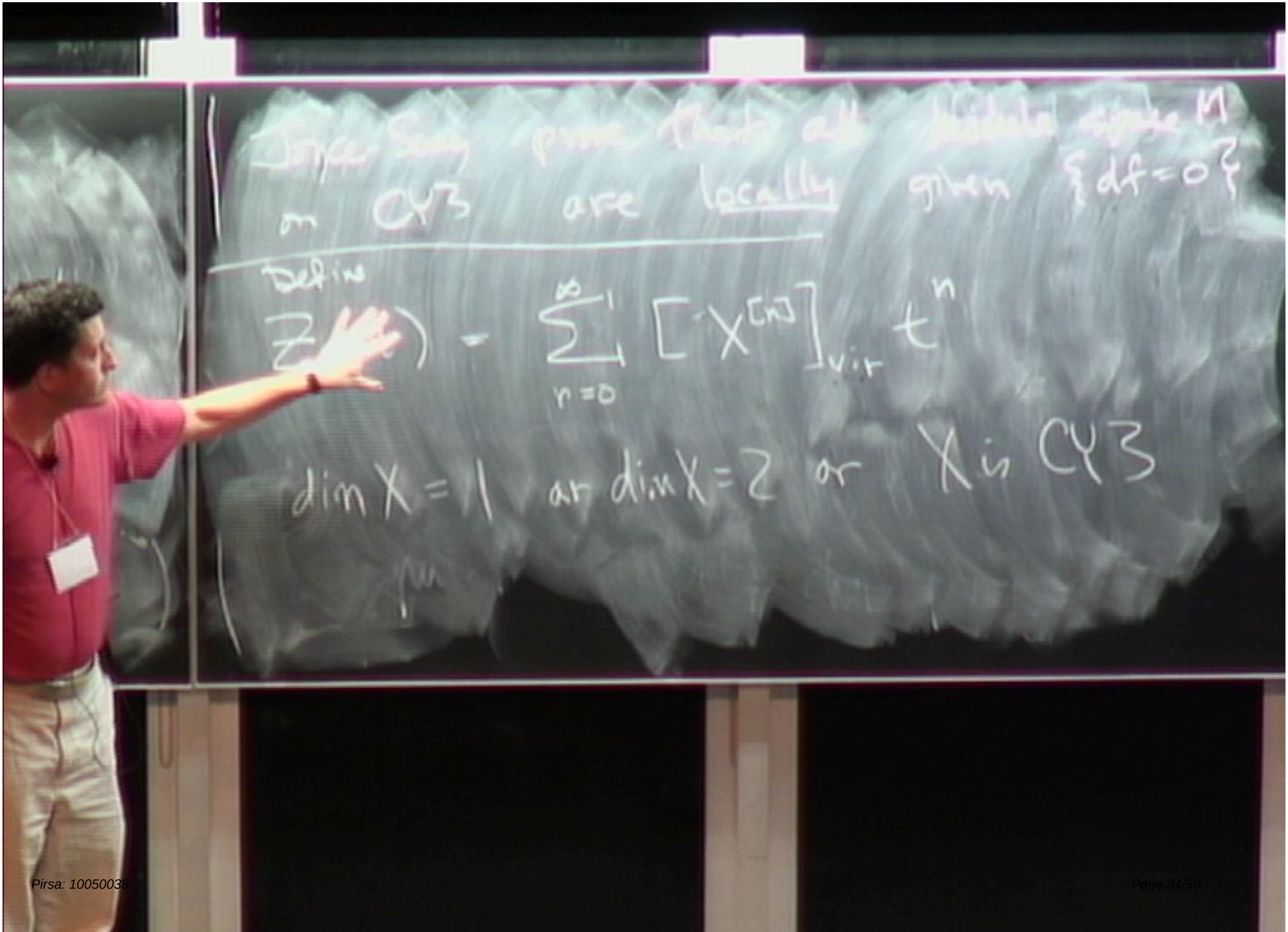
e.g.

$$[M]_{vir} = \mathbb{L}^{-\dim/2}[M]$$

e.g. if $f : V \to \mathbb{C}$ is $\mathbb{C}^\times$ equiv. wrt. to a weight 1 action on $\mathbb{C}$, then

$$[\phi_f] = [f^{-1}(1)] - [f^{-1}(0)]$$


wrt.
on CY3 are locally given {df=0}


Joyce Song prove that all moduli space M
on CY3 are locally given {df = 0}
Define
$Z(\ \ ) = \sum_{n=0}^{\infty} [X^{[n]}]_{vir}\, t^n$
dim X = 1 or dim X = 2 or X is CY3

wrt.

