Title: Representations of Generalized Clifford Algebras

Date: May 08, 2010 01:00 PM

URL: http://pirsa.org/10050037

Abstract: Clifford algebras arose in Dirac's work on the relativistic wave equation in quantum mechanics. Using the Clifford algebra associated to a quadratic form on a finite dimensional vector space, one can reduce the relativistic wave equation, a PDE of order two, to a system of linear PDEs. Similarly, one can use matrix representations of generalized (i.e. higher degree) Clifford algebras to reduce a PDE of higher degree. These generalized Clifford algebras have been the subject of ongoing research since late 1980s. In this talk, we will discuss generalized Clifford algebras, known results about their representations, and results of ongoing work in this direction.

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

Representations of Generalized Clifford Algebras

Emre Coskun
ecoskun@uwo.ca

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7 CANADA

Connections in Geometry and Physics 2010

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

The Classical Case

Assume that the base field k has $char(k) \neq 2$. Let (V, q) be a quadratic space.

Definition

The Clifford algebra of (V, q) is defined as $C(V, q) = T(V)/I$, where $T(V)$ is the tensor algebra of V and I is the two-sided ideal generated by elements of the form $x \otimes x - q(x)1$ for $x \in V$.

Generalized Clifford Algebras

Let $f(u, v)$ be a binary form of degree d. We assume that f is nondegenerate, i.e. has no repeated factors over the algebraic closure of k. We also assume that $char(k)$ does not divide d.

Generalized Clifford Algebras

Let $f(u, v)$ be a binary form of degree d. We assume that f is nondegenerate, i.e. has no repeated factors over the algebraic closure of k. We also assume that $char(k)$ does not divide d.

Generalized Clifford Algebras

Let $f(u, v)$ be a binary form of degree d. We assume that f is nondegenerate, i.e. has no repeated factors over the algebraic closure of k. We also assume that $char(k)$ does not divide d.

Definition

The generalized Clifford algebra C_f is defined to be the k-algebra $k\{u, v\}/I$, where I is the two-sided ideal generated by elements of the form $(\alpha u + \beta v)^d - f(\alpha, \beta)$ with $\alpha, \beta \in k$.

rreducible Representations

Definition

A representation of C_f is a k-algebra homomorphism $\phi : C_f \to M_m(L)$ where L is a field extension of k. n is called the dimension of the representation. ϕ is called irreducible if $\phi(C_f)L = M_m(L)$.

- $d = 2$. Assume $dim(V) = n$. Then $dim(C_f) = 2^n$. So there always exists a (finite-dimensional) representation of C_f.

- $d = 2$. Assume $dim(V) = n$. Then $dim(C_f) = 2^n$. So there always exists a (finite-dimensional) representation of C_f.
- $d \geq 3$. C_f is infinite-dimensional ([Childs]), so the above argument does not work.
- The dimension of a representation of C_f is divisible by d. [Haile-Tesser]

Definition

Let R be a commutative ring with identity. An Azumaya algebra is an R-algebra A that is a faithful, projective R-module of finite rank such that the canonical map $\phi_A : A \otimes A^\circ \to End_{R-mod}(A)$ that is given by $a \otimes a' \mapsto (x \mapsto axa')$ is an isomorphism.

Definition

Let R be a commutative ring with identity. An Azumaya algebra is an R-algebra A that is a faithful, projective R-module of finite rank such that the canonical map $\phi_A : A \otimes A^\circ \to End_{R-mod}(A)$ that is given by $a \otimes a' \mapsto (x \mapsto axa')$ is an isomorphism.

- Another way to define an Azumaya algebra is an R-algebra A such that R has a ring extension R' with $A \otimes_R R' \cong M_n(R')$ for some $n \in \mathbb{N}$.

- Another way to define an Azumaya algebra is an R-algebra A such that R has a ring extension R' with $A \otimes_R R' \cong M_n(R')$ for some $n \in \mathbb{N}$.
- Hence, Azumaya algebras are a generalization of central simple algebras.

- Another way to define an Azumaya algebra is an R-algebra A such that R has a ring extension R' with $A \otimes_R R' \cong M_n(R')$ for some $n \in \mathbb{N}$.
- Hence, Azumaya algebras are a generalization of central simple algebras.
- Example: Consider $\mathbb{H}$ as an $\mathbb{R}$-algebra. It can be shown that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

- $d = 3$, $char(k) \neq 2, 3$. C_f is Azumaya over its center. [Haile]

- $d = 3$, $char(k) \neq 2, 3$. C_f is Azumaya over its center. [Haile]
- Let X be the projective plane curve defined by the equation $w^3 = f(u, v)$. Let $E = Jac(X)$.

