Title: Normal Forms for Lattice Polarized K3 Surfaces and the Kuga-Satake Hodge Conjecture

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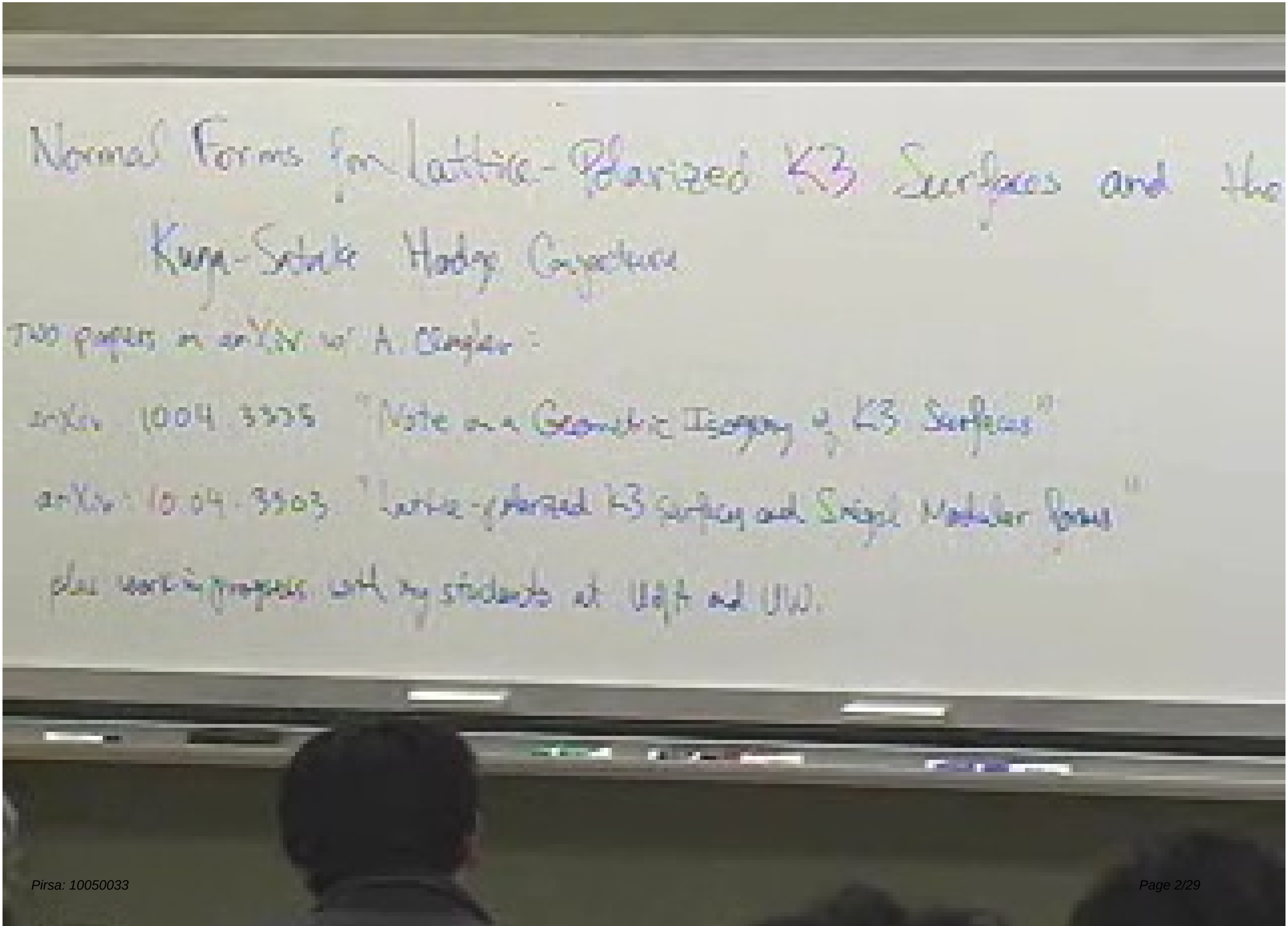
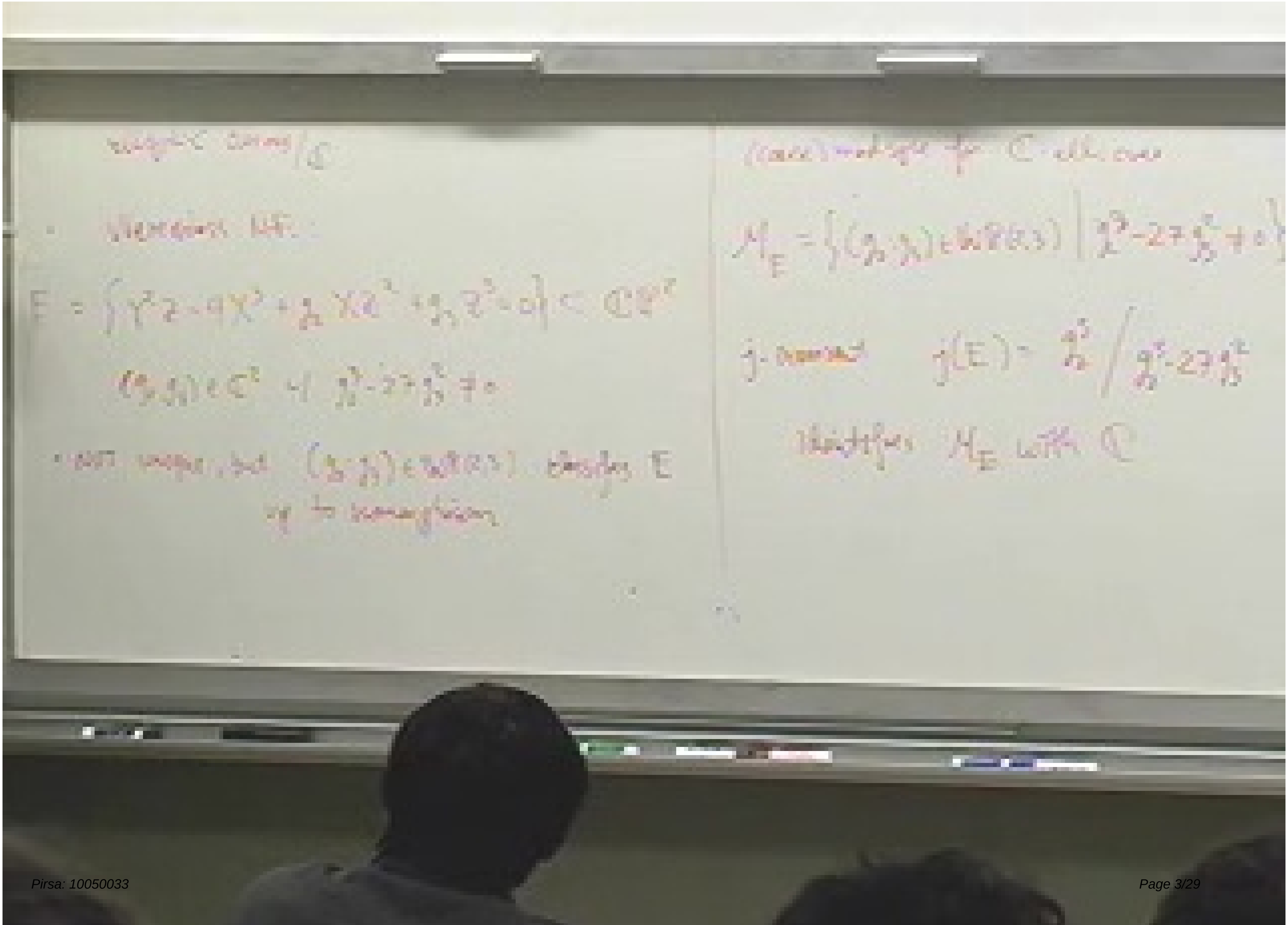
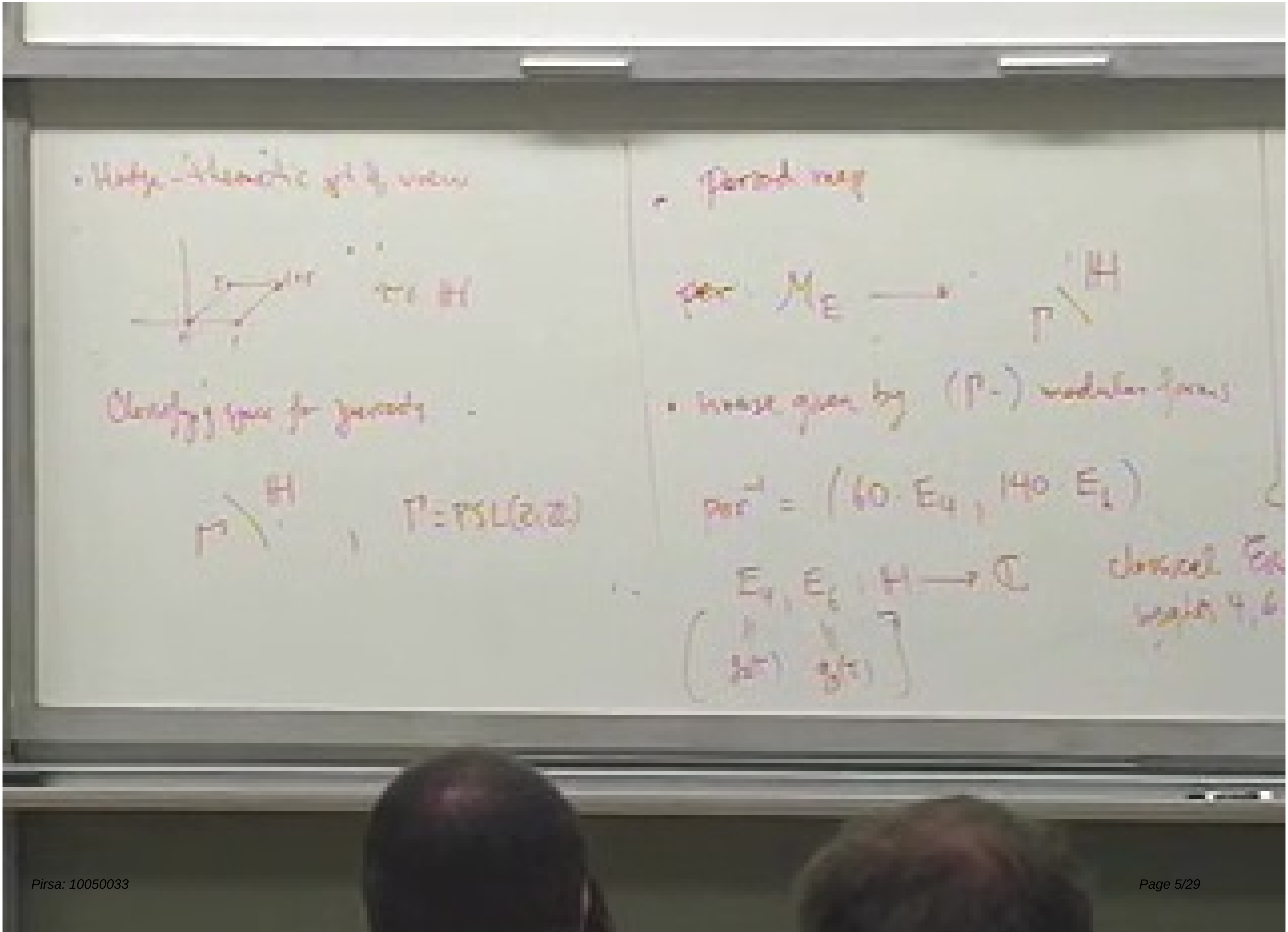
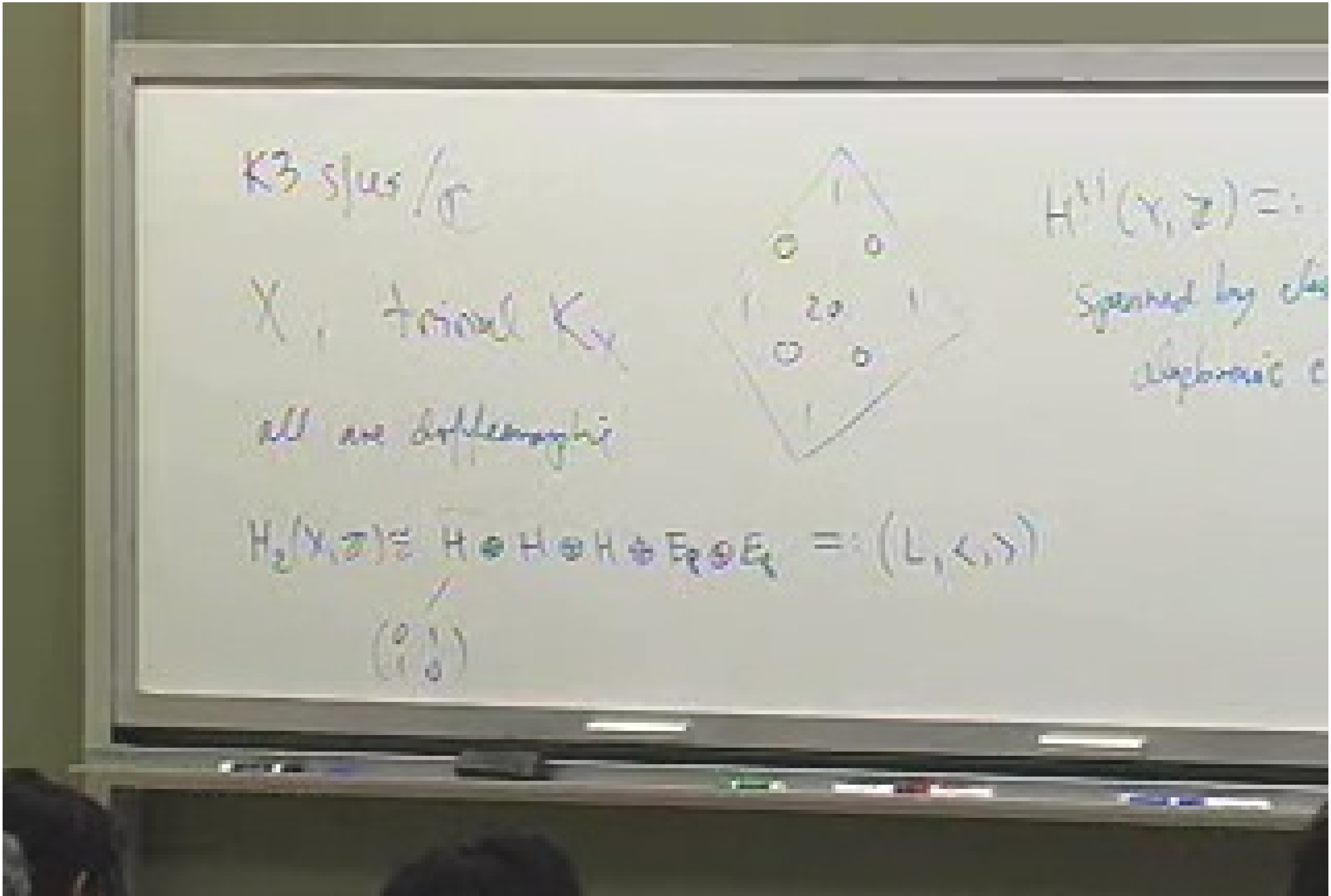
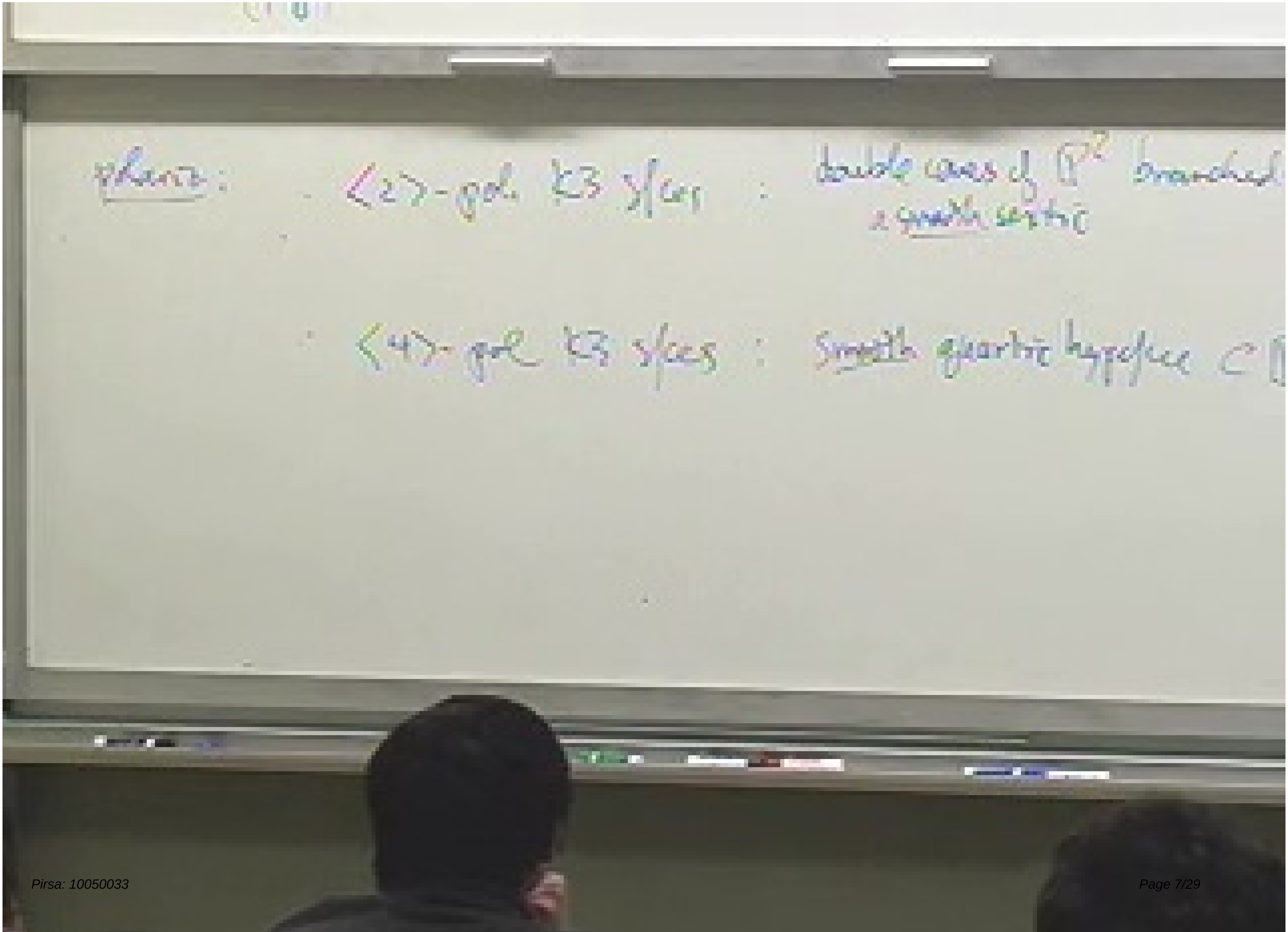
Abstract: We introduce a projective hypersurface "normal form" for a class of K3 surfaces which generalizes the classical Weierstrass normal form for complex elliptic curves. A geometric two-isogeny relates these K3 surfaces to the Kummer K3 surfaces of principally polarized abelian surfaces, with the normal form coefficients naturally identifying with the Igusa basis of Siegel modular forms of degree two. These results are reinterpreted through the lens of the Kuga-Satake Hodge Conjecture, and seen as a prediction coming from mirror symmetry.

