

Title: Normal Forms for Lattice Polarized K3 Surfaces and the Kuga-Satake Hodge Conjecture

Date: May 07, 2010 04:30 PM

URL: <http://pirsa.org/10050033>

Abstract: We introduce a projective hypersurface "normal form" for a class of K3 surfaces which generalizes the classical Weierstrass normal form for complex elliptic curves. A geometric two-isogeny relates these K3 surfaces to the Kummer K3 surfaces of principally polarized abelian surfaces, with the normal form coefficients naturally identifying with the Igusa basis of Siegel modular forms of degree two. These results are reinterpreted through the lens of the Kuga-Satake Hodge Conjecture, and seen as a prediction coming from mirror symmetry.

# Normal Forms for Lattice-Polarized K3 Surfaces and the Kuga-Satake Hodge Conjecture

Two papers in an LN of A. Cingolani:

arXiv: 1004.3335 "Note on a Geometric Theory of K3 Surfaces"

arXiv: 10.04.3303 "Lattice-polarized K3 surfaces and Siegel Modular forms"

plus work in progress with my students at UqA and UW.

weight  $\in \mathbb{C}^*$

Newton's lift:

$$E = \{y^2 - 4x^2 + 2xy + 3z^2 = 0\} \subset \mathbb{C}P^2$$

$$(y, z) \in \mathbb{C}^2 \mid y^2 - 27z^2 = 0$$

not important  $(y, z) \in \mathbb{C}P^1$  divides  $E$   
by normalization

curve  $\in \mathbb{C}$  all over

$$M_E = \{(y, z) \in \mathbb{C}P^1 \mid y^2 - 27z^2 = 0\}$$

$y$ -normal  $f(E) = \frac{y^2}{z^2} - 27 = 0$

identifies  $M_E$  with  $\mathbb{C}$

Activity:  $\mathbb{R}^2$  as a vector space



vector

Classify your  $\mathbb{R}^2$  vectors



Resultant

periodicity

for  $M_E$

$H$

• inverse image

to  $H$

part

• Study arithmetic g.l.g. over



• Classify g.l.g. for groups

$$\mathbb{P}^1 \setminus \mathbb{H} \quad \Gamma = \text{PSL}(2, \mathbb{Z})$$

• Periodicity

$$\text{gen } M_E \longrightarrow \mathbb{P}^1 \setminus \mathbb{H}$$

• Inverse given by (P-) modular forms

$$\text{gen}^d = (60 \cdot E_4, 140 \cdot E_2)$$

$$\begin{matrix} E_4, E_2 : \mathbb{H} \longrightarrow \mathbb{C} \\ \left( \begin{matrix} 1 & \\ 2\pi i & 1 \end{matrix} \right) \quad \left( \begin{matrix} 1 & \\ & -1 \end{matrix} \right) \end{matrix}$$

classical form  
weight 4, 6

$K3$  surfaces

$X$ , formal  $K3$

all are deformation



$H^1(X, \mathbb{Z}) = 0$

Spanned by algebraic classes

$$H_2(X, \mathbb{Z}) \cong H \oplus H \oplus H \oplus E_1 \oplus E_1 = (L_1, L_2)$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

plans:

$\langle 2 \rangle$ -gen K3 spec

double cover of  $\mathbb{P}^2$  branched  
a smooth sextic

$\langle 4 \rangle$ -gen K3 spec

smooth quartic hypersurface  $\subset \mathbb{P}^3$

all are polynomials

$$E_2(x, y) = x^2 + y^2 = (x+iy)(x-iy)$$

Example 1:  $E_2(x, y) = x^2 + y^2$  is irreducible over  $\mathbb{R}$  but reducible over  $\mathbb{C}$

Example 2:  $E_3(x, y) = x^2 + y^2 + z^2$  is irreducible over  $\mathbb{C}$

Factorization



is irreducible over  $\mathbb{C}$  because it is a sum of squares of linear forms in  $x, y, z$  and is not a perfect square



(19-4) - level mod. spec for M-pl. K3 spec.

$M_{11} = H \oplus E_2 \oplus E_3 \oplus \langle \omega \rangle$

( $M_2 =$  mod. quartic K3 spec)

$H \oplus E_2 \oplus E_3$



$H \oplus E_2 \oplus E_7$



$H \oplus E_2 \oplus E_7$

Curve



Surface



Threefold



4-fold

Algebraic N.C.

