Title: Categorical Lie algebra actions and braid group actions

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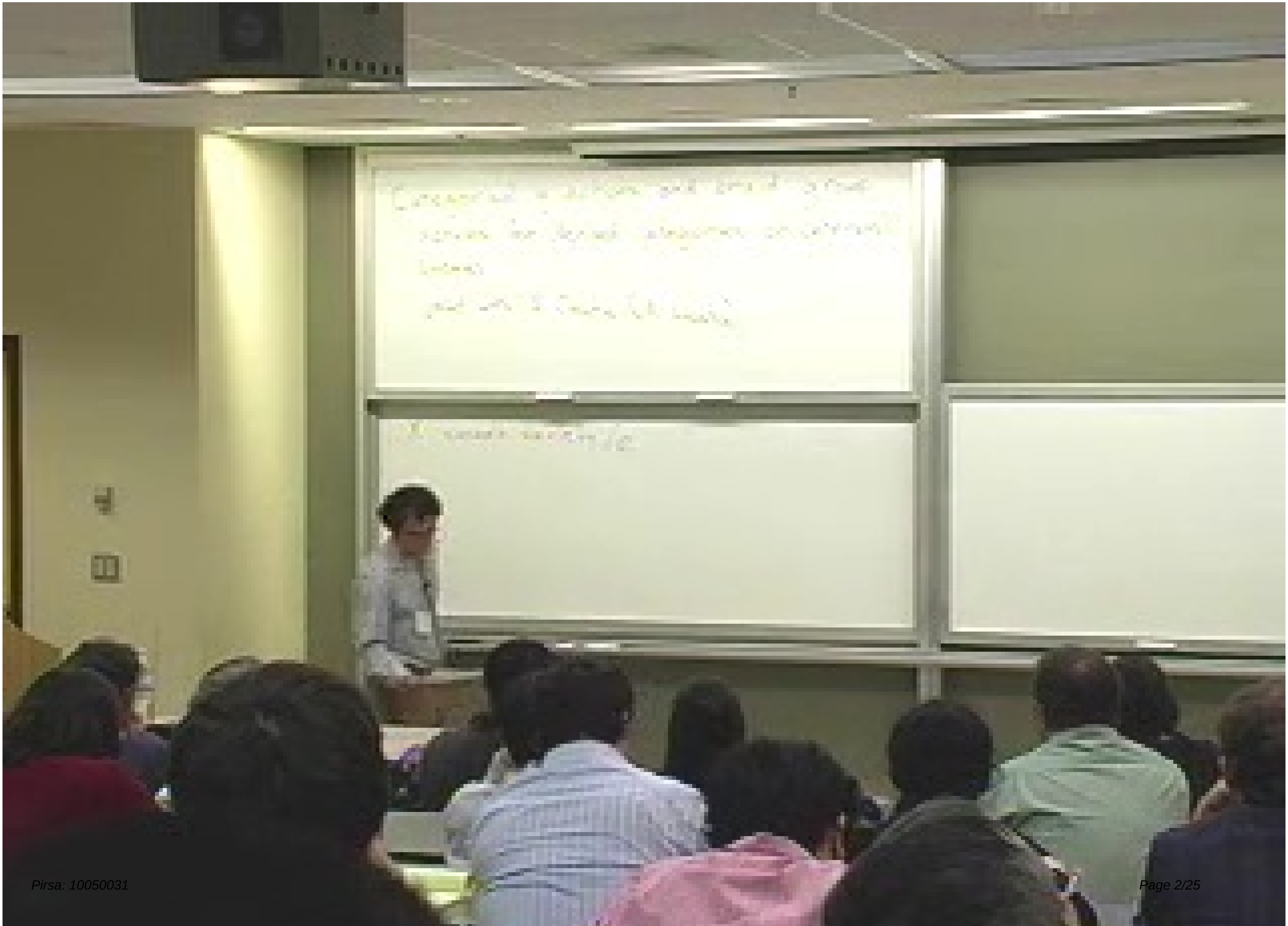
Abstract: We will discuss the notion of categorical Lie algebra actions, as introduced by Rouquier and Khovanov-Lauda. In particular, we will give examples of categorical Lie algebra actions on derived categories of coherent sheaves. We will show that such categorical Lie algebra actions lead to actions of braid groups.



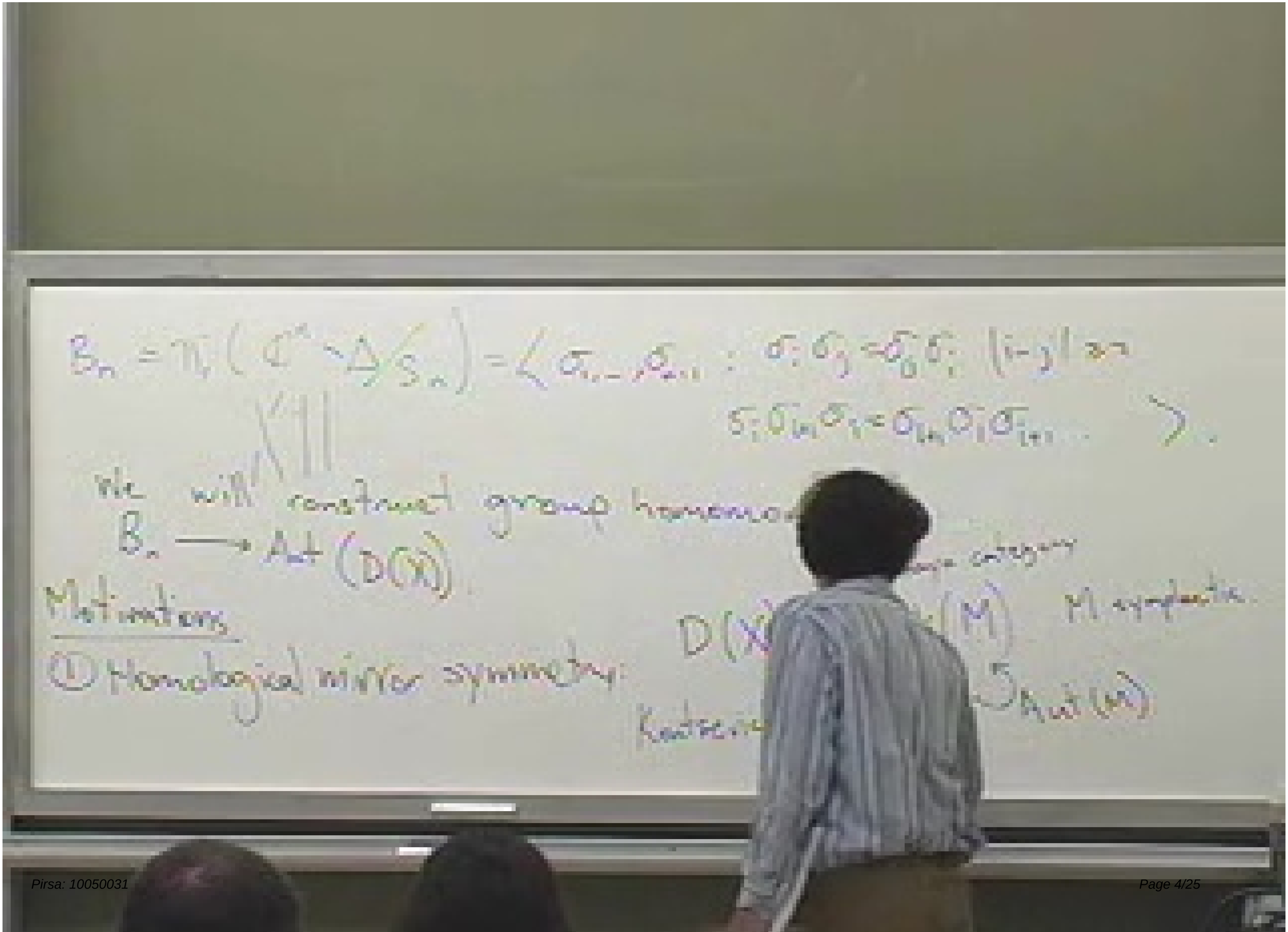
joint with S. Cautis (A. Licata)