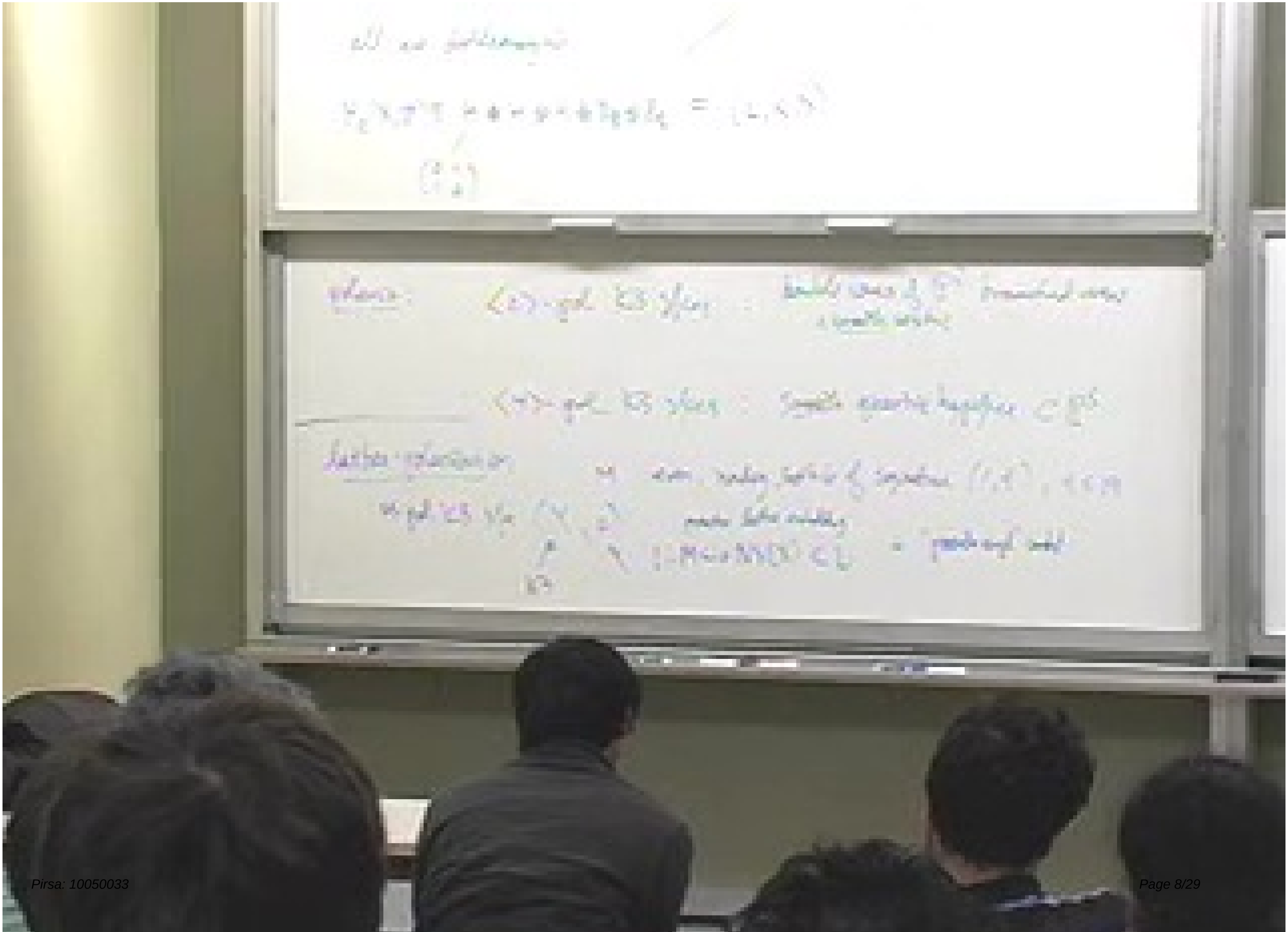

Normal Forms for Lattice-Polarized K3 Surfaces and the
Kuga-Satake Hodge Conjecture
Two papers on arXiv w/ A. Clingher:
arXiv 1004.3335 "Note on a Geometric Isogeny of K3 Surfaces"
arXiv 1004.3503 "Lattice-polarized K3 Surfaces and Siegel Modular Forms"
plus work in progress with my students at Utah and UW.

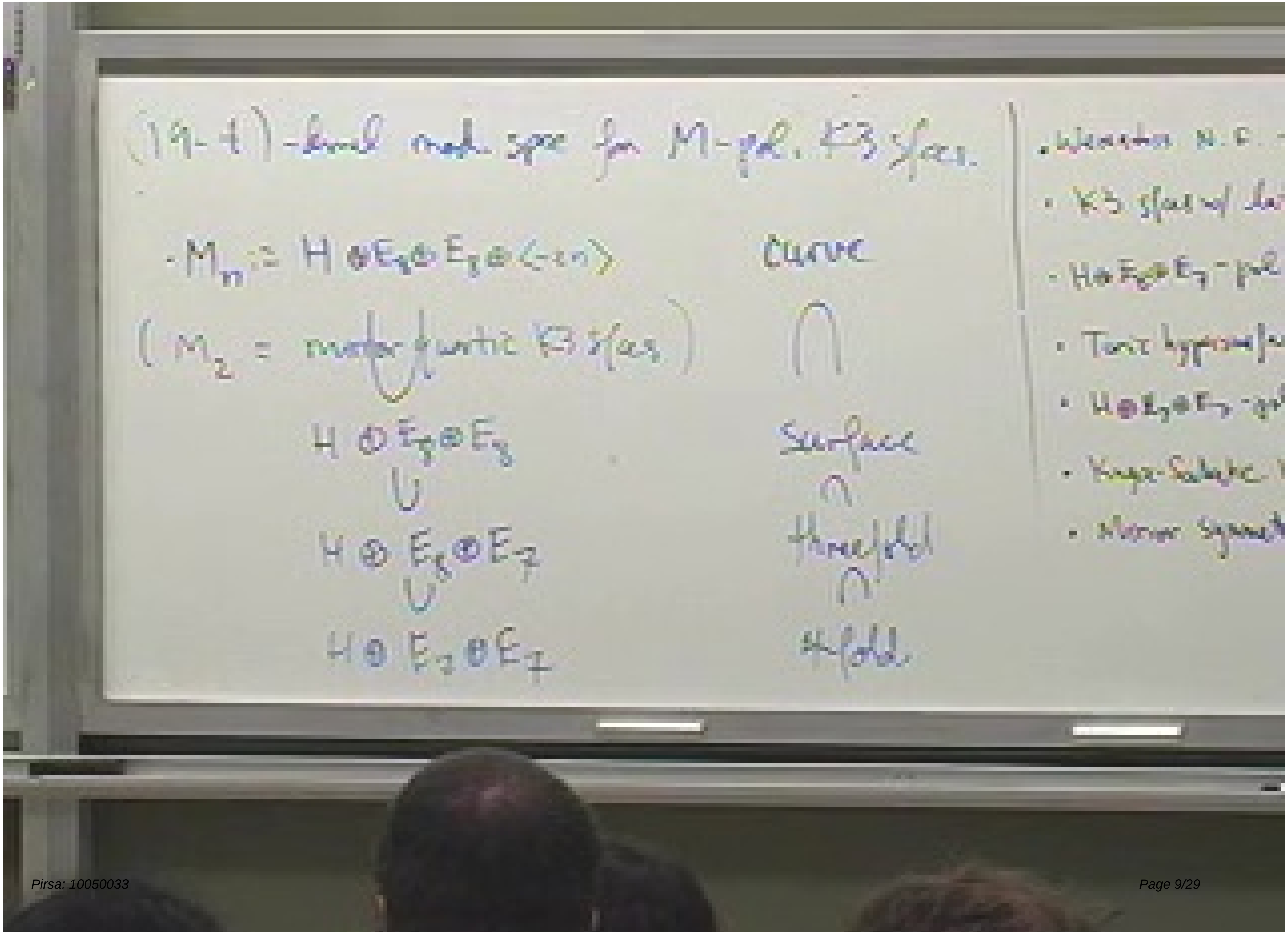

elliptic curves / ℂ
• Weierstrass NF:
E = {Y²Z - 4X³ + g₂XZ² + g₃Z³ = 0} ⊂ ℂℙ²
(g₂, g₃) ∈ ℂ² s.t. g₂³ - 27g₃² ≠ 0
• not unique, but (g₂, g₃) ∈ ℂ²/ℂ* classifies E
up to isomorphism
(coarse) moduli space for ℂ ell. curves
M_E = {(g₂, g₃) ∈ ℂ²/ℂ* | g₂³ - 27g₃² ≠ 0}
j-invariant j(E) = g₂³ / g₂³ - 27g₃²
identifies M_E with ℂ

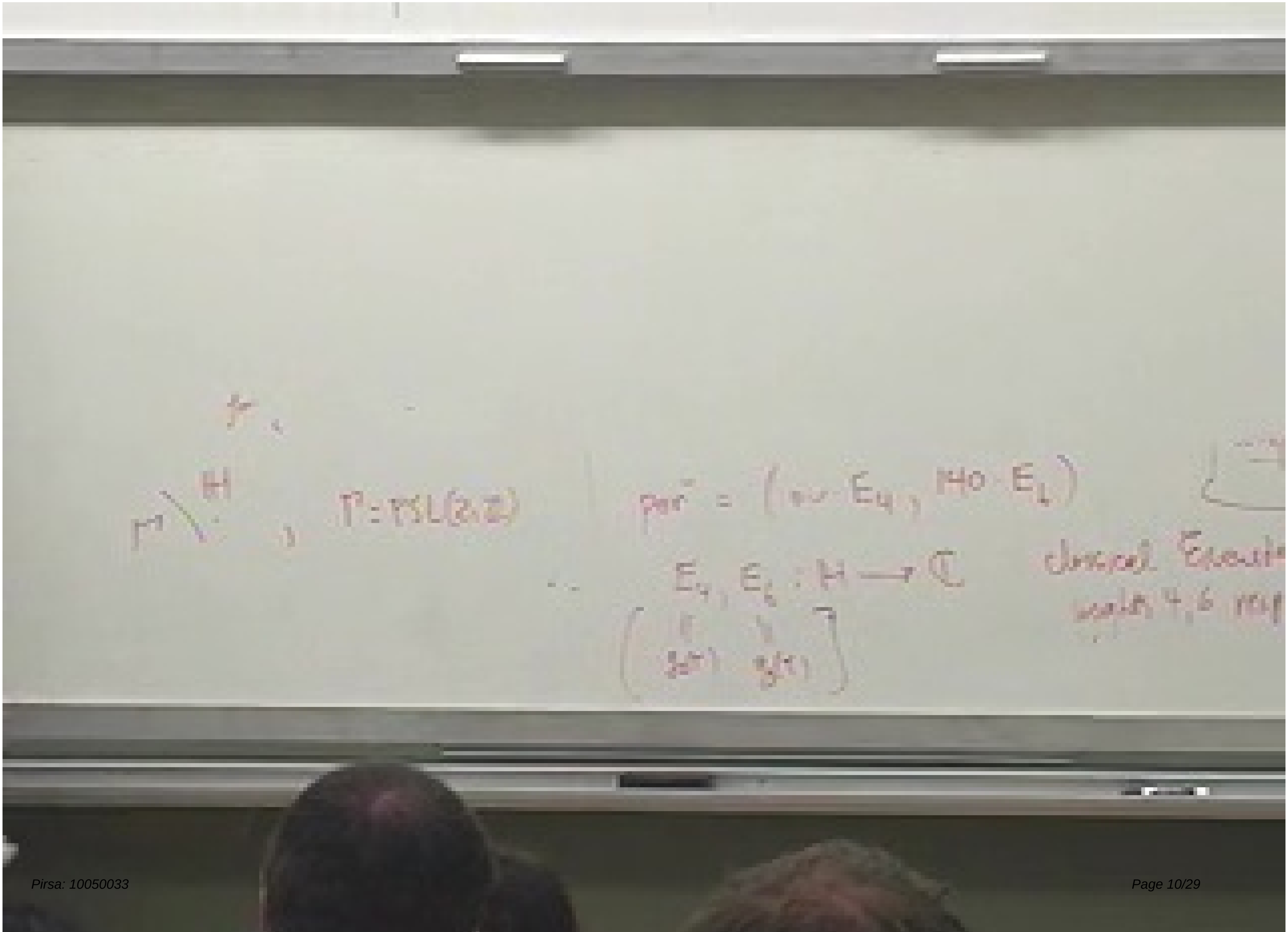

Period map
$M_E \longrightarrow \Gamma \backslash \mathbb{H}$
• inverse given by (Γ-) modular forms
$per^{-1} = (60 \cdot E_4, 140 E_6)$
$E_4, E_6 : \mathbb{H} \longrightarrow \mathbb{C}$
$\Gamma \backslash \mathbb{H}$, $\Gamma = PSL(2,\mathbb{Z})$
$\tau \in \mathbb{H}$

