Title: Heterotic string and complex Monge-Ampere equation

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Abstract: I will talk about the work that I did with Jixiang Fu and Jun Li on the Strominger system and their role in string theory.

Heterotic String and Complex Monge-Ampère Equation

Shing-Tung Yau

Harvard University

with Jixiang Fu and Jun Li

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Kähler geometry

The basic foundation of Kähler geometry is that the holonomy group of the metric is $U(n)$ and the Laplacian operator acting on the differential form Ω^m commutes with the projection operator: $\Omega^m \to \Omega^{p,q}$ with $p+q=m$. This is due to the Kähler form

$$\omega = \sqrt{-1} \sum g_{i\bar{j}} \, dz_i \wedge d\bar{z}_j$$

being covariantly constant.

ω, Ω can actually be constructed from two parallel spinors. In type II string theory, having two parallel spinors results in $N = 2$ supersymmetry. This allows the possibility of mirror symmetry and leads to many important consequences in algebraic geometry of Calabi-Yau manifolds.

M. Reid made a proposal based on the construction of Clemens-Friedman. Clemens wanted to take a rational curve in CY manifold M whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and contract such a curve to a rational double point. Friedman proposed the condition to deform such a manifold into a smooth complex manifold. By blowing down enough such rational curves, $H^2(M)$ can be killed and we end up with a complex manifold which is diffeomorphic to a connected sum of $S^3 \times S^3$, but it is not Kähler.

Reid conjectured that one can connect any CY three-fold to any other through such conifold transitions. It is a nice picture and can be checked in many cases. However, one needs to understand the geometry of such non-Kähler manifolds.

In order to do this, we find that the most suitable structure is the hermitian metric with torsion introduced by Strominger.

The important point here is that supersymmetry still exists. (There are still parallel spinors.) There are four equations in the Strominger system. (We will write them down explicitly.)

The last equation of Strominger's system is equivalent to the existence of a certain hermitian form ω that satisfies

$$d(\omega^2) = 0 \ .$$

Such class of metrics were studied by Michelsohn and Alessandrini-Bassanelli. They called them balanced metrics.

Balanced metrics are found on non-Kähler manifolds such as Iwasawa manifolds and twistor spaces of self-dual Riemannian four-manifolds. Furthermore, it is known that balanced manifolds respect fiber bundle construction, and importantly, a complex manifold birational to a balanced manifold must admit a balanced metric.

Hence, we believe that the balanced manifold as required for the Strominger system is a good class of manifolds. We shall now introduce the Strominger's equations and prove the existence of solutions.

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Strominger's system

Let X be a compact complex threefold X with a trivial canonical bundle, i.e. there is a non-vanishing holomorphic three-form Ω.

Let V be a holomorphic vector bundle V over X.

Consider the pair (ω, h), where ω is a hermitian metric on X, and h is a hermitian metric on V.

The following equations are required to admit supersymmetry.

(1) $d(\| \Omega \|_\omega \, \omega^2) = 0$

(2) $F_h^{2,0} = F_h^{0,2} = 0\,, \quad F_h \wedge \omega^2 = 0$

(3) $\sqrt{-1}\partial\bar{\partial}\,\omega = \frac{\alpha'}{4}\left(\mathrm{tr}(R_\omega \wedge R_\omega) - \mathrm{tr}(F_h \wedge F_h)\right)$

The first equation is equivalent to the existence of a balanced metric. The second is the Hermitian-Yang-Mills equations. And the third equation is the anomaly equation.

When V is the tangent bundle TX and ω is Kähler, the system is solved by the Calabi-Yau metric. So Strominger's system should be viewed as a generalization of Calabi's conjecture for the case of non-Kähler Calabi-Yau threefolds.

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Since 2004, with Jun Li and Jixiang Fu, we have solved the Strominger system in two cases:

- Using the perturbation method, Li and Yau constructed irreducible smooth solutions on a class of Calabi-Yau's with $U(4)$ and $U(5)$ principle bundles.
- Fu and Yau constructed solutions to this system on non-Kähler threefolds that are T^2-bundles over a $K3$ surface.

For the Reid's conjecture, Fu-Li-Yau have also shown the existence of balanced metrics on connected sums of $S^3 \times S^3$. We expect that the Strominger system can also be solved in this case.

I shall describe our work in the following.