X smooth variety $/\mathbb{C}$

$D(X)$ = bounded derived category of coherent sheaves on X.

(complexes of vector bundles)

We are interested in equivalence $\Phi : D(X) \to D(Y)$


$B_n = \pi_1(\mathbb{C}^n \setminus \Delta / S_n) = \langle \sigma_1 \ldots \sigma_{n-1} : \sigma_i\sigma_j = \sigma_j\sigma_i \;\; |i-j| \geq 2$
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ldots \rangle$.
We will construct group
$B_n \longrightarrow Aut(D(X))$
Motivations
① Homological mirror symmetry
$D(X)$
M symplectic
$Aut(M)$

$(D(X))$.

Motivations

① Homological mirror symmetry: $D(X) \cong F.K(M)$ M symplectic.

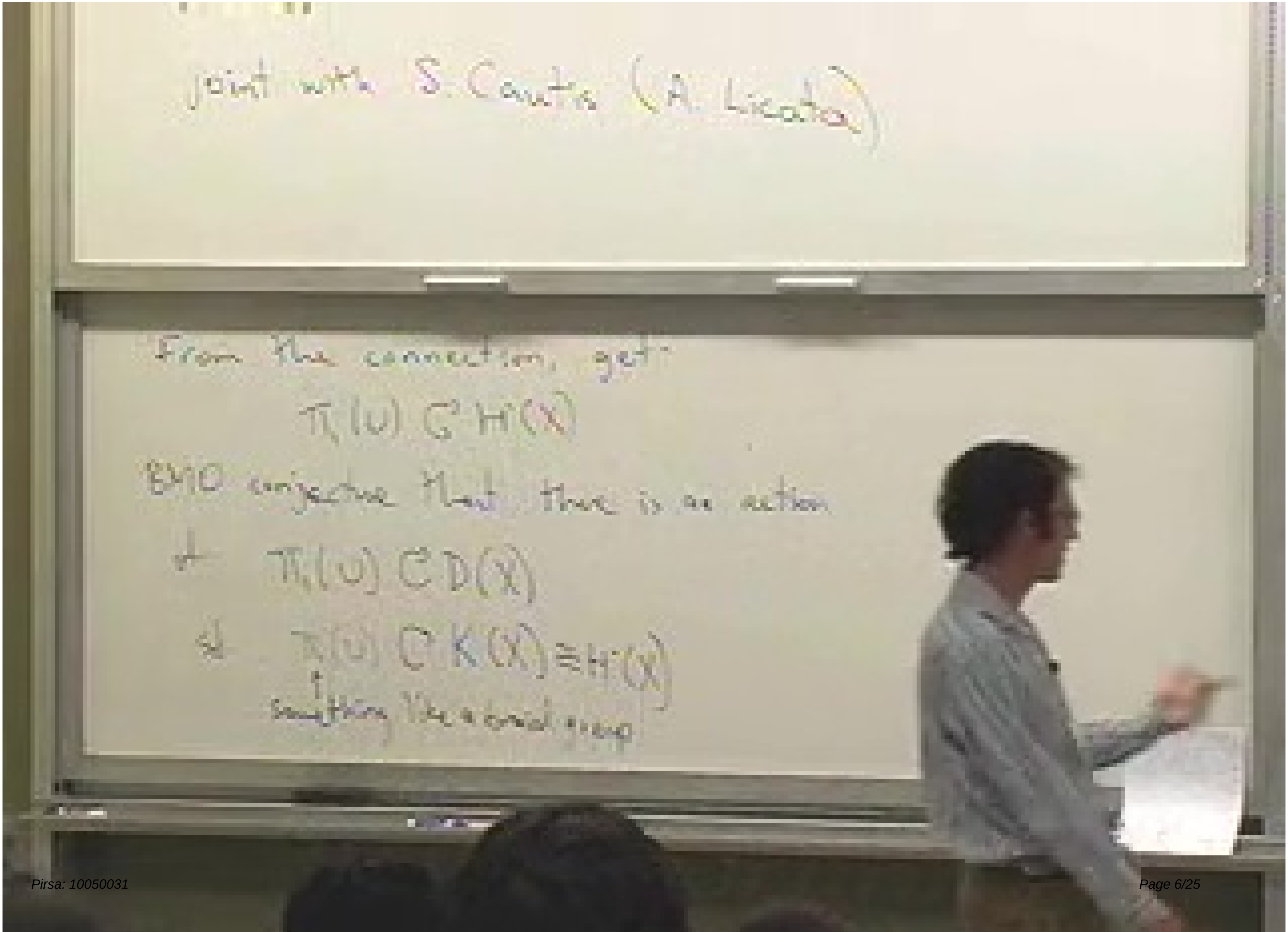
Kontsevich, Seidel-Thomas $\circlearrowleft \mathrm{Aut}(M)$

Braverman-Maulik-Okounkov:

$QH^*(X)$

consider action of $H^2(X) \subset H^*(X)$
by quantum mult.

Gives rise to a connection on trivial vector bundle on U
with fiber $H^*(X)$, $\underset{\text{torus}}{U \subset T}$ T torus with character lattice
$H_2(X, \mathbb{Z})$.


joint with S. Cautis (A. Licata)
From the connection, get
$\pi_1(U) \curvearrowright H(X)$
BMO conjecture that there is an action
of $\pi_1(U) \curvearrowright D(X)$
& $\pi_1(U) \curvearrowright K(X) \cong H(X)$
something like a braid group

d $\pi_1(U) \subset D(X)$

d. $\pi(U) \subset K(X) \cong H^{\cdot}(X)$

something like a braid group

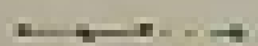
② Homological knot invariant (Khovanov homology).

come from actions of groups on categories

Γ any graph.