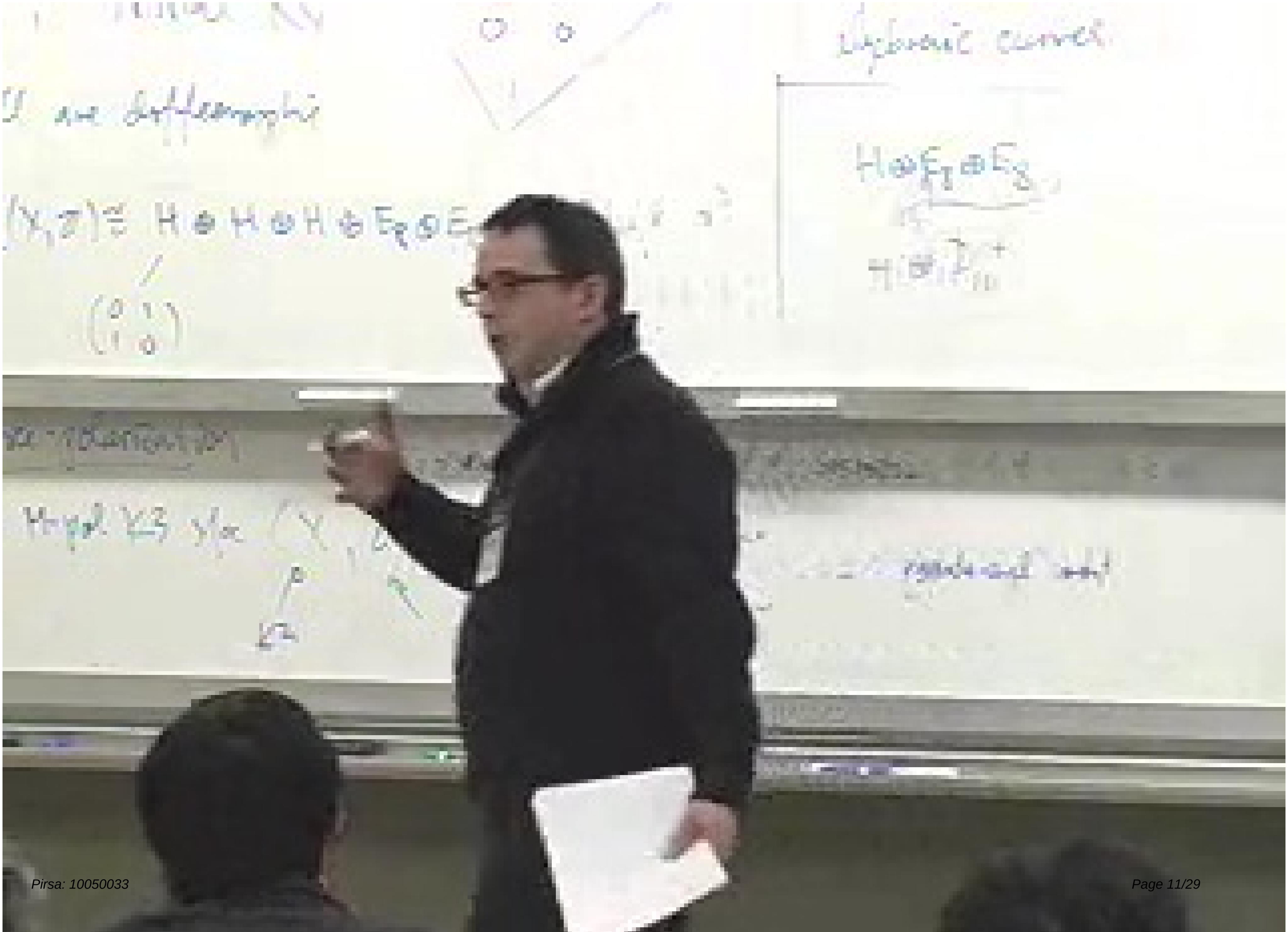

K3 s/ces / ℂ
X, trivial K_X
all are diffeomorphic
H_2(X,Z) ≅ H⊕H⊕H⊕E_8⊕E_8 =: (L, <,>)
H^{1,1}(X,Z) =:
Spanned by
algebraic


⟨2⟩-pol. K3 s/ces : double cover of P² branched
a smooth sextic
⟨4⟩-pol. K3 s/ces : smooth quartic hypsfce ⊂ P




(19-t)-dim'l mod. spc for M-pol. K3 s'faces.
· M_n := H ⊕ E_8 ⊕ E_8 ⊕ ⟨-2n⟩
Curve
(M_2 = mirror quartic K3 s'faces)
H ⊕ E_8 ⊕ E_8
Surface
H ⊕ E_8 ⊕ E_7
threefold
H ⊕ E_7 ⊕ E_7
4-fold
· K3 s'faces w/
· Toric hypersurf
· Kuga-Satake
· Mirror symmet


Γ\H , Γ = PSL(2,Z)
E4, E6 : H → C
classical



$N = H \oplus E_8 \oplus E_8$

$\ldots \alpha\, xzw^2 + \beta\, zw^3 + \gamma\, xz^2w - \frac{1}{2}(\delta\, z^2w^2 + w^4) = 0\} \subset \mathbb{CP}^3$

...s minimal resolution • If $\gamma \neq 0$ or $\delta \neq 0$, then $X(\alpha,\beta,\gamma,\delta)$ is a K3 s'ce w/ a canonical N-polarization