Joyce Song prove that all moduli space M on CY3 are locally given $\{df=0\}$

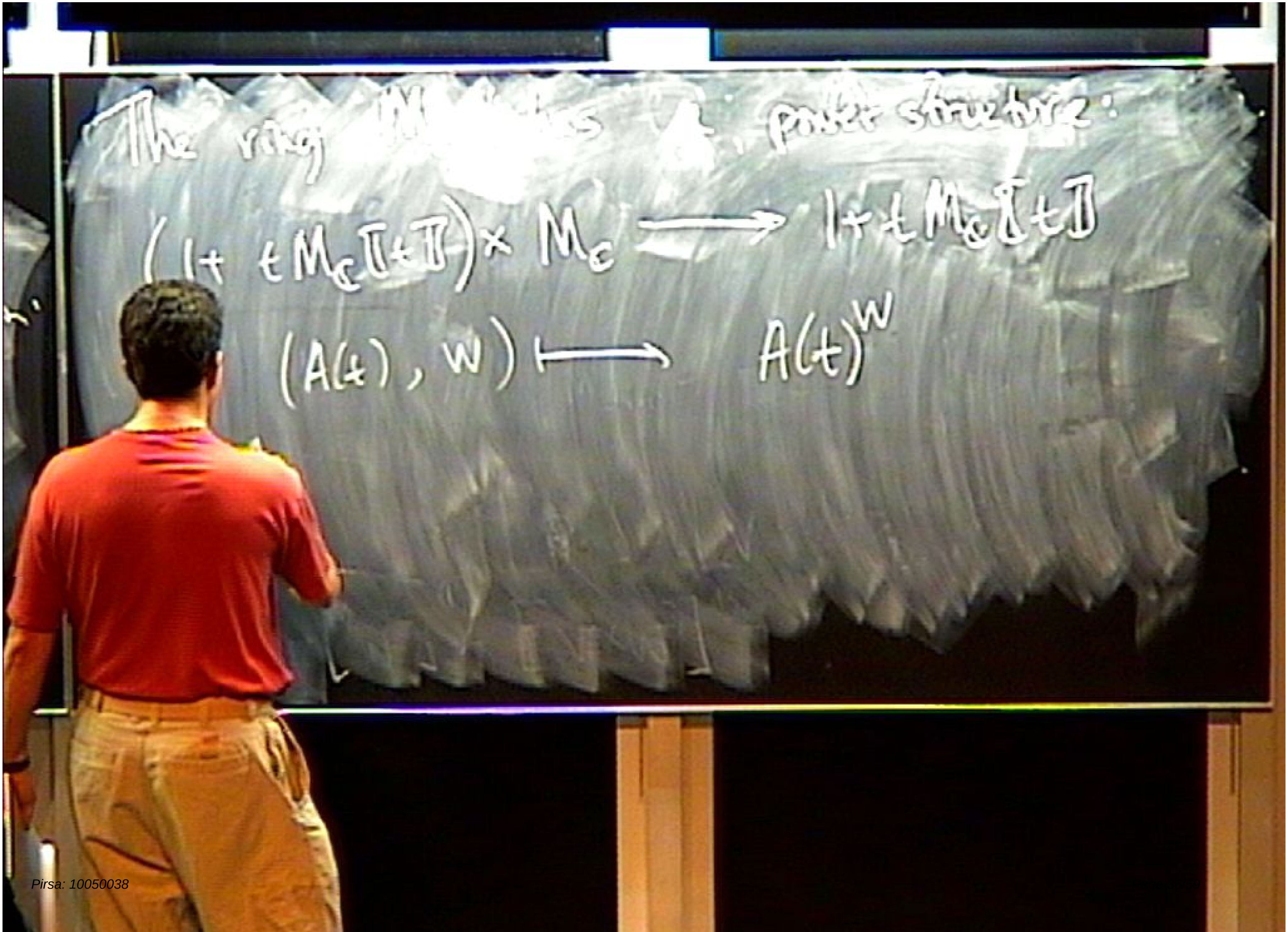