• K3 spec of det

•  $H \oplus E_2 \oplus E_7$  - pol

• Toric hypersurface

•  $H \oplus E_2 \oplus E_7$  - pol

• Hyper-Salvatore

• Mirror Symmetry

$\mathbb{P}^1 \xrightarrow{H} \mathbb{P}^2$   
 $\mathbb{P} = \text{PSL}(2, \mathbb{C})$

$$\text{par}^{\pm} = (100 \cdot E_4, 140 \cdot E_L)$$

$$E_4, E_L : \mathbb{H} \rightarrow \mathbb{C}$$

$$\left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \quad \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$$

classical Example  
 water 4, 6 map

$\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$

is a diffeomorphism

$(x, z) \cong \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{E}_2 \oplus \mathbb{E}_2$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



Algebraic curve

$\mathbb{H} \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$

$\mathbb{H} \oplus \mathbb{F}_2^+$

is a diffeomorphism

Prop. 4.3 spec



$$N = H^0(F_2) \oplus H^0(F_1)$$

$$\left\{ \alpha z^2 w^2 + \beta z w^3 + \gamma x z^2 w - \frac{1}{2} (\alpha z^2 w^2 + w^4) = 0 \right\} \subset \mathbb{CP}^3$$

is minimal resolution. • If  $\gamma \neq 0$  or  $\delta \neq 0$ , then  $X(\alpha, \beta, \gamma, \delta)$  is a K3 spec of a canonical K3 polarization.

Thm (Clifford - 3)

$N = 4 \oplus C_2 \oplus C_2$

$\mathbb{P}^3 \quad (X, Y, Z, W)$

$$\begin{cases} y^2zw - 4x^2z^2 + 3 \approx xz^2w^2 + \frac{5}{4}z^2w^2 + \frac{3}{2}xz^2w - \frac{1}{2} \end{cases}$$

$X(a, y, z, w)$  be its minimal resolution

• If  $\dots$

• Conversely, given a 3-pol  $F \in k[X, Y, Z, W]$

$$(a, y, z, w) \in X \iff y \neq 0 \text{ or } z \neq 0 \text{ or } w \neq 0$$

$$(a, y, z, w) \in X$$

Thm: (Chap. 1) : Case includes space for  $N$ -qd  $K=3$  w/  $e$

$$M_{N\text{-qd}} = \{ (\alpha, \beta, \gamma, \delta) \in \mathbb{W}(\mathbb{Z}, 3, 5, 6) \mid \gamma \neq 0 \text{ or } \delta \neq 0 \}$$

$$\text{per}^{-1} = (\alpha, \beta, \gamma, \delta) \text{ where } \alpha = E_4, \quad \gamma = 2^{12} \cdot 3^5 \cdot E_1$$

$$\beta = E_6, \quad \delta = 2^{12} \cdot 3^6 \cdot E_1$$

$$Sp(4, \mathbb{Z})$$

gen. functions

Local Hypersurface N.F.

$\Delta \rightsquigarrow \mathbb{P}_\Delta$ 

 general hypersurface  $\subset \mathbb{P}_\Delta$

$\exists$  3-D affine patch  $K_3 \subset \mathbb{P}_\Delta \rightsquigarrow \mathbb{P}_\Delta \supset K_3$

$\exists$  At st. for each of the pd. lattices we consider, the general affine hypersurface is called  $\mathbb{P}_\Delta$ 's local Normal form

$X$   $\mathbb{C}$  algebra KB spec

Def: A nilradical (nilpotent) ideal in  $X$  is an algebra extension of

$\alpha: X \rightarrow X$  s.t.  $\alpha^2(\omega) = \omega$ , for any ideal  $Z$  in  $X$

Properties: (a) ideal locus of  $\alpha$  consists of  $S$  distinct pts

(b) the spec  $Y = \widetilde{X}/\alpha$  is still a KB spec

(c) there is a dgc called by  $\rho_\alpha: X \dashrightarrow Y$  w/ br



Def: A nilpotent (or nilpotent) map on  $X$  is an analytic automorphism of order 2  
 $\alpha: X \rightarrow X$  st  $\alpha^2(\omega) = \omega$ , for any holo  $\omega$  on  $X$

- Properties:
- ① fixed locus of  $\alpha$  consists of  $\delta$  distinct pts
  - ② the spec  $Y = \overline{X/\alpha}$  is still a  $\mathbb{C}^3$  spec.
  - ③ there is a diag pulling  $P_\alpha: X \dashrightarrow Y$  w/ branch locus given by  $\delta$  pts  
(discriminant)
  - ④ there is a push forward map  $(p_\alpha)_* : H^2(X, \mathbb{C}) \rightarrow H^2(Y)$