Li-Yau: Perturbation method

Assume that X is a Calabi-Yau threefold and ω_0 is a Calabi-Yau metric. Take the vector bundle $V = E = TX \oplus \mathbb{C}_X^{\oplus r}$ and $h = \omega_0 \oplus h_1$, where h_1 is a standard constant metric on $\mathbb{C}_X^{\oplus r}$, then (X, E, ω_0, h) is a solution, which is called a reducible solution by Li-Yau.

For any small deformations $\overline{\partial}_s$ of the holomorphic structure of $TX \oplus \mathbb{C}_X^{\oplus r}$, Li and Yau derived a sufficient condition for Strominger system being solvable for $(X, \overline{\partial}_s)$: it is that the Kodaira-Spencer class of the family $\overline{\partial}_s$ at $s = 0$ satisfies certain non-degeneracy condition.

By showing this sufficient condition to hold on following Calabi-Yau manifolds, we provided the first example of regular irreducible solution to Strominger system with gauge group $U(4)$ and $U(5)$:

1. $X \subset \mathbb{P}^4$: a smooth quintic threefold;
2. $X \subset \mathbb{P}^3 \times \mathbb{P}^3$: cut out by three homogeneous polynomials of bi-degree $(3, 0)$, $(0, 3)$ and $(1, 1)$.

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Let E_s be a smooth family of holomorphic vector bundles over a Calabi-Yau space X. Let h_0 be a Hermitian-Yang-Mills metric on E_0.

We would like to extend h_0 to a smooth family of Hermitian-Yang-Mills metrics.

The interesting case is when h_0 is reducible.

Let (X, ω_0) be Kähler.

Let (E_1, D_1'') and (E_2, D_2'') be degree zero and slope-stable vector bundles.

Let h_1 and h_2 be the hermitian metrics on E_1 and E_2 respectively.

Then $h_1 \oplus e^t h_2$ is still a hermitian metric that corresponds to the connection $D_0'' = D_1'' \oplus D_2''$.

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Suppose we are given a deformation of holomorphic structure D''_s of D''_0. Then Kodaira-Spencer identifies the first order deformation of D''_s at 0 to an element

$$k \in H^1(X, \varepsilon^* \otimes \varepsilon)$$

where ε is the sheaf of holomorphic section s of (E, D''_0).

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Theorem

Suppose k_{12} and k_{21} are nonzero. Then there is a unique t so that for s sufficiently small $h_0(t) = h_1 \oplus e^t h_2$ extends to a smooth family of Hermitian-Yang-Mills metrics on (E, D_s'').

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We can construct irreducible solutions to Strominger's system perturbatively.

Start with a Calabi–Yau manifold,

$$(E, D_0'') = \mathbb{C}_X^{\oplus(r-3)} \oplus TX,$$

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Let X be a Calabi-Yau three-fold with ω a Ricci-flat Kähler form. Let D''_s be a smooth deformation of holomorphic structure D''_0 on $E = \mathbb{C}_X \oplus TX$. Suppose the associated cohomology classes $[C_{12}]$ and $[C_{21}]$ are non-zero. Then for sufficiently large c, there is a family of pairs of hermitian metrics and hermitian forms (h_s, ω_s) for $0 \leq s < \varepsilon$ such that

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Let

$$D_s'' = D_0'' + A_3, \quad A_s \in \Omega^{0,1}(\mathrm{End}\, E)$$

$$A_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \Omega^{0,1}(\mathrm{End}\, E).$$

We can assume C_{ij} are D_0'' harmonic. Since $H^1(X, \mathcal{O}_X) = 0$, $C_{11} = 0$.

In general, we consider the $r+3$ holomorphic vector bundle $\mathbb{C}_X^{\oplus r} \oplus TX$. We also have

$$D_0'' = \begin{pmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$C_{12} = (\alpha_1, \ldots, \alpha_r)^t \in \Omega^{0,1}(TX)^{\oplus j}$$
$$C_{21} = (\beta_1, \ldots, \beta_r) \in \Omega^{0,1}(TX^r)^{\oplus j}$$
$$C_{22} \in \Omega^{0,1}(\mathrm{End}\, TX).$$

Suppose $[\alpha_1], \ldots, [\alpha_r] \in H^1(X, TX^*)$ are linearly independent and $[\beta_1], \ldots, [\beta_r] \in H^1(X, TX^*)$ are linearly independent. Then the above theorem holds.