$B_\Gamma = \langle \sigma_i, i \in \mathrm{Vert}(\Gamma) \rangle$: $\sigma_i\sigma_j = \sigma_j\sigma_i$ i,j not conn

$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ i,j are conn

$B_n = B_{A_{n-1}}$ 

A_{n-1}

Motivations

① Homological mirror symmetry $D(X) \cong \mathrm{Fuk}(M)$ M symplectic

Kontsevich, Seidel-Thomas $\circlearrowleft \mathrm{Aut}(M)$

Ex

$H \subset SL_2\mathbb{C}$ finite

$X = \widetilde{\mathbb{C}^2/H}$, Γ graph of components of the exceptional locus of X

Theorem (Seidel-Thomas)

The group B_Γ acts on $D(X)$

The generators $\sigma_i \in B_\Gamma$ will act by a correspondence:

$\mathcal{E}$

σ_i acts as $\Phi_{\mathcal{E}}$

The group B_n acts on $D(X)$

The generators $\sigma_i \in B_n$ will act by a correspondence:

σ_i acts as $q_{i*} p_i^*$

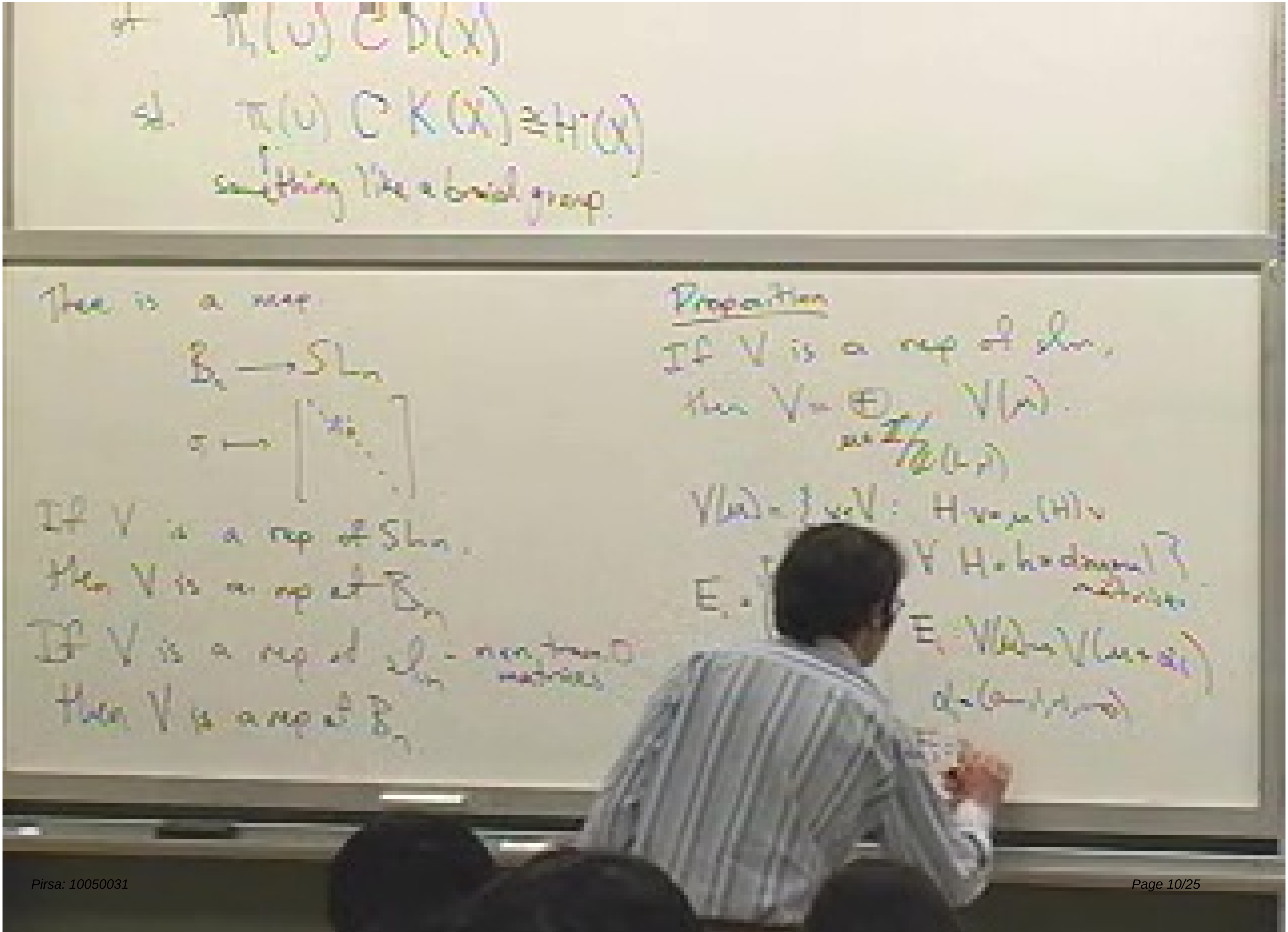
Eg ($X = T^*\mathbb{P}^1$, $n = 2$)

$X = T^* G(2,4)$

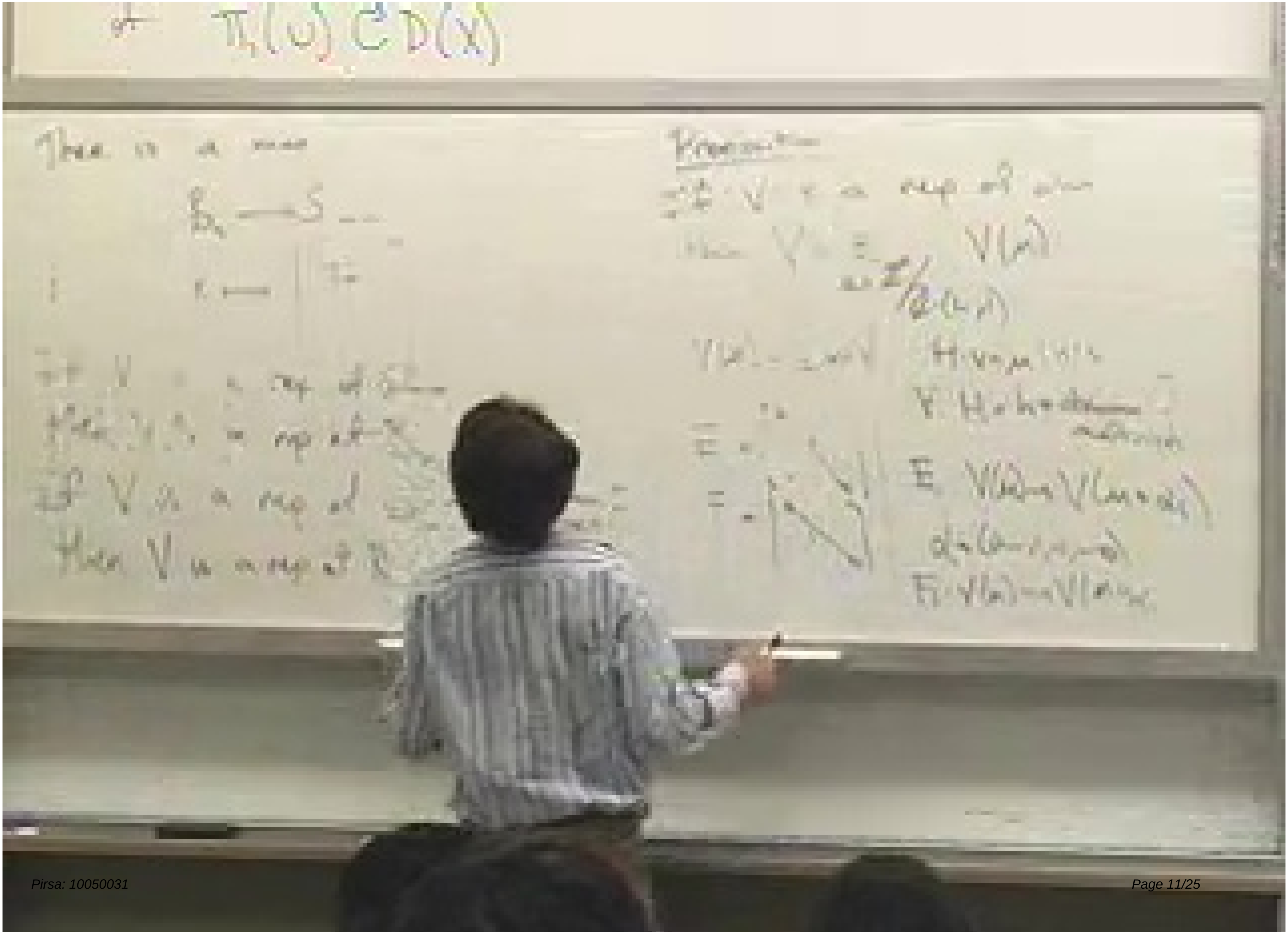
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ T^* G(2,4) \arrow[dr] & & T^* G(2,4) \arrow[dl] \\ & \{A \in M_2(\mathbb{C}^4) : A^2 = 0\} & \end{tikzcd}$$