Thm (Clingher–D) $N = H \oplus E_8 \oplus E_7$

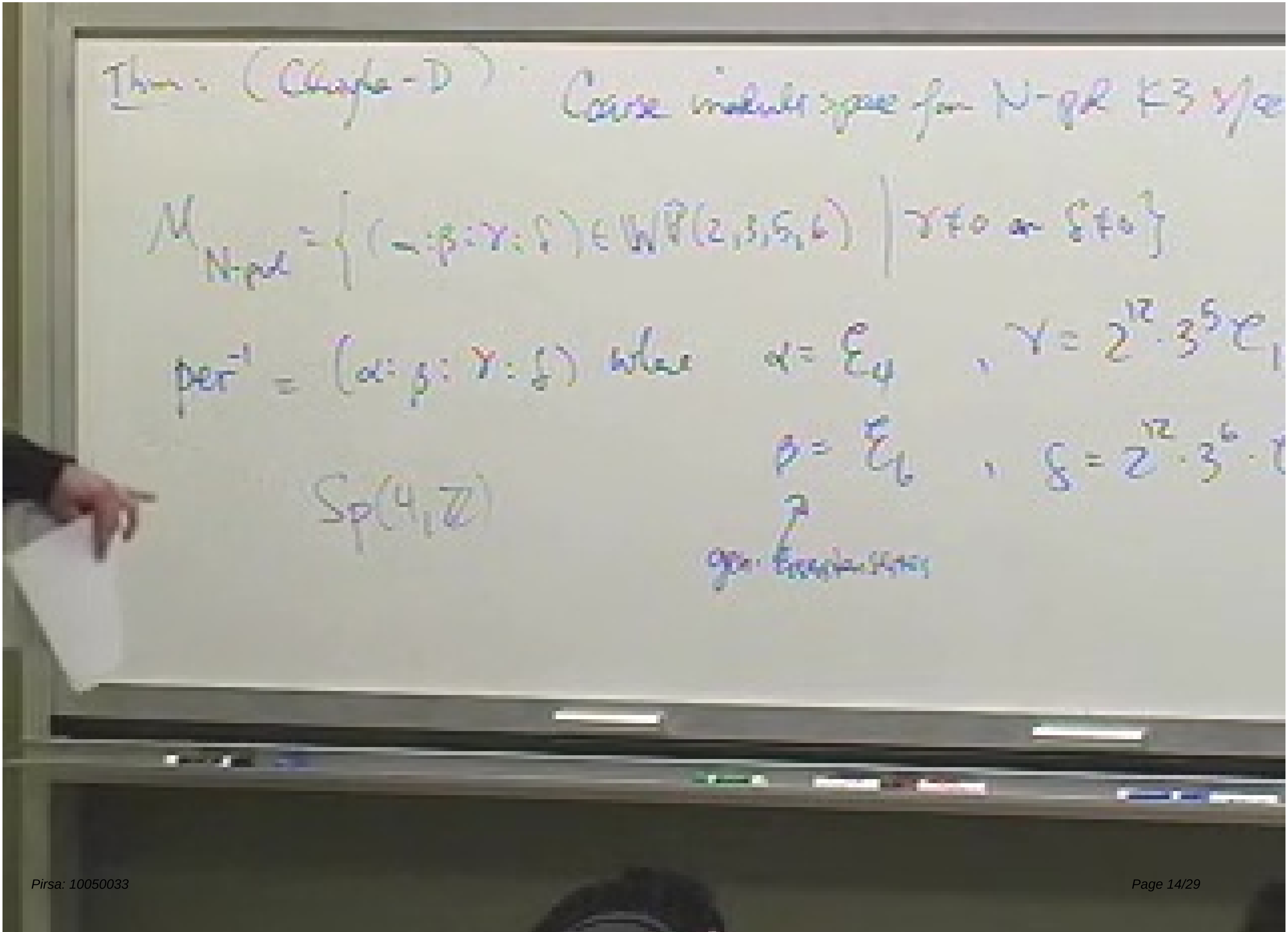
$\mathbb{P}^3 \quad (x, y, z, w)$

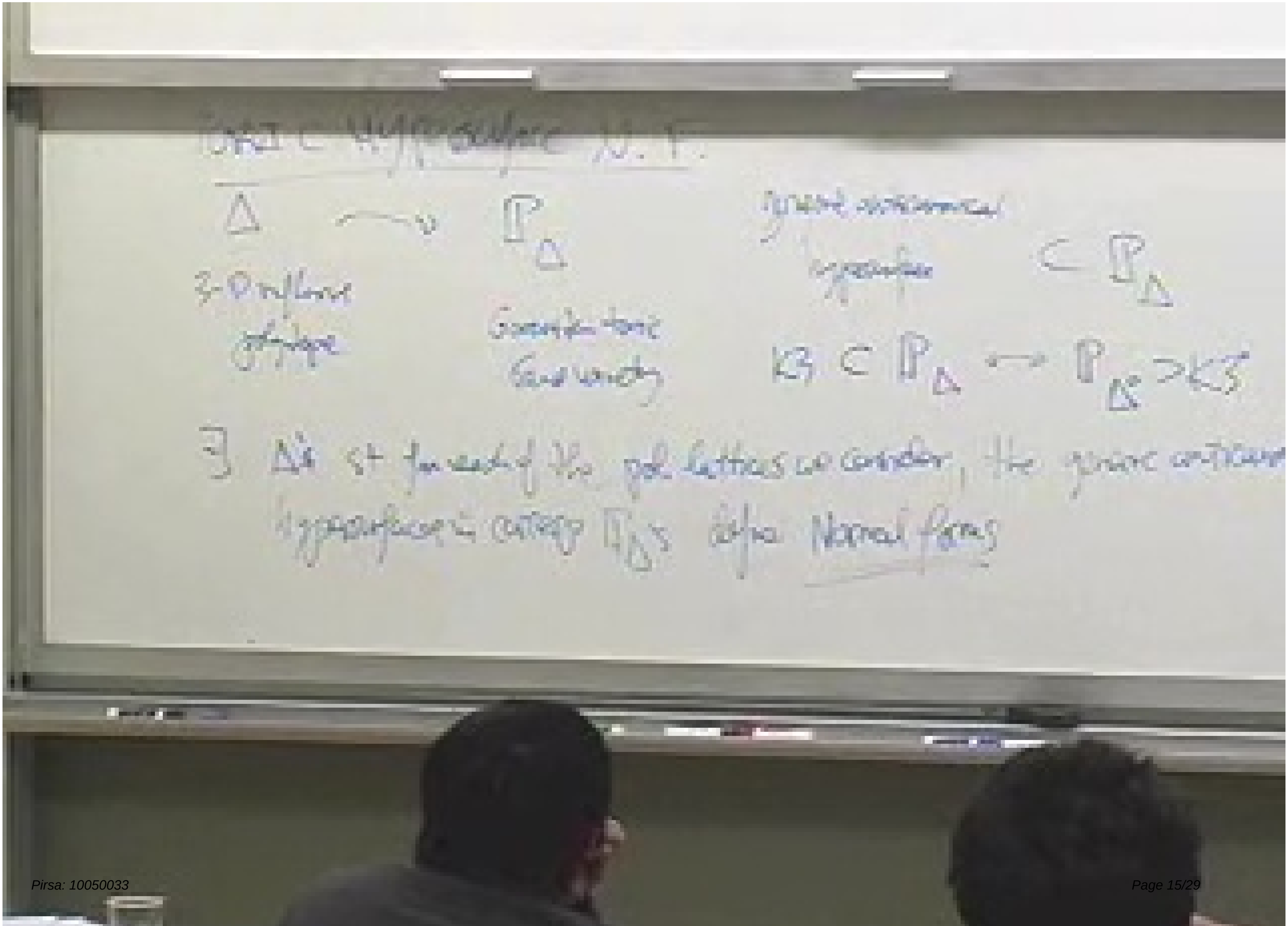
$$\left\{ y^2 z w - 4x^3 z + 3\alpha x z w^2 + \beta z w^3 + \gamma x z^2 w - \frac{1}{2} \ldots \right.$$

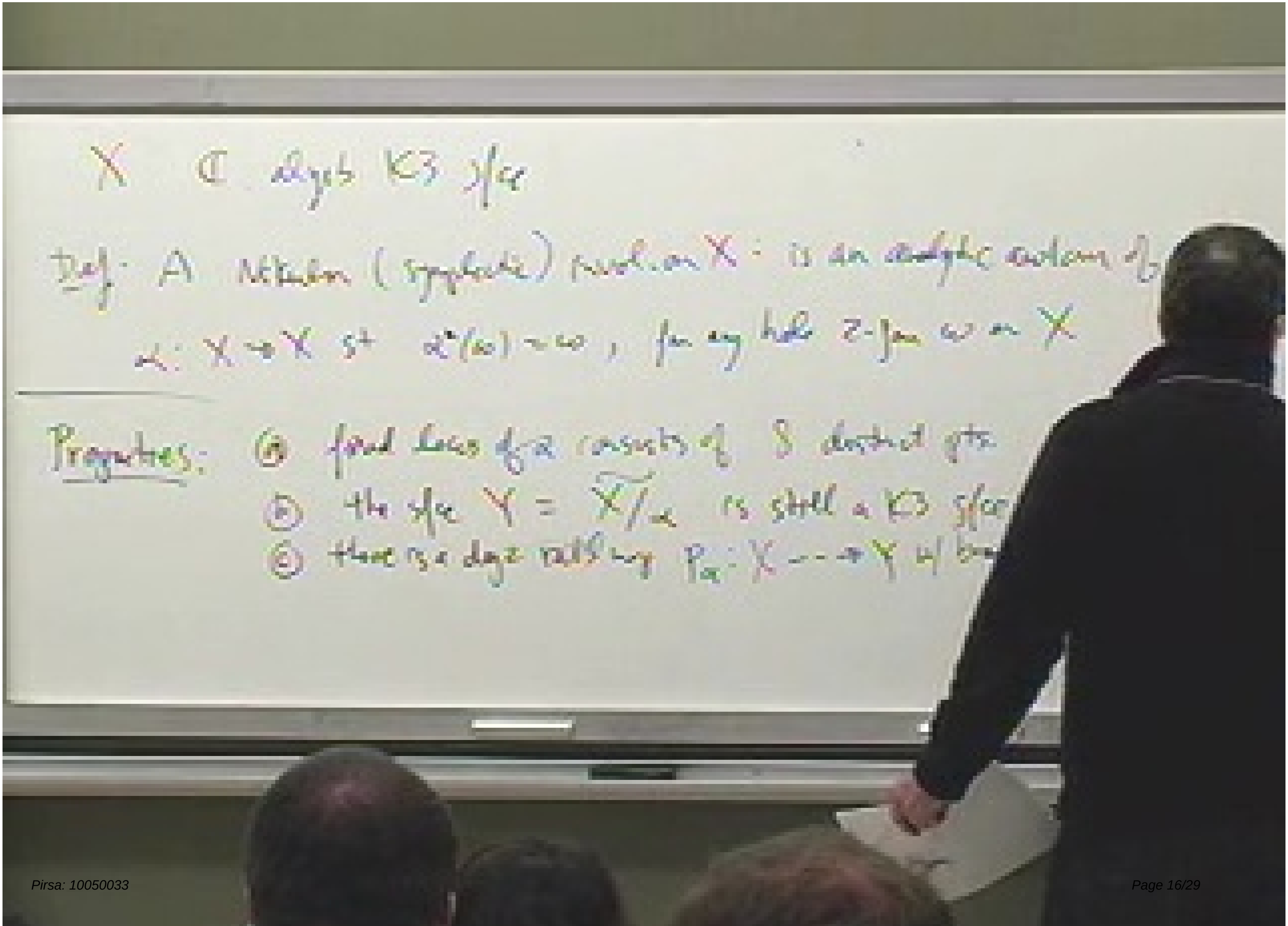
$X(\alpha, \beta, \gamma, \delta)$ be its minimal resolution   • If ...

• Conversely given an N-pol K3 sfce Y $\exists$ $\alpha, \beta, \gamma, \delta$ w/ $\gamma \neq 0$ or $\delta \neq 0$ s.t.

$X(\alpha, \beta, \gamma, \delta) \cong Y$


Thm. (Clingher-D)
Coarse moduli space for N-pol K3 s/
M_N-pol = {(α:β:γ:δ) ∈ WP(2,3,5,6) | γ≠0 or δ≠0}
per^-1 = (α:β:γ:δ) where α = E_4, γ = 2^12 · 3^5
β = E_6, δ = 2^12 · 3^6
Sp(4,Z)
gen. Eisenstein series


Normal forms


X ℂ algeb K3 s/ce
Def: A Nikulin (symplectic) invol. on X : is an analytic autom of
α: X → X s.t. α*(ω) = ω, for any hol 2-form ω on X
Properties:
(a) fixed locus of α consists of 8 distinct pts
(b) the s/ce Y = X̃/α is still a K3 s/ce
(c) there is a deg 2 rat'l map p_α : X - - → Y w/ br

Def: A Nikulin (symplectic) invol. on X: is an analytic autom of order 2

$\alpha: X \to X$ s.t. $\alpha^*(\omega) = \omega$, for any hol 2-form ω on X

Properties:

(a) fixed locus of α consists of 8 distinct pts

(b) the sfc $Y = \overline{X/\alpha}$ is still a K3 sfce.

(c) there is a deg 2 rat'l map $p_\alpha: X \dashrightarrow Y$ w/ branch locus given by 8 ... ("even-eight")

(d) there is a push-forward morphism $(p_\alpha)_*: H^2(X,\mathbb{Z}) \to H_Y$

$Y(\alpha,\beta,\gamma,\delta) \cong Y$

only to an eqn in $H^2(X,\mathbb{Z})$

Thm (Clingher–D)

$\mathbb{P}^3 \quad (x,y,z,w)$

$$\{\, y^2zw - 4x^3z + 3\alpha xzw^2 + \beta zw^3 + \gamma xz^2w - \tfrac{1}{2}(\delta z^2w^2 + w^4) = 0$$

$X(\alpha,\beta,\gamma,\delta)$ be its minimal resolution

- If $\gamma \neq 0$ or $\delta \neq 0$, then $X(\alpha,\beta,\gamma,\delta)$ is a K3 sfce w/ canonical M-polarization
- Conversely, given an M-pol K3 sfce X, $\exists$ $(\alpha,\beta,\gamma,\delta)$ w/ $\gamma \neq 0$ or $\delta \neq 0$ s.t. $X(\alpha,\beta,\gamma,\delta) \cong X$

orthog to Lie alg in $H^2(X,\mathbb{Z})$

Ex: Shioda-Inose structures on Kummer surfaces of A×E ...

① $Y = \mathrm{Km}(A) := \widetilde{A/\{\pm 1\}}$

② $(\pi_X)_*$ induces Hodge isometry between lattices of transc cycles

$T_X(2)$ and T_Y

Note: Dave Morrison gave an effective lattice-theoretic criterion for when a generic K3 surface admits a S-I structure.

