Title: On the notion quasilocal mass in general relativity

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Abstract: There have been many attempts to define quasilocal mass for a spacelike 2-surface in a spacetime by the Hamilton-Jacobi method. The essential difficulty in this approach is the choice of the background configuration to be subtracted from the physical Hamiltonian. The quasilocal mass should be positive for general surfaces, but on the other hand should be zero for surfaces in the flat spacetime. In this talk, I shall describe how to use isometric embeddings into the Minkowski space to overcome this difficulty and propose a new definition of gauge-independent quasi-local mass that has the desired properties, in addition to other natural requirements for a mass. This talk is based on a joint work with Shing-Tung Yau at Harvard.

On the notion of quasilocal mass in general relativity

Mu-Tao Wang

Columbia University

Connections in Geometry and Physics

Perimeter Institute, Waterloo, May 8, 2010

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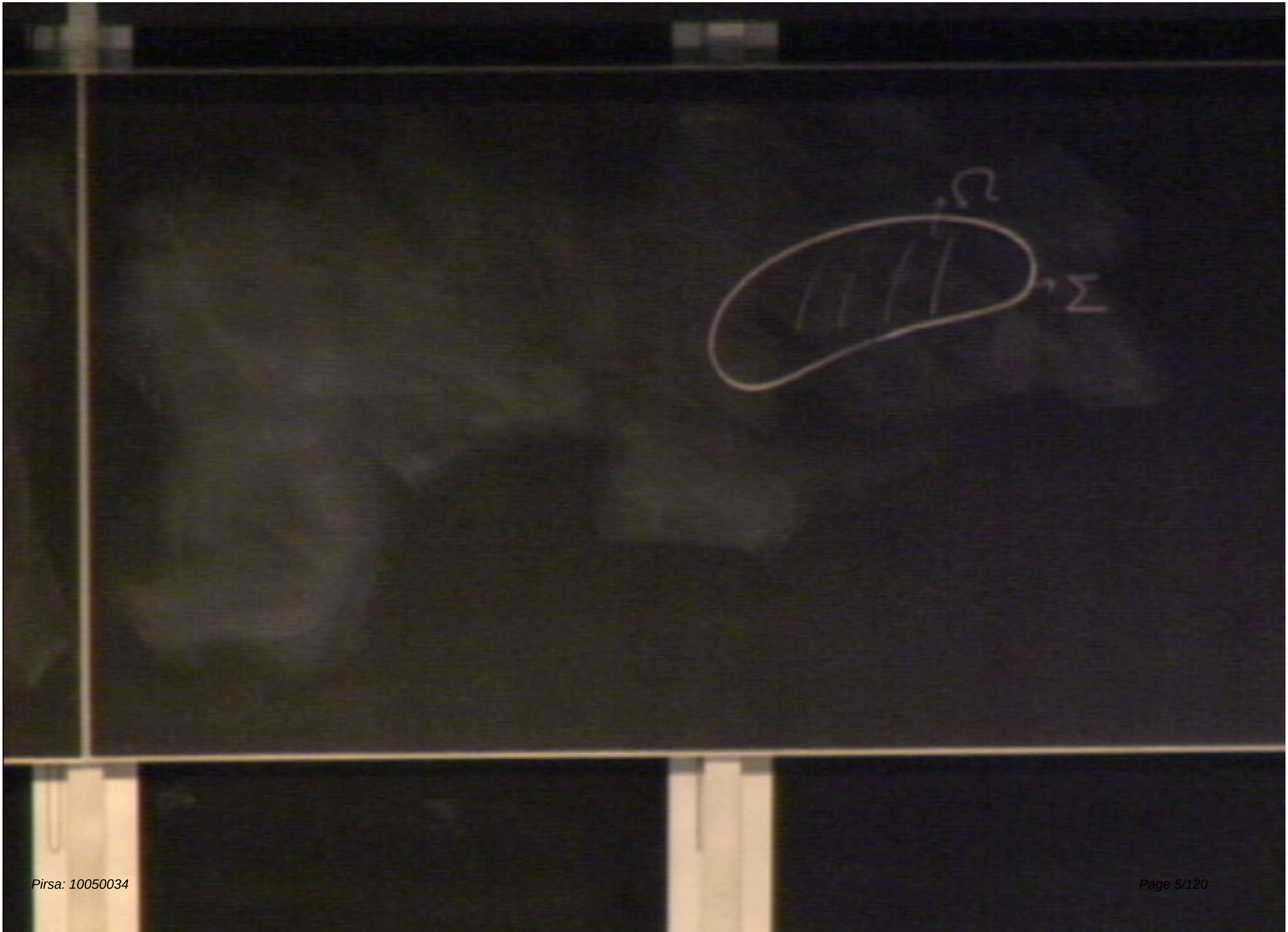
Σ a spacelike closed 2-surface in spacetime.


Ω
Σ

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$g_{\mu\nu}$: Lorentzian $(-,+,+,+)$ metric, gravitational field.

Φ: matter fields.

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Einstein's equation is obtained by taking the variation of

$$\frac{1}{16\pi}\int R + \int L(g, \Phi)$$

where R is the scalar curvature of $g_{\mu\nu}$ and L is the Lagrangian of matter coupled to gravity.

Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}.$$

where $T_{\mu\nu}$ is the energy momentum tensor of matter density.

Relation between matter fields and gravitational field.

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Relation between matter fields and gravitational field.

Ω bounded spacelike region in spacetime.

Question: How do we measure the total energy contained in Ω, counting contributions from both matter fields and gravitational field?

Special relativity in $\mathbb{R}^{3,1}$, Ω a spacelike hypersurface and u^μ its unit normal.

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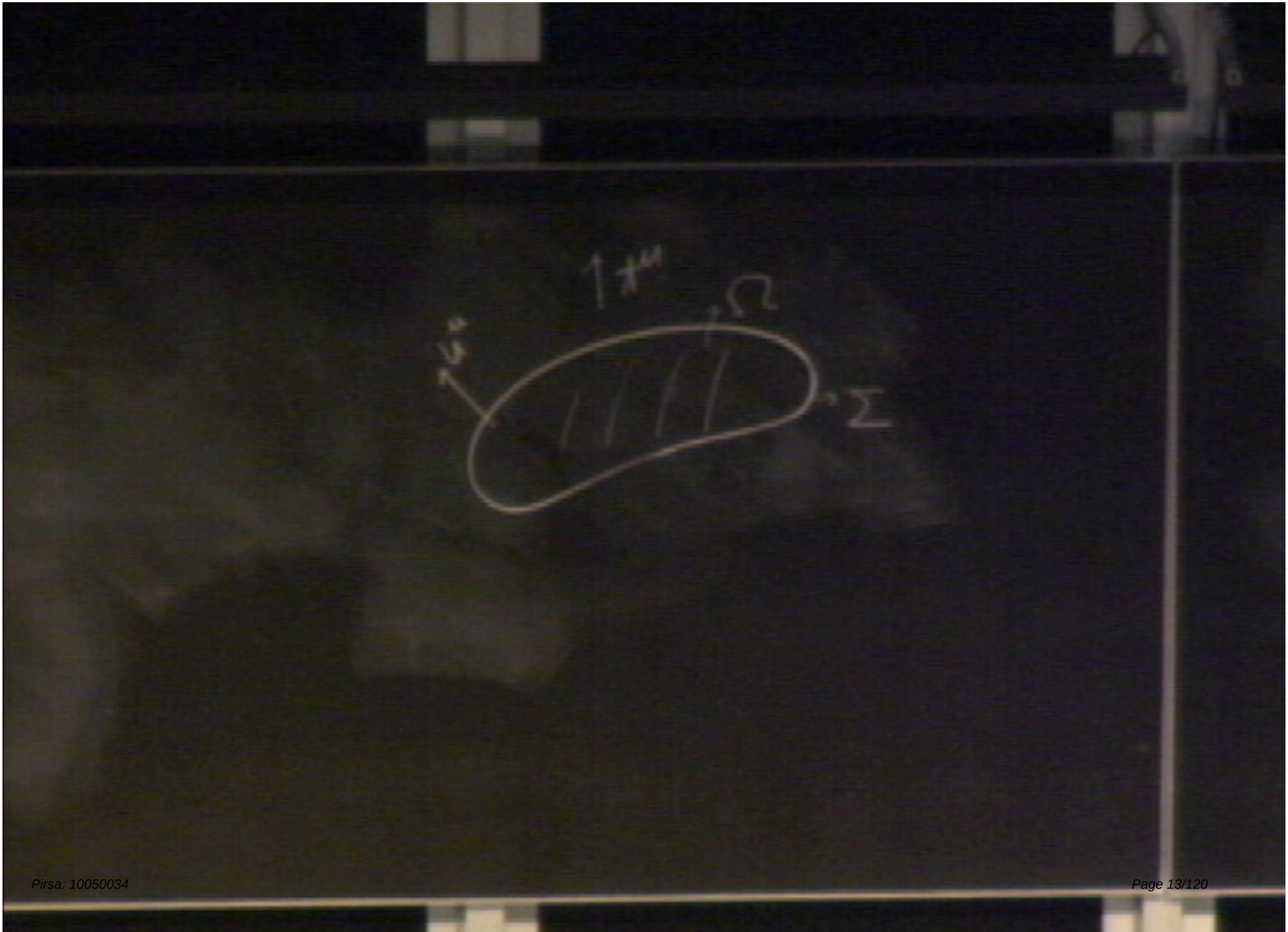
Given any t^ν constant (Killing) timelike unit vector,

$$\int_\Omega T_{\mu\nu} u^\mu t^\nu$$

represents energy intercepted by Ω (observed by t^ν).



