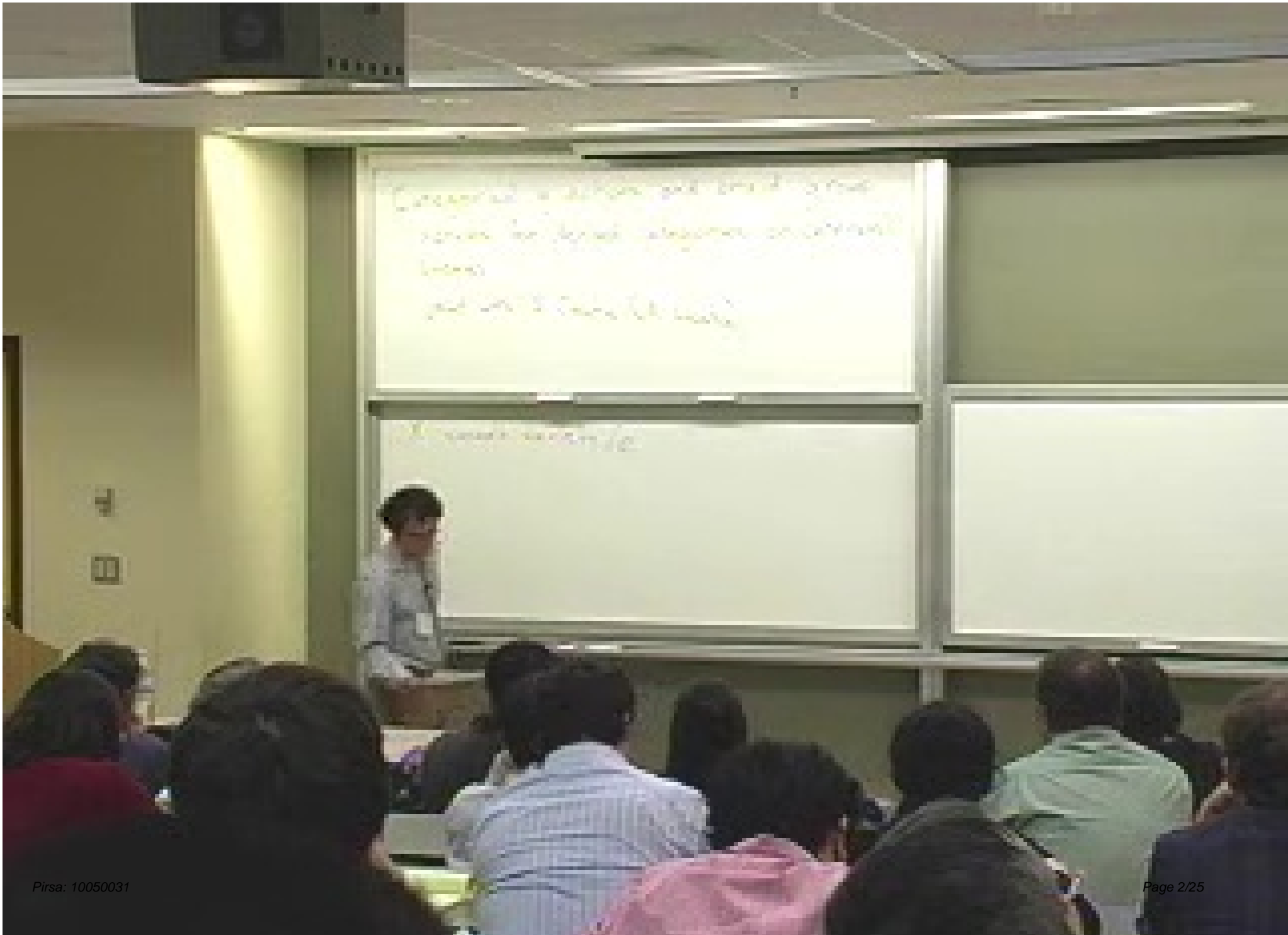


Title: Categorical Lie algebra actions and braid group actions

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Abstract: We will discuss the notion of categorical Lie algebra actions, as introduced by Rouquier and Khovanov-Lauda. In particular, we will give examples of categorical Lie algebra actions on derived categories of coherent sheaves. We will show that such categorical Lie algebra actions lead to actions of braid groups.



joint with S. Cautra (A. Licata)

$X$  smooth variety /  $\mathbb{C}$

$D(X)$  = bounded derived category  
of coherent sheaves on  $X$ .