Theorem (Namikawa).

$q_* p^*$ is not an equivalence



something like a braid group
There is a map
Proposition
If V is a rep of SL_n,
then V is a rep of B_n
If V is a rep of
then V is a rep of B_n



then V is a rep of B_n

If V is a rep of $\mathfrak{sl}_n$ - min transl matrices

then V is a rep of B_n

$E_i =$

$F_i =$

$E_i : V(\lambda) \to V(\lambda + \alpha_i)$

$\dim(\ldots)$

$F_i : V(\lambda) \to V(\lambda - \alpha_i)$

$E_iF_i - F_iE_i\big|_{V(\lambda)} = \langle \lambda, \alpha_i \rangle I_{V(\lambda)}$

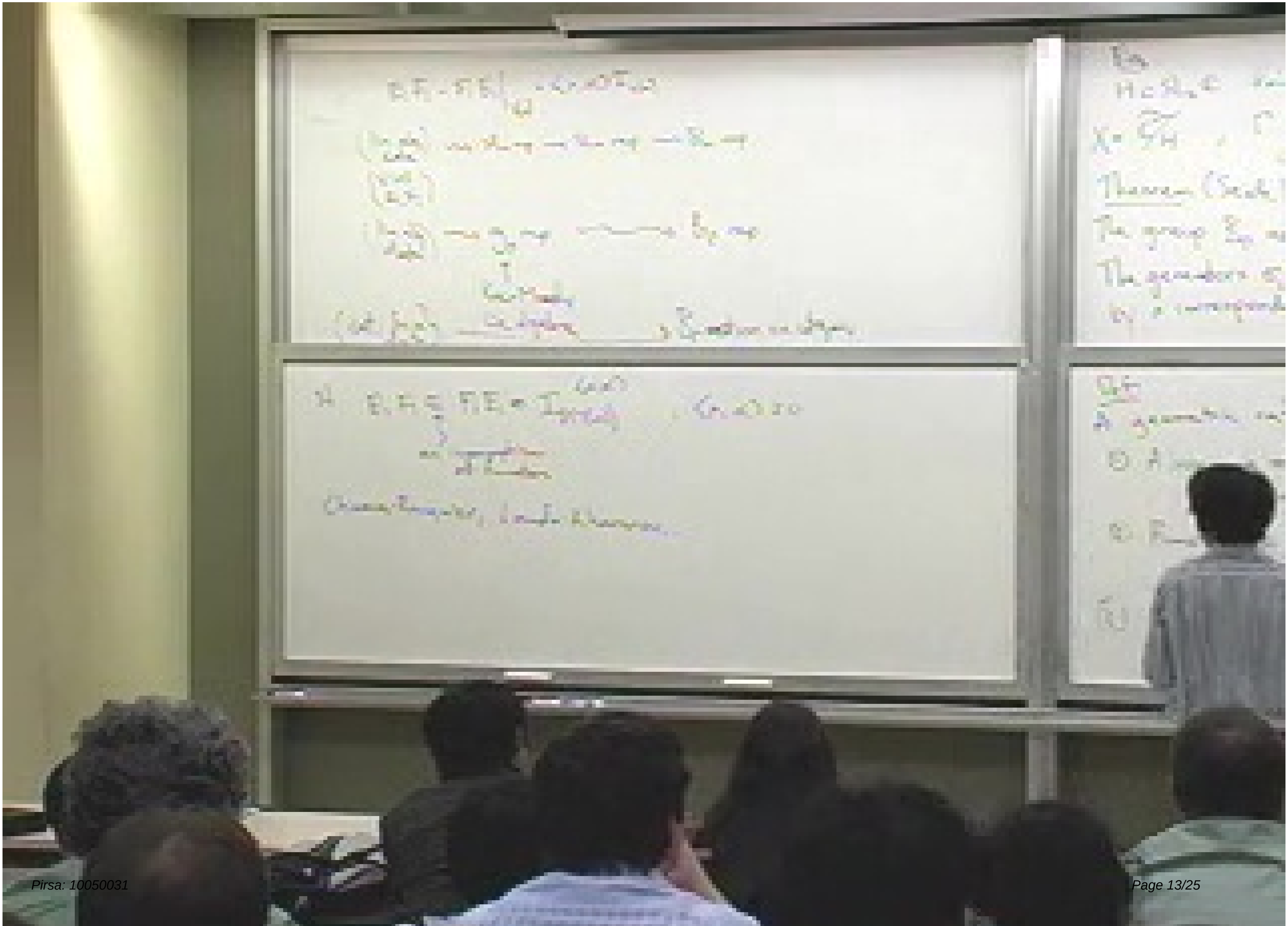
(lin alg data) $\rightsquigarrow$ $\mathfrak{h}_n$ rep $\rightsquigarrow$ $\mathfrak{sl}_n$ rep $\rightsquigarrow$ B_n rep