- The center of C_f is the affine coordinate ring of E minus the identity point. [Haile]

- The center of C_f is the affine coordinate ring of E minus the identity point. [Haile]
- The dimension 3 representations of C_f are in one-to-one correspondence with the points of E minus the identity point.

- $d > 3$. Define X to be the projective plane curve $w^d = f(u, v)$. Let $g = genus(X)$.

- $d > 3$. Define X to be the projective plane curve $w^d = f(u, v)$. Let $g = genus(X)$.
- There is a one-to-one correspondence between rd-dimensional representations of C_f and certain vector bundles on X.

- $d > 3$. Define X to be the projective plane curve $w^d = f(u, v)$. Let $g = genus(X)$.
- There is a one-to-one correspondence between rd-dimensional representations of C_f and certain vector bundles on X.
- These are vector bundles E on X such that $\pi_*(E) \cong \bigoplus_{rd} \mathcal{O}_{\mathbb{P}^1}$, where $\pi : X \to \mathbb{P}^1$ is the map $[u : v : w] \mapsto [u : v]$. [van den Bergh]

Another way to describe the vector bundles E as described above is rank r, degree $r(d+g-1)$ bundles over X such that $H^0(X, E(-1)) = 0$. These bundles are all semistable, and the stable ones correspond to irreducible representations. [van den Bergh]

quick example

- Assume $k = \mathbb{C}$, and $d = 2$ for simplicity. We also assume we have a binary form.

A quick example

- Assume $k = \mathbb{C}$, and $d = 2$ for simplicity. We also assume we have a binary form.
- In this case, it can be shown that C_f isomorphic to $M_2(\mathbb{C})$.
- Hence, up to equivalence, there is only one irreducible representation of C_f. This representation has dimension 2.

ontinued

- Hence, by the correspondence, we must demonstrate a unique line bundle L on the curve X of degree $d + g - 1$ and such that $H^0(X, L(-1)) = 0$.

ontinued

- Hence, by the correspondence, we must demonstrate a unique line bundle L on the curve X of degree $d+g-1$ and such that $H^0(X, L(-1)) = 0$.
- X is isomorphic to $\mathbb{P}^1$. Hence $g = 0$ and L must have degree $d+g-1 = 2+0-1 = 1$. There is only one line bundle on X with this property.

ontinued

- Hence, by the correspondence, we must demonstrate a unique line bundle L on the curve X of degree $d+g-1$ and such that $H^0(X, L(-1)) = 0$.
- X is isomorphic to $\mathbb{P}^1$. Hence $g = 0$ and L must have degree $d+g-1 = 2+0-1 = 1$. There is only one line bundle on X with this property.
- To check that $H^0(X, L(-1)) = 0$, recall that X is embedded as a conic in $\mathbb{P}^2$, hence the degree of $L(-1)$ is -1.

Outline of the Correspondence

Let $\eta : C_f \to M_{rd}(k)$ be a representation. Denote $\eta(u) = \alpha_u$ and $\eta(v) = \alpha_v$. We use these $rd \times rd$-matrices to define a graded ring homomorphism $S = k[u, v, w]/(w^d - f(u, v)) \to M_{rd}(k[u, v])$ by taking u and v to uI_{rd} and vI_{rd} respectively, and w to $u\alpha_u + v\alpha_v$.

This makes $N = \bigoplus_{rd} k[u, v]$ into a graded S-module. Hence $\widetilde{N}$ is a coherent sheaf on $X = Proj(S)$ such that $\pi_*(\widetilde{N}) \cong \bigoplus_{rd} \mathcal{O}_{\mathbb{P}^1}$.

- $d > 3$. Define $\widetilde{C}_f = C_f / \cap I$, where I runs over the kernels of all the (irreducible) d-dimensional representations.

- $d > 3$. Define $\widetilde{C}_f = C_f / \cap I$, where I runs over the kernels of all the (irreducible) d-dimensional representations.
- $\widetilde{C}_f$ is Azumaya of rank d^2 over its center. This center is reduced. [Haile-Tesser]

- Recall that X is the projective plane curve defined by $w^d = f(u, v)$.

- Recall that X is the projective plane curve defined by $w^d = f(u, v)$.
- Consider the morphism $X^{(g-1)} \to Pic^{g-1}_{X/k}$ that takes a collection of $g - 1$ points to their sum as a divisor.