$\omega_1, \omega_2, \dots, \omega_\delta \in H^2(X, \mathbb{C})$   
 $X(\delta, \gamma, \delta) \in Y$

getting the same result in  $H^1(X, \mathcal{O}_X)$

Ex (Cubic)  $N = H^0(\mathcal{O}_X(3))$   
 $\mathbb{P}^3 (x, y, z, w)$

$$\left\{ \begin{aligned} & y^2zw - 4x^3z + 3xz^2w^2 + \alpha zw^3 + \beta xz^2w - \frac{1}{2} (\gamma z^2w^2 + w^4) = 0 \end{aligned} \right.$$

- $X(\alpha, \beta, \gamma, \delta)$  be its minimal resolution
- if  $\gamma \neq 0$  or  $\delta \neq 0$ , then  $X(\alpha, \beta, \gamma, \delta)$  is a K3 spec of canonical  $\mathbb{P}^3$ -polarization
- Conversely, open as  $\mathbb{P}^3$ -pol K3 spec  $X$   $\exists$   $(\alpha, \beta, \gamma, \delta)$  w/  $\gamma \neq 0$  or  $\delta \neq 0$  st  $X(\alpha, \beta, \gamma, \delta) \cong X$

only to be used in  $H^1(Y, \mathbb{Z})$

Def. Shioda-Tate conjecture: the Néron-Severi group  $\underline{NS}(Y)$  is

$$\textcircled{1} \quad \gamma = \text{rank}(A) = \dim A$$

$\textcircled{2} \quad (P_X)_*$  induces Hodge isomorphism between lattices of forms on

$$T_X(z) \quad \text{and} \quad T_X$$

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Note: Don't Mess up on effective lattice-theoretic criteria for when a given  $K3$  surface admits a S-T structure.

Def: Van Geemen-Sarti involution is an auto  $\alpha_X: X \rightarrow X$  for which  $\exists$

a triple  $(\varphi_X, S_1, S_2)$

①  $\varphi_X: X \rightarrow \mathbb{P}^1$  cell-fiber on  $X$

②  $S_1, S_2$  disjoint sections of  $\varphi_X$

③  $S_2$  is a class of order 2 in  $H^2(X, \mathbb{Z})$

④  $\alpha_X$  involution obtained by twisting the fibrewise involutions by  $S_2$  on the small fibres of  $\varphi_X$  using any  $\sigma \in \text{Aut } \mathbb{P}^1 \neq \text{id} = S_1$

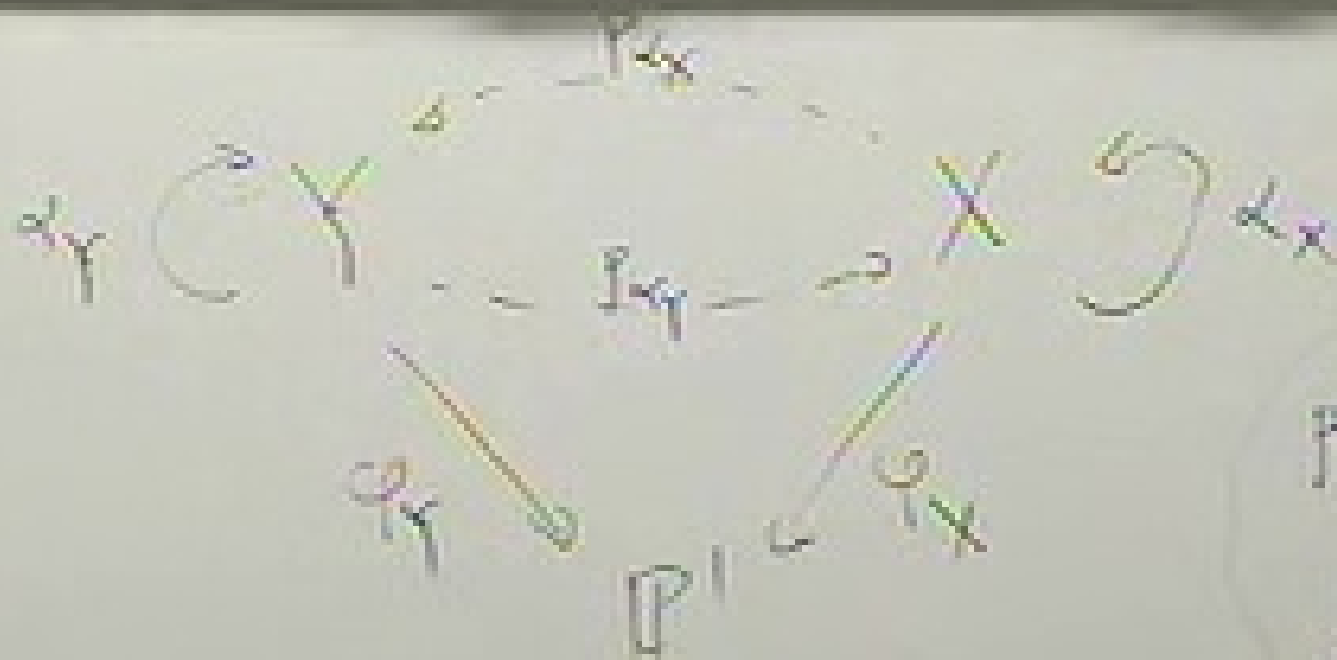
$\alpha_X$ -fixed = fibrewise 2-torsion  
data original KS sfc  
& new KS sfc  $Y$