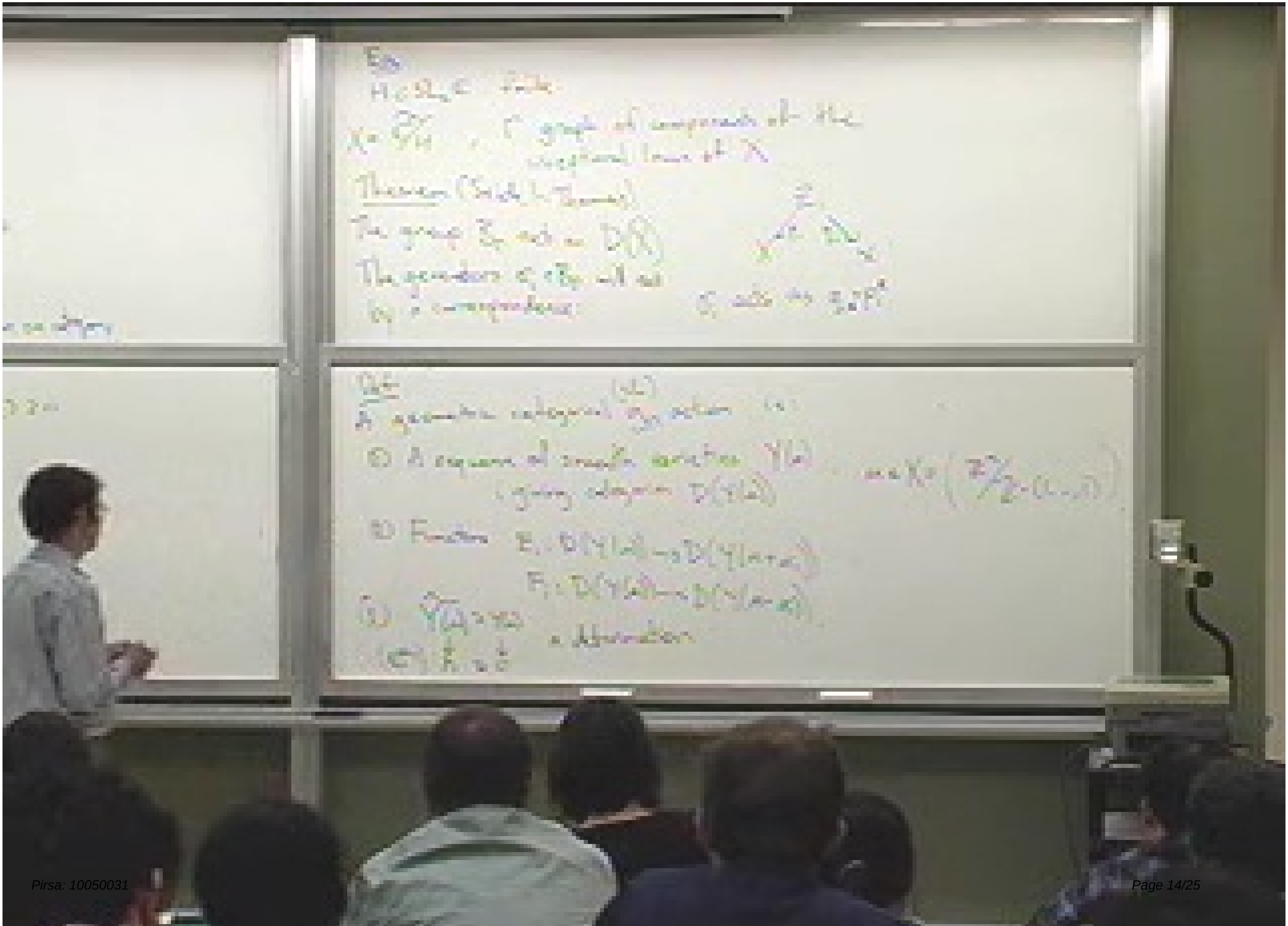
$\begin{pmatrix} V(\lambda) \\ E, F \end{pmatrix}$

(lin alg data) $\rightsquigarrow$ $\mathfrak{g}$ rep $\rightsquigarrow$ $B_{\mathfrak{g}}$ rep

Kac-Moody Lie algebra

(lin alg data) $\rightsquigarrow$ $B_{\mathfrak{g}}$ action on category





Kac-Moody

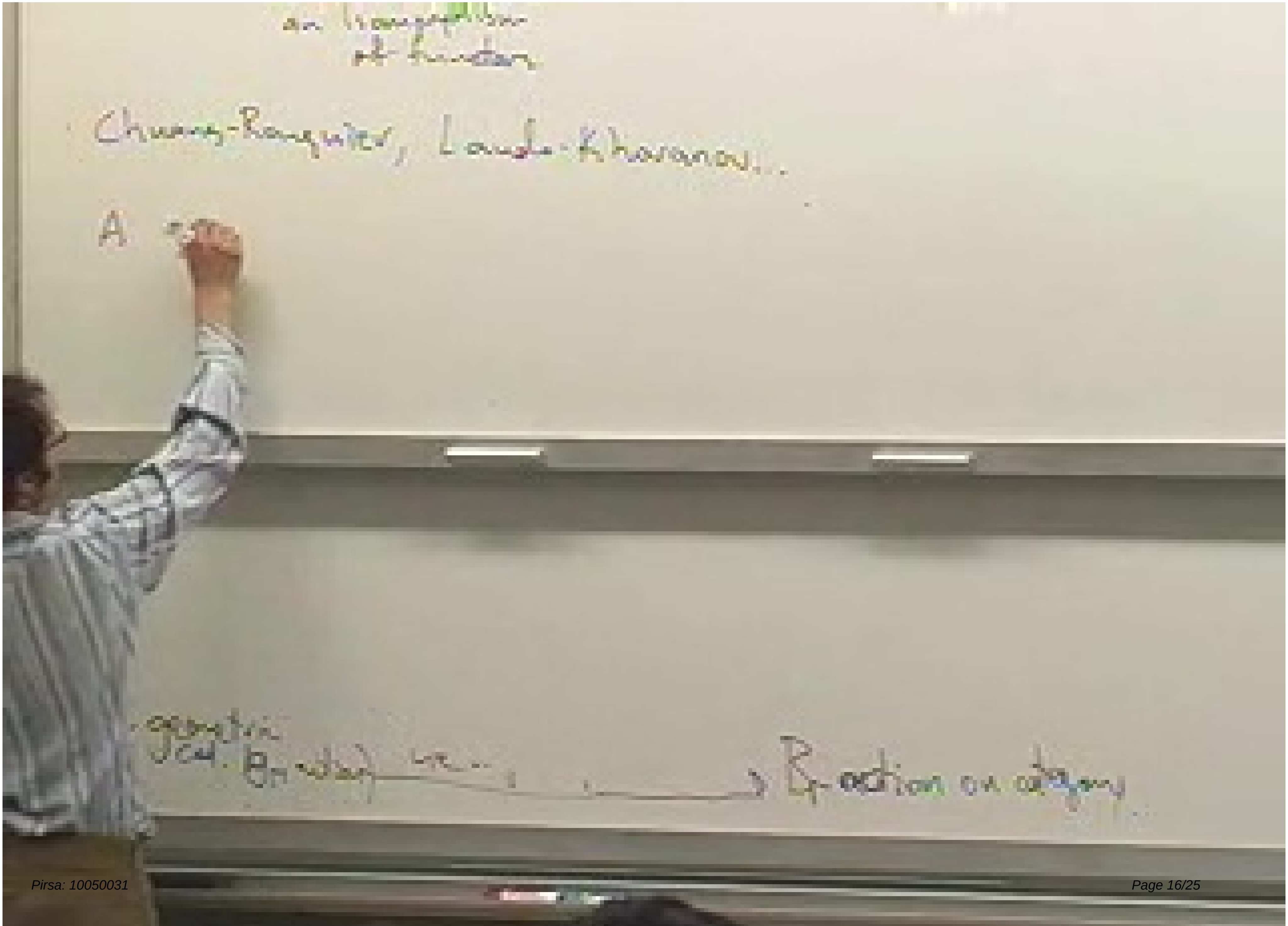
(cat. [illegible]) Lie algebra $\longrightarrow$ B_n action on category