Def: Van Geemen–Sarti involution is an auto $\alpha_X : X \to X$ for which $\exists$ a triple (φ_X, S_1, S_2)

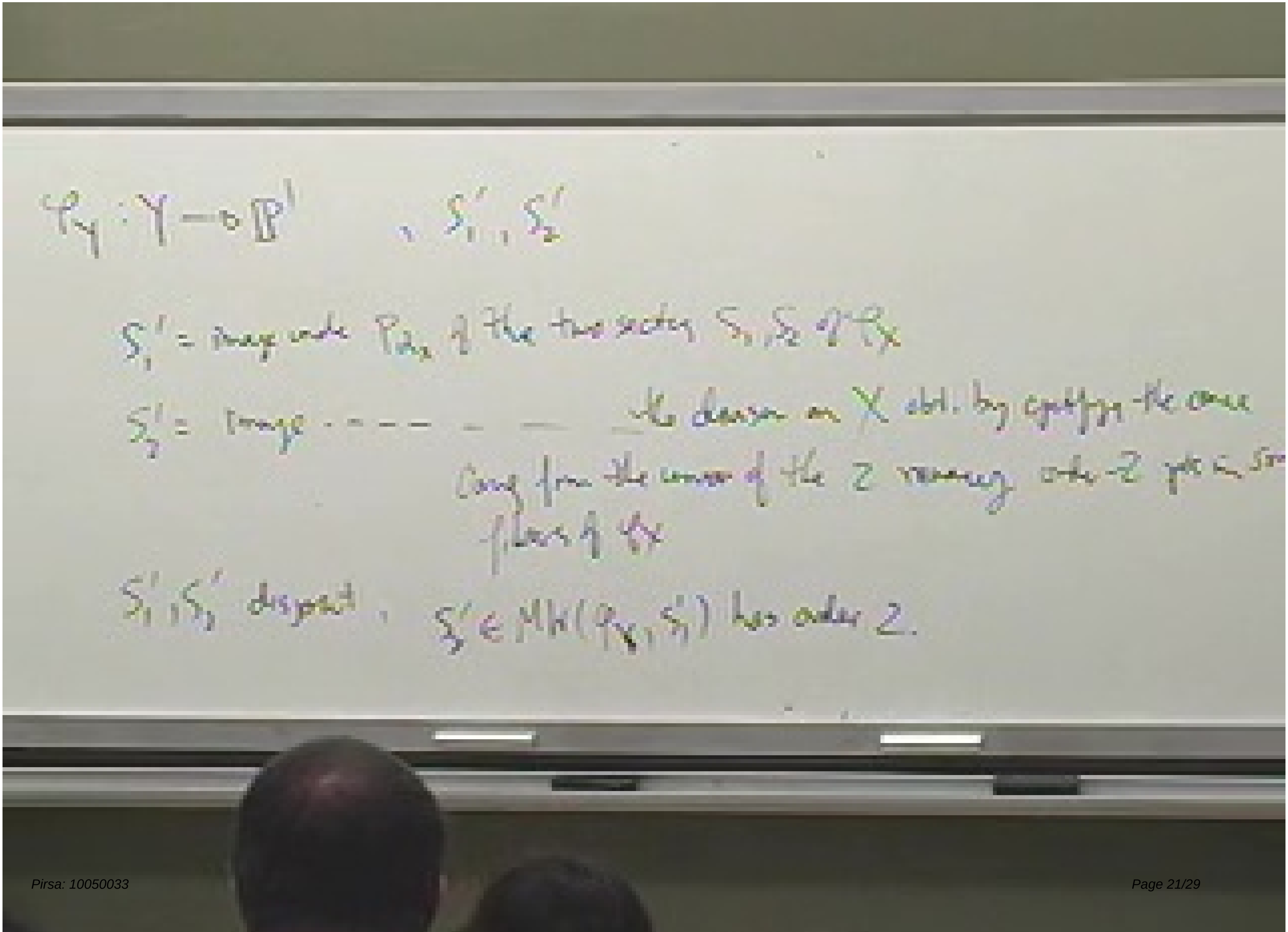
(a) $\varphi_X : X \to \mathbb{P}^1$ ell. fibr. on X

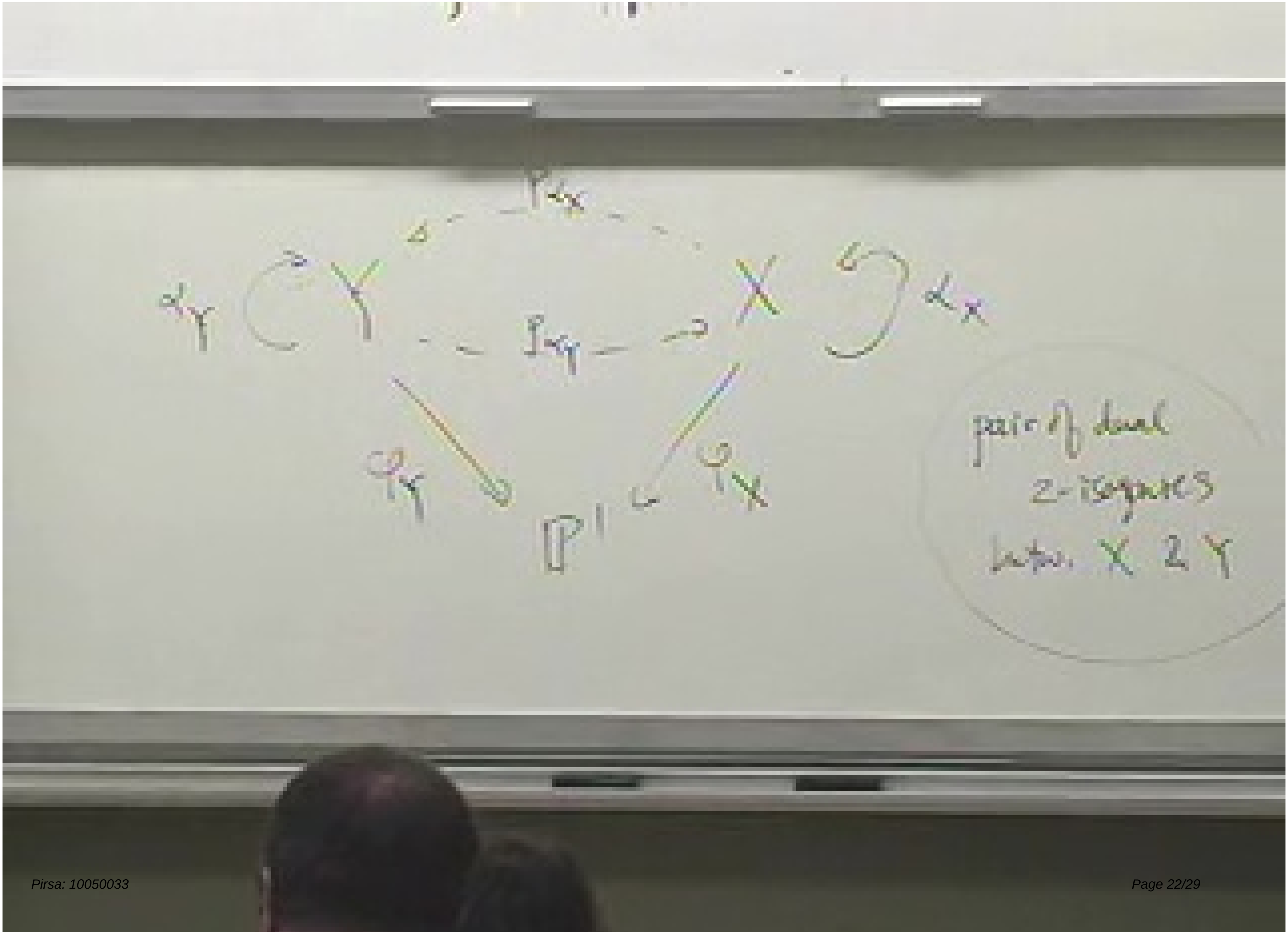
(b) S_1, S_2 disjoint sections of φ_X

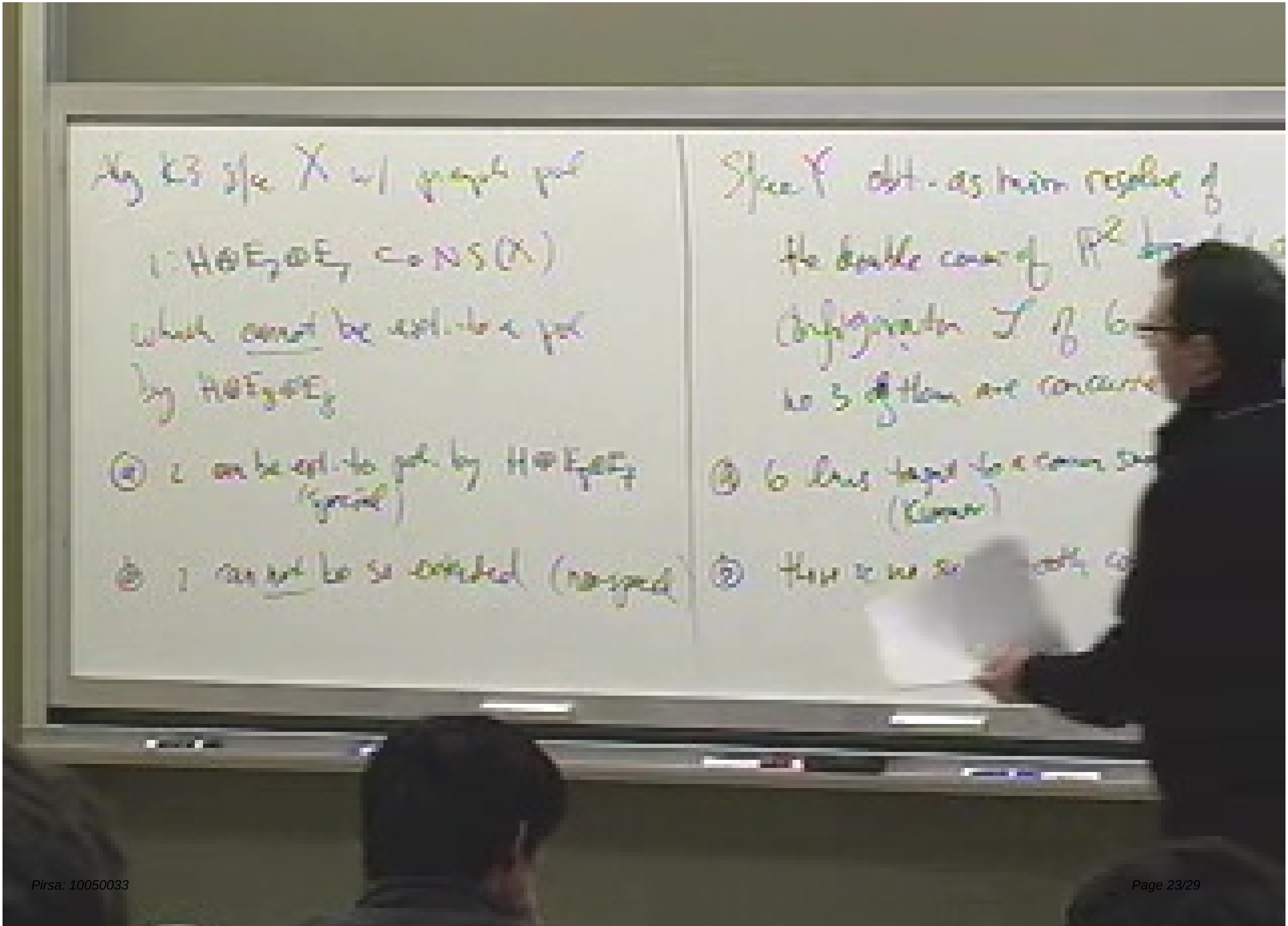
(c) S_2 is an element of order 2 in $MW(\varphi_X, S_1)$