Example: Consider $X = \{z_0^5 + \cdots + z_x^5 = 0\} \in \mathbb{P}^4$

$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & TX \arrow[r] \arrow[u] & T_X\mathbb{P}^4 \arrow[r] \arrow[u] & \mathcal{O}_X(5) \arrow[r] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[u] & \mathcal{O}_X(1)^{\oplus 5} \arrow[r] \arrow[u] & \mathcal{O}_X(5) \arrow[r] \arrow[u, equal] & 0 \\
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Here F is the cokernel of $\mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{O}_X(5)$ and fill in

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The above sequence is a non-split extension.

Making use of this element in $\mathrm{Ext}^1(TX, \mathcal{O}_X)$ we can perform a deformation of the holomorphic structure D_t'' with $C_{12} \neq 0$ and $C_{21} \neq 0$.

Hence we have proved:

Let X be a smooth quintic three-fold and ω be any Kähler form on X. Then for large $c > 0$, there is a smooth deformation of $\mathbb{C}_X \oplus TX$ such that for small s, there are pairs (h_s, ω_s) of hermitian metrics on E and hermitian forms ω_s on X that solves Strominger's system.

For the Calabi–Yau manifold with three generations that I constructed:

$$X \subset \mathbb{P}^3 \times \mathbb{P}^3$$

given by

$$\sum x_i^3 = 0$$
$$\sum y_i^3 = 0$$
$$\sum x_i y_i = 0$$

quotient by $\mathbb{Z}_3$. One can also construct irreducible solution to Strominger's system on $TX \oplus \mathbb{C}_X^{\oplus 2}$.

Fu-Yau: Non-Kähler manifolds

Fu and Yau constructed the solution of Strominger system on a class of non-Kähler three-dimensional complex manifolds, especially on a class of non-Kähler Calabi-Yau threefolds. These are T^2-bundles over $K3$-surfaces which were constructed by Goldstein and Prokushkin.

On these manifolds, Goldstein and Prokushkin observed that there exist natural metrics:

$$\omega_u = e^u \omega_{K3} + \frac{i}{2} \theta \wedge \overline{\theta},$$

which satisfy the first equation of Strominger system. Here u is any function of $K3$ surface, θ is the connection 1-form on the T^2-bundle. Similar ansatz were also considered by Dasgupta-Rajesh-Sethi and Becker-Becker-Dasgupts-Green earlier.

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By explicit calculation on $\mathrm{tr}(R \wedge R)$, Fu and Yau reduced the third equation of the Strominger system to the following Monge-Ampère equation:

$$\triangle(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

where f and μ are functions on $K3$ surface satisfying $f \geq 0$ and $\int_S \mu \omega_{K3}^2 = 0$.

This equation is more complicated than the equation for the Calabi conjecture. For example, the estimate of volume form gives extra complications. We obtained some crucial a priori estimates up to third order in derivatives and then used the continuity method to solve the equation.

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Fu-Yau thus found the existence theorem on Strominger's system:

Theorem

Let S be a $K3$ surface with Calabi-Yau metric ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Let X be a T^2-bundle over S constructed (twisted) by ω_1 and ω_2. Let E be a stable bundle over S with the gauge group $SU(r)$. Suppose ω_1, ω_2 and $c_2(E)$ satisfy the topological constraint

$$\alpha'(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right).$$

Then there exists a smooth function u on $K3$ surface and a Hermitian-Yang-Mills metric h on E such that $(X, \pi^* E, \omega_u, \pi^* h)$ is a solution of Strominger's system.

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From Mukai's theory of stable vector bundles over $K3$ surface, we know that a sufficient condition for the existence of a stable bundle E with $(r, c_1(E) = 0, c_2(E))$ on $K3$ surface is given by the inequality

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So we can determine all $(\omega_1, \omega_2, c_2(E))$ which satisfy topological constraint in the theorem.

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Let (S, ω_S, Ω_S) be the K3 surface.

Let $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S, \mathbb{Z})$ and let ω_1 and ω_2 be anti-self-dual (1,1)-forms.

Then there is a non-Kähler manifold X such that $\pi : X \to S$ is a holomorphic T^2 bundle over S.

If we write locally $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$, then there are coordinates of the T^2 fiber, x and y, such that $dx + \pi^*\alpha_1$ and $dy + \pi^*\alpha_2$ are globally defined 1-forms on X.

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So we can determine all $(\omega_1, \omega_2, c_2(E))$ which satisfy topological constraint in the theorem.