- Recall that X is the projective plane curve defined by $w^d = f(u, v)$.
- Consider the morphism $X^{(g-1)} \to Pic^{g-1}_{X/k}$ that takes a collection of $g - 1$ points to their sum as a divisor.
- Composing this morphism with $Pic^{g-1}_{X/k} \to Pic^{d+g-1}_{X/k}$, given by tensoring with $\mathcal{O}_X(1)$, gives us a map whose scheme-theoretic image is the so-called Θ-divisor. This divisor is ample. Hence, its complement is affine.

- Recall that X is the projective plane curve defined by $w^d = f(u, v)$.
- Consider the morphism $X^{(g-1)} \to Pic^{g-1}_{X/k}$ that takes a collection of $g-1$ points to their sum as a divisor.
- Composing this morphism with $Pic^{g-1}_{X/k} \to Pic^{d+g-1}_{X/k}$, given by tensoring with $\mathcal{O}_X(1)$, gives us a map whose scheme-theoretic image is the so-called Θ-divisor. This divisor is ample. Hence, its complement is affine.
- The center of $\widetilde{C}_f$ is given by the affine coordinate ring of this complement. [Kulkarni]

- We are interested in irreducible rd-dimensional representations.

- We are interested in irreducible rd-dimensional representations.
- Consider the moduli space $\mathcal{M}(r, r(d+g-1))$ of stable vector bundles of rank r and degree $r(d+g-1)$ over X.

ontinued

- Hence, by the correspondence, we must demonstrate a unique line bundle L on the curve X of degree $d+g-1$ and such that $H^0(X, L(-1)) = 0$.
- X is isomorphic to $\mathbb{P}^1$. Hence $g = 0$ and L must have degree $d+g-1 = 2+0-1 = 1$. There is only one line bundle on X with this property.
- To check that $H^0(X, L(-1)) = 0$, recall that X is embedded as a conic in $\mathbb{P}^2$, hence the degree of $L(-1)$ is -1.

Another way to describe the vector bundles E as described above is rank r, degree $r(d+g-1)$ bundles over X such that $H^0(X, E(-1)) = 0$. These bundles are all semistable, and the stable ones correspond to irreducible representations. [van den Bergh]

- $d > 3$. Define X to be the projective plane curve $w^d = f(u, v)$. Let $g = genus(X)$.
- There is a one-to-one correspondence between rd-dimensional representations of C_f and certain vector bundles on X.
- These are vector bundles E on X such that $\pi_*(E) \cong \bigoplus_{rd} \mathcal{O}_{\mathbb{P}^1}$, where $\pi : X \to \mathbb{P}^1$ is the map $[u : v : w] \mapsto [u : v]$. [van den Bergh]
- We will call these Clifford bundles.

Another way to describe the vector bundles E as described above is rank r, degree $r(d+g-1)$ bundles over X such that $H^0(X, E(-1)) = 0$. These bundles are all semistable, and the stable ones correspond to irreducible representations. [van den Bergh]

- Recall that X is the projective plane curve defined by $w^d = f(u,v)$.
- Consider the morphism $X^{(g-1)} \to Pic^{g-1}_{X/k}$ that takes a collection of $g-1$ points to their sum as a divisor.
- Composing this morphism with $Pic^{g-1}_{X/k} \to Pic^{d+g-1}_{X/k}$, given by tensoring with $\mathcal{O}_X(1)$, gives us a map whose scheme-theoretic image is the so-called Θ-divisor. This divisor is ample. Hence, its complement is affine.

- We are interested in irreducible rd-dimensional representations.
- Consider the moduli space $\mathcal{M}(r, r(d+g-1))$ of stable vector bundles of rank r and degree $r(d+g-1)$ over X.

- We are interested in irreducible rd-dimensional representations.
- Consider the moduli space $\mathcal{M}(r, r(d+g-1))$ of stable vector bundles of rank r and degree $r(d+g-1)$ over X.
- By a result of van den Bergh, there is a nonempty open subset U corresponding to vector bundles E over X such that $H^0(X, E(-1)) = 0$.

Definition

If S is a k-scheme then an S-representation of dimension n of B is a pair $(\phi, \mathcal{O}_A)$, where $\mathcal{O}_A$ is a sheaf of Azumaya algebras of rank n^2 over S and $\phi : B \to H^0(S, \mathcal{O}_A)$ is a ring homomorphism. Two representations $(\phi_1, \mathcal{O}_{A_1})$ and $(\phi_2, \mathcal{O}_{A_2})$ are called *equivalent* if there is an isomorphism $\theta : \mathcal{O}_{A_1} \to \mathcal{O}_{A_2}$ of sheaves of rings such that $\phi_2 = H^0(S, \theta) \circ \phi_1$. A representation of B is called *irreducible* if the image of B generates $\mathcal{O}_A$ locally.