x^μ
n^μ
Ω
Σ

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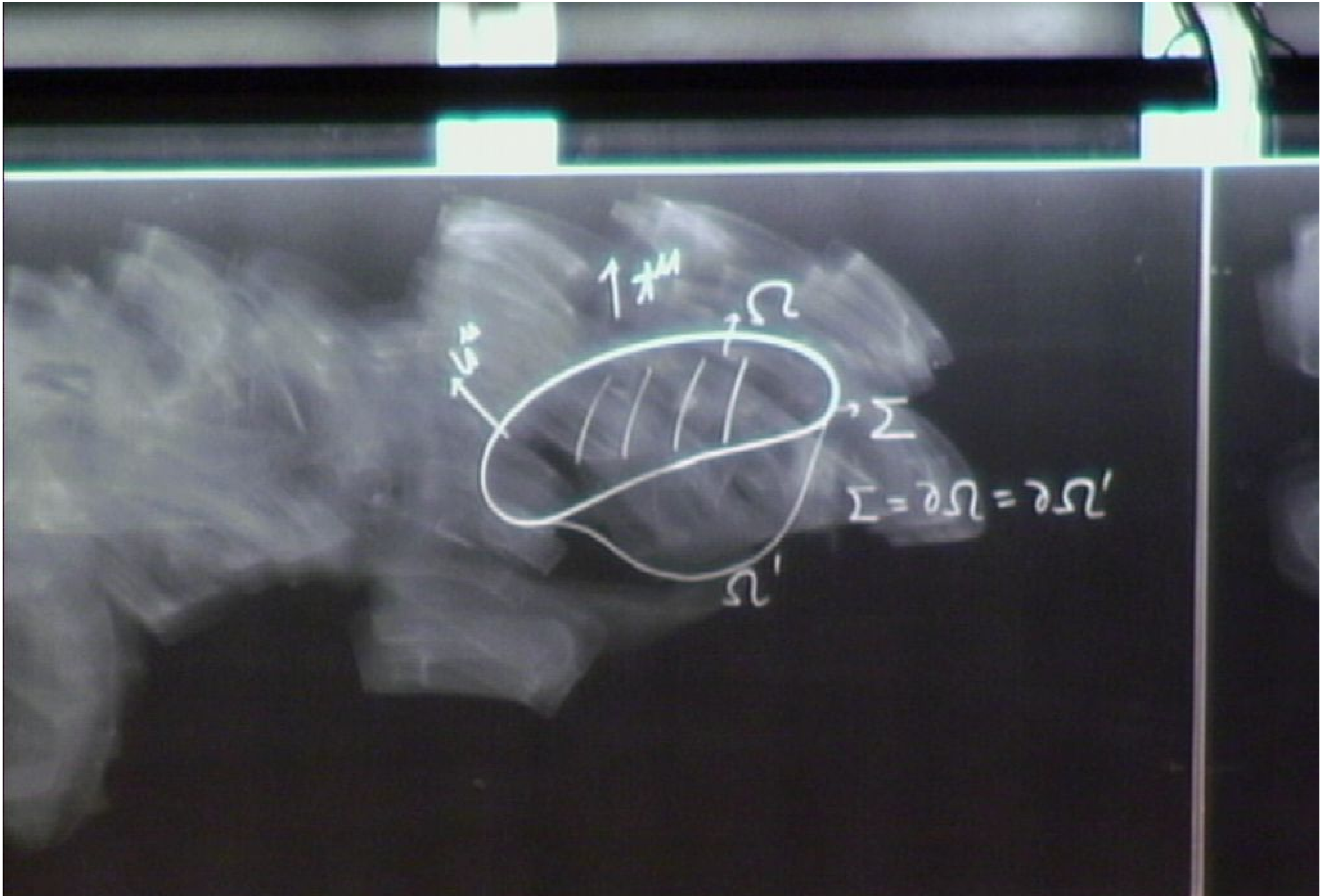
t^ν Killing implies $T_{\mu\nu} t^\nu$ is divergence free.

Energy depends only on $\partial\Omega = \Sigma$!


Ω
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Σ = ∂Ω = ∂Ω'
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Does not work in general relativity.

No Killing vector field in general spacetime.

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Gravitation energy must have contribution, as seen in binding energy (depending on distance apart).

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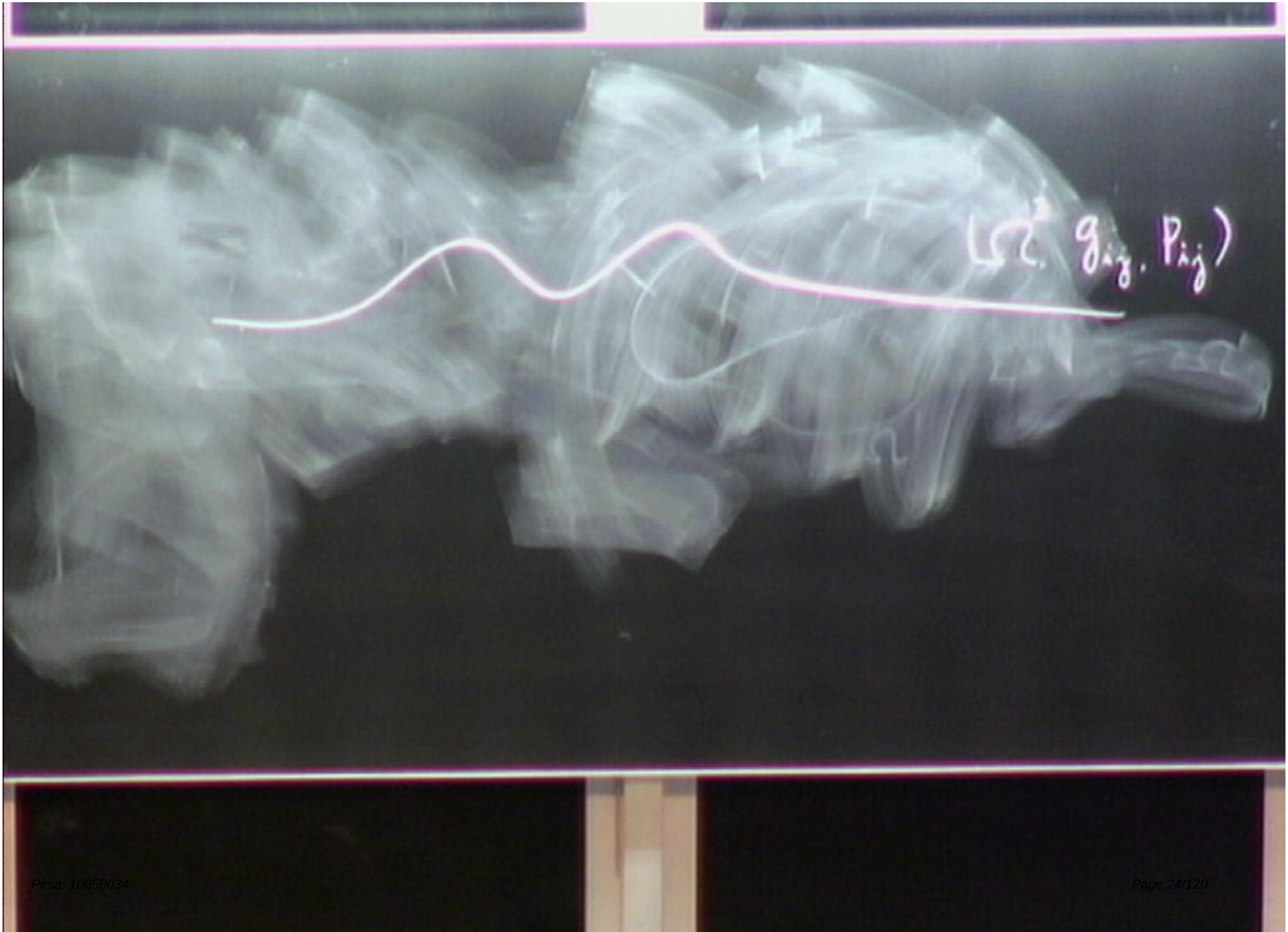
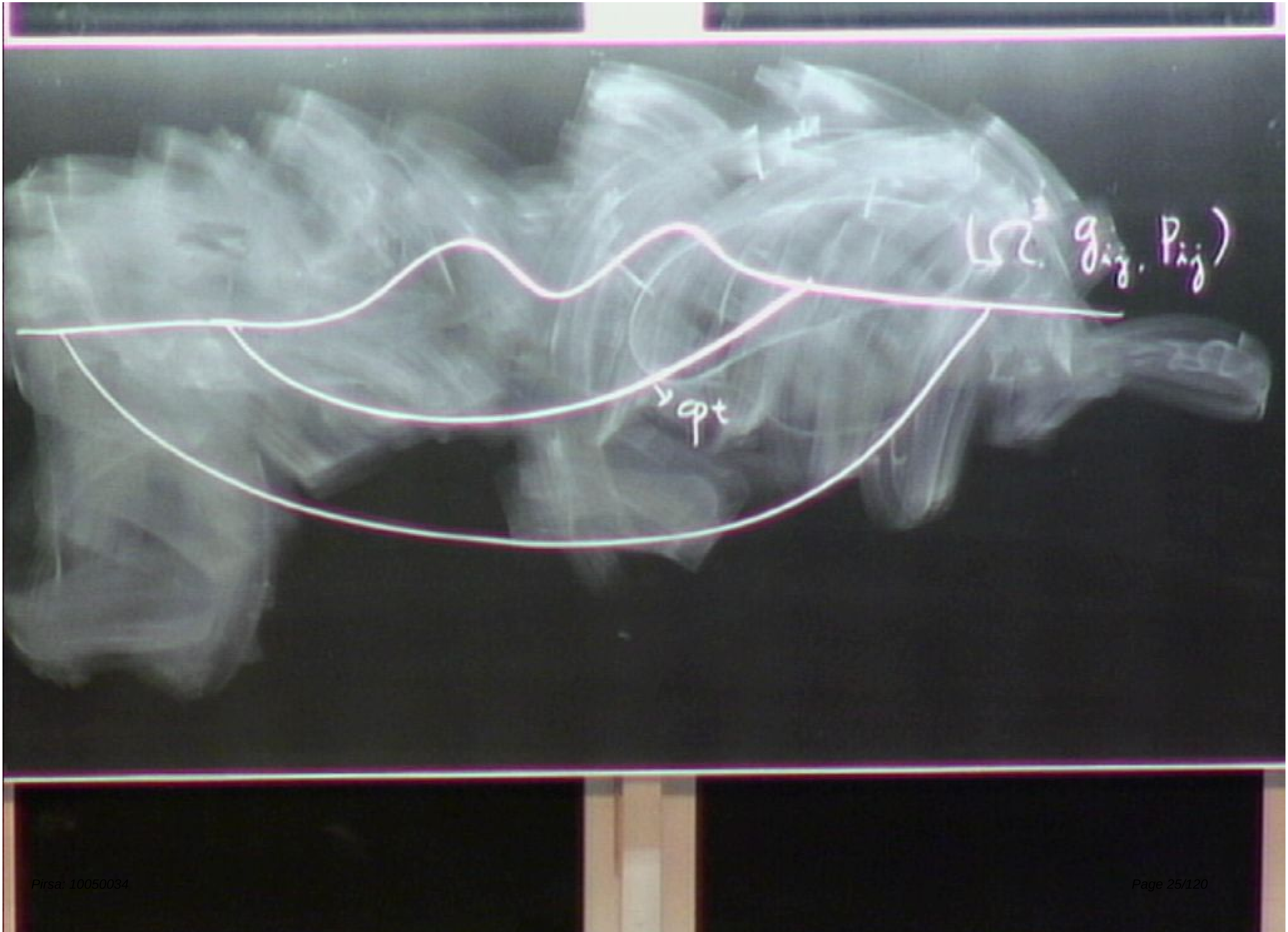
Gravitational energy depends on underlying geometry which is distorted by the field itself, thus more nonlinear.

