Title: Topological defects in gauge theories.

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Abstract: We discuss recent progress on the rigorous description of the dynamics of the energy concentration sets in the abelian Higgs model. This is joint work with R. Jerrard.

Topological defects in gauge theories

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Introduction

Goal: we would like to describe dynamics of solutions of wave equations arising from gauge theory.
So heuristically we are interested in equations of the form

$$\boxed{\Box_A u + \frac{1}{\varepsilon^2} V(u) = 0} \qquad \text{(NLW)}$$

coupled to equations for electromagnetic fields, so resulting in

- Abelian Higgs Model
- Yang Mills Higgs

Motivation:

1. cosmology
2. *many* parallel results about elliptic and parabolic PDE.

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some cosmology I

Consider Einstein's equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}},$$

here $T_{\mu\nu} = T_{\mu\nu}(g, U)$ is the stress-energy tensor corresponding to solution U of the equation of roughly of the form (NLW) above (with respect to a metric coming from Einstein's equation).

- Then a cosmologist might be interested in studying the equation for U (while making a specific choice for the metric g).
- Many variants of (NLW) are considered.

some cosmology II

Studies of equations of the form (NLW) have led to predictions/speculation about

- (smooth) dynamics of cosmic strings and domain walls
- cosmic strings: collisions, radiation and scattering
- exotic cosmic strings — superconducting, nonabelian, fermionic....
- decay of a false vacuum
- cosmic strings formation in the early universe
- any of the above coupled to Einstein's equations
- the ekpyrotic universe
- $\cdots$

Motivation: Vacuum Manifold I

We start with Goldstone model-1961 (introduced in the context of the spontaneous symmetry breaking) :

$$\mathcal{L}(\phi) = \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle - V(p),$$

where $u : \mathbb{R}^{N+1} \to \mathbb{R}^k \quad k < N$
$V : \mathbb{R}^k \to [0, \infty], \quad V = (|p|^2 - 1)^2.$
and the corresponding E-L equations are

$$-\partial_t^2 + \triangle \phi - \phi(|\phi|^2 - 1) = 0$$

Observe: if $p \in \mathbb{R}^k$ and $V(p) = 0$ then $\phi(x, t) = p$ is a solution.

Definition

The set $\mathcal{M} = \{p \in \mathbb{R}^k : V(p) = 0\}$ is called the vacuum manifold.

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Motivation: Vacuum Manifold Examples

Suppose: $k = 1$
so $u : \mathbb{R}^{3+1} \to \mathbb{R}$ and if $V(p) = (|p|^2 - 1)^2$, then
$\mathcal{M} = \{-1, +1\}$.
Look for a special solution: $u(t, x) = q(x_1)$.
We can show $q(x_1) = \tanh(\frac{x}{\sqrt{2}})$. ($q$ connects $+1$ and -1.) This leads to a domain wall.

Suppose $k = 2 : u : \to \mathbb{R}^{3+1} \to \mathbb{R}^2$, then
$\mathcal{M} = S^1$.
Special solutions: $\phi(t, x) = q(x_1, x_2) = f(r)e^{in\theta}$. This leads to a cosmic string.
Domain walls, strings are two examples of topological defects.

Motivation: Minimal Surfaces

In particular the general expectation is that, there exists initial data, such that the energy of the solutions of (NLW) concentrates on sets evolving approximately via the equation for timelike Minkowski minimal surface.

We say $\Gamma = \mathrm{Image}(H)$ is timelike if

$$\boxed{\gamma(DH) = \det(DH^T\, \eta\, DH) < 0}.$$

Γ is called a timelike minimal surface if H is a critical point of

$$\boxed{\mathcal{A}(\Gamma) := \int \sqrt{-\gamma}}$$

For us $H : (-T, T) \times \mathbb{T}^n \to (-T, T) \times \mathbb{R}^N$.

Let $k = N - n$.