Let $\mathcal{R}ep_{rd}(C_f, S)$ be the set of equivalence classes of irreducible S-representations of dimension rd of C_f. This defines a contravariant functor $\mathcal{R}ep_{rd}(C_f, -)$ from the category of k-schemes to the category of sets.

Theorem

$\mathcal{R}ep_{rd}(C_f, S)$ is represented by U. [C]

In other words, there exists an irreducible U-representation $(\phi, \mathcal{A})$ such that given any other irreducible S-representation $(\psi, \mathcal{O}_{\mathbf{A}})$, there is a unique morphism $f : S \to U$ such that $(\psi, \mathcal{O}_{\mathbf{A}}) \cong f^*(\phi, \mathcal{A})$.

)ngoing Research

- (with Rajesh Kulkarni and Yusuf Mustopa) Let $f(u, v, w)$ be a ternary cubic form. We define a cubic surface $X \subset \mathbb{P}^3$ by the equation $z^3 = f(u, v, w)$. We assume f to be such that X is nonsingular.

Ongoing Research

- (with Rajesh Kulkarni and Yusuf Mustopa) Let $f(u, v, w)$ be a ternary cubic form. We define a cubic surface $X \subset \mathbb{P}^3$ by the equation $z^3 = f(u, v, w)$. We assume f to be such that X is nonsingular.
- Then irreducible representations of the Clifford algebra C_f of dimension $3r$ correspond to (Gieseker) stable ACM (arithmetically Cohen-Macaulay) vector bundles E over X of rank r with Hilbert polynomial $P_E(n) = \frac{3r(n+2)(n+1)}{2}$.

Ongoing Research

- (with Rajesh Kulkarni and Yusuf Mustopa) Let $f(u, v, w)$ be a ternary cubic form. We define a cubic surface $X \subset \mathbb{P}^3$ by the equation $z^3 = f(u, v, w)$. We assume f to be such that X is nonsingular.
- Then irreducible representations of the Clifford algebra C_f of dimension $3r$ correspond to (Gieseker) stable ACM (arithmetically Cohen-Macaulay) vector bundles E over X of rank r with Hilbert polynomial $P_E(n) = \frac{3r(n+2)(n+1)}{2}$.
- Recall that the ACM condition means that for a vector bundle E over X, $H^1(X, E(tH)) = 0$ for all $t \in \mathbb{Z}$, where H denotes the hyperplane section.

Known Examples

- Rank 1: There are 72 isomorphism classes of line bundles satisfying this property.

Known Examples

- Rank 1: There are 72 isomorphism classes of line bundles satisfying this property.
- These correspond to the twisted cubics T on the surface X.
- Rank 2: The first Chern classes $c_1(E)$ can be $2H$, $H + T$, $H + C + L$ (where $C.L = 0$). There are 1, 72, 270 possible values respectively. Here, L stands for one of the 27 lines on X and C stands for a conic on X. (This is a complete classification.)

Known Examples

- Rank 1: There are 72 isomorphism classes of line bundles satisfying this property.
- These correspond to the twisted cubics T on the surface X.
- Rank 2: The first Chern classes $c_1(E)$ can be $2H$, $H + T$, $H + C + L$ (where $C.L = 0$). There are 1, 72, 270 possible values respectively. Here, L stands for one of the 27 lines on X and C stands for a conic on X. (This is a complete classification.)
- Rank r: Only known case is $c_1(E) = rH$, in which case there is a connected family of Clifford bundles of dimension $r^2 + 1$. (The classification is not complete in this case.)

)ngoing Research

- (with Rajesh Kulkarni and Yusuf Mustopa) Let $f(u, v, w)$ be a ternary cubic form. We define a cubic surface $X \subset \mathbb{P}^3$ by the equation $z^3 = f(u, v, w)$. We assume f to be such that X is nonsingular.
- Then irreducible representations of the Clifford algebra C_f of dimension $3r$ correspond to (Gieseker) stable ACM (arithmetically Cohen-Macaulay) vector bundles E over X of rank r with Hilbert polynomial $P_E(n) = \frac{3r(n+2)(n+1)}{2}$.
- Recall that the ACM condition means that for a vector bundle E over X, $H^1(X, E(tH)) = 0$ for all $t \in \mathbb{Z}$, where H denotes the hyperplane section.