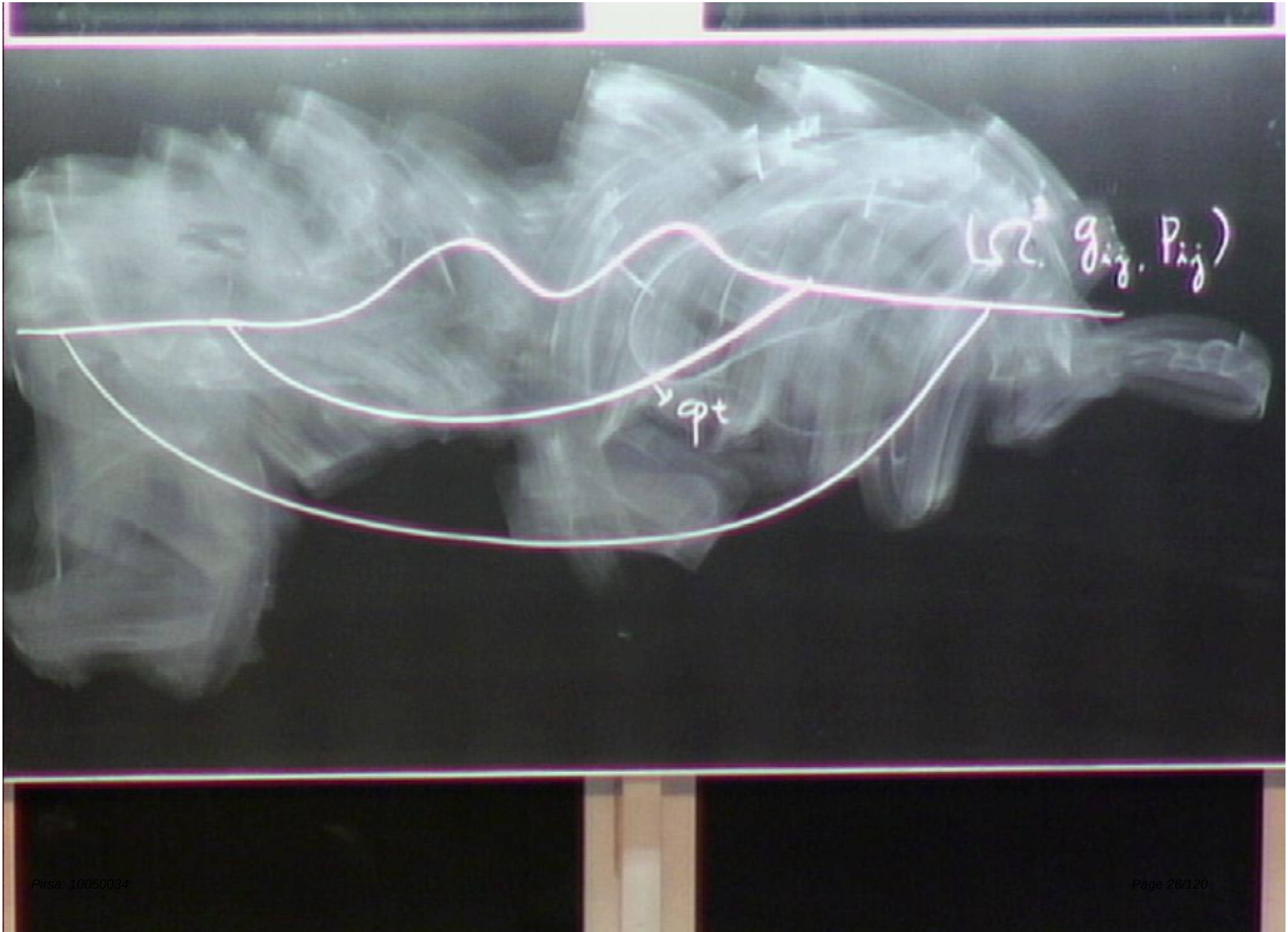
In general, no "background" unless gravitation is weak (or asymptotic symmetry).

(Ω, g_{ij}, p_{ij}) asymptotically flat spacelike hypersurface in spacetime.
$\Omega \backslash cpt$ diffeomorphic to union of ends (complements of balls in $\mathbb{R}^3$).


g_{ij}, P_{ij})



$(\Omega^3, g_{ij}, P_{ij})$
φ_t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On each end, $g_{ij} - \delta_{ij} = O(\frac{1}{r})$, $p_{ij} = O(\frac{1}{r^2})$, etc.

$$E = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,j} - g_{jj,i}) \nu^i dS_r$$

$$P_i = \lim_{r \to \infty} \frac{1}{8\pi} \int_{S_r} (p_{ij} - \delta_{ij} p_{kk}) \nu^j dS_r.$$

where S_r are coordinate spheres approaching infinity as $r \to \infty$.

(E, P_1, P_2, P_3) ADM energy momentum vector.

Future timelike 4-vector (positive mass) under dominant energy condition by PMT (Schoen-Yau, Witten).

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Many important statements in general relativity make sense only with the presence of a good definition of quasilocal mass. For example, binding energy of two bodies rotating around each other.

More importantly, a good definition of quasilocal mass may help us to control the dynamics of the gravitational field.

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Various attempts to define quasilocal mass (Hawking, Bartnik, Penrose, Huisken, etc.), will focus on the Hamilton-Jacobi analysis approach in this lecture.

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Recall that for a Lorentzian manifold D with boundary ∂D, the action in general relativity should be

$$I(g,\Phi) = \int_D \left(\frac{1}{16\pi} R + L(g,\Phi) \right) + \frac{1}{8\pi} \int_{\partial D} K$$

K is the trace of the second fundamental form (extrinsic curvature) of ∂D.

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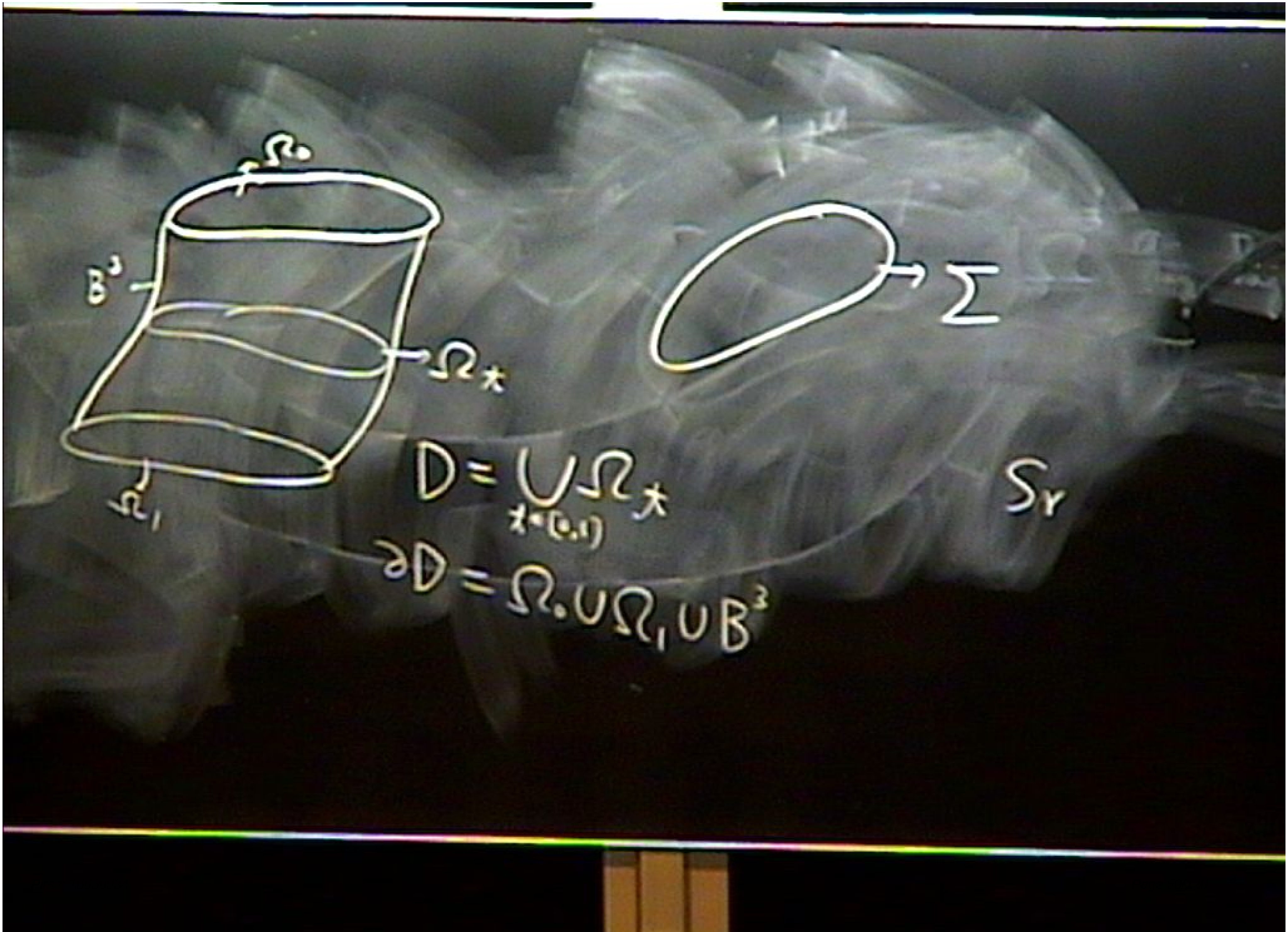
Apply Hamilton-Jacobi analysis to the time history $D = \cup_{t\in[0,1]} \Omega_t$ of spatially bounded region with $\partial D = \Omega_0 \cup {}^3B \cup \Omega_1$.