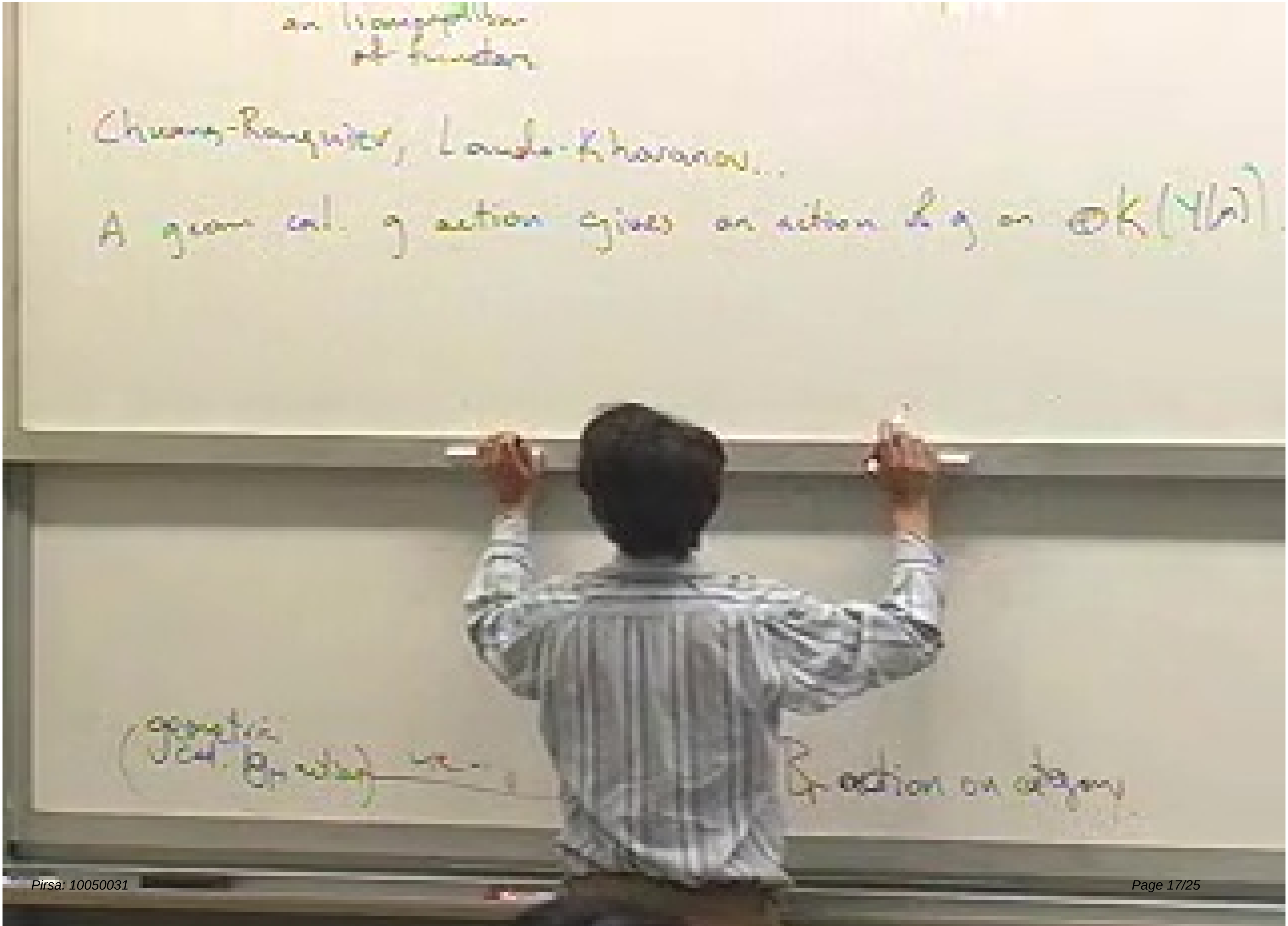
sl $\quad E_i F_i \cong F_i E_i \oplus I_{D(Y(\lambda))}^{\langle \lambda, \alpha_i \rangle}$, $\langle \lambda, \alpha_i \rangle \geq 0$

an isomorphism of functors

.... and relations...

Chuang-Rouquier, Lauda-Khovanov...


Chuang-Rouquier, Lauda-Khovanov, ..
A
action on category



Chuang-Rouquier, Lauda-Khovanov...
A given cat. g action gives an action of g on ⊗K(Y(λ))
action on category

Theorem (CR Chuang-Rouquier).

A geometric categorical $\mathfrak{g}$ action gives rise to a action of $B_{\mathfrak{g}}$.

Eg $\mathfrak{g} = \mathfrak{sl}_m$. Fix N.

Consider $Fl_m(\mathbb{C}^N) = \{0 \subseteq V_1 \subseteq \dots \subseteq V_m = \mathbb{C}^N\} = \coprod_{\substack{\mu = (\mu_1, \dots, \mu_m) \\ \mu_1 + \dots + \mu_m = N}} Fl_\mu(\mathbb{C}^N)$.

$Y(\lambda) = T^* Fl_m(\mathbb{C}^N)$

$F_i : D(Y(\mu)) \to D(Y(\mu - \alpha_i))$

(i) $\widetilde{Y(\mu)} \supset Y(\mu)$ a deformation

$(\mathbb{C}^\times)$ h $\downarrow$ $\downarrow$ 0

$T^* \cdot$ $\quad T^*\mathbb{P}^2 \quad T^*G(2,4) \quad T^*\mathbb{P}^3 \quad T^*\cdot$