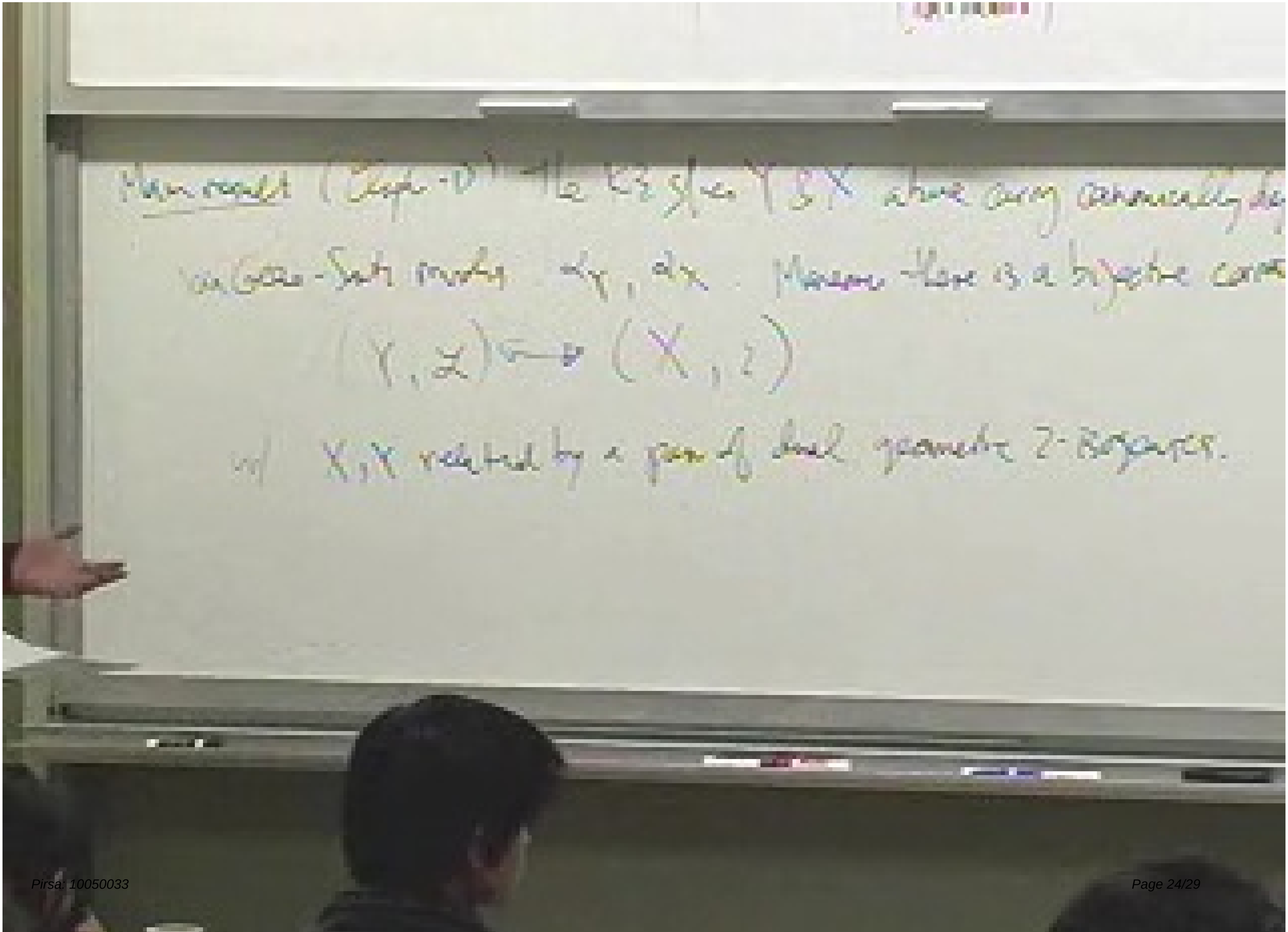
(d) α_X involution obtained by fiberwise translation by S_2 in the smooth fibers of the ellip. grp. w/ id $= S_1$

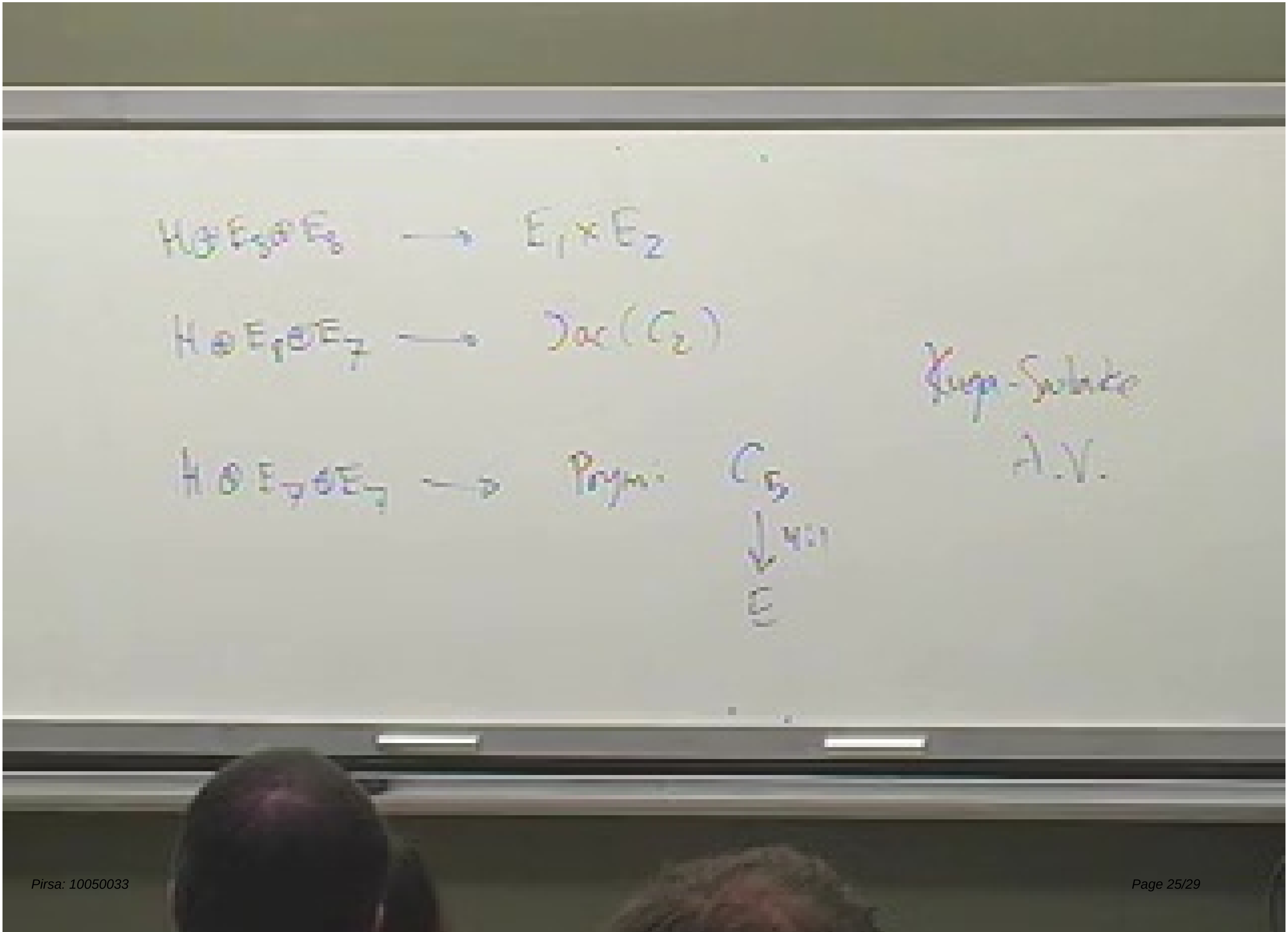
VG-S invol $\equiv$ fiberwise 2-isogeny between original K3 s/c & new K3 s/c Y