$$\pi_Y: Y \rightarrow \mathbb{P}^1 \quad \text{with } S_1', S_2'$$

$S_1'$  = image under  $\pi_{X_2}$  of the two sectors  $S_1, S_2$  of  $\mathbb{P}^1$

$S_2'$  = image of the divisor on  $X$  obt. by cutting the cone along from the center of the 2 remaining order-2 poles in  $S_1$  fibers of  $\pi_X$

$S_1', S_2'$  disjoint,  $\xi' \in \text{MK}(\pi_X, S_1')$  has order 2.



Any  $k \geq 3$  spec  $X$  w/ graph  $\text{pt}$

$$H^0(E_1 \oplus E_2) \cong NS(X)$$

which cannot be est. to a  $\text{pt}$   
by  $H^0(E_1 \oplus E_2)$

①  $c$  can be est. to  $\text{pt}$  by  $H^0(E_1 \oplus E_2)$   
(good)

②  $\gamma$  cannot be so est. (nonsense)

Spec  $X$  obt. as main residue of

the double cover of  $\mathbb{P}^2$  branched

configuration  $\Gamma$  of 6

pts of them are concurrent

③  $c$  does not map to a curve  $D$   
(curve)

④ How is  $c$  so smoothly est.

Main result (Prop. 1) - The KS spaces  $(X, \mathcal{X})$  above carry canonically defined  $\sigma$ -finite sub-measures  $\mu_X, \nu_X$ . Moreover there is a bijective correspondence

$$(Y, \mathcal{Y}) \longleftrightarrow (X, \mathcal{X})$$

with  $(X, \mathcal{X})$  related by a pair of dual geometric 2-Borel spaces.



$$H \otimes E_3 \otimes E_3 \rightarrow E_1 \otimes E_2$$

$$H \otimes E_1 \otimes E_7 \rightarrow \text{Jac}(G_2)$$

$$H \otimes E_7 \otimes E_7 \rightarrow \text{Payni } C_6$$

$\downarrow \text{Hilb}$   
 $(\mathbb{P}^1)$

Yoga-Subalgebra  
A.V.

