Title: Current algebra in conformal sigma models on supergroups (Part II)

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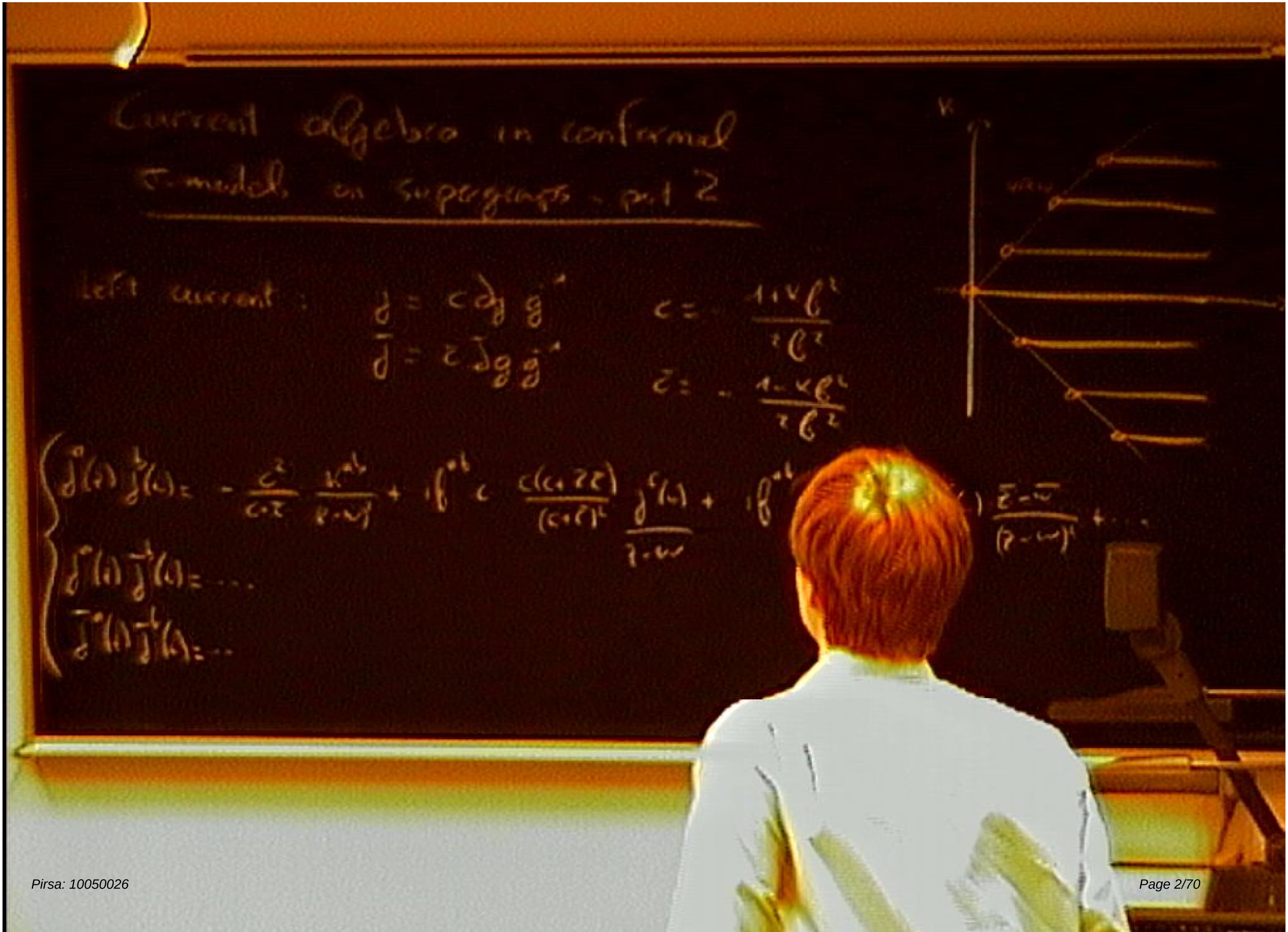
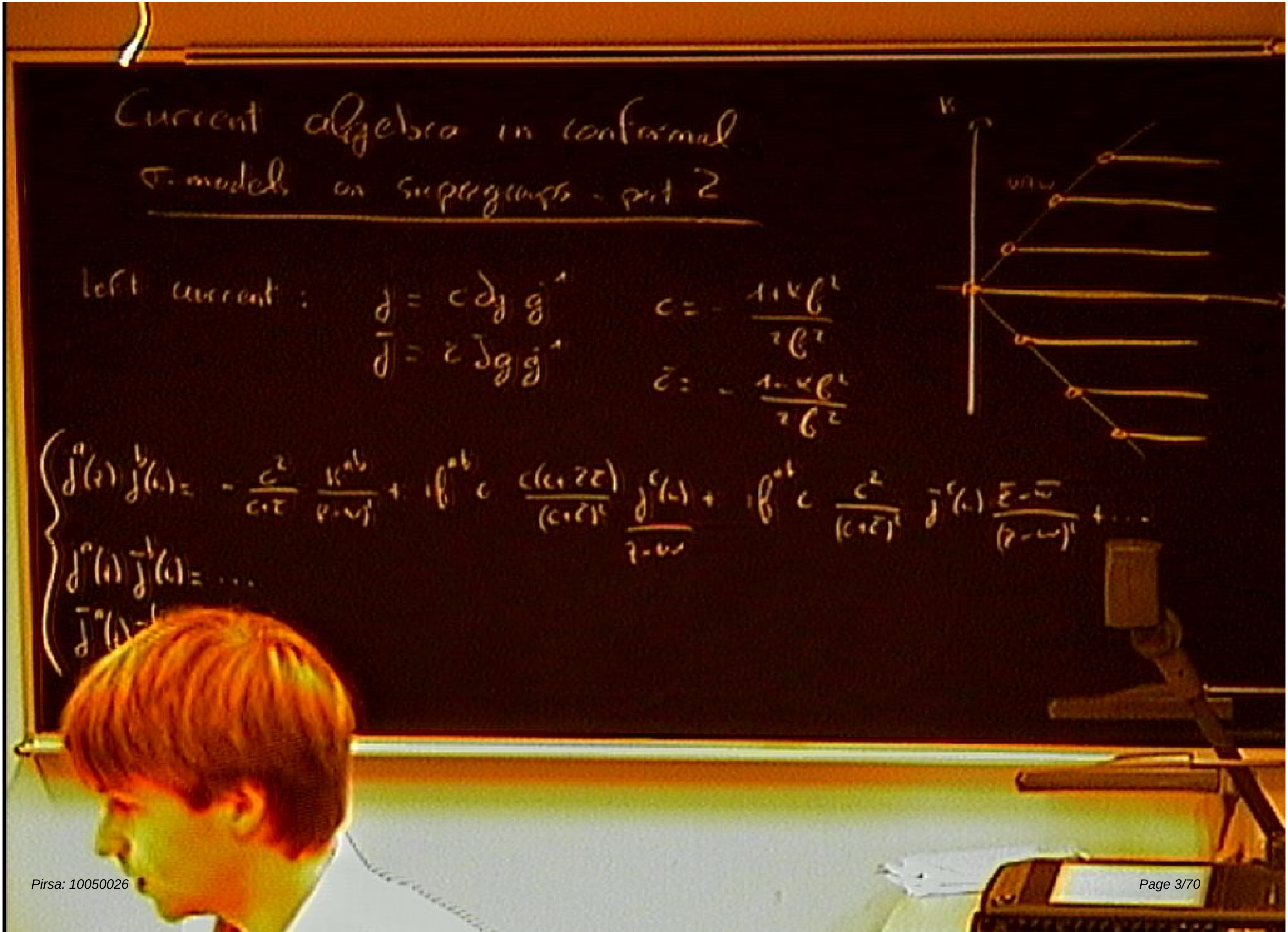
Abstract: Two-dimensional non-linear sigma models on some supergroup manifolds are conformal field theories whether the action includes a Wess-Zumino term or not. These models are relevant for the worldsheet description of string theory in Anti-de Sitter backgrounds with Ramond-Ramond fluxes. The current algebra is an useful tool to study these theories. In these lectures I will review the construction of the current algebra. Then I will discuss some applications to the computation of the spectrum and integrability.