[redacted]


Ω_0
B^3
Ω_t
Ω_1
Σ
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S_r
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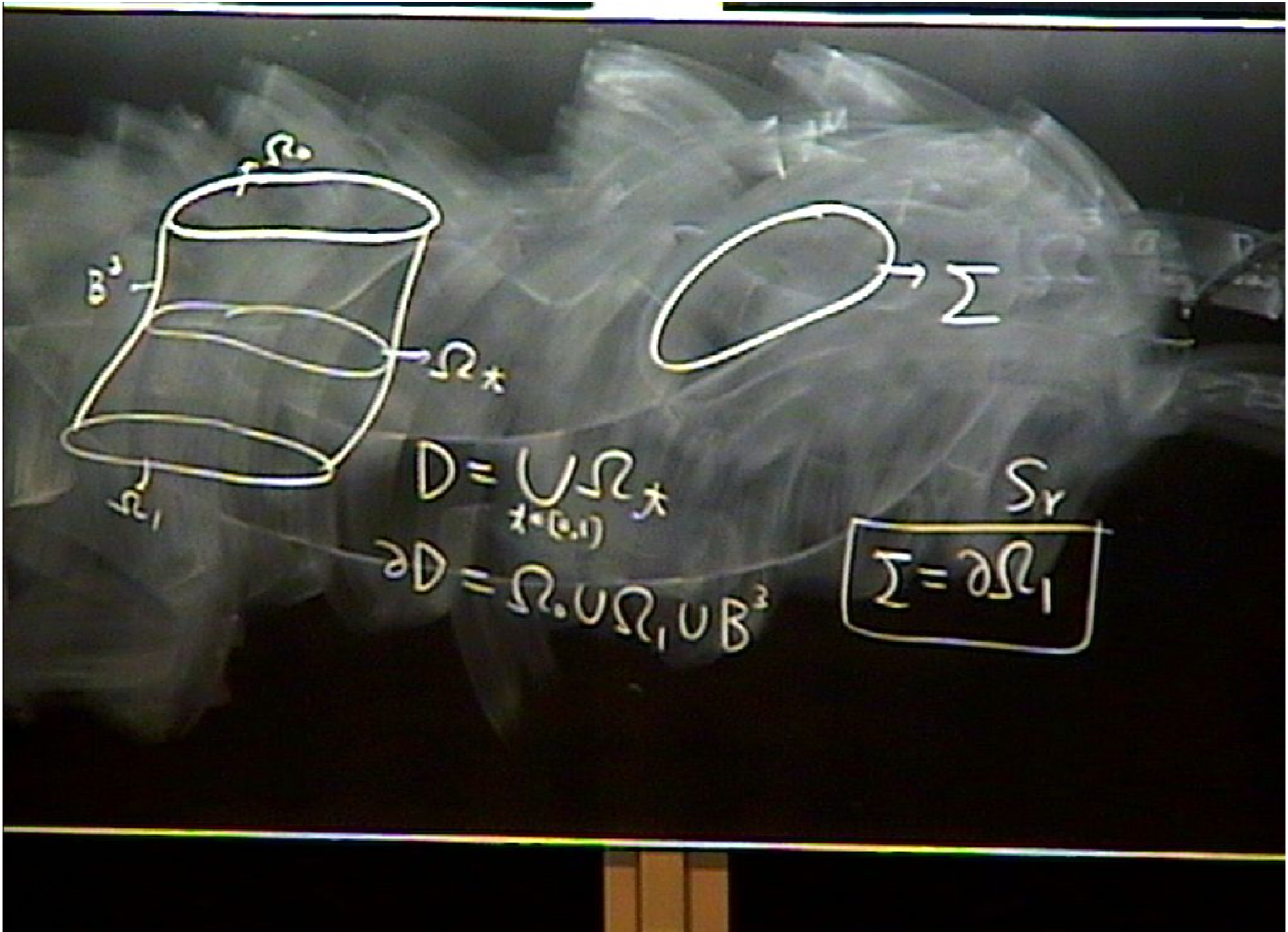

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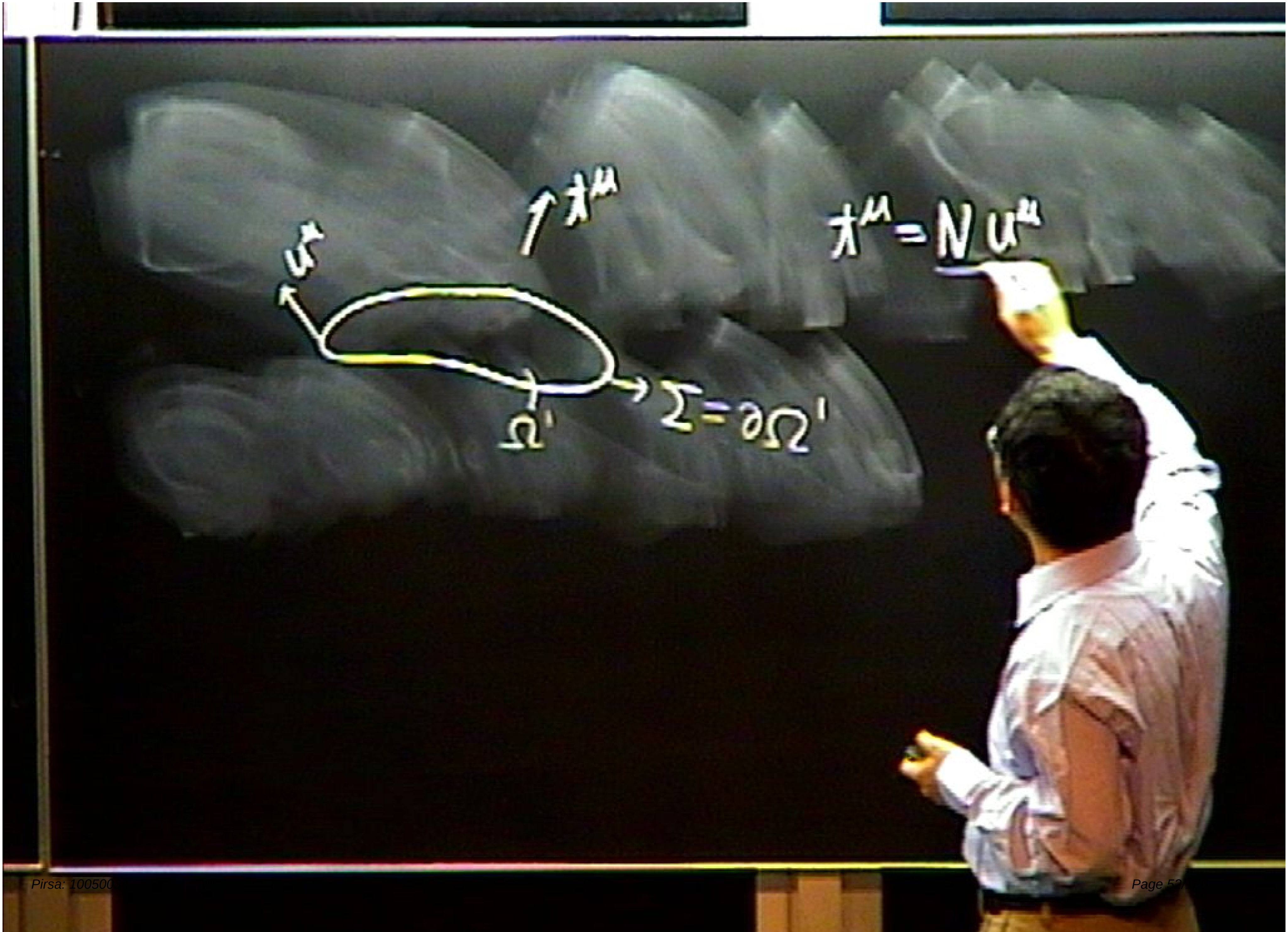
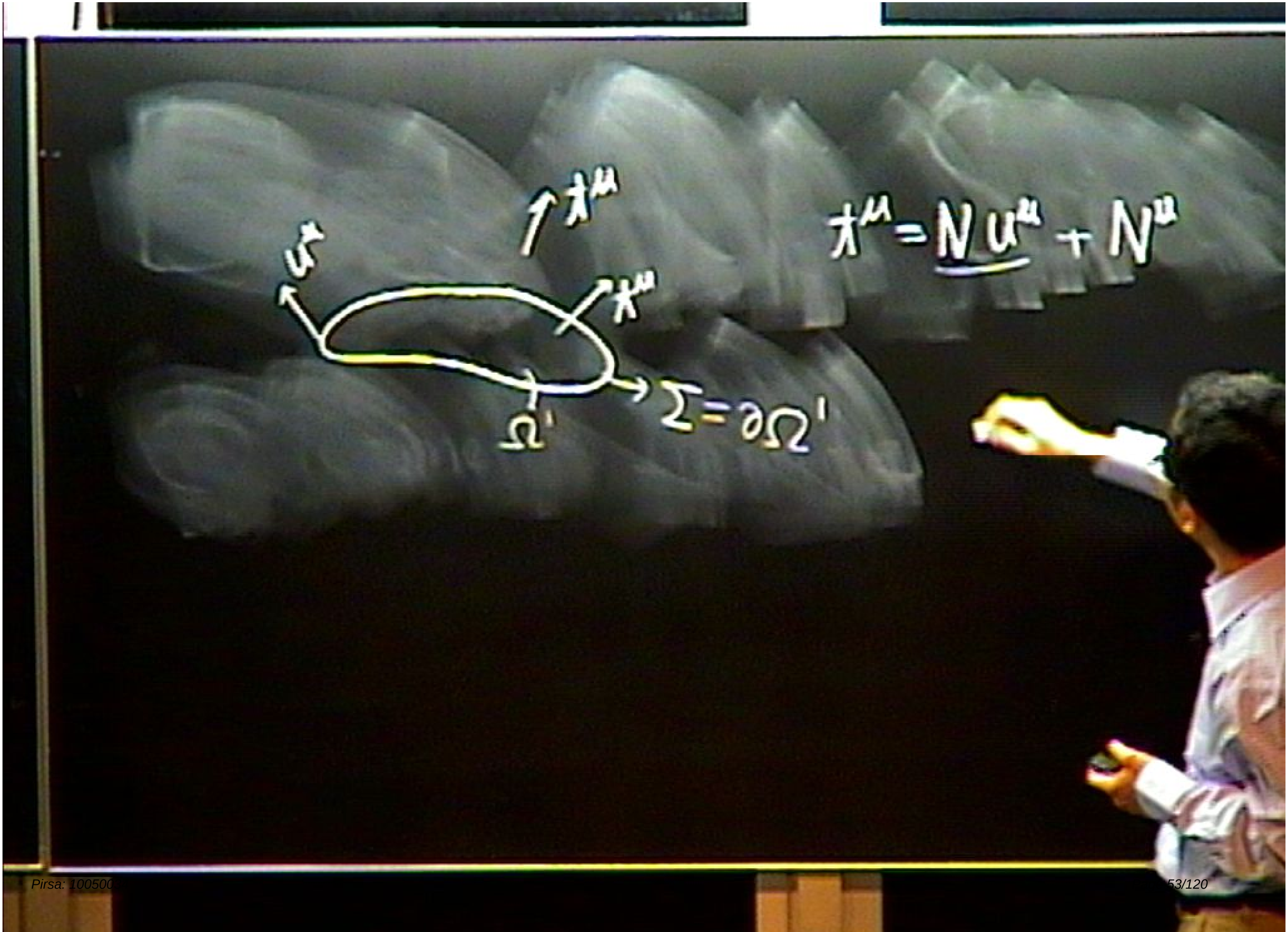