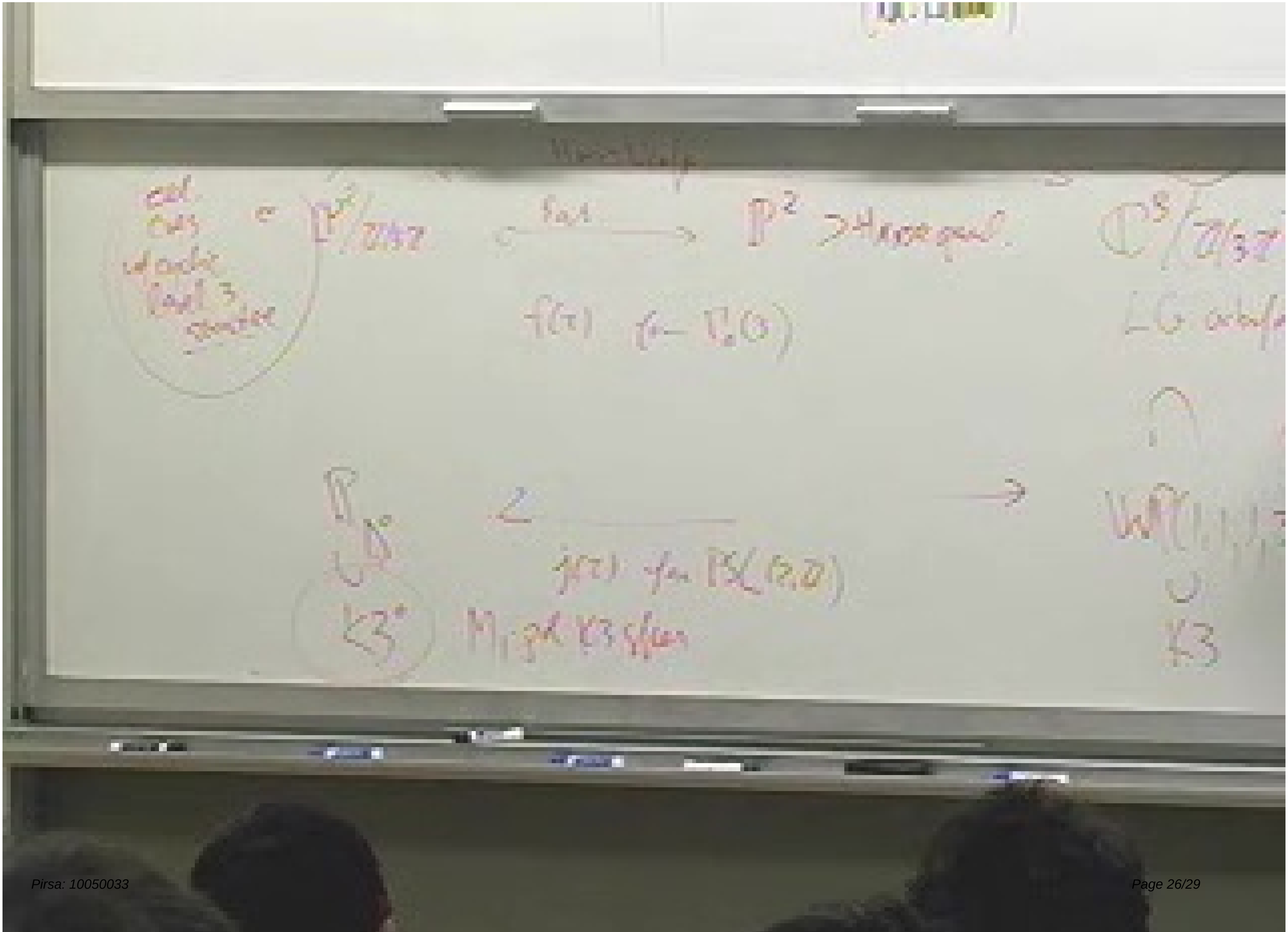

$\varphi_Y : Y \to \mathbb{P}^1$, S_1' , S_2'
S_1' and S_2' disjoint

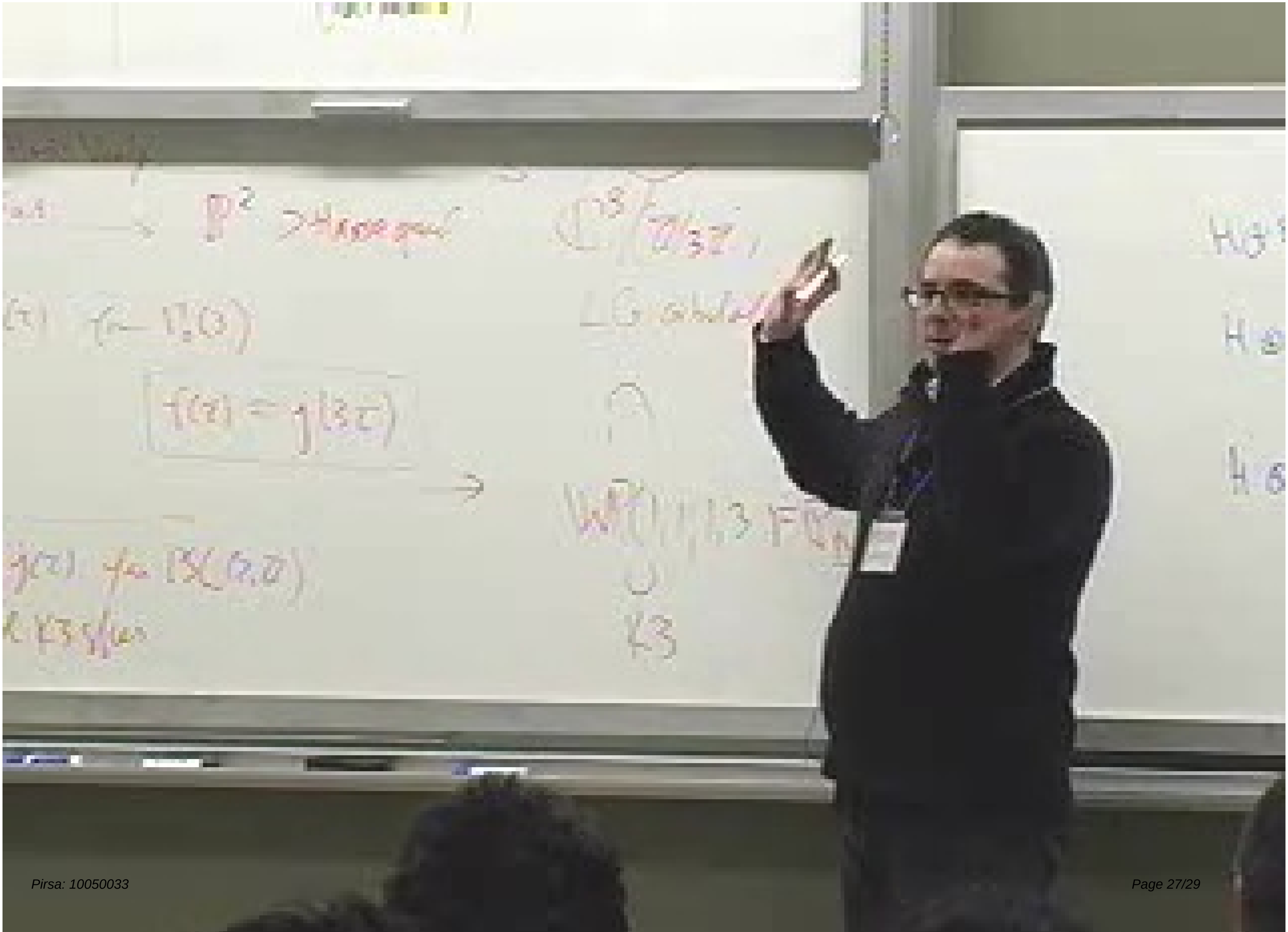

Y
X
$\mathbb{P}^1$
pair of dual
2-isogenies
betw. X & Y

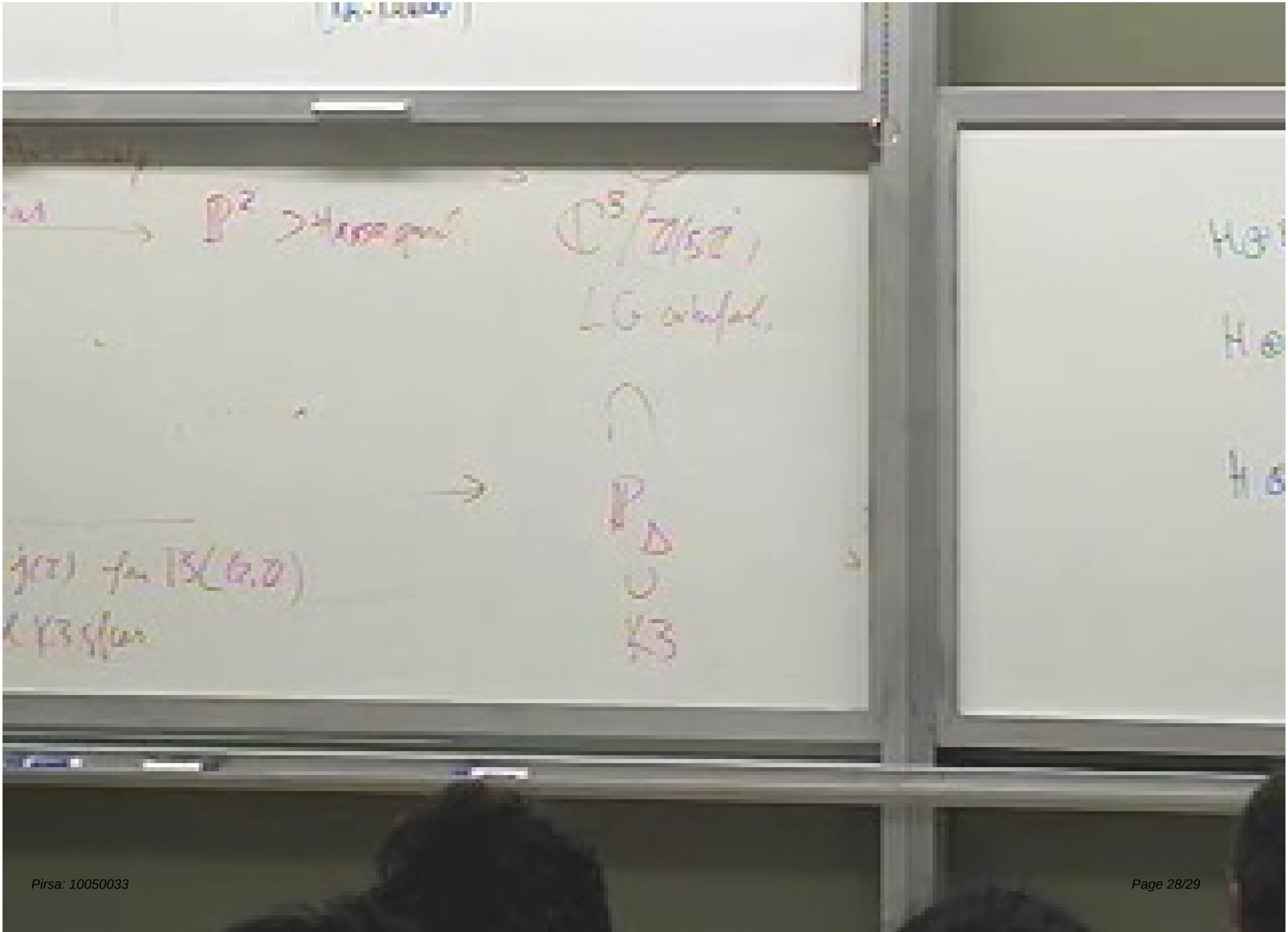



Main result


E1 × E2
Jac(C2)
Prym
Kuga-Satake
A.V.






LG orbifold
K3