Current algebra in conformal
σ-models on supergroups - part 2
left current :


Current algebra in conformal
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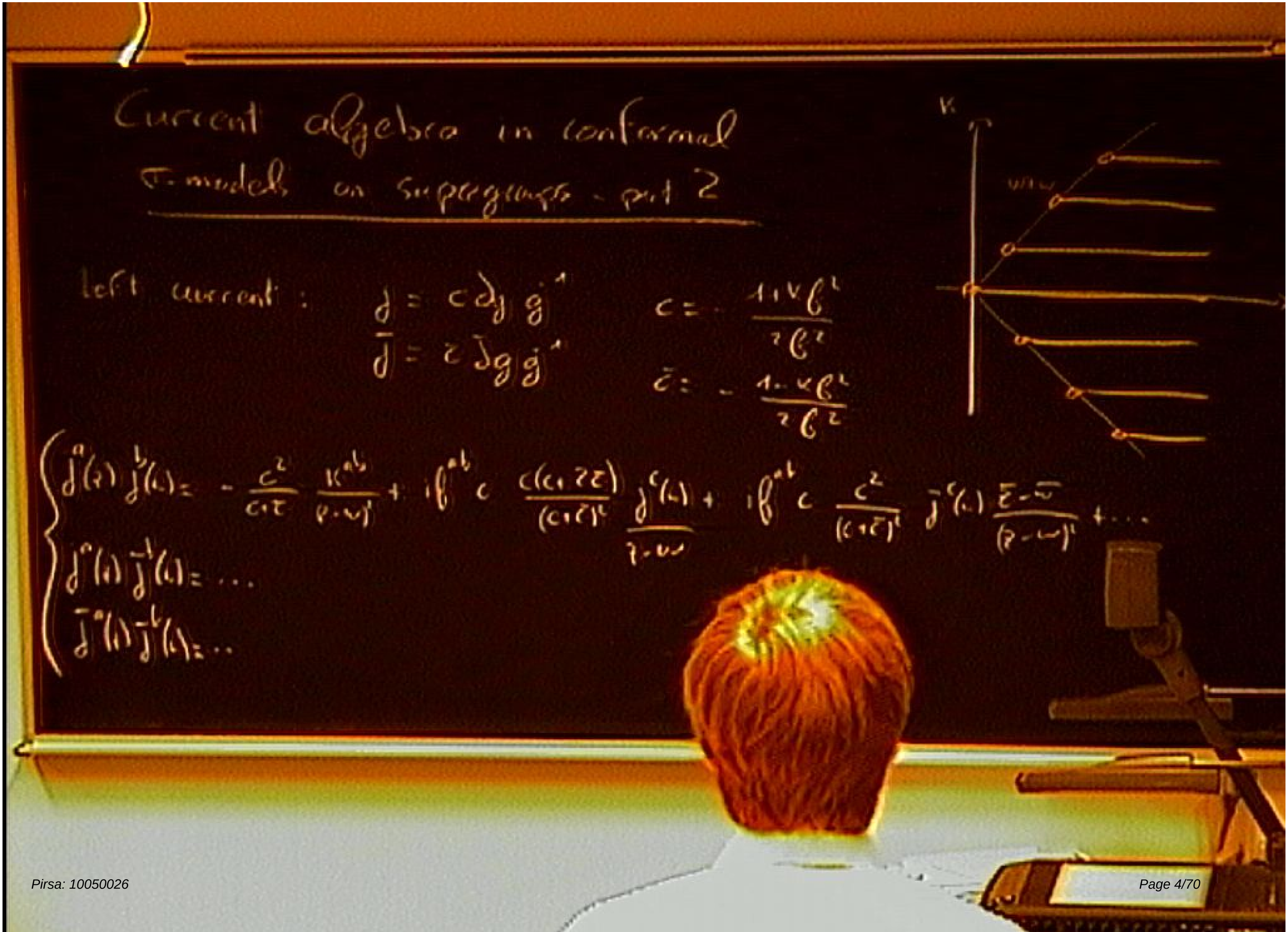