u^μ
J^μ
$J^\mu = N u^\mu$
Ω'
$\Sigma = \partial\Omega'$



u^μ
J^μ
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Ω'
$\Sigma = \partial\Omega'$
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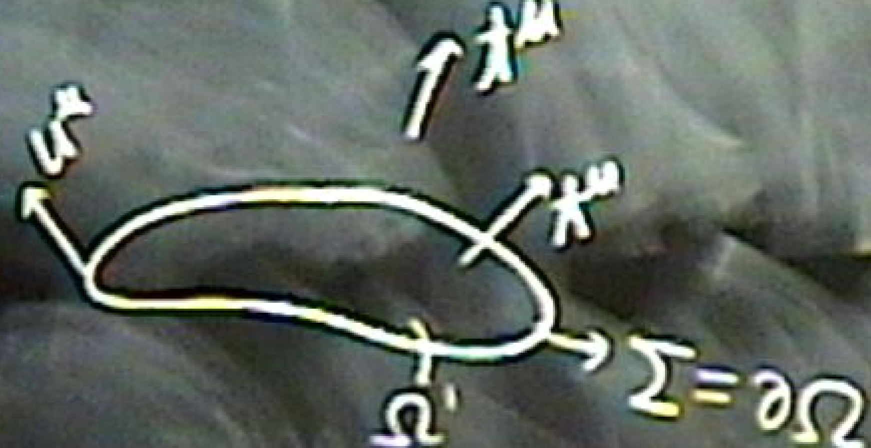
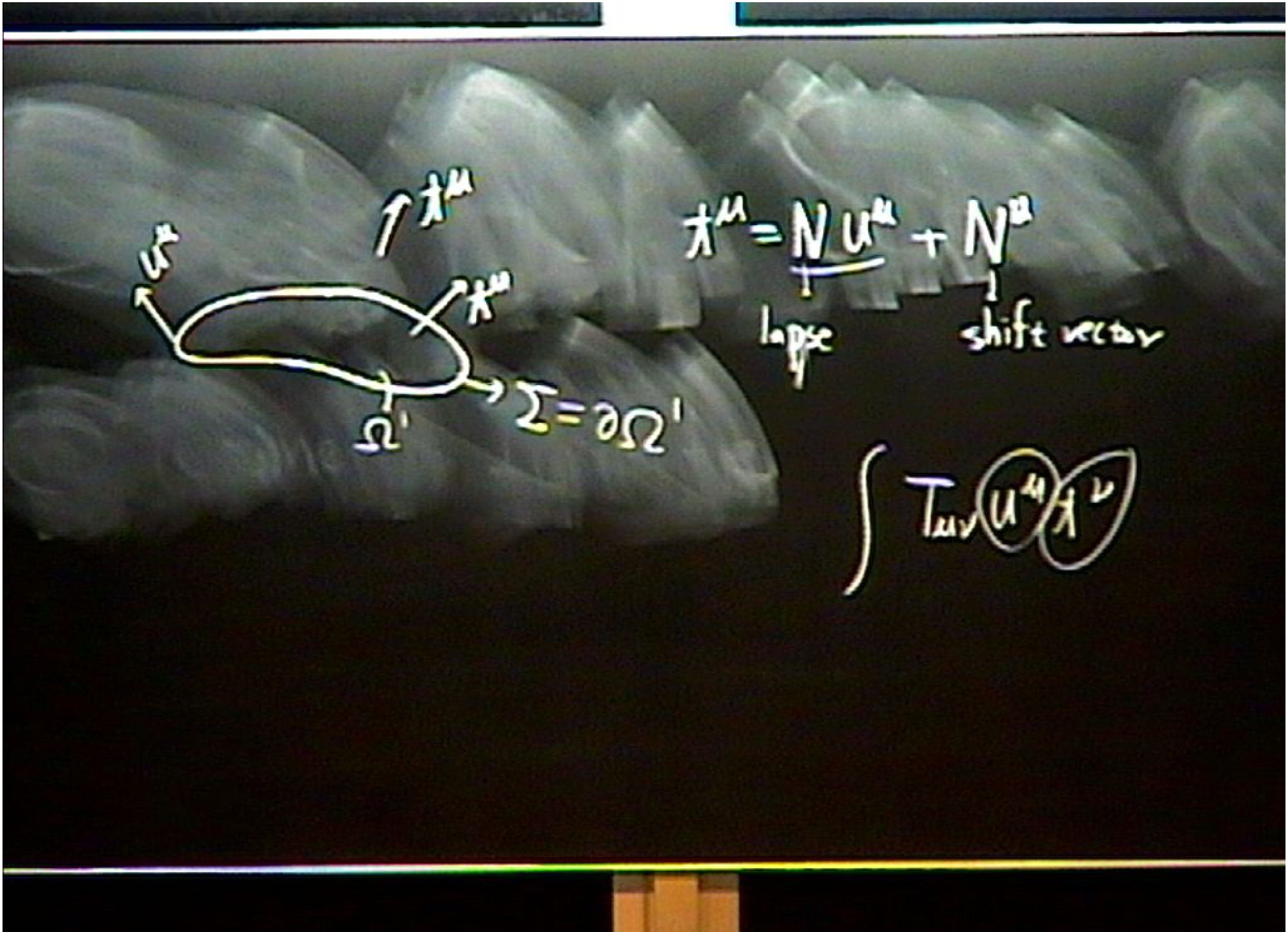


$$t^\mu = N u^\mu + N^\mu$$

N: lapse

N^μ: shift vector


λ^μ
u^μ
λ^μ
Ω'
Σ = ∂Ω'
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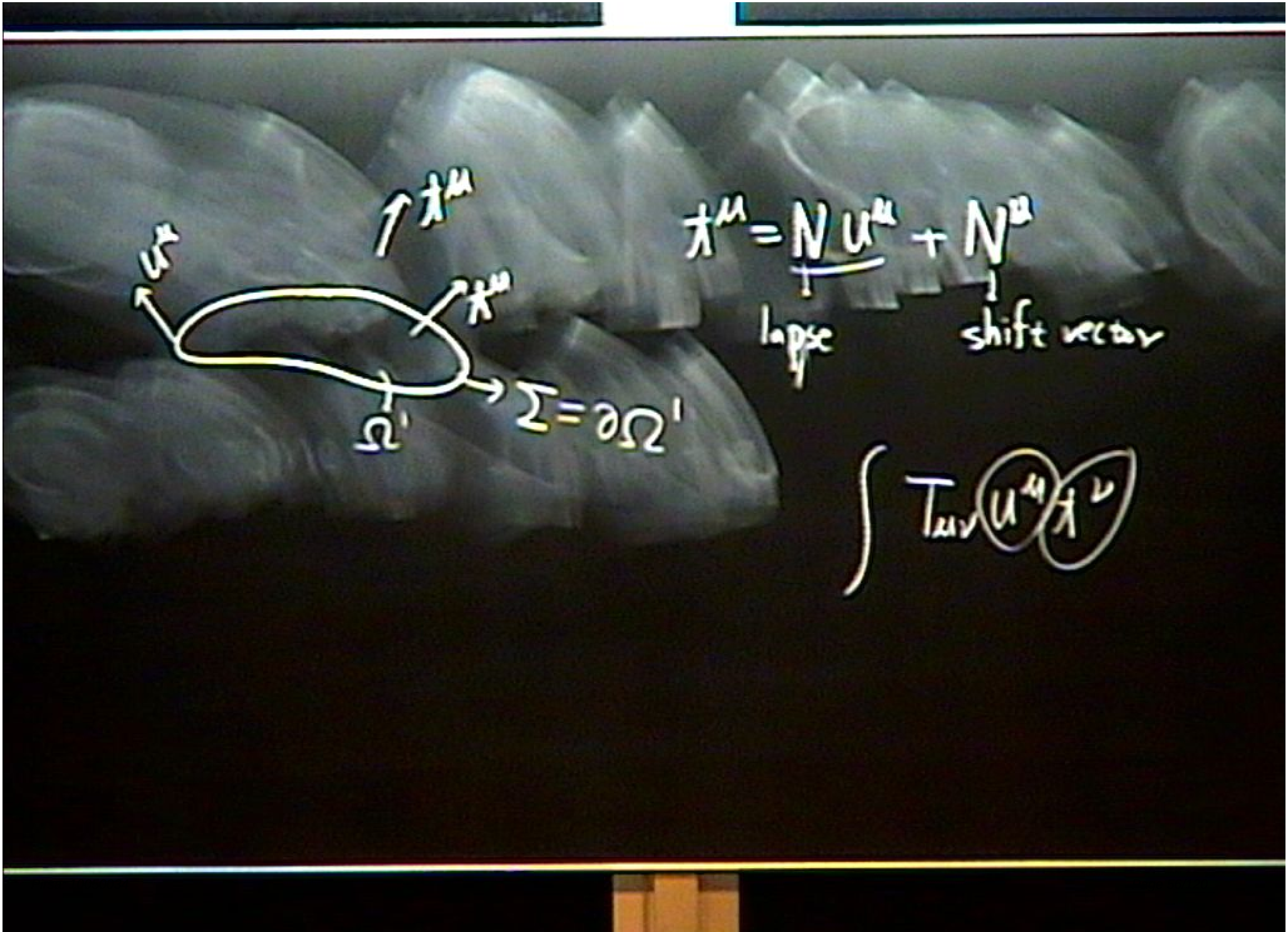
lapse
shift vector
∫ T_μν u^μ λ^ν