Let

$$\theta = dx + \pi^*\alpha_1 + \sqrt{-1}(dy + \pi^*\alpha_2)\ .$$

Then the hermitian form on X is

$$\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \overline{\theta}$$

and the holomorphic 3-form is

$$\Omega = \pi^*\Omega_S \wedge \theta\ .$$

ω_0 satisfies the forth equation $d(\| \Omega \|_{\omega_0} \omega_0^2) = 0$.

Let u be any smooth function on S and let

$$\omega_u = \pi^*(e^u\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \overline{\theta}\,.$$

Then ω_u is a hermitian metric on X and (ω_u, Ω) also satisfies

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As ω_1 and ω_2 are harmonic, $\overline{\partial}\omega_1 = \overline{\partial}\omega_2 = 0$. Then according to $\overline{\partial}$-Poincare Lemma, we can write ω_1 and ω_2 locally as

$$\omega_1 = \overline{\partial}\xi = \overline{\partial}(\xi_1 dz_1 + \xi_2 dz_2)$$

and

$$\omega_2 = \overline{\partial}\zeta = \overline{\partial}(\zeta_1 dz_1 + \zeta_2 dz_2),$$

where (z_1, z_2) is the local coordinate on S.

Let

$$B = \begin{pmatrix} \xi_1 + \sqrt{-1}\zeta_1 \\ \xi_2 + \sqrt{-1}\zeta_2 \end{pmatrix}.$$

If we let R_u to be the curvature of the hermitian connection of the metric ω_u on the holomorphic tangent bundle, then

$$\begin{aligned}\operatorname{tr} R_u \wedge R_u = \operatorname{tr} R_S \wedge R_S &+ 2\,\partial\overline{\partial} u \wedge \partial\overline{\partial} u \\ &+ 2\,\partial\overline{\partial}(e^{-u}\operatorname{tr}(\overline{\partial} B \wedge \partial B^* \cdot g^{-1})).\end{aligned}$$

So the third equation in Strominger's system can be reduced to

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial} e^{u} \wedge \omega_S - 2\,\partial\overline{\partial} u \wedge \partial\overline{\partial} u - 2\,\partial\overline{\partial}(e^{-u}\operatorname{tr}(\overline{\partial} B \wedge \partial B^* \cdot g^{-1})) \\ = \operatorname{tr} R_S \wedge R_S - \operatorname{tr} F_h \wedge F_h - (\|\omega_1\|^2 + \|\omega_2\|^2)\frac{\omega_S^2}{2!}\,. \quad (1)\end{aligned}$$

Since $\operatorname{tr} F_h \wedge F_h \geq 0$ and $\operatorname{tr} R_S \wedge R_S = 0$ for the case where the base $S = T^4$ base, we obtain

Proposition

There is no solution of Strominger's system on the torus bundle X over T^4 for the metric ansatz

$$e^u \omega_S + \frac{\sqrt{-1}}{2} \theta \wedge \overline{\theta}.$$

We consider the case of a $K3$ surface base. Let (E, h) be the Hermitian-Yang-Mills vector bundle over S with the gauge group $SU(r)$. Then $(V = \pi^* E, h)$ is also the Hermitian-Yang-Mills vector bundle over X. We can consider equation (1) as the equation on the $K3$ surface S. Integrating equation (1) over S,

$$\int_S \{\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_h \wedge F_h\} = \int_S (\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S}) \frac{\omega_S^2}{2!}.$$

We use $Q(\frac{\omega_i}{2\pi})$ to denote the intersection number of anti-self-dual (1,1)-form $\frac{\omega_i}{2\pi}$. As $\frac{1}{8\pi^2}\int_S \mathrm{tr} R_S \wedge R_S = 24$, the above condition can be written as

$$2(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right). \qquad (2)$$

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Certainly we can choose ω_1 and ω_2 and $SU(r)$ vector bundle E such that they satisfy the condition (2). Then there is a smooth function μ such that

$$\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_h \wedge F_h - \left(\|\omega_1\|^2 + \|\omega_2\|^2\right) \frac{\omega_S^2}{2!} = -\mu \frac{\omega_S^2}{2!}.$$

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So we obtain the following equation:

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}e^{u}\wedge\omega_S-\partial\overline{\partial}u\wedge\partial\overline{\partial}u&\\ -\partial\overline{\partial}(e^{-u}\mathrm{tr}(\overline{\partial}B\wedge\partial B^*\cdot g^{-1}))+\mu\frac{\omega_S^2}{2!}=0\,.&\end{aligned}\tag{3}$$