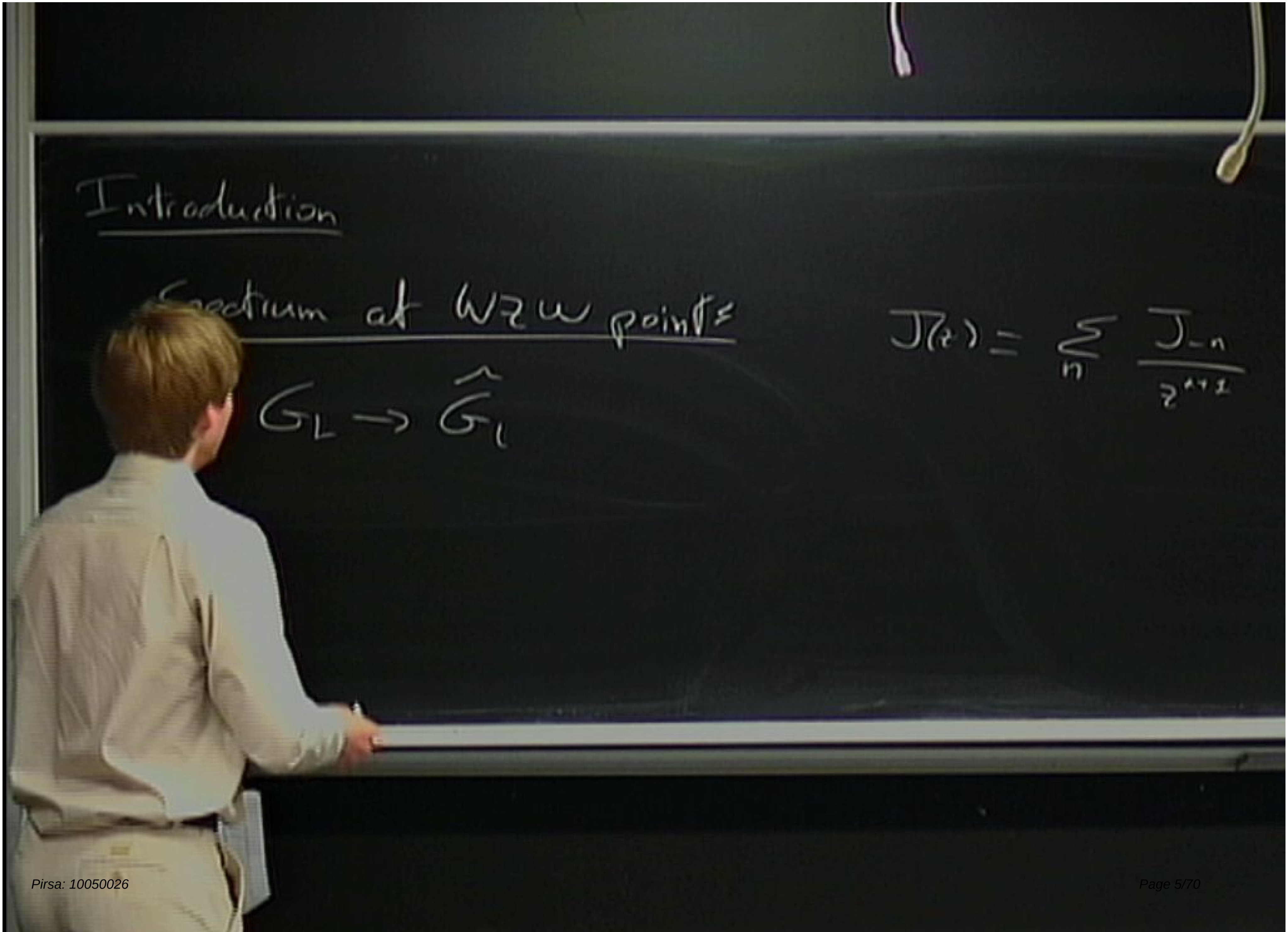
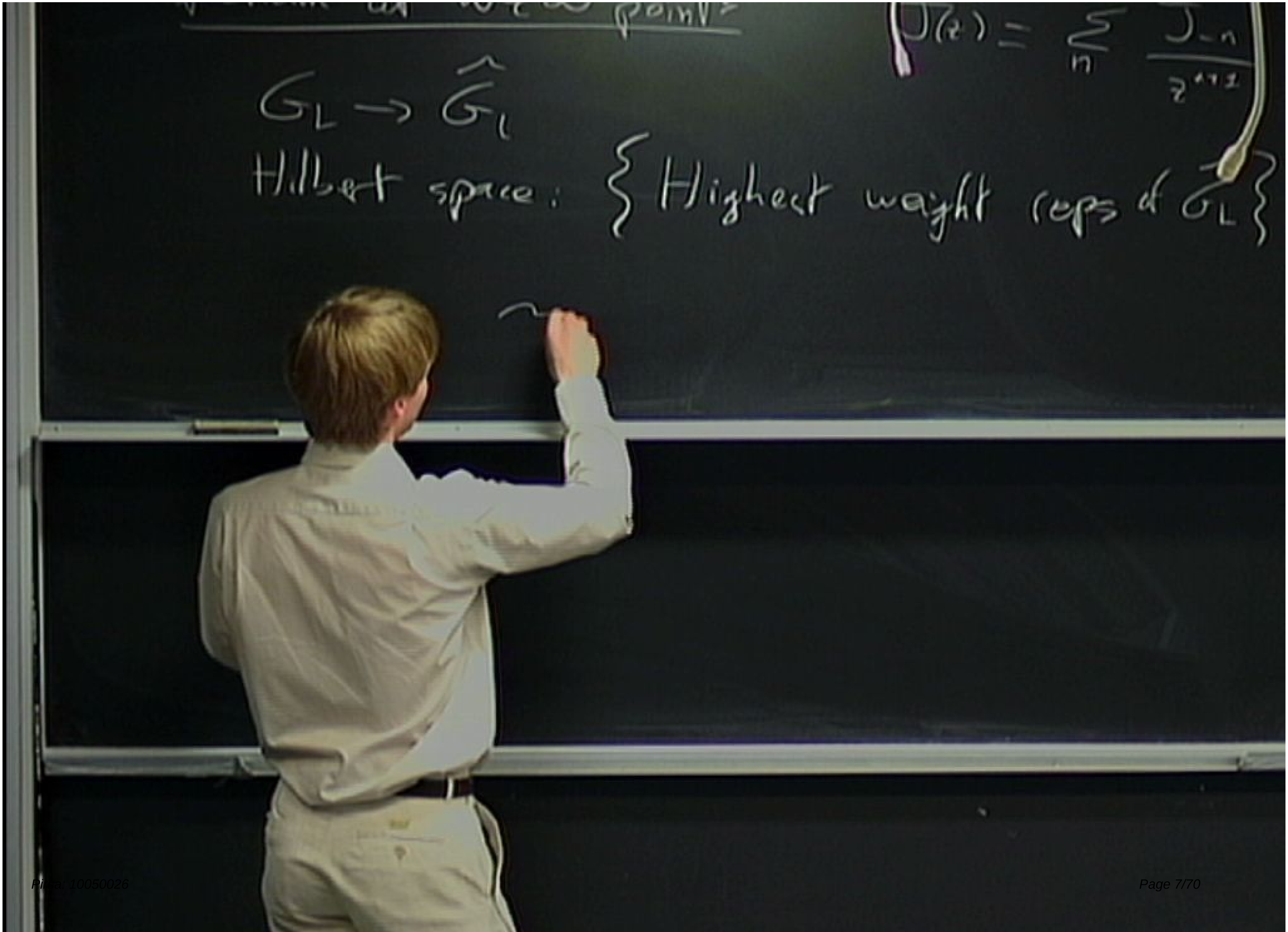