t^μ
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lapse
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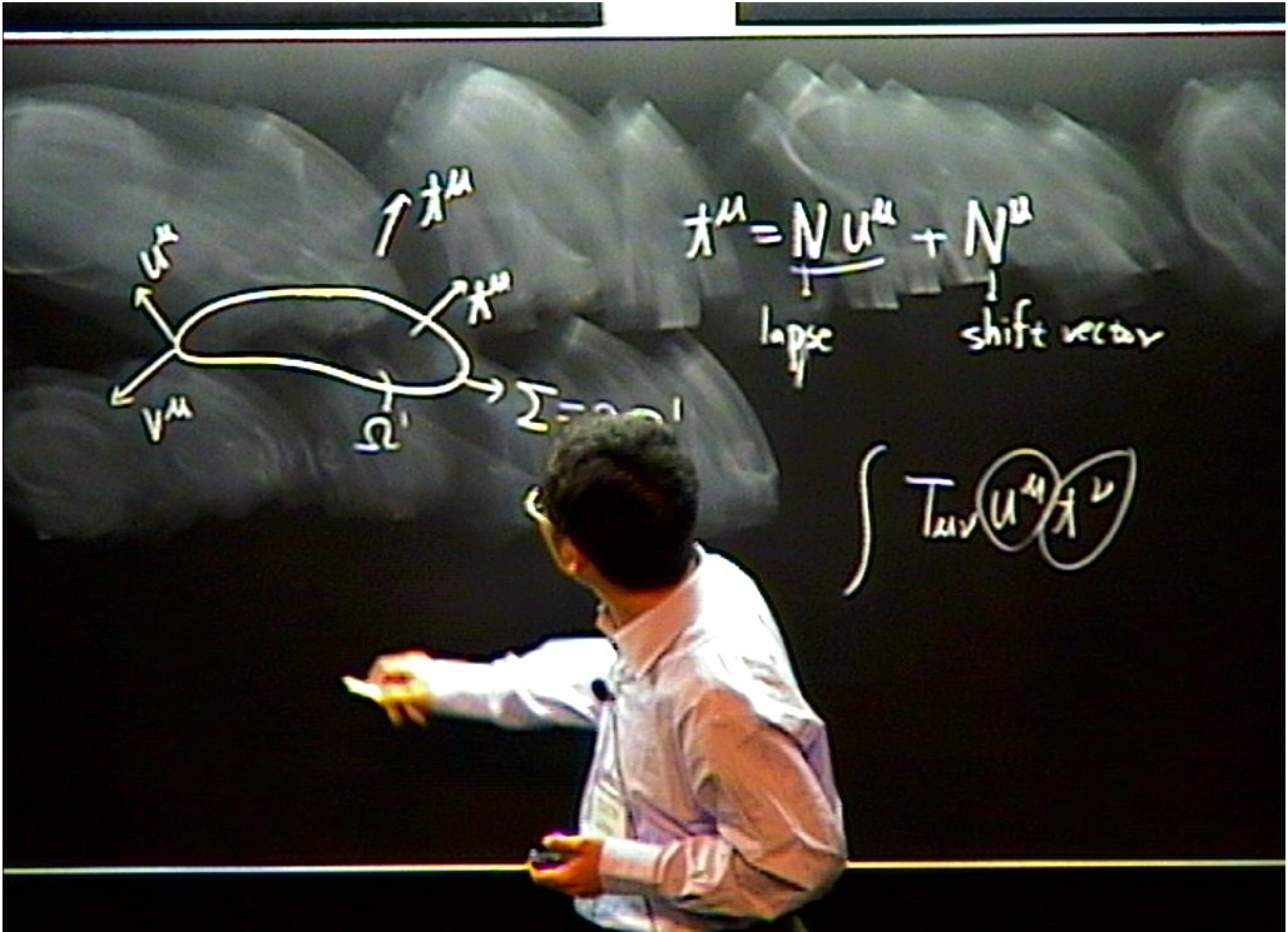

t^μ = N u^μ + N^μ
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u^μ
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v^μ
Ω'
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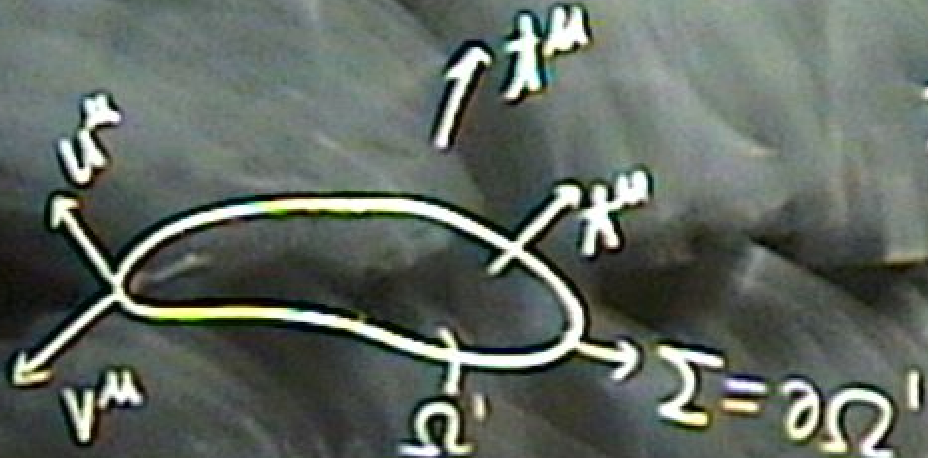

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$$\int T_{\mu\nu} u^\mu \chi^\nu$$

The Hamiltonian on a solution is a surface integral on $\Sigma = \Sigma_1 = \partial\Omega_1$.

t^μ a future timelike unit vector (observer).

u^μ a future timelike unit normal vector (unit normal of spacelike hypersurface Ω bounded by Σ)

$$t^\mu = Nu^\mu + N^\mu,\ \mathcal{H}(t^\mu, u^\mu) = \frac{1}{8\pi}\int_\Sigma Nk - N^\mu(p_{\mu\nu} - p^\lambda_\lambda g_{\mu\nu})v^\nu .$$

k is the mean curvature of Σ as boundary of Ω.

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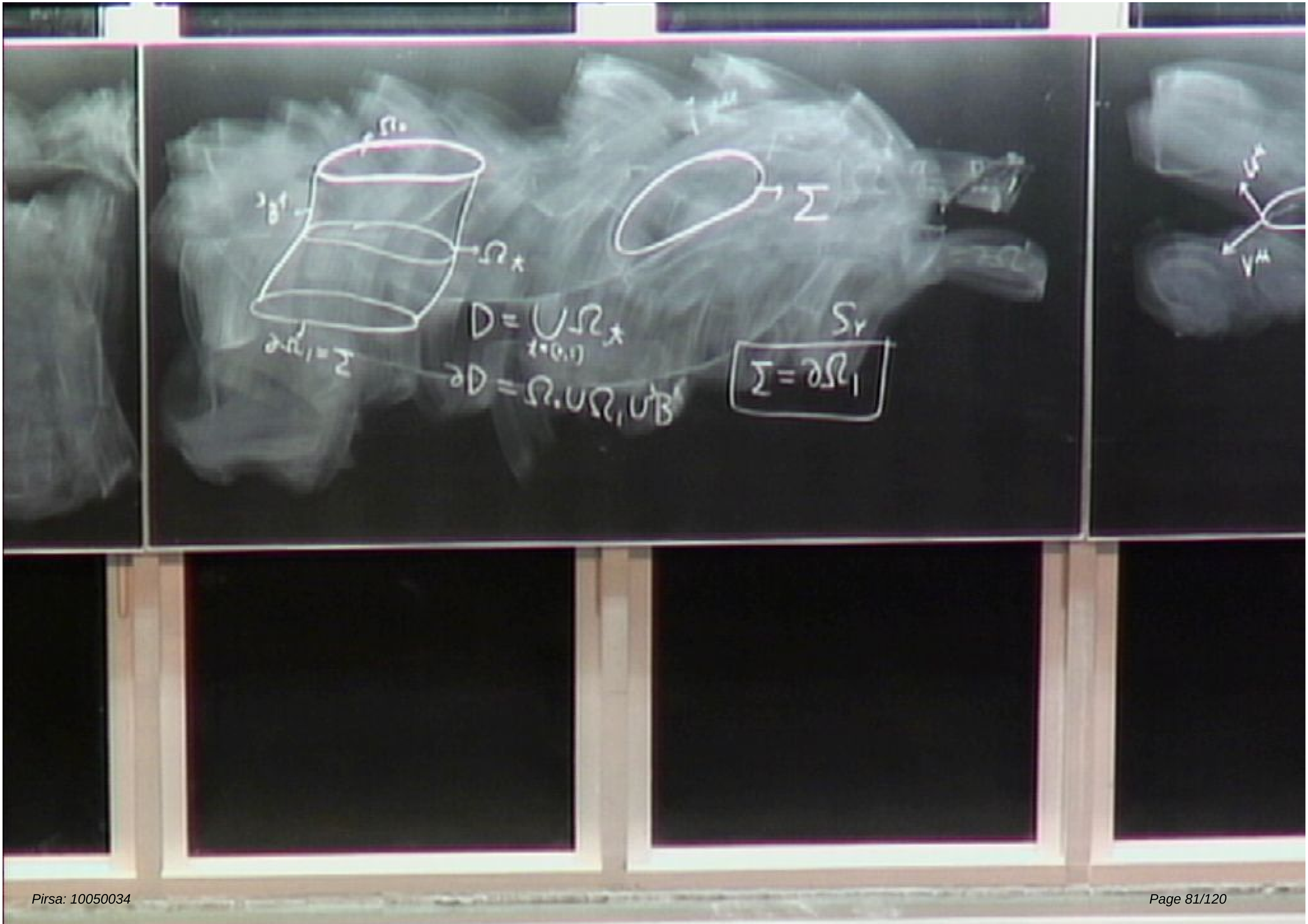
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D = ∪ Ω
∂D = Ω ∪ Ω ∪ B'
Σ = ∂Ω

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$$\text{physical Hamiltonian} - \text{reference Hamiltonian}$$

Reference Hamiltonian should come from data associated with isometric embedding of the time history of the boundary ${}^3B$ into a reference spacetime ($\mathbb{R}^{3,1}$). An over-determined equation.

Isometric embedding of Σ into $\mathbb{R}^3$ has been used to define Brown-York mass and Liu-Yau mass (Kijowski, Booth-Mann, Epp) with $u^\mu = t^\mu$ to be specified.

Important positivity was proved by Shi-Tam and Liu-Yau.