### Homomorphism

col  
 C<sub>13</sub>  
 id cubic  
 level 3  
 quotient

$\mathbb{F}_3[x]$   
 $\mathbb{F}_3[x]$

$\xrightarrow{\text{fact}}$

$\mathbb{P}^2$

$\xrightarrow{\text{Hesse quad}}$

$\mathbb{C}^3$   
 $\mathbb{F}_3[x]$

LG orbifold

$f(t) = t^3 - 1$

$\mathbb{F}_3$

$\mathbb{F}_3$

$\xrightarrow{\quad}$

$f(t) = t^3 - 1$

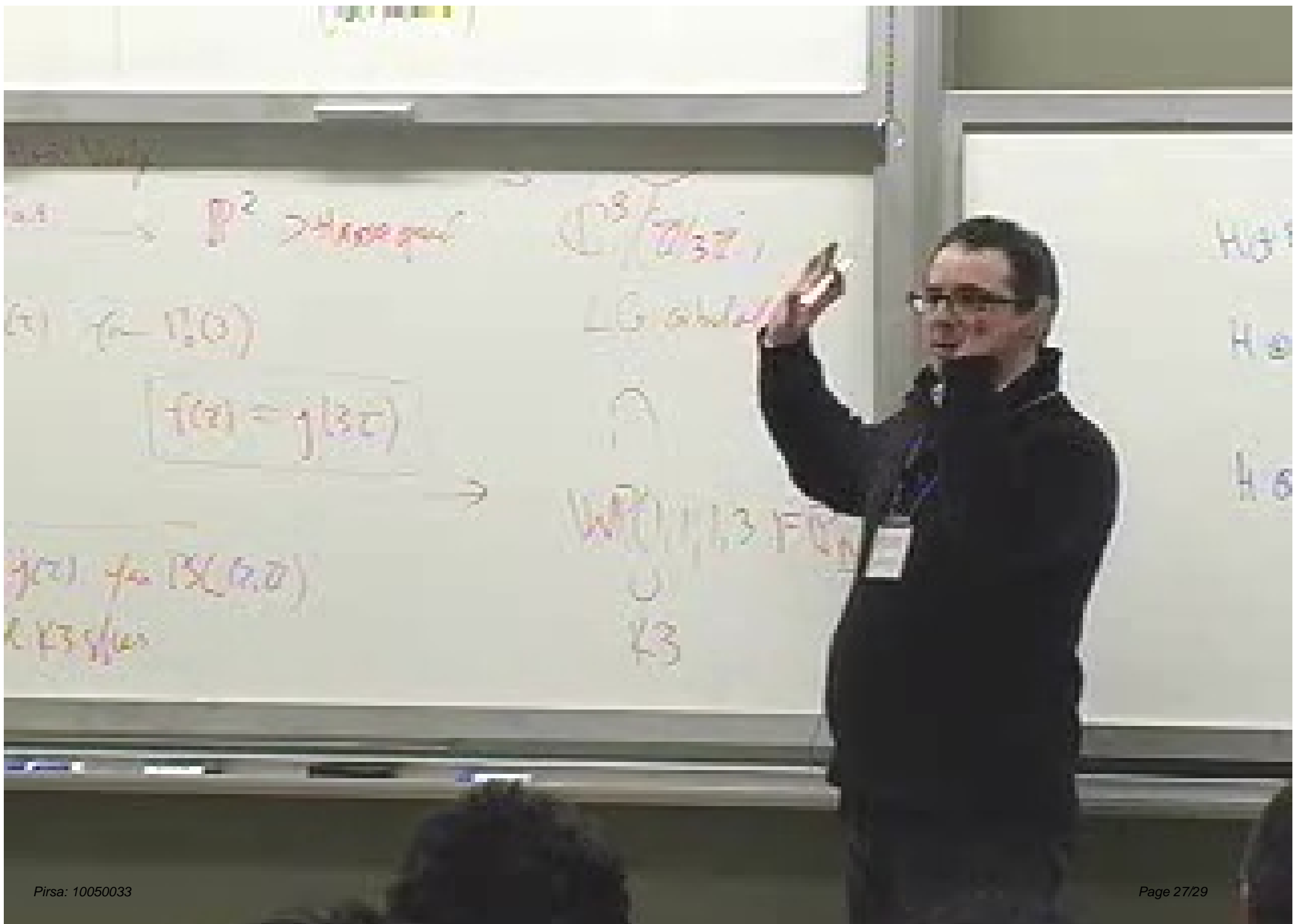
Might  $\mathbb{F}_3$  span

$\rightarrow$

$\mathbb{F}_3$

$\mathbb{F}_3$

$\mathbb{F}_3$



$P^2$   $\rightarrow$   $\dots$   $\dots$   $\dots$   
 $(x) \rightarrow (x, 0)$   
 $f(x) = y(x)$   
 $\rightarrow$   
 $f(x) = y(x)$   
 $\dots$

$P^2 \rightarrow$  *Handwritten text*

$C^5$  *Handwritten text*  
LG *Handwritten text*

$f(t)$  for  $B(0,0)$   
 $L(1,3)$  *Handwritten text*



$\mathbb{Z}_5$

Fact  $\rightarrow$

$\mathbb{P}^2$

Homomorphism

$\mathbb{C}^3$  (Hilbert)  
LG algebra

$\mathbb{Z}_3$

$\leftarrow$   $\rightarrow$

$\mathbb{H} \oplus \mathbb{E}_7 \oplus \mathbb{P}^3$

$\mathbb{P}^3$