Define

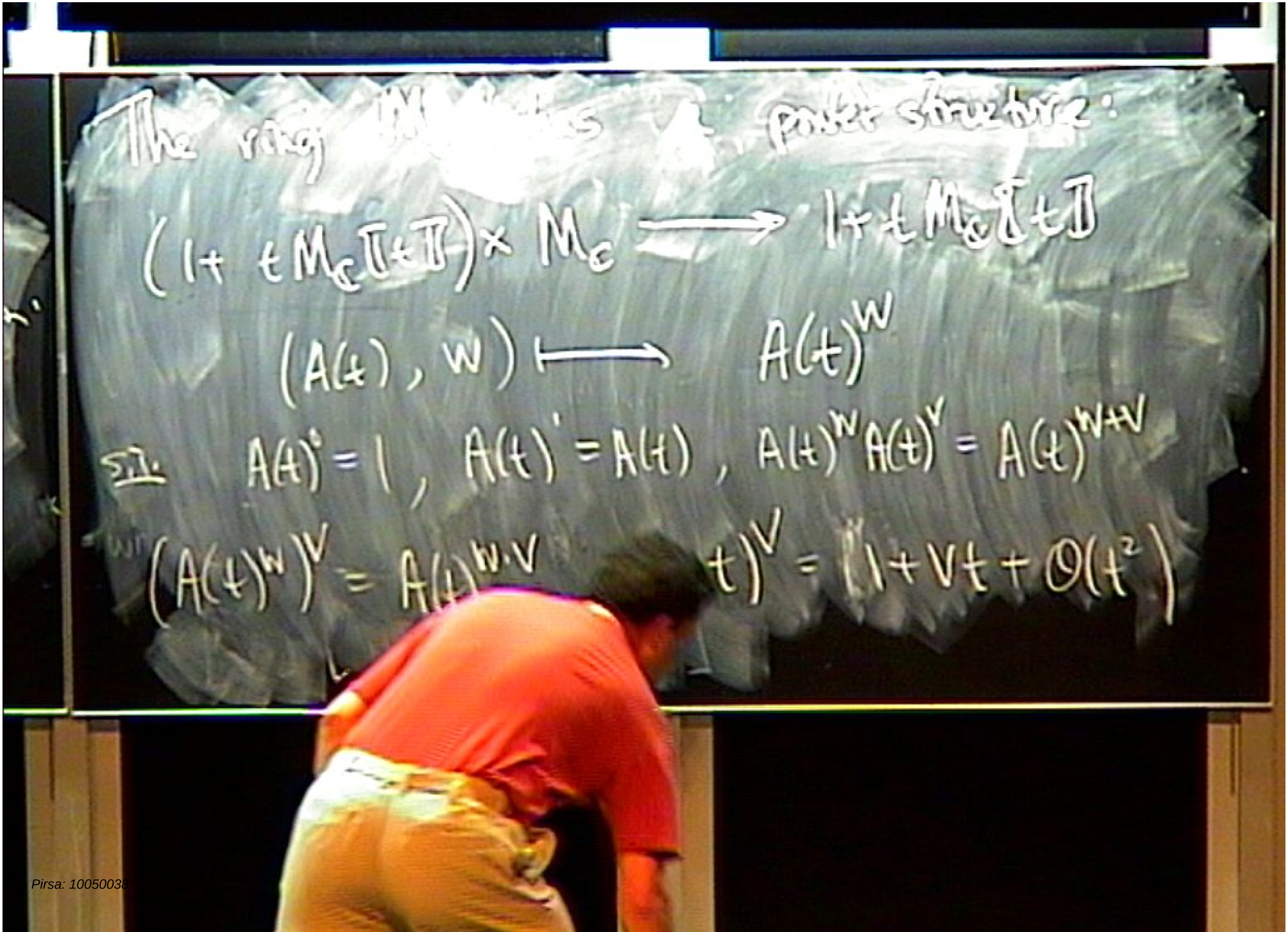
$$Z_X(t) = \sum_{n=0}^{\infty} [X^{[n]}]_{vir} \, t^n$$