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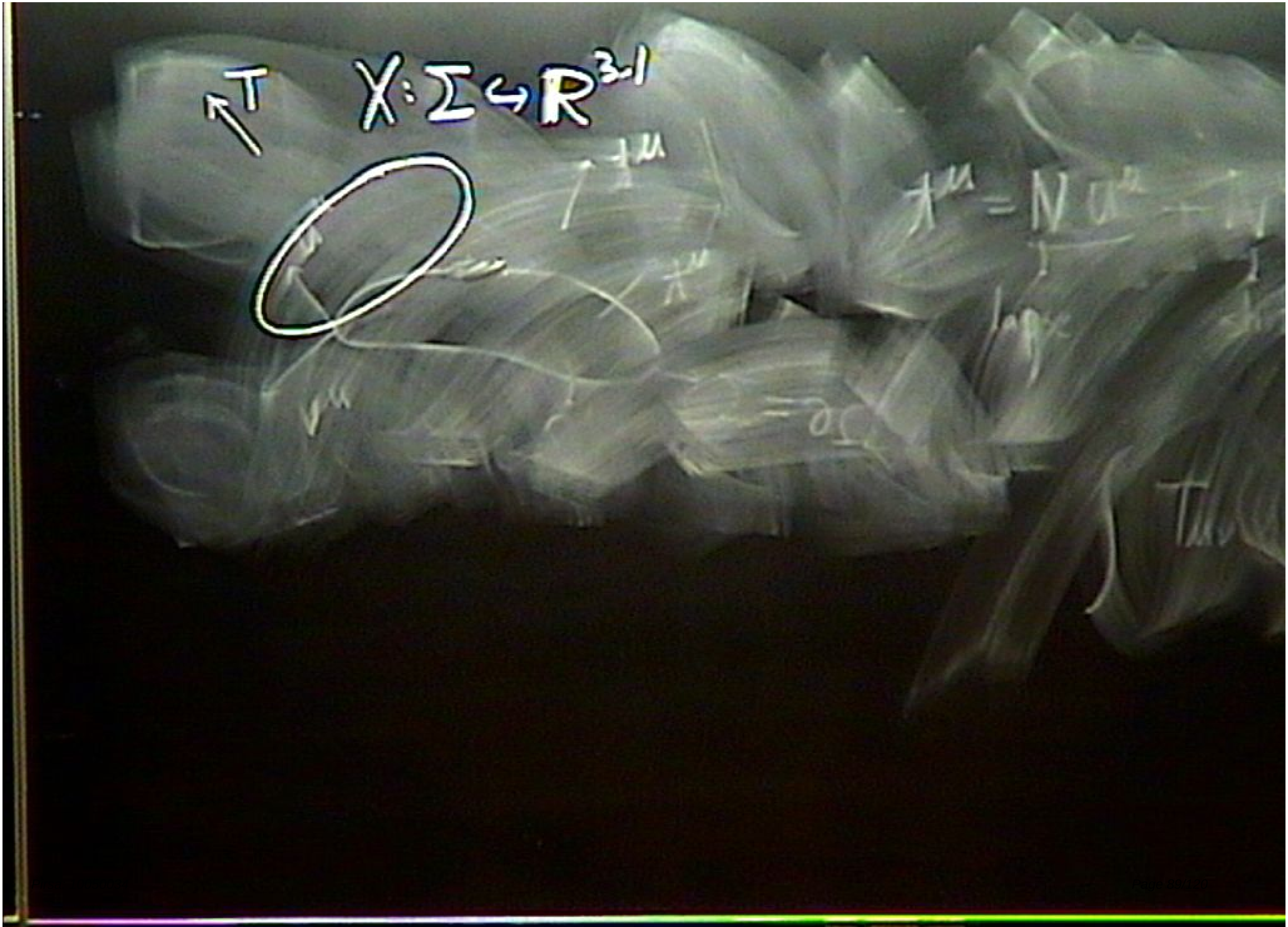
New definition of quasilocal energy (W-Yau)

Given $\Sigma \subset N$, for an isometric embedding $X : \Sigma \to \mathbb{R}^{3,1}$ and $T \in \mathbb{R}^{3,1}$, we define the quasilocal energy to be


T
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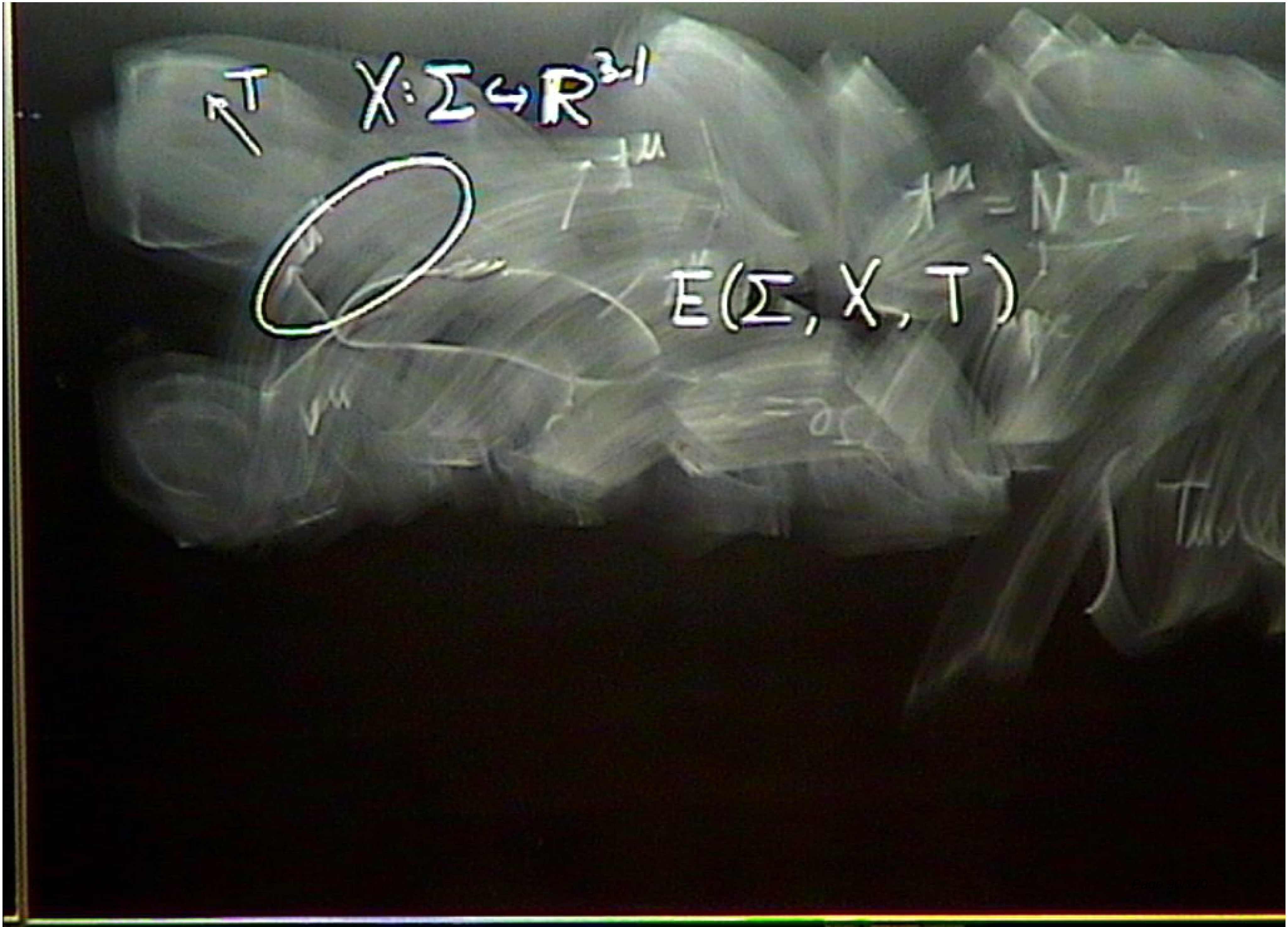
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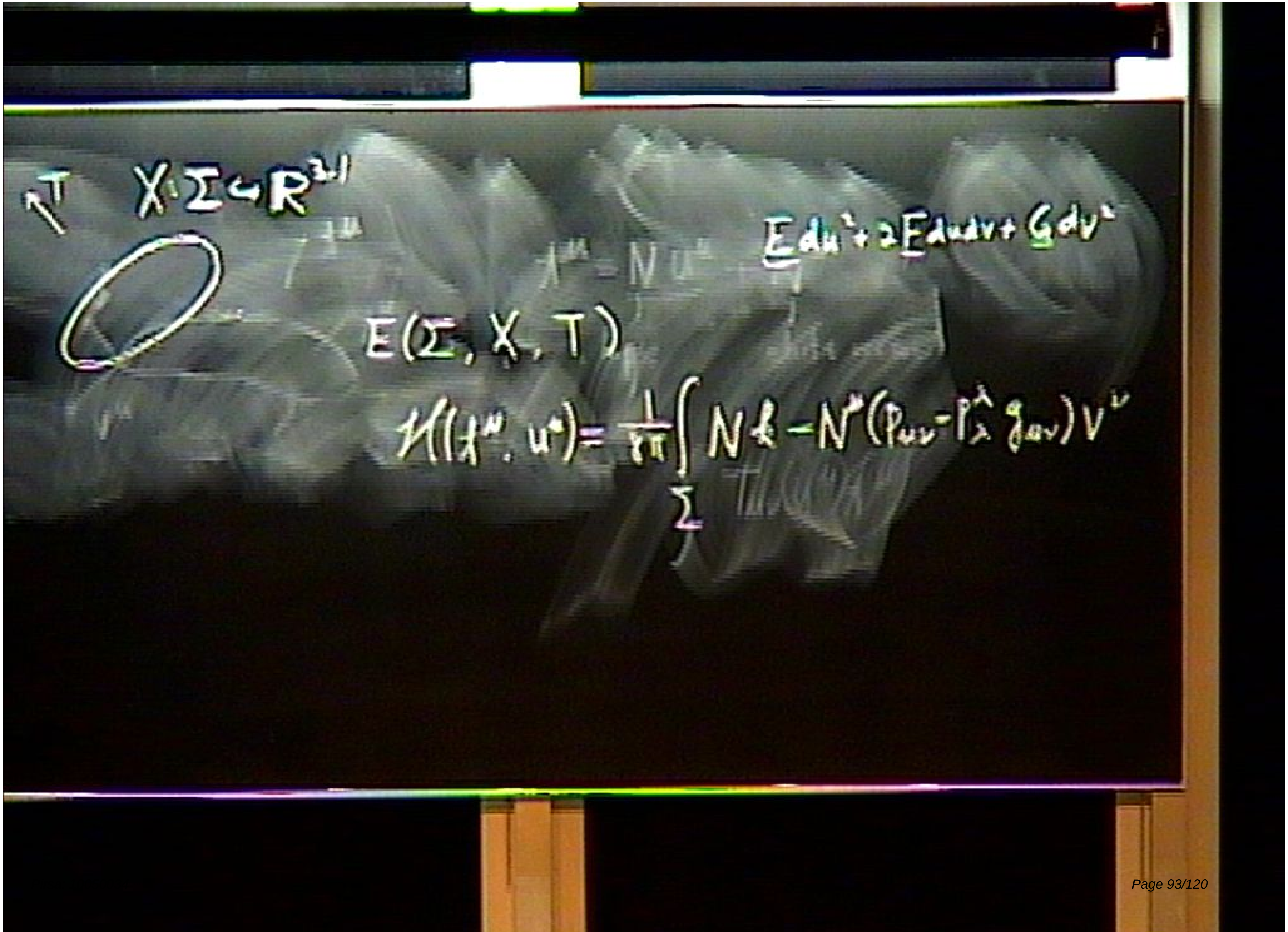
T
X:Σ→R^3,1
E(Σ, X, T)
Edu² + 2Edudv + Gdv²

Three new ingredients:

(1) Isometric embedding in $\mathbb{R}^{3,1}$ with fixed expansion.

Given an metric σ on Σ and a constant timelike vector $t_0^\mu \in \mathbb{R}^{3,1}$ suppose ρ is a smooth function on Σ that satisfies $\int_\Sigma \rho = 0$. Suppose a "convexity" condition holds, then there exists a unique isometric embedding of Σ into $\mathbb{R}^{3,1}$ such that the expansion of Σ in the direction of t_0^μ is ρ.

(2) In order to be gauge independent (of the 3-manifold Σ bounds), we choose u^μ to be in the direction of the normal part of t^μ along Σ. Likewise for u_0^μ and t_0^μ in the reference isometric embedding.

Expression in term of the mean curvature vector field of Σ in N.