Milbredt '08: local in time existence of smooth timelike minimal surfaces. See also works of Deck, Gu, Müler for results on motion of strings.

Abelian Higgs Model

We let

$$\phi : \mathbb{R}^{n+1} \to \mathbb{C}, \quad A_\alpha : \mathbb{R}^{n+1} \to \mathbb{R}, \;\; \alpha = 0, 1, \ldots n. \tag{1}$$

Then the covariant derivative is

$$D_\alpha \phi = (\partial_\alpha - iA_\alpha)\phi$$

and the curvature is

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

The Lagrangian is given by

$$\mathcal{L}(\phi, A) = \frac{1}{2} D_\alpha \phi \overline{D^\alpha \phi} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{\lambda}{8}(|\phi|^2 - 1)^2,$$

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related mathematical work: wave equations

- Gustafson and Sigal '08 for point defects in the abelian Higgs Model.
- Jerrard, Lin dynamics also of point vortices in solutions of

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}(|u|^2 - 1)u = 0 \quad \text{for } u : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$$

Also

- soliton dynamics for somewhat different equation: *relativistic* dynamics of point particles D Stuart.
- asymptotic stability of flat kink: for single equation when $N = 3$, small perturbations (in $H^{24} \times H^{23}$) of flat kink converge to 0 in L^∞, with control over rate. Cuccagna 2008.
- Conditional results on behavior of (NLW) with the double-well potential when $\varepsilon \to 0$. Bellettini, Novaga, Orlandi 2008.
- Jerrard '09 k=1 and k=2 the solutions of (NLW) do in fact exhibit concentration sets that follow Γ.

Abelian Higgs Model result

Theorem (C & Jerrard, '10)

Let Γ be a codimension 2 timelike minimal surface, smooth in $(-T, T) \times \mathbb{R}^3$. Let the initial velocity of Γ at $t = 0$ be equal to 0. Then there exists $T_0 > 0$, independent of ϵ, and a solution $U = (\phi, A)$ of the critical ($\lambda = 1$) Abelian Higgs model such that

$$\boxed{\int_{(-T_0,T_0)\times\mathbb{R}^3} \tilde{d}^2 e_\epsilon(U)\,dxdt \le C\epsilon},$$

where

$$\tilde{d}(t,x) = min\{1, dist(x, \Gamma_t)\},$$

and

$$\boxed{\|U - U^*\|_{L^2} \le C\sqrt{\varepsilon}},$$

where U^* is an m-vortex.

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About the proof

The main ingredients needed are:

- change of variables rewrite the equations in a frame that follows the timelike minimal surface
- energy estimates in the new coordinates obtain approximately conserved energy density exploiting the new coordinates
- defect confinement need a functional that if it stays small it can detect that our defect is confined near Γ.

proof I: transformed equation

In the new coordinates the abelian Higgs model becomes:

$$-\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\phi)+\frac{\lambda}{4\epsilon^2}(|\phi|^2-1)^2=0$$

$$-\frac{1}{\sqrt{-g}}\partial_\alpha(g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}\sqrt{-g})+\mathcal{I}m(\phi g^{\beta\nu}\overline{D_\nu\phi})=0$$

or equivalently

$$-D_\alpha(g^{\alpha\beta}D_\beta\phi)-b\cdot D\phi+\frac{1}{2}V'(\phi)=0, \qquad \text{(AHM1)}$$

$$-\partial_\alpha(F^{\alpha\beta})-\frac{F^{\alpha\beta}\partial_\alpha\sqrt{-g}}{\sqrt{-g}}+\mathcal{I}m(\phi g^{\beta\alpha}\overline{D_\alpha\phi})=0, \qquad \text{(AHM2)}$$

where

$$b^\beta=\frac{\partial_\alpha\sqrt{-g}}{\sqrt{-g}}g^{\alpha\beta}, \quad V'(\phi)=\frac{\lambda}{4\epsilon^2}(|\phi|^2-1)\phi,$$

$$F^{\alpha\beta}=g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}.$$

proof II: transformed advection term

Lemma (Jerrard '09)