Motivic DT partition fnc.

dim $X = 1$ or dim $X = 2$ or X is CY3

$\chi(Z_X(t))$ usual DT part fnc.


The ring
power structure:
$(1+tM_c[[t]]) \times M_c \longrightarrow 1+tM_c[[t]]$
$(A(t), w) \longmapsto A(t)^w$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s.t. $A(t)^0 = 1$, $A(t)^1 = A(t)$, $A(t)^w A(t)^v = A(t)^{w+v}$
$(A(t)^w)^v = A(t)^{w\cdot v}$
$t)^v = 1+vt+\mathcal{O}(t^2)$

The ring … , power structure:

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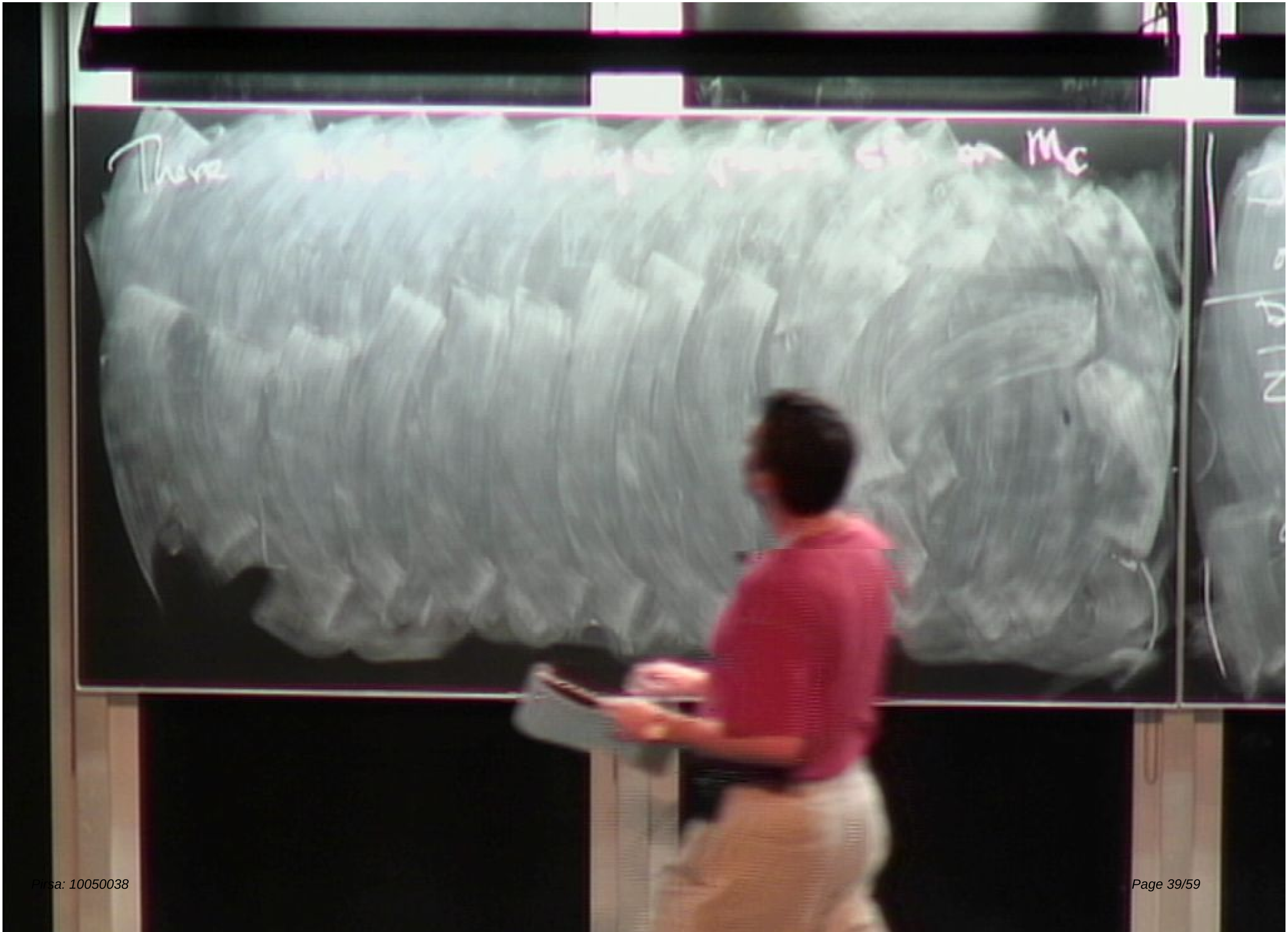
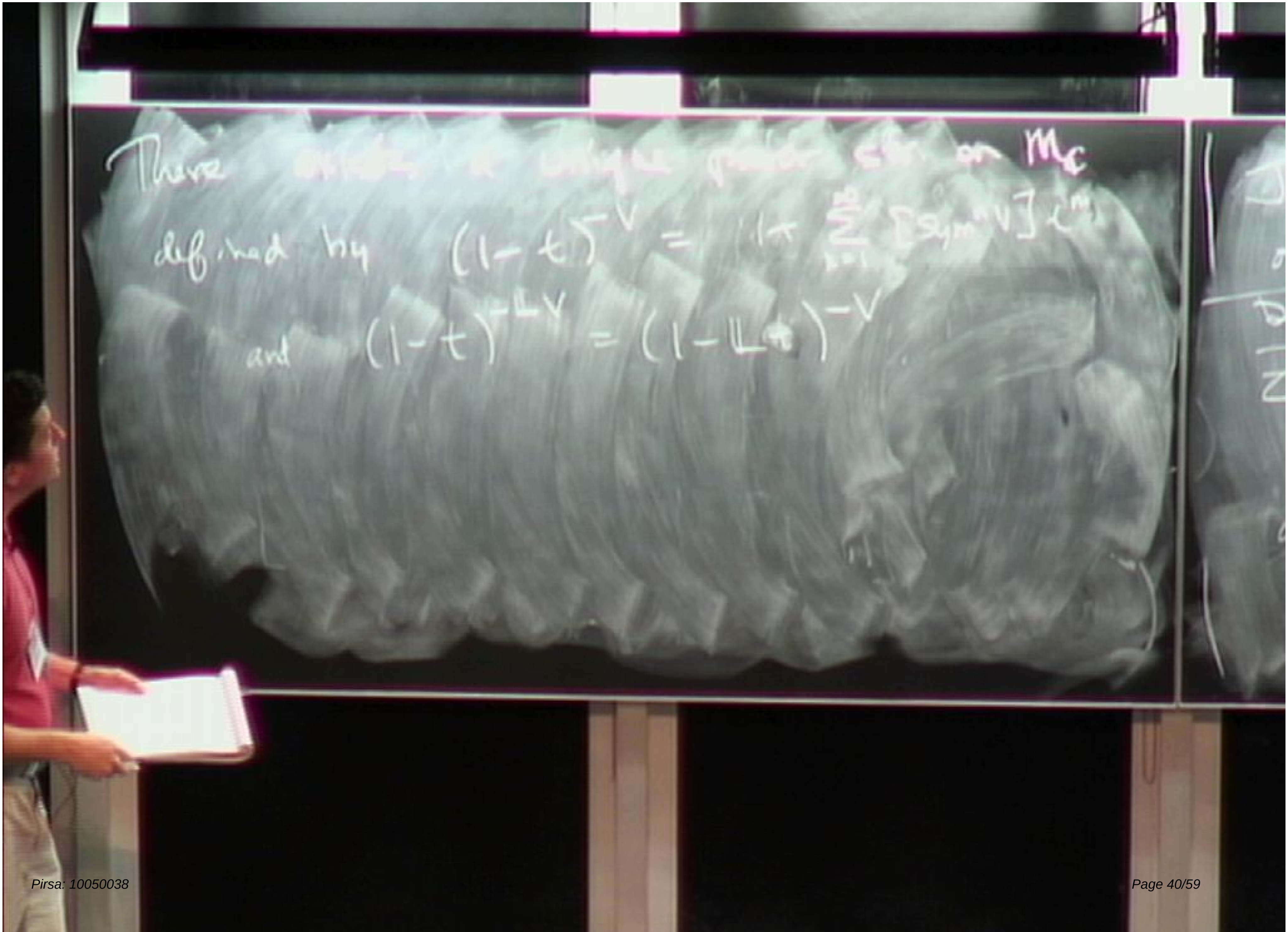
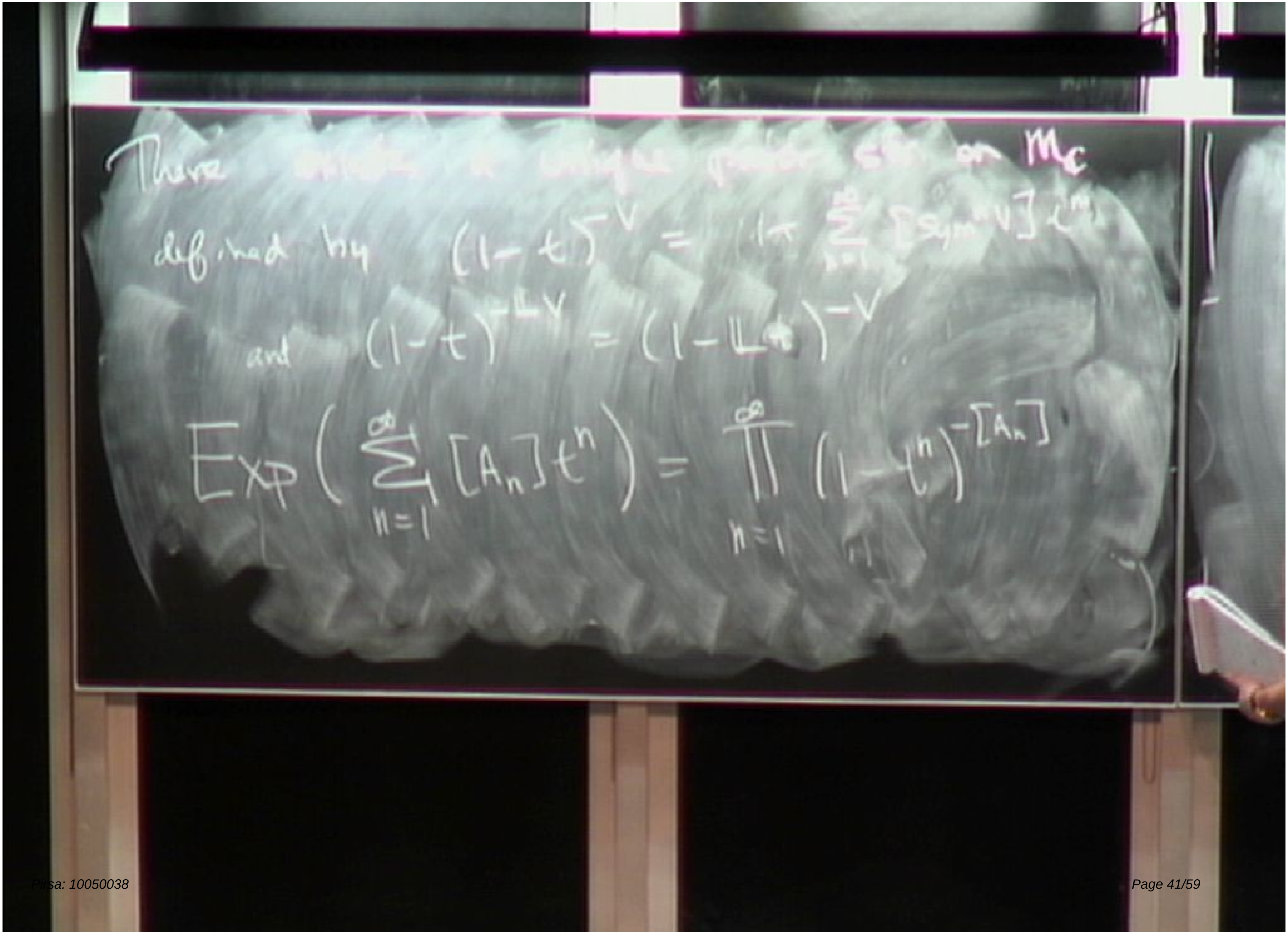
$$(A(t)^{w})^{v} = A(t)^{w \cdot v} \qquad (1+t)^{v} = 1 + vt + \mathcal{O}(t^{2})$$