For any spacelike surface Σ in N, the mean curvature vector is

$$H = -kv^{\mu} + pu^{\mu}$$

where k is the mean curvature of Σ in a spacelike Ω with $\partial\Omega = \Sigma$ and p is the trace of the restriction of p_{ij} to Σ.

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Fact: the mean curvature vector of X is $H_0 = \Delta X$, and thus the expansion in the direction of T is $\langle H_0, T \rangle = -\Delta \tau$

∇ and Δ the gradient and Laplace operator for functions on Σ with respect to the induced metric.

We then use the canonical gauge condition to find the corresponding future timelike vector T along Σ in N.

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$$\frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{k} - \frac{1}{8\pi}\int_{\Sigma} [\sqrt{|H|^2(1+|\nabla\tau|^2)+(\Delta\tau)^2} - \Delta\tau \sinh^{-1}\frac{\Delta\tau}{\sqrt{1+|\nabla\tau|^2}|H|} - \langle \nabla^N_{\nabla\tau}\frac{J}{|H|}, \frac{H}{|H|}\rangle]$$

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Quasilocal mass is defined to be the minimum of quasilocal energy among (X, T). Proof of positivity (under convexity assumptions) and rigidity of this mass (W-Yau).

Asymptotic behavior:

Quasilocal mass approaches the ADM mass and Bondi mass at spatial and null infinity, respectively.

In an asymptotically space time,

t slice is an asymptotically flat spacelike hypersurface, let S_r be coordinate spheres.

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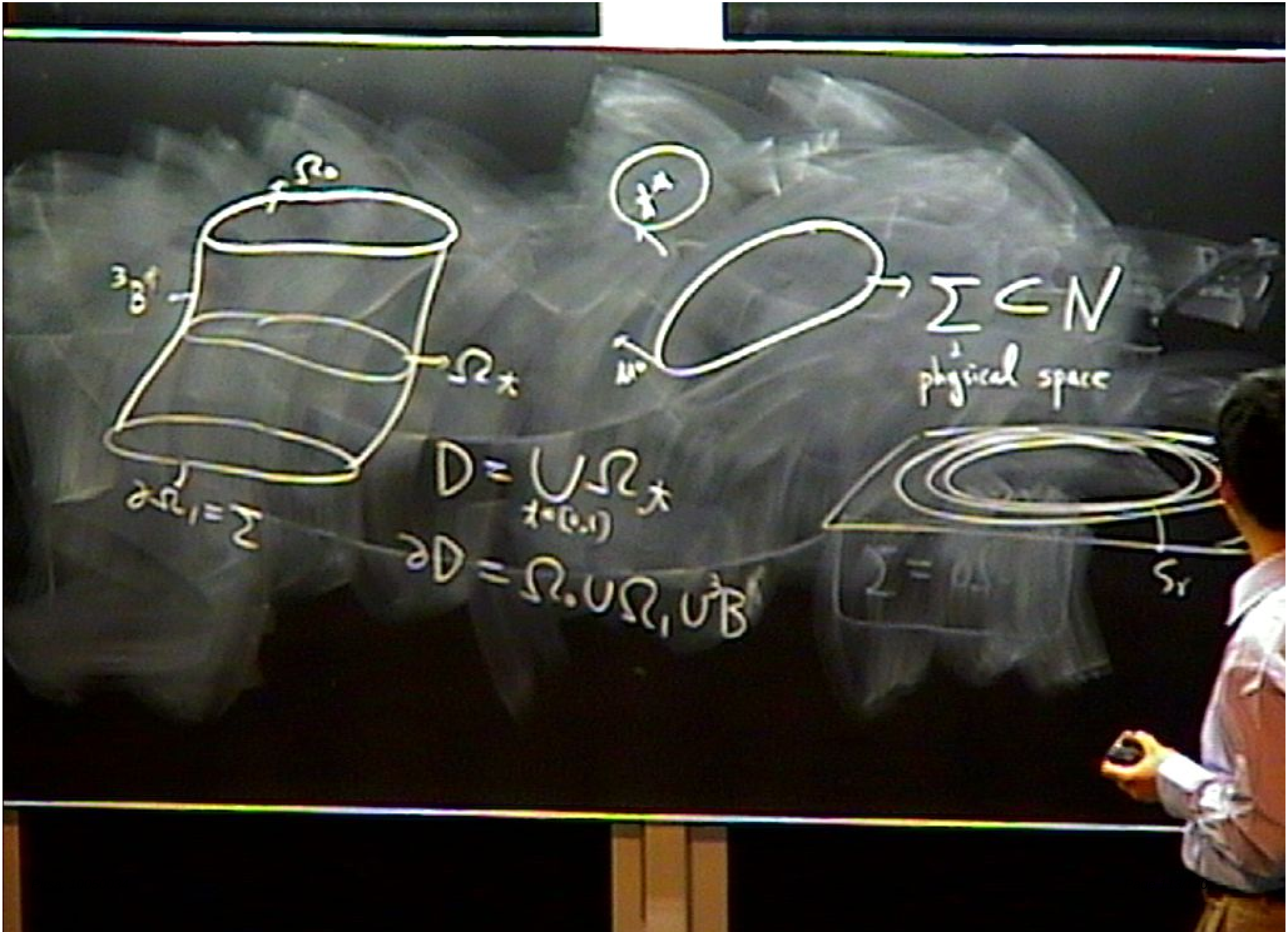
In Bondi coordinates, w(retarded time) slice is null, let S_r be (Bondi) coordinate spheres.


Ω₀
³B¹
Ω t
∂Ω₁ = Σ
D = ∪ Ω t
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∂D = Ω₀ ∪ Ω₁ ∪ ³B¹
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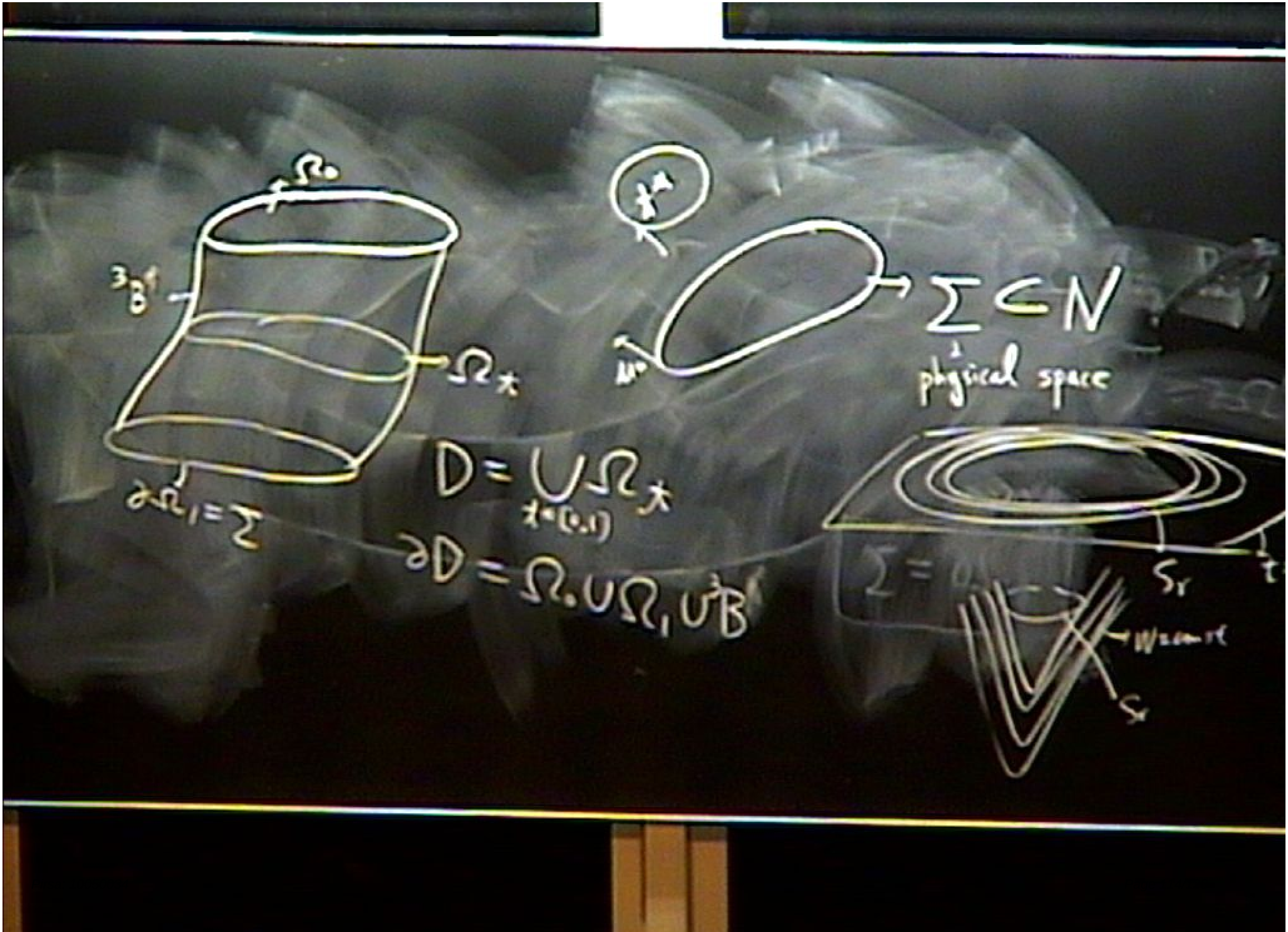

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Let $X_r : S_r \to \mathbb{R}^3$ be the (essentially unique) isometric embedding into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. T any future timelike constant vector in $\mathbb{R}^{3,1}$.

We prove that

$$\lim_{r \to \infty} E(S_r, X_r, T) = T^\mu P_\mu$$

where $P_\mu = (P_0, P_1, P_2, P_3)$ is the AMD / Bondi-Sachs energy-momentum 4-vector, in spatial/null infinity.

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In other words, the quasilocal energy $E(S_r, X_r, T)$ gets linearized and acquires the Lorentzian symmetry at infinity.

X_r together with the energy-momentum 4 vector (boost) coincide with the optimal isometric embedding that minimizes $E(S_r, \cdot, \cdot)$ in top order.

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The actual optimal isometric embedding is given by

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All $X_0, X_1, \cdots$ can be solved from the linearized equation of the Euler-Lagrange equation of the quasi-local energy functional.

Putting together $\cup_{r \text{ large}} X_r$ gives an "optimal reference" hypersurface (spacelike or null) in $\mathbb{R}^{3,1}$.

Summary:

Isometric embedding of surfaces into $\mathbb{R}^{3,1}$ with convex shadows.

A canonical gauge choice so the physical surface and reference surface have the same expansion with respect to the observer.

A new definition of quasilocal mass that has both positivity and rigidity properties and approaches the ADM mass and Bondi mass at spatial and null infinity, respectively.