To define E_i, F_i consider

$T^*_{(\ldots)}(Fl_n \times Fl_{n,\ldots})$

$T^*Fl_n(\mathbb{C}^N) \qquad T^*Fl_{\ldots}(\mathbb{C}^N)$

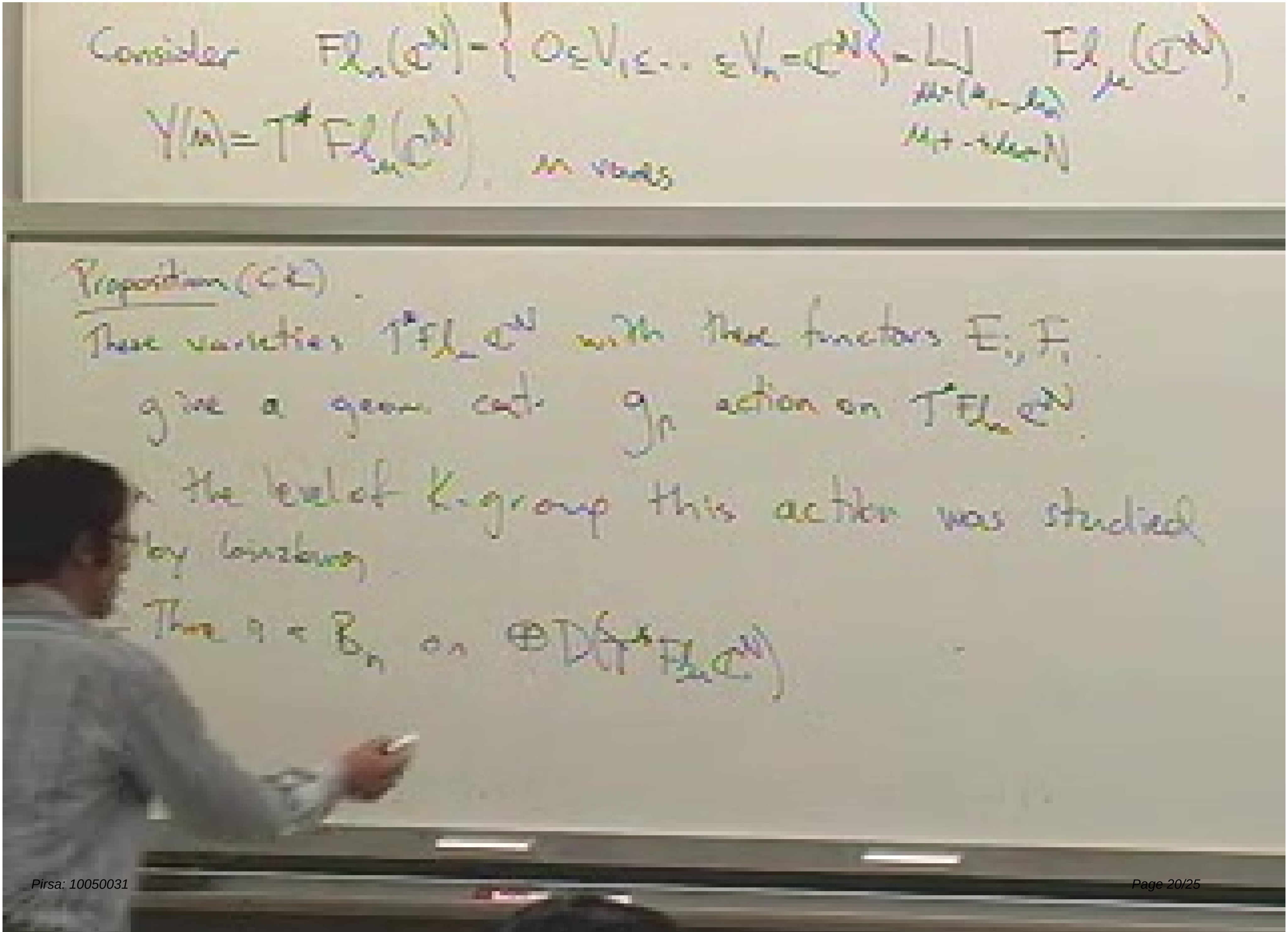
$E_i = q_* p^*$

$0 \subset V_1 \subset V_1 \subset \cdots$

$W = \{(V, V') \in Fl \ldots$

s.t. $V_j = V'_j$

$V_i \subset V'_i$


Consider Fl_n(C^N) = { 0 ⊂ V_1 ⊂ ... ⊂ V_n = C^N } = ⊔ Fl_μ(C^N).
Y(n) = T*Fl_n(C^N)
n vars
Proposition (CK).
These varieties T*Fl_n C^N with these functors E_i, F_i
give a geom cat. g_n action on T*Fl_n C^N.
the level of K-group this action was studied
by Ginzburg
Thm ∃ a B_n on ⊕ D(T*Fl_n C^N)

$F_i : D(Y(\mu)) \to D(Y(\mu - \alpha_i))$

(2) $\tilde{Y}(\mu) \supset Y(\mu)$ a deformation

$(\mathbb{C}^m) \ni h \supset 0$

eg $n=2$, $N=4$

$T^*\cdot \qquad T^*\mathbb{P}^3 \qquad T^*G(2,4) \qquad T^*\mathbb{P}^3 \qquad T^*\cdot$