(considered as vector bundles)

We are interested in equivalence  $\Psi: D(X) \rightarrow D(Y)$

$$B_n = \pi_1(\mathbb{C}^n \setminus \Delta / S_n) = \langle \sigma_i, \sigma_j : \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \dots \rangle$$

We will construct group homomorphism

$$B_n \rightarrow \text{Aut}(D(X))$$

Motivations

① Homological mirror symmetry:



Motivations

① Homological mirror symmetry:

$D(X) \cong \text{Fuk}(M)$   $M$  symplectic.  
Kadetschik, Seidel-Thomson, Aurif (M)

Brauer-Manifolds-Obeunkos:

$QH(X)$

consider action of  $H^1(X) \subset H^1(X)$   
by quantum wall.

Gives rise to a connection on trivial vector bundle on  $U$

with fibre  $H^1(X)$ ,  $U \in T$  torsion.  $T$  has with character lattice  $H_2(X, \mathbb{Z})$ .

joint with S. Cantat (A. Liatao)

From the connection, get:

$$\pi(U) \subset H(X)$$

BMC conjecture that this is an action

$$\star \pi(U) \subset D(X)$$

$$\star \pi(U) \subset K(X) \cong H(X)$$

something like a central group

$$\text{st. } \pi_1(U) \cong D(X)$$

$$\text{st. } \pi_1(U) \cong K(X) \cong H^1(X)$$

something like a braid group

② Homological knot invariant (Khovanov homology).

come from actions of groups on categories

↑ any graph.

$$B_n = \langle \sigma_i, \text{all } i \in [n] \rangle, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i-j| = 1$$

$$B_n \cong B_{n-1}$$



Motivations

① Homological mirror symmetry

$$D(X) \cong F \cdot K(M) \quad M \text{ symplectic}$$

Katsevich, Seidel-Thomas, Kontsevich

Ex

$H \subset SL_2(\mathbb{C})$  finite

$X = \mathbb{C}P^1/H$ ,  $\Gamma$  graph of components of the exceptional locus of  $X$

Theorem (Seidel-Thomas)

The group  $B_1$  acts on  $D(X)$

The generators  $\sigma_i \in B_1$  will act by a correspondence:



$\sigma_i$  acts as  $g_i^{-1}$



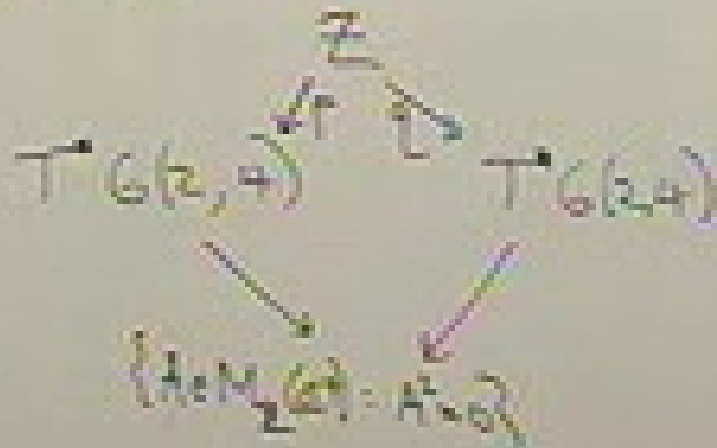
The group  $B_n$  acts on  $D(X)$

The generators  $\sigma_i \in B_n$  will act  
by a correspondence:

$$\sigma_i \text{ acts as } q_{i+1} p_i^{\pm 1}$$

Ex  $(X = T^*G, H = \frac{1}{2}g^2)$

$$X = T^*G(2,4)$$



Theorem (Nanikani).

$q_{i+1} p_i^{\pm 1}$  is not an equivalence

$$\# \pi_1(U) \subset D(X)$$

$$\# \pi_1(U) \subset K(X) \cong H(X)$$

something like a braid group.