In particular, when $\omega_2 = n\omega_1$, $n\in\mathbb{Z}$, we have

$$\mathrm{tr}(\overline{\partial}B\wedge\partial B^*\cdot g^{-1})=\sqrt{-1}\frac{1+n^2}{4}\,\|\omega_1\|^2_{\omega_S}\,\omega_S\,.$$

Hence in this case, if we set $f=\frac{1+n^2}{4}\,\|\omega_1\|^2_{\omega_S}$, we can rewrite equation (3) as the standard complex Monge-Ampère equation:

$$\Delta(e^{-u}-fe^{-u})+8\,\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}+\mu=0\,.$$

We solve equation (3) by the continuity method and get

Theorem

Equation (3) has a smooth solution u such that

$$\omega' = e^u\omega_S + 2\sqrt{-1}\partial\bar{\partial}u - \sqrt{-1}e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})$$

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Theorem

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Theorem

Let S be a $K3$ surface with Calabi-Yau metric ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that $\frac{\omega_1}{2\pi} \in H^2(S, \mathbb{Z})$ and $\frac{\omega_2}{2\pi} \in H^2(S, \mathbb{Z})$. Let X be a T^2-bundle over S constructed by ω_1 and ω_2. Let E be a stable bundle over S with the gauge group $SU(r)$. Suppose ω_1, ω_2 and $c_2(E)$ satisfy the topological constraint (2). Then there exists a smooth function u and a Hermitian-Yang-Mills metric h on E such that $(X, V = \pi^* E, \omega_u, \pi^* h)$ is a solution of Strominger's system.

For simplicity, we consider the equation

$$\Delta(e^{-u} - fe^{-u}) + 8\,\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0.$$

We impose the following elliptic condition

$$\omega' = (e^{u} + fe^{-u})\omega + 2\sqrt{-1}\partial\bar{\partial}u > 0$$

and the normalization condition

$$\left(\int_S e^{-4u}\,\frac{\omega_S^2}{2!}\right)^{\frac{1}{4}} = A, \quad \int_S 1\,\frac{\omega_S^2}{2!} = 1\,.$$

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Zeroth order estimate

Let $P = 2g'^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$. We have two methods of calculating

$$\int_S P(e^{ku}) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \frac{\omega_S^2}{2!} .$$

Then using the Sobolev inequality, Moser iteration and Poincaré inequality, we obtain

Proposition: *If $A < 1$, then there is a constant C_1 which depends on f, μ and the Sobolev constant of ω_S such that*

$$\inf_S u \geq -\ln(C_1 A) .$$

Moreover, if A is small enough such that $A < (C_1)^{-1}$, then there is an upper bound of $\sup_S u$ which depends on f, μ, Sobolev constant of ω_S and A.

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An estimate of the determinant

We need to estimate the lower bound of the determinant

$$F = \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}.$$

We apply the maximum principle to the function

$$G = 1 - e^{-u} \mid \nabla u \mid^2 + 2e^{-\epsilon u} - 2e^{-\epsilon \inf u}$$

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Proposition: *Given any constant $\kappa \in (0,1)$, we fix some positive constant ϵ satisfying*

$$\epsilon < \min\{1, 2^{-1}\kappa\} \ .$$

Suppose that A satisfies

$$A < \min\{1, C_1^{-1}, \{2(1+\sup f)\}^{-\frac{1}{2}} C_1^{-1}, \left(\frac{1-\kappa}{2C_3}\right)^{\frac{1}{\epsilon}} C_1^{-1}, \frac{3-6\epsilon}{C_4} C_1^{-1}, C_5\} \ ,$$

where C_3 and C_4 depend on f and μ, C_4 also depends on the curvature bound of ω_S, and C_5 depends on κ, ϵ and C_3. Then $F > \kappa e^{2u} \geq \kappa (C_1 A)^{-2}$.

