Title: A black hole uniqueness theorem

Date: May 07, 2010 08:30 AM

URL: http://pirsa.org/10050027

Abstract: I will discuss recent joint work with A. Ionescu and S. Klainerman on the black hole uniqueness problem. A classical result of Hawking (building on earlier work of Carter and Robinson) asserts that any vacuum, stationary black hole exterior region must be isometric to the Kerr exterior, under the restrictive assumption that the space-time metric shouldbe analytic in the entire exterior region. We prove that Hawking's theorem remains valid without the assumption of analyticity, for black hole exteriors which are apriori assumed to be "close" to the Kerr exterior solution in a very precise sense. Our method of proof relies on certain geometric Carleman-type estimates for the wave operator. Time permitting, some more recent developments will also be surveyed.

A black hole uniqueness theorem.

Spyros Alexakis

Microsoft Windows xp Professional

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Spyros Alexakis


Microsoft
Windows xp
Professional


Microsoft
Windows xp
Professional

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Conjecture 1 *Let* (M_{ext}, g) *be a stationary, vacuum* $(Ric(g) = 0)$ *black hole exterior region, with connected, regular, non-degenerate bifurcate event horizon* $\mathcal{H}^+ \cup \mathcal{H}^-$.

Then (M_{ext}, g) *is isometric to a Kerr exterior* $(M_{Kerr}, \mathrm{g}_{M,a})$, $a < M$.

The Conjecture is *proven*, under the *additionnal* assumption that g is real analytic *up to* the event horizon.

Main Ideas:

Hawking, Friedrich-Racz-Wald: Existence of a jet of a *second, rotational* Killing field Z, tangent to $\mathcal{H}^+ \cup \mathcal{H}^-$.

Analyticity: Implies that Z can be extended to the entire manifold (M_{ext}, g).

Carter-Robinson Theorem: A stationary *and* axisymmetric (M_{ext}, g) is isometric to a Kerr exterior $(M_{Kerr}, \mathrm{g}_{M,a})$, $a < M$.

Theorem 1 (A.-Ionescu-Klainerman.) *Assume* (M_{ext}, g) *be as in the Conjecture. Assume additionnaly that* (M_{ext}, g) *small perturbation of a Kerr space-time* ($|S_{ijkl}| < \epsilon$, *for some given* $\epsilon > 0$.*)*

Then (M_{ext}, g) *isometric to a Kerr exterior* $(M_{Kerr}, g_{M,a})$, $a < M$.

Main Idea: Prove existence of a rotationnal Killing field Z throughout (M_{ext}, g), *without analyticity*. Appeal to Carter-Robinson Theorem.

Main Tools: Unique continuation for vacuum Einstein equations (Carleman estimates; Use $|S| < \epsilon$) $\Rightarrow$ Extension of Killing fields.

Two classical results:

Theorem 2 (Carleman–Calderón) $P(x,D)$ *a second-order uniformly elliptic operator (principal symbol $a^{ij}(x)\xi_i\xi_j$), over $\Omega \subset \mathbb{R}^n$. Suppose:*

$P(x,D)u = 0$, where $u \in C^\infty(\Omega)$.

$\Sigma \subset \Omega$ smooth hypersurface, and $u = 0$ in Σ^-.

Then $u = 0$ in Σ^+.

Theorem 3 (Hörmander) $P(x,D)$ *a second-order wave operator (principal symbol* $g^{ij}(x)\xi_i\xi_j$*), over* $\Omega \subset \mathbb{R}^n$*. Suppose:*

$P(x,D)u = 0$, *where* $u \in C^\infty(\Omega)$.

$\Sigma \subset \Omega$ *time-like hypersurface,* $u = 0$ *in* Σ^-*, and* Σ^+ *strongly pseudo-convex at* $O \in \Sigma$*.*

Then $u = 0$ *in an open neighborhood of* Σ^+*, near* O*.*

Main steps:

Step 1: (M, g) space-time, $Ric(\mathrm{g}) = 0$. Σ is a time-like (or bifurcate null) hypersurface in M. Suppose g admits a Killing field Z in Σ^-. Then: **If** Σ^- is strongly pseudo-convex, the Killing field Z extends into Σ^+.

Consequence of Step 1: Hawking's rotational Killing field Z extends to a *real* Killing field in a (small) neighborhood of $\mathcal{H}^+ \cup \mathcal{H}^-$.

Step 2: (M, g) space-time, $Ric(\mathrm{g}) = 0$. Σ is a time-like hypersurface in M. Suppose (M, g) admits a Killing field T in M. Suppose (M, g) admits a Killing field Z in Σ^-. Then if Σ^- is strongly **conditionally** pseudo-convex, then the Killing field Z can be extended in Σ^+.

Step 3: Suppose (M_{ext}, g) vaccum stationary black hole exterior as in our theorem (with $|S| < \epsilon$). Then there exists a function y on *all* of M_{ext} whose level sets are conditionaly pseudo-convex.

Consequence of Steps 2 and 3: The rotational Killing field Z extends to entire exterior region (M_{ext}, g). Therefore by Carter-Robinson, (M_{ext}, g) is isometric to a Kerr exterior.