There is a map

$$B_n \rightarrow SL_n$$

$$\sigma \mapsto \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

If  $V$  is a rep of  $SL_n$ ,

then  $V$  is a rep of  $B_n$

If  $V$  is a rep of  $B_n$  - non-trivial matrices

then  $V$  is a rep of  $B_n$

Proposition

If  $V$  is a rep of  $SL_n$ ,

$$\text{then } V \in \mathcal{O}_{\mathbb{C}} \text{ at } \frac{1}{2}(n, 0)$$

$$V(\lambda) = \text{trivial} : H_{\text{non-trivial}}(H_n)$$

$$E \cdot \begin{cases} V \text{ is a } H_n \text{ dominant } \\ \text{matrix} \end{cases}$$

$$E \cdot V(\lambda) = V(\lambda + \alpha)$$

$$d(\lambda - \mu)$$

# \* $\mathbb{T}_1(\omega) \subset \mathbb{D}(\lambda)$

There is a map

$$\mathbb{R} \rightarrow \mathbb{S}^1$$

$$t \mapsto e^{it}$$

If  $V$  is a rep of  $\mathbb{R}$

then  $V$  is a rep of  $\mathbb{S}^1$

If  $V$  is a rep of  $\mathbb{S}^1$

then  $V$  is a rep of  $\mathbb{R}$

Proposition

If  $V$  is a rep of  $\mathbb{R}$

then  $V \cong \mathbb{R} \oplus V(i)$

where  $V(i) = \{v \in V \mid \rho(v) = i v\}$

$V(i)$  is a rep of  $\mathbb{R}$

$V(i)$  is a rep of  $\mathbb{S}^1$

$V(i)$  is a rep of  $\mathbb{R}$

$V(i)$  is a rep of  $\mathbb{S}^1$

$V(i)$  is a rep of  $\mathbb{R}$

$V(i)$  is a rep of  $\mathbb{S}^1$

Then  $V$  is a rep of  $B_n$   
 If  $V$  is a rep of  $\mathfrak{sl}_n$  - non-trivial  
 matrices  
 Then  $V$  is a rep of  $B_n$



Let  $\alpha_i$   
 $E_i \rightarrow V(\lambda) \rightarrow V(\lambda - \alpha_i)$   
 $F_i \rightarrow V(\lambda) \rightarrow V(\lambda + \alpha_i)$

$$E_i F_i - F_i E_i = (n_i + 1)E_i$$

$\begin{pmatrix} \text{lin. alg} \\ \text{data} \end{pmatrix}$  is the rep on the rep  $\rightarrow B_n$  rep.

$\begin{pmatrix} \text{sl}_n \\ \mathfrak{g}, F_i \end{pmatrix}$

$\begin{pmatrix} \text{lin. alg} \\ \text{data} \end{pmatrix}$  is a rep  $\rightarrow B_n$  rep

Let  $\mathfrak{h}$  be the  
 Lie algebra

$\rightarrow B_n$  action on  $\mathfrak{h}$

$E = \{E_1, E_2, \dots, E_n\}$   
 (1)  $E_1, E_2, \dots, E_n$  are disjoint  
 (2)  $E_1 \cup E_2 \cup \dots \cup E_n = \Omega$   
 (3)  $P(E_i) > 0$   
 (4)  $\sum_{i=1}^n P(E_i) = 1$   
 (5)  $E_1, E_2, \dots, E_n$  are independent

4.  $E_1, E_2, \dots, E_n$  are independent  
 (1)  $P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1)P(E_2)\dots P(E_n)$   
 (2)  $P(E_1 \cap E_2) = P(E_1)P(E_2)$   
 (3)  $P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$   
 (4)  $P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1)P(E_2)P(E_3)P(E_4)$   
 (5)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)$

(6)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)$   
 (7)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)$   
 (8)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)$   
 (9)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)$   
 (10)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9 \cap E_{10}) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)P(E_{10})$

(11)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9 \cap E_{10} \cap E_{11}) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)P(E_{10})P(E_{11})$   
 (12)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9 \cap E_{10} \cap E_{11} \cap E_{12}) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)P(E_{10})P(E_{11})P(E_{12})$   
 (13)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9 \cap E_{10} \cap E_{11} \cap E_{12} \cap E_{13}) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)P(E_{10})P(E_{11})P(E_{12})P(E_{13})$   
 (14)  $P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6 \cap E_7 \cap E_8 \cap E_9 \cap E_{10} \cap E_{11} \cap E_{12} \cap E_{13} \cap E_{14}) = P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)P(E_7)P(E_8)P(E_9)P(E_{10})P(E_{11})P(E_{12})P(E_{13})P(E_{14})$