For simplicity, we consider the equation

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We impose the following elliptic condition

$$\omega' = (e^{u} + fe^{-u})\omega + 2\sqrt{-1}\partial\bar{\partial}u > 0$$

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So we obtain the following equation:

$$\sqrt{-1}\partial\overline{\partial}e^{u}\wedge\omega_S-\partial\overline{\partial}u\wedge\partial\overline{\partial}u \\ -\partial\overline{\partial}(e^{-u}\mathrm{tr}(\overline{\partial}B\wedge\partial B^{*}\cdot g^{-1}))+\mu\frac{\omega_S^{2}}{2!}=0\,. \tag{3}$$

In particular, when $\omega_2=n\omega_1$, $n\in\mathbb{Z}$, we have

$$\mathrm{tr}(\overline{\partial}B\wedge\partial B^{*}\cdot g^{-1})=\sqrt{-1}\frac{1+n^{2}}{4}\,\|\omega_1\|^{2}_{\omega_S}\,\omega_S\,.$$

Hence in this case, if we set $f=\frac{1+n^{2}}{4}\,\|\omega_1\|^{2}_{\omega_S}$, we can rewrite equation (3) as the standard complex Monge-Ampère equation:

$$\Delta(e^{-u}-fe^{-u})+8\,\frac{\det u_{i\overline{j}}}{\det g_{i\overline{j}}}+\mu=0\,.$$

Since tr $F_h \wedge F_h \geq 0$ and tr $R_S \wedge R_S = 0$ for the case where the base $S = T^4$ base, we obtain

Proposition

There is no solution of Strominger's system on the torus bundle X over T^4 for the metric ansatz

$$e^u \omega_S + \frac{\sqrt{-1}}{2} \theta \wedge \overline{\theta}.$$

We consider the case of a $K3$ surface base. Let (E,h) be the Hermitian-Yang-Mills vector bundle over S with the gauge group $SU(r)$. Then $(V=\pi^*E,h)$ is also the Hermitian-Yang-Mills vector bundle over X. We can consider equation (1) as the equation on the $K3$ surface S. Integrating equation (1) over S,

$$\int_S \{\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_h \wedge F_h\} = \int_S (\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S}) \frac{\omega_S^2}{2!}.$$

We use $Q(\frac{\omega_i}{2\pi})$ to denote the intersection number of anti-self-dual (1,1)-form $\frac{\omega_i}{2\pi}$. As $\frac{1}{8\pi^2}\int_S \mathrm{tr} R_S \wedge R_S = 24$, the above condition can be written as

$$2(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right). \qquad (2)$$

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Third order estimate

Let

$$
\begin{aligned}
\Gamma &= g^{i\bar{j}} g^{k\bar{l}} u_{,ik} u_{,\bar{j}\bar{l}} \\
\Theta &= g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}} \\
\Xi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} u_{,ikp} u_{,\bar{j}\bar{l}\bar{q}} \\
\Phi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}pr} u_{,\bar{j}k\bar{q}\bar{s}} \\
\Psi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}p\bar{s}} u_{,\bar{j}k\bar{q}r},
\end{aligned}
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where indices preceded by a comma indicate covariant differentiation with respective to the metric ω_S.

Second order estimate

Since

$$e^u + fe^{-u} + \triangle u \geq F^{\frac{1}{2}} > \kappa^{\frac{1}{2}}(C_1 A)^{-1} > 0,$$

it is sufficient to have an upper estimate of $e^u + fe^{-u} + \triangle u$.

Applying the maximum principle to the function

$$e^{-\lambda_1 u + \lambda_2 |\nabla u|^2} \cdot \left(e^u + fe^{-u} + \triangle u\right),$$

where λ_1 and λ_2 are positive constants that can be determined, we can obtain the second order estimate.

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where indices preceded by a comma indicate covariant differentiation with respective to the metric ω_S.

Again, we apply the maximum principle to the function

$$(\kappa_1 + \triangle u)\Theta + \kappa_2(m + \triangle u)\Gamma + \kappa_3 \mid \bigtriangledown u \mid^2 \Gamma + \kappa_4\Gamma,$$

where all κ_i are positive constants that can be determined and m is a fixed constant such that $m + \triangle u > 0$. We can then obtain the third order estimate.

Topology of the total space

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\begin{aligned}
(1)\ h^{0,1}(X) &= h^{0,1}(S) + 1 \\
(2)\ h^{1,0}(X) &= h^{1,0}(S) \\
(3)\ b_1(X) &= b_1(S) + 1 \\
(4a)\ b_2(X) &= b_2(S) - 1 \\
& \text{if } \omega_1 \text{ is a multiple of } \omega_2 \\
(4b)\ b_2(X) &= b_2(S) - 2 \\
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Fu-Li-Yau: Connected sums of $S^3 \times S^3$

Main Theorem

Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0$ be a contraction of mutually disjoint $(-1,-1)$-curves. Suppose X_0 can be smoothed to a family of smooth complex manifolds X_t. Then for sufficiently small t, X_t admit smooth balanced metrics.