There
on Mc


There ... on M_C
defined by $(1-t)^{-V} = 1 + \sum_{n \geq 1} [\mathrm{Sym}^n V] t^n$
and $(1-t)^{-\mathbb{L} V} = (1-\mathbb{L}t)^{-V}$

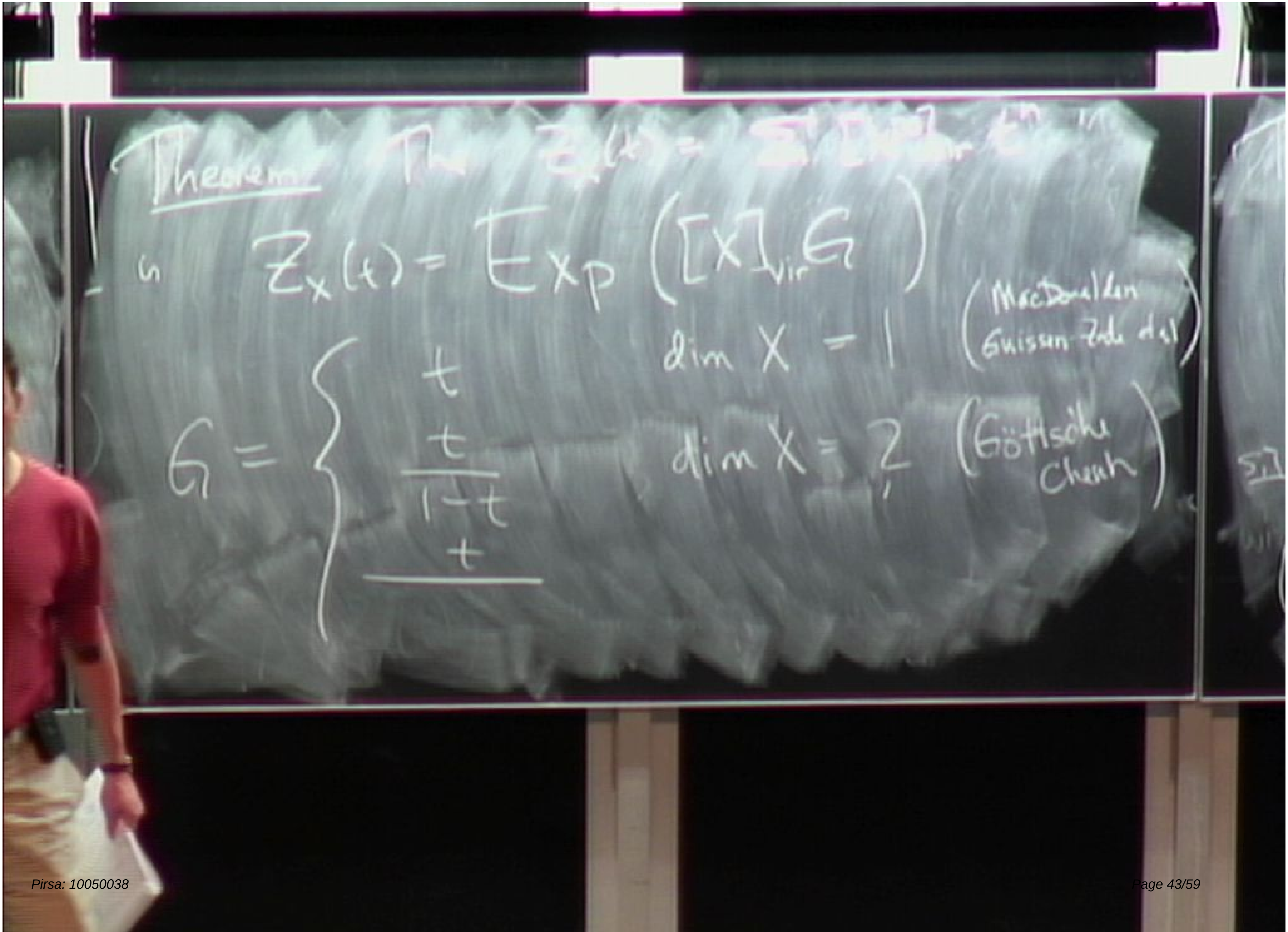

There
defined by (1-t)^-V = 1+ Σ [Sym^n V] t^n
and (1-t)^-LV = (1-Lt)^-V
Exp(Σ_{n=1}^∞ [A_n] t^n) = Π_{n=1}^∞ (1-t^n)^-[A_n]

Theorem

$$Z_X(t) = \mathrm{Exp}\left([X]\, G\right)$$

$\dim X = 1$

$$G = \begin{cases} t \\ \end{cases}$$



Theorem
Zx(t) = Exp([X]vir G)
dim X = 1
(MacDonald Guisun Zade et al)
dim X = 2
(Göttsche Cheah)
G = t
t / 1-t
t

Theorem

$$Z_X(t) = \mathrm{Exp}\left([X]_{vir}\, G\right)$$

$$G = \begin{cases} t & \dim X = 1 \quad (\text{MacDonald, Gusein-Zade et al}) \\ \dfrac{t}{1-t} & \dim X = 2 \quad (\text{Göttsche, Cheah}) \\ \dfrac{t}{(1-\mathbb{L}^{1/2}t)(1-\mathbb{L}^{-1/2}t)} & X \ \text{CY3} \quad \text{Behrend-B-Szendroi.} \end{cases}$$

Key example: $X = \mathbb{C}^3$:

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-m/2}\, t^m\right)^{-1}$$

$$\mathbb{L}^{1/2} \mapsto -1$$

$$\chi Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \left(1 - (-t)^m\right)^{-m}$$

generating fn for 3D partitions

The moduli space $(\mathbb{C}^3)^{[n]}$ has explicit description

$$(\mathbb{C}^3)^{[n]} = \left\{ I \subset \mathbb{C}[x,y,z] \;:\; \dim_{\mathbb{C}} \mathbb{C}[x,y,z]/I = n \right\}$$

claim $(\mathbb{C}^3)^{[n]} = \{ df = 0 \}$ $\quad f : V \longrightarrow \mathbb{C}$

$$V = \left\{ (A,B,C,v) \in \mathrm{Mat}^3_{n\times n} \times \mathbb{C}^n \right\}^{st} \Big/ Gl_n(\mathbb{C})$$

$$f: V \longrightarrow \mathbb{C}$$

$$f(A,B,C,u) = \mathrm{tr}(A[B,C])$$

$$df = 0 \iff [B,C]=0,\ [A,B]=0,\ [A,C]=0$$

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fix an isomorphism $\mathbb{C}[x,y,z]/I \cong \mathbb{C}^n$