Proof of Step 1: Unique continuation for Einstein metrics, via Carleman estimates:

Main Lemma: Let (M, g), $(\tilde{M}, \tilde{\mathrm{g}})$ be vacuum space-times, $Ric(\mathrm{g}) = 0, Ric(\tilde{\mathrm{g}}) = 0$.

Let $\Sigma^-, \tilde{\Sigma}^-$ be isometric domains in $M, \tilde{M}$: There exists an isometry $\Psi : M \to \tilde{M}$ s.t. $\Psi^* \tilde{\mathrm{g}} = \mathrm{g}$ in Σ^-.

Claim: If Σ^- is strongly pseudo-convex at $P \in \Sigma^-$, then $\mathrm{g}, \tilde{\mathrm{g}}$ are isometric in $\Sigma^+, \tilde{\Sigma}^+$, near P.

Consequence: If Z is Killing for g in Σ^-, then Z extends into Σ^+, near P.

Proposition 1 (Hörmander:) *Let* (M, g) *Lorentzian and* $\Sigma \subset M$ *time-like hypersurface. Suppose* Σ^{-} *is strongly pseudo-convex at* P*. Suppose* ϕ *compactly supported in a small neighborhood of* P *in* Σ^{+}*. Then there exists weight function* $f \geq 0$ *such that:*

Let $||\phi||_{L^2_\lambda} := ||e^{-\lambda \cdot f} \cdot \phi||_{L^2}$*. Then for* $\lambda >> 0$*:*

$$\lambda \cdot ||\phi||_{L^2_\lambda} + ||D\phi||_{L^2_\lambda} \leq \frac{C}{\sqrt{\lambda}} ||\Box_g \phi||_{L^2_\lambda}$$

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Proposition 1 (Hörmander:) *Let (M, g) Lorentzian and $\Sigma \subset M$ time-like hypersurface. Suppose Σ^- is strongly pseudo-convex at P. Suppose ϕ compactly supported in a small neighborhood of P in Σ^+. Then there exists weight function $f \geq 0$ such that:*

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The PDE-ODE system: Let $\vec{\nu}$ be vector field s.t.

$$D_{\vec{\nu}}\vec{\nu} = 0, \tilde{D}_{\vec{\nu}}\vec{\nu} = 0.$$

Let $v^i, \tilde{v}^i$ be frames s.t.:

$$D_{\vec{\nu}}v^i = 0, \tilde{D}_{\vec{\nu}}\tilde{v}^i = 0.$$

Let $\Gamma_{ab,c}, \tilde{\Gamma}_{ab,c}$ be the connection coefficients for $D, \tilde{D}$ for these frames. Define:

$(d\mathrm{R})_{abcd} = (\mathrm{R})_{abcd} - (\tilde{\mathrm{R}})_{abcd}$,

$(d\Gamma)_{ab,c} = (\Gamma)_{ab,c} - (\tilde{\Gamma})_{ab,c}$, $(dg)_{ab} = g_{ab} - \tilde{g}_{ab}$.

Main equations:

$$\Box_g(d\mathrm{R}) = m(d\mathrm{R}) + m(dg) + m(d\Gamma) + m(\partial(d\Gamma_{sa,b})).$$

$$\vec{\nu}(dg) = m(d\Gamma) + m(dg).$$

$$\vec{\nu}(d\Gamma_{ab,c}) = m(d\mathrm{R}_{\vec{\nu}abc}) + m(d\Gamma).$$

Main Claim:

$$||\Box_g(d\mathrm{R})\cdot\chi||_{L^2_\lambda} \leq C[||d\mathrm{R}\cdot\chi||_{L^2_\lambda} + ||\partial(d\mathrm{R})\cdot\chi||_{L^2_\lambda} + ||\partial_0\chi||_{L^2_\lambda}]$$

Lemma 1 *If* $\phi = 0$ *on* Σ, $\phi \in C_0^\infty$ *then for* $\lambda >> 0$*:*

$$||\phi||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}} ||\vec{\nu}(\phi)||_{L^2_\lambda}.$$

Estimates:

$$||d\mathrm{g} \cdot \chi||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}} [||d\Gamma \cdot \chi||_{L^2_\lambda} + ||d\mathrm{g} \cdot \chi||_{L^2_\lambda}] + ||\vec{\nu}(\chi)||_{L^2_\lambda}]$$

$$||d\Gamma \cdot \chi||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}} [||d\mathrm{R} \cdot \chi||_{L^2_\lambda} + ||(d\Gamma) \cdot \chi||_{L^2_\lambda} + ||\vec{\nu}(\chi)||_{L^2_\lambda}]$$

$$||\partial d\Gamma \cdot \chi||_{L^2_\lambda} \le \frac{C}{\sqrt{\lambda}} [||\partial(d\mathrm{R}) \cdot \chi||_{L^2_\lambda} + ||(d\mathrm{R}) \cdot \chi||_{L^2_\lambda} + ||d\Gamma \cdot \chi||_{L^2_\lambda} + ||\vec{\nu}(\chi)||_{L^2_\lambda}] \quad (1)$$

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