Our construction provides balanced metrics on a large class of threefolds. In particular,

Corollary

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Let

$$\theta = dx + \pi^*\alpha_1 + \sqrt{-1}(dy + \pi^*\alpha_2) \ .$$

Then the hermitian form on X is

$$\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \overline{\theta}$$

and the holomorphic 3-form is

$$\Omega = \pi^*\Omega_S \wedge \theta \ .$$

ω_0 satisfies the forth equation $d(\| \Omega \|_{\omega_0} \omega_0^2) = 0$.

Let u be any smooth function on S and let

$$\omega_u = \pi^*(e^u\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \overline{\theta} \, .$$

Then ω_u is a hermitian metric on X and (ω_u, Ω) also satisfies

$d(\| \Omega \|_{\omega_u} \omega_u^2) = 0$.

On these manifolds, Goldstein and Prokushkin observed that there exist natural metrics:

$$\omega_u = e^u \omega_{K3} + \frac{i}{2}\theta \wedge \overline{\theta},$$

which satisfy the first equation of Strominger system. Here u is any function of $K3$ surface, θ is the connection 1-form on the T^2-bundle. Similar ansatz were also considered by Dasgupta-Rajesh-Sethi and Becker-Becker-Dasgupts-Green earlier.

In general, we consider the $r+3$ holomorphic vector bundle $\mathbb{C}_X^{\oplus r} \oplus TX$. We also have

$$D_0'' = \begin{pmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$C_{12} = (\alpha_1, \ldots, \alpha_r)^t \in \Omega^{0,1}(TX)^{\oplus j}$$
$$C_{21} = (\beta_1, \ldots, \beta_r) \in \Omega^{0,1}(TX^r)^{\oplus j}$$
$$C_{22} \in \Omega^{0,1}(\mathrm{End}\, TX).$$

Suppose $[\alpha_1], \ldots, [\alpha_r] \in H^1(X, TX^*)$ are linearly independent and $[\beta_1], \ldots, [\beta_r] \in H^1(X, TX^*)$ are linearly independent. Then the above theorem holds.

Let E_s be a smooth family of holomorphic vector bundles over a Calabi-Yau space X. Let h_0 be a Hermitian-Yang-Mills metric on E_0.

We would like to extend h_0 to a smooth family of Hermitian-Yang-Mills metrics.

The first equation is equivalent to the existence of a balanced metric. The second is the Hermitian-Yang-Mills equations. And the third equation is the anomaly equation.

When V is the tangent bundle TX and ω is Kähler, the system is solved by the Calabi-Yau metric. So Strominger's system should be viewed as a generalization of Calabi's conjecture for the case of non-Kähler Calabi-Yau threefolds.

Reid conjectured that one can connect any CY three-fold to any other through such conifold transitions. It is a nice picture and can be checked in many cases. However, one needs to understand the geometry of such non-Kähler manifolds.

In order to do this, we find that the most suitable structure is the hermitian metric with torsion introduced by Strominger.

Balanced metrics are found on non-Kähler manifolds such as Iwasawa manifolds and twistor spaces of self-dual Riemannian four-manifolds. Furthermore, it is known that balanced manifolds respect fiber bundle construction, and importantly, a complex manifold birational to a balanced manifold must admit a balanced metric.

Hence, we believe that the balanced manifold as required for the Strominger system is a good class of manifolds. We shall now introduce the Strominger's equations and prove the existence of solutions.

Strominger's system

Let X be a compact complex threefold X with a trivial canonical bundle, i.e. there is a non-vanishing holomorphic three-form Ω.

Let V be a holomorphic vector bundle V over X.

Consider the pair (ω, h), where ω is a hermitian metric on X, and h is a hermitian metric on V.

The following equations are required to admit supersymmetry.

(1) $d(\| \Omega \|_\omega \, \omega^2) = 0$

(2) $F_h^{2,0} = F_h^{0,2} = 0 , \quad F_h \wedge \omega^2 = 0$

(3) $\sqrt{-1}\partial\bar{\partial}\,\omega = \frac{\alpha'}{4}\left(\mathrm{tr}(R_\omega \wedge R_\omega) - \mathrm{tr}(F_h \wedge F_h)\right)$

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