description

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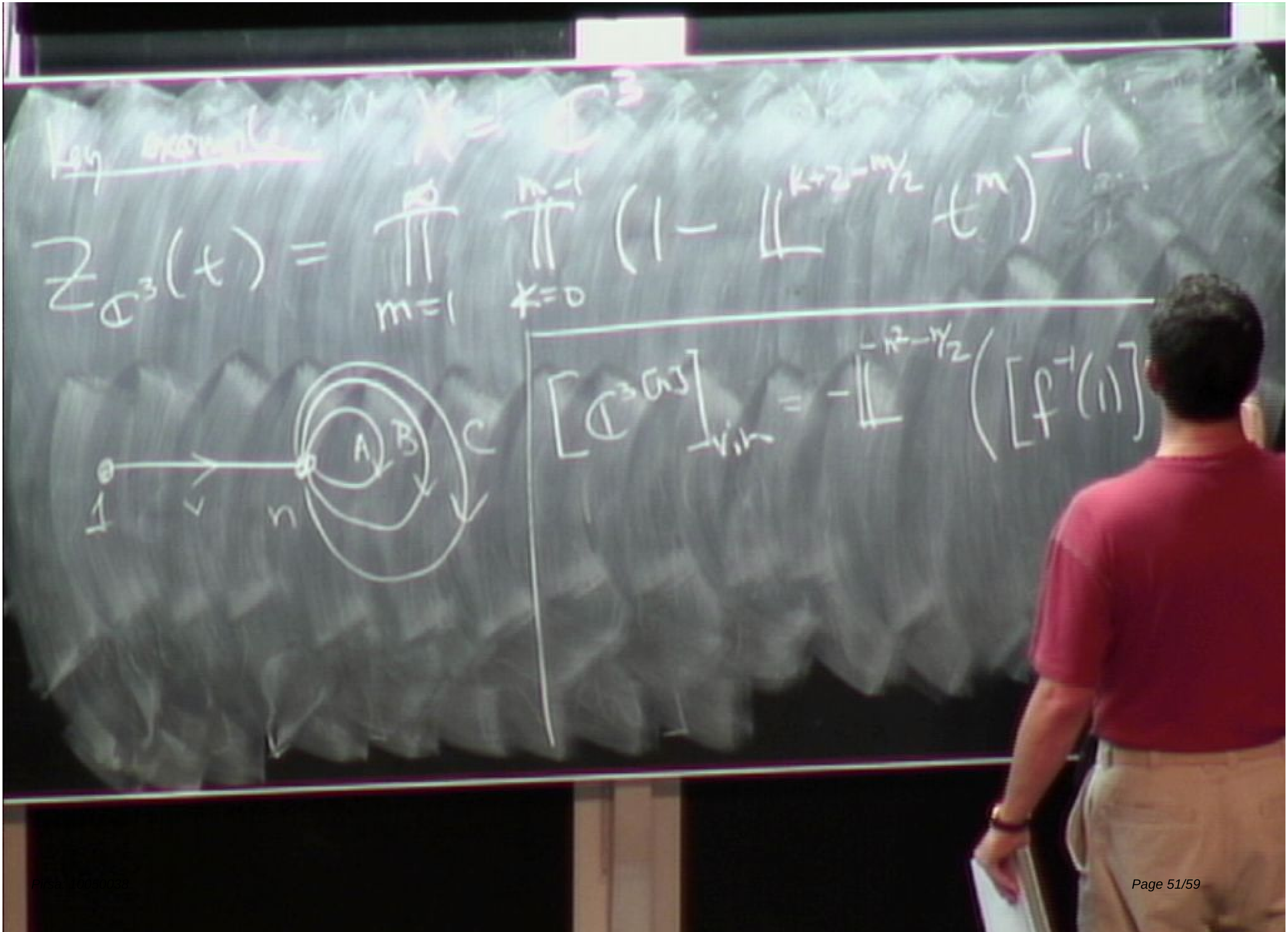
A, B, C correspond to multiplication by x, y, z