Ex  
 $H = \mathbb{Z}_2 \times \mathbb{Z}_2$

$X = \mathbb{P}^1$  graph of components of the exceptional locus of  $X$

Theorem (Saito, 1980)

The group  $\mathbb{Z}_2$  acts on  $D(X)$

The members of  $\mathbb{Z}_2$  will act by  $\pm$  correspondences



Def  
 A geometric integral  $\mathbb{Z}_2$  action is:

(1) A square of smooth varieties  $(Y, \sigma)$  over  $K$  ( $\mathbb{Z}_2$ -action)

(2) Functions  $E: D(Y/\sigma) \rightarrow D(Y/\sigma)$   
 $F: D(Y/\sigma) \rightarrow D(Y/\sigma)$

(3)  $\sigma(Y) = Y$  a fibration

(4)  $K = \mathbb{Z}_2$

(cat. prod)  $\xrightarrow{\text{Kac-Moody Lie algebra}}$  Reaction on algebra

St  $E_1 \otimes E_2 \cong F_1 \otimes E_2 \oplus I_2(\mathbb{Z})$       $(\rho, \sigma) \neq 0$   
 an isomorphism of bundles     ... and relations...

Chern-Langmuir, Landau-Khalatnikov...

an  $\mathbb{R}$ -module  
of functions

Chuang-Rangier, Landi-Kharanov...

A

operator

act. (on what) ...

Reaction on others



an  $\mathbb{R}$ -algebra  
of functions

Chuang-Rangier, Lands-Kharanov...

A geom. cal. of action gives an action  $\mathcal{L}_g$  on  $\mathcal{O}_K(Y/W)$

$\mathcal{O}_K(Y/W)$   
 $\mathcal{L}_g$  action on  $\mathcal{O}_K(Y/W)$

$\mathcal{L}_g$  action on  $\mathcal{O}_K(Y/W)$

A group action of a Lie group on a manifold

Theorem (Chern-Simons-Regge)

A geometric categorical action gives rise to a  
action of  $\mathfrak{B}_g$ .

Eg gauge  $F \times N$ .

Consider  $Fl_n(\mathbb{C}) = \{O \in V_1 \oplus \dots \oplus V_n = \mathbb{C}^n\} \cong \mathbb{C}^n$   $Fl_n(\mathbb{C})$   
 $Y(N) = T^* Fl_n(\mathbb{C})$   
 $H^1(N, \mathbb{C}) = \mathbb{C}$   
 $H^2(N, \mathbb{C}) = \mathbb{C}$

$\textcircled{1} \quad \gamma(t) = Y(t)$   
 $\textcircled{2} \quad k = 0$

$F: D(Y(t)) \rightarrow D(Y(t-c))$   
 a deformation

$\mu = 0, \mu = 4$

$\tau: \mathbb{R}P^2 \rightarrow T^*(\mathbb{R}P^2)$       $\tau: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$   
 $\mu = 0, \mu = 4$

To define  $E, F$  consider

$E = \mathbb{R}P^2$   
 $T^*(\mathbb{R}P^2)$   
 $T^*(\text{Flux} \times \text{Flux})$   
 $T^*(\text{Flux} \times \mathbb{R}P^2)$

$\omega = \langle \alpha, \nu \rangle \in \text{Flux}$   
 $\text{str } \nu_1 = \nu_2$   
 $\nu_1 \neq \nu_2$

Consider  $FL_n(\mathbb{C}^N) = \{0 \in V, \dots, \varepsilon V_n = \mathbb{C}^N\} = \mathbb{C}^N \times FL_n(\mathbb{C}^N)$ .  
 $Y(A) = T^* FL_n(\mathbb{C}^N)$  in cases  $M \times \text{ident}$ .

Proposition (CZ)

Three varieties  $T^* FL_n(\mathbb{C}^N)$  with three functors  $E, F, G$  give a geom. cont.  $g_n$  action on  $T^* FL_n(\mathbb{C}^N)$ .

In the level of  $K$ -group this action was studied by Lusztig.