The term $b \cdot D\phi$ in the rewritten PDE satisfies

$$\boxed{b \cdot D\phi \leq C|D_\tau \phi|^2 + |y_\nu|^2 |D_\nu \phi|^2}.$$

Recall that

$$b^\beta = -\frac{\partial_{y_\alpha}\sqrt{-g}}{\sqrt{-g}} g^{\alpha\beta}$$

The lemma follows from the fact that Γ is a minimal surface.
This is the **only** place in our argument where we explicitly invoke the fact that Γ is minimal. It implies good energy estimates for solutions.

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proof 3: Energy Estimates

Let

$$e_\epsilon(\phi, A) = \frac{1}{2}\mathcal{R}e(D_\alpha\phi a^{\alpha\beta}\overline{D_\beta\phi}) + \frac{1}{4}F_{\alpha\beta}a^{\alpha\mu}a^{\beta\nu}F_{\mu\nu} + \frac{\lambda}{8\epsilon^2}(|\phi|^2 - 1)^2$$

Then

Lemma

$$\partial_0 e_\epsilon(\phi, A) \le C\left(|D_\tau\phi|^2 + |F_\tau|^2 + |y^\nu|^2(|D_\nu\phi|^2 + |F_\nu|^2)\right) + \nabla\cdot\Phi,$$

where

$$|F_\tau|^2 = \sum_{0\le\alpha,\beta\le n}|F_{\alpha\beta}|^2 \quad \textit{and} \quad |F_\nu|^2 = \sum_{n+1\le\alpha\le N, 0\le\beta\le N}|F_{\alpha\beta}|^2$$

$$\Phi^j = g^{jk}\left(\mathcal{R}e(D_k\phi\overline{D_0\phi}) + F_{km}g^{ml}F_{0l}\right).$$

proof 3: Energy Estimates

At some point we need lower energy bounds in the NORMAL variables. This forces us to consider the $2-D$ problem, for which the energy is

$$e(\phi, A) = \frac{1}{2}|D_A\phi|^2 + \frac{1}{2}|F_{12}|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2,$$

Then using the observation of Bogomol'nyi we can rewrite the energy as

$$\begin{aligned} e(\phi, A) = & \pm\frac{1}{2}\nabla \times j + \frac{1}{2}|(D_1 \pm iD_2)\phi|^2 \\ & + \frac{1}{2}\left(F_{12} \pm \frac{1}{2}(|\phi|^2 - 1)\right)^2 \pm \frac{1}{2}F_{12} + \frac{\lambda - 1}{8}(|\phi|^2 - 1)^2 \end{aligned}$$

proof 4: Bogomol'nyi observation

Therefore we can write

$$e(\phi, A) \geq \pm(\nabla \times j + F_{12}) + \frac{\lambda - 1}{8}(|\phi|^2 - 1)^2$$

We note the identity holds pointwise. The identity and the quantization of the magnetic charge leads to global minimizers of the energy functional among fields of the fixed degree m [See Jaffe-Taubes]. The lower energy bound is: $\geq 2\pi\,|m|$.

Properties of the defect confinement

We have

Lemma (Lower energy bounds)

There exists κ_1 such that if

$$D_\nu(U,\rho) < \kappa_1$$

then

$$\int_{B_\rho} e_{\epsilon,\nu}(U) \geq 2\pi - C\sqrt{\epsilon}$$

as $\epsilon \to 0$.

and

Lemma (Stability)

$$|D_\nu(U(\tau)) - D_\nu(U(0))| \leq C \int_{(0,\tau)\times B_\nu(\rho)} |D_\tau \phi|^2 + \epsilon |F_\tau|^2 + |y|^2 e_{\epsilon,\nu} dy^\nu d\tau$$

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