v corresponds to 1

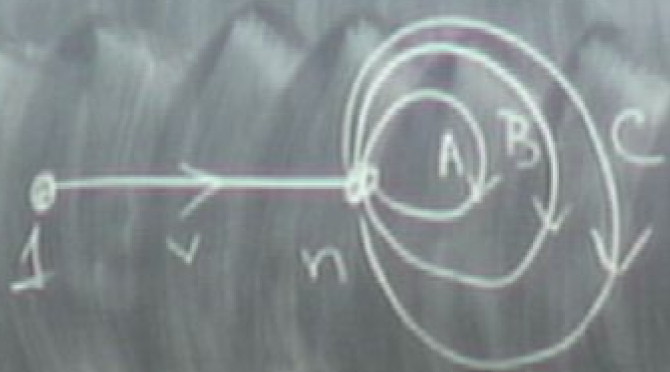
Key example: $X = \mathbb{C}^3$

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-m/2}\, t^m\right)^{-1}$$

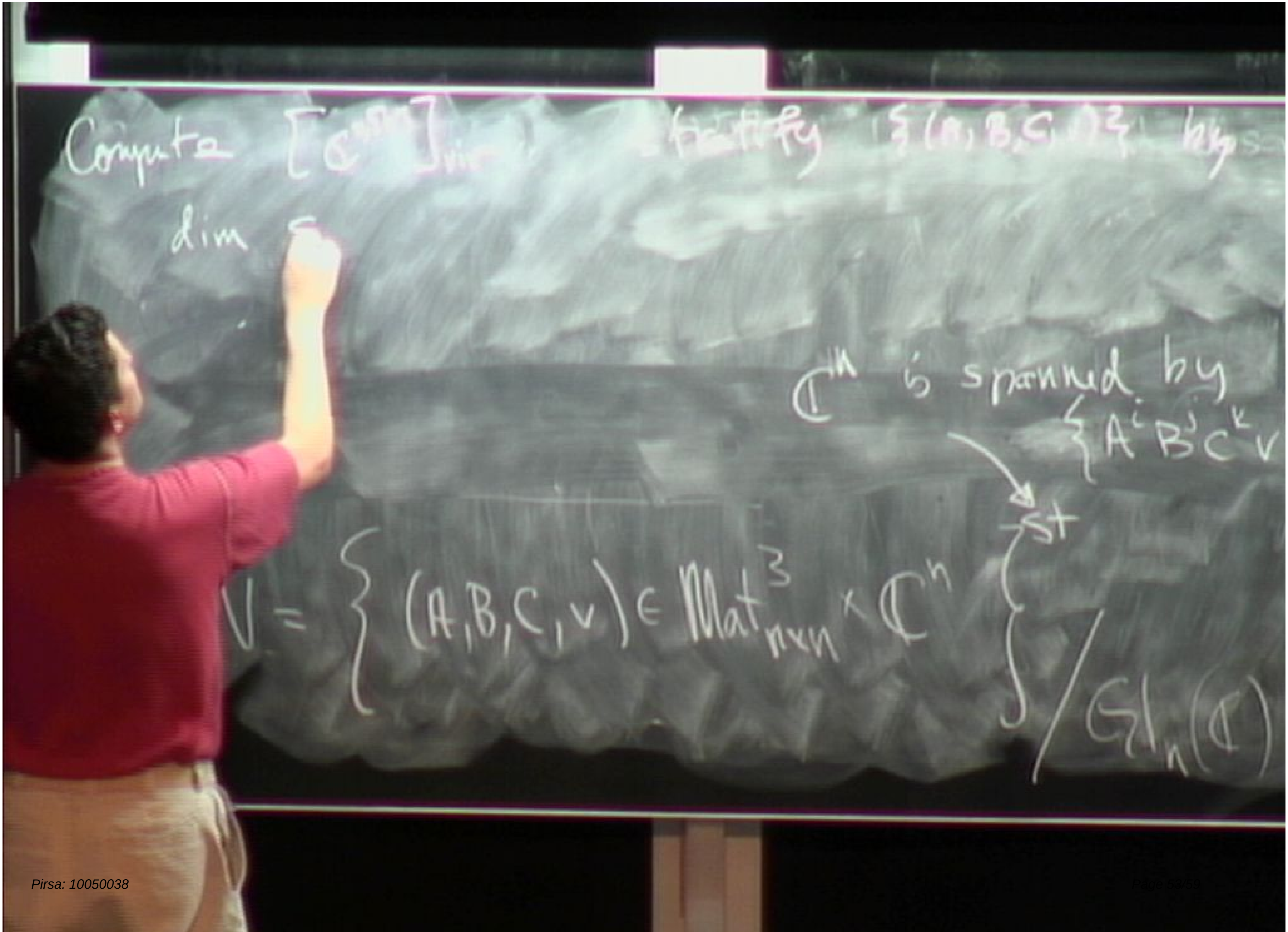

Key example: X = C^3
Z_{C^3}(t) = ∏_{m=1}^{∞} ∏_{k=0}^{m-1} (1 − L^{k+2−m/2} t^m)^{−1}
[C^3[n]]_{v,h} = −L^{−n^2−n/2} ([f^{−1}(1)]
1
v
n
A
B
C

Key example: $X = \mathbb{C}^3$

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-m/2}\, t^m\right)^{-1}$$



$$\left[\mathbb{C}^3[n]\right]_{vir} = -\mathbb{L}^{-n^2 - n/2}\left(\left[f^{-1}(1)\right] - \left[f^{-1}(0)\right]\right)$$


Compute
dim
$\mathbb{C}^n$ is spanned by
$\{A^i B^j C^k v$
st
$V = \{(A,B,C,v) \in \mathrm{Mat}^3_{n\times n} \times \mathbb{C}^n\} / Gl_n(\mathbb{C})$

Compute $[\mathbb{C}^{3[n]}]_{vir}$, satisfy $\{(A,B,C,v)\}$ ...

$\dim \operatorname{span}\{A^iB^jC^kv\}_{i,j,k} \rightsquigarrow$ recursive form...

for $[\mathbb{C}^{3[n]}]_{vir}$

$\mathbb{C}^n$ is spanned by $\{A^iB^jC^kv\}$

$$V = \left\{ (A,B,C,v) \in \mathrm{Mat}^3_{n\times n} \times \mathbb{C}^n \right\}^{st} \Big/ Gl_n(\mathbb{C})$$

Key input in the motive of the commuting variety:

$$C_n = \{ (A,B) \in Mat^2_{n\times n} : [A,B] \}$$

$$\sum_{n=0}^{\infty} \frac{[C_n]}{[Gl_n]} t^n = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} (1 - \mathbb{L}^{1-j} t^m)^{-1}$$

(Feit-Fine 40's

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(Feit-Fine 40's

Recently: Andrew Thomson extended these results to CY3 orbifolds, with local models $\mathbb{C} \times (\mathbb{C}^2/G)$ $\quad G \subset SU(2)$

McKay Quiver

$$Z_{\mathbb{C}\times\mathbb{C}^2/G}(t, t_1, \dots, t_k) = \mathrm{Exp}\left([\mathbb{C}^2/G \times \mathbb{C}]_{vir,orb} \right.$$

non-triv
repr of G

$$\frac{t}{(1-\mathbb{L}^{1/2}t)(1-\mathbb{L}^{-1}t)}$$

$$\underbrace{\mathbb{L}^{3/2} + \mathbb{L}^{1/2}\left(k-1 + \sum_{\alpha\in R}{}' \, t^{\alpha}\right)}_{[I(\mathbb{C}^2/G \times \mathbb{C})]_{vir}}$$

ADE root system

View repr of G on elts of root lattice.