$\mu = (0,4) \qquad \mu = (2,2)$

To define E_i, F_i consider

$$T^*_V(Fl_\mu \times Fl_{\mu+\alpha_i})$$

$$T^*Fl_\mu(\mathbb{C}^N) \qquad T^*Fl_{\mu+\alpha_i}(\mathbb{C}^N)$$

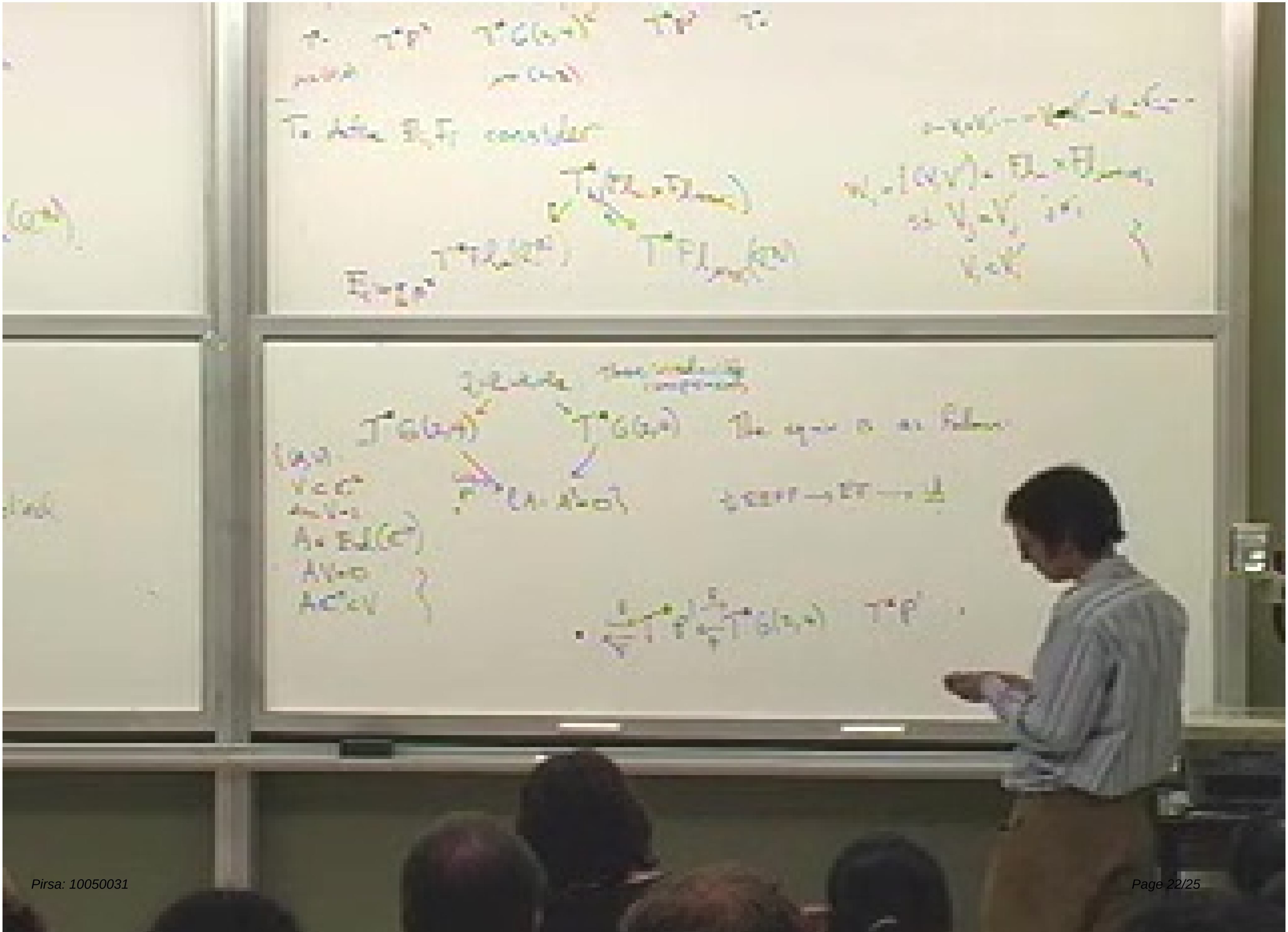
$E_i = \ldots$

$0 - V_1 = V_i \ldots$

$V_i = \ldots$

s.t. $V_j = V \ldots$

$V_i = V \ldots$



$A\mathbb{C}^* \times V$

$\bullet \; \overset{E}{\underset{F}{\leftrightarrows}} \; T^*\mathbb{P} \; \overset{E}{\underset{F}{\leftrightarrows}} \; T^*G(2,4)$ $\quad T^*\mathbb{P}$

Associated to our graph Γ are Nakajima quiver varieties (includes $T^*Fl_n(\mathbb{C}^n)$, $\widetilde{\mathbb{C}^2/\mu}$)

There is an action of $\mathfrak{g}_\Gamma$ on H^*(Nak quiver ...) constructed by Nakajima.

<u>Theorem</u> (CKL)

There is a geometric categorical $\mathfrak{g}_\Gamma$ action on Nakajima quiver varieties

Consider $Fl_{\mu}(\mathbb{C}^N) = \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^N \} \sqcup \bigsqcup Fl_{\mu}(\mathbb{C}^N)$.

$\mu = (\mu_1, \ldots, \mu_n)$

$\mu_1 + \cdots + \mu_n = N$

$Y(\mu) = T^* Fl_{\mu}(\mathbb{C}^N)$ n vars

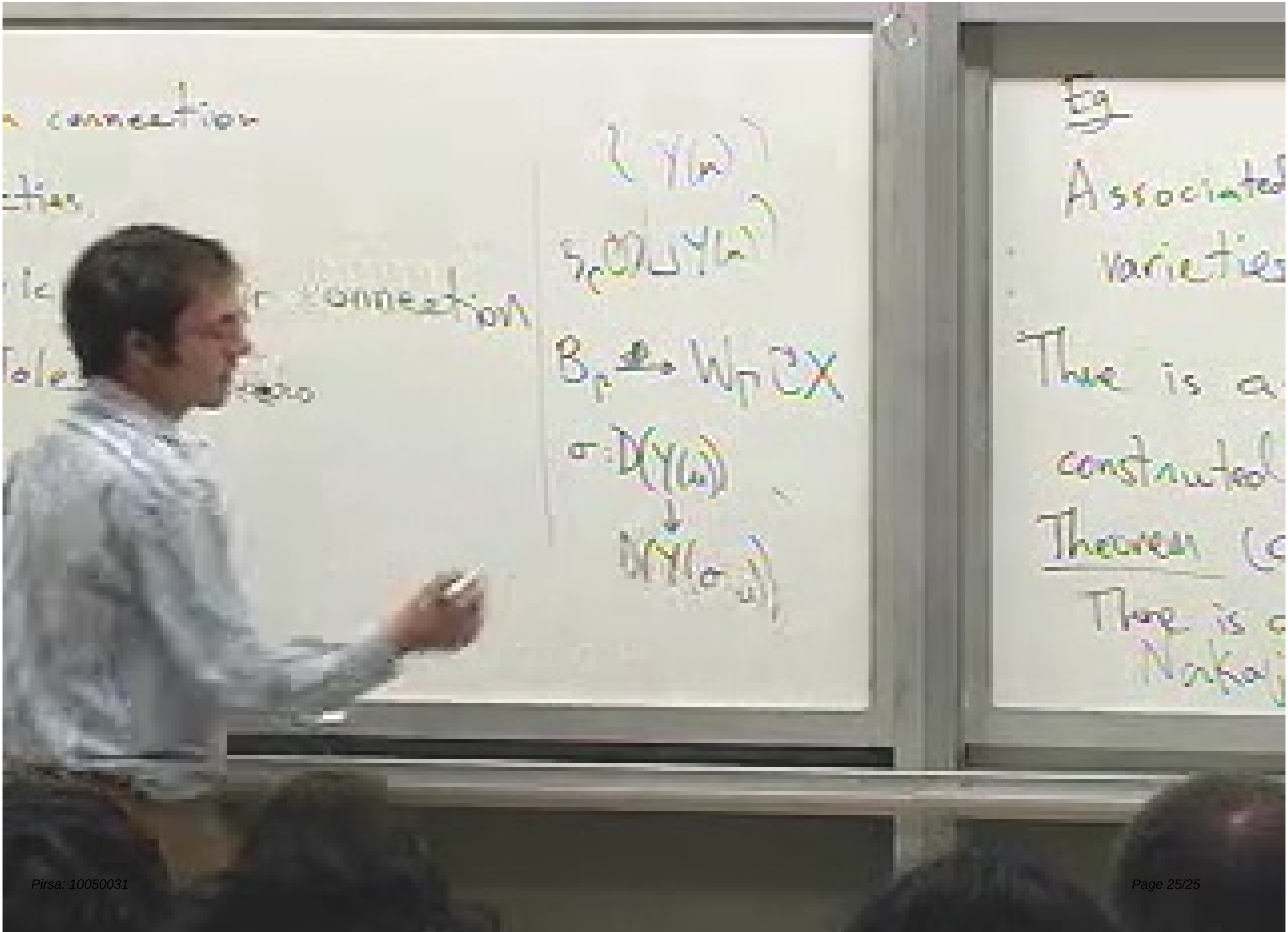
BMO studied the quantum connection on Nakajima quiver varieties.

It is the trigonometric Casimir connection

{ conj of Toledano Laredo

of the quantum Weyl group

this action can be lifted to D(Nakajima quiver varieties).


connection
connection
Eg
Associated
varieties
There is a
constructed
Theorem
There is