Then  $g_n \in \mathbb{C}_n$  on  $\oplus D(T^* FL_n(\mathbb{C}^N))$ .

$$F_1: D(Y(x)) \rightarrow D(Y(x-\alpha))$$

①  $Y(x) = Y(x)$

②  $h = 0$

a deformation

ex.  $N=4$

$\pi$

$T^*P^2$

$T^*(S^2)$

$T^*P^2$

$\pi$

$\mu = 0$

$\mu = 0$

To define  $E, F_1$  consider

$$T^*(\mathbb{R}^2 \times \mathbb{R}^2)$$

$$E = \mathbb{R}^2 \times T^*\mathbb{R}^2$$

$$T^*\mathbb{R}^2$$

$$0 = \langle \alpha, V \rangle$$

$$W_1 = \{(\alpha, V) \in \dots\}$$

$$\text{st. } \langle \alpha, V \rangle = 0$$

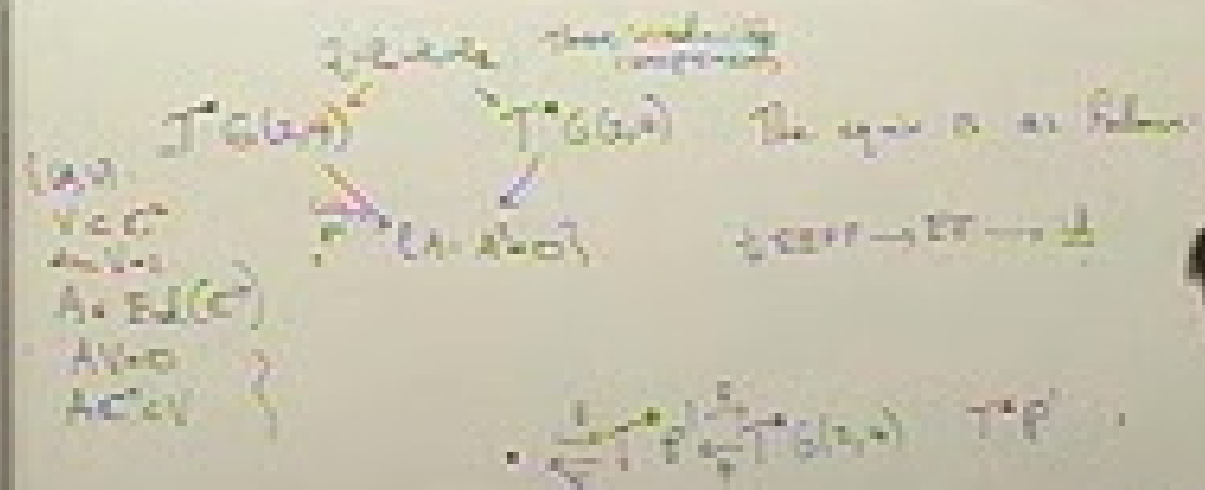
$$V = \alpha$$

$T^* \rightarrow T^* \rightarrow T^* \rightarrow T^* \rightarrow T^*$   
 $T^* \rightarrow T^* \rightarrow T^* \rightarrow T^* \rightarrow T^*$

To define  $T^*$ , consider



$T^*(A, B, C, D) = T^*(A, B, C) \cdot T^*(A, D)$   
 $T^*(A, B, C) = T^*(A, B) \cdot T^*(A, C)$   
 $T^*(A, B) = T^*(A) \cdot T^*(B)$   
 $T^*(A, C) = T^*(A) \cdot T^*(C)$





Consider  $Fl_{\mathbb{C}}(\mathbb{C}^N) = \{O \in U(N) \mid O^T = -O\} \cong U(N)$   $Fl_{\mathbb{C}}(\mathbb{C}^N)$   
 $Y(A) = T^* Fl_{\mathbb{C}}(\mathbb{C}^N)$  in vases

BMO studied the quantum connection  
 on Nakajima quiver varieties.

It is the trigonometric Casimir connection  
 { any of Toledo-Langolo

the quantum Weil group  
 this action can be lifted  
 to  $\mathcal{D}$  (Nakajima quiver varieties).



a connection

ities

le connection

Tele

$$\begin{aligned}
 & (Y^{\omega}) \\
 & \mathcal{G}_p(Y^{\omega})
 \end{aligned}$$

$$B_p \xrightarrow{\sigma} W_p \subset X$$

$$\begin{aligned}
 & \sigma: D(Y^{\omega}) \\
 & \downarrow \\
 & W_p(Y^{\omega})
 \end{aligned}$$

Eg

Associated varieties

There is a construction

Theorem (C)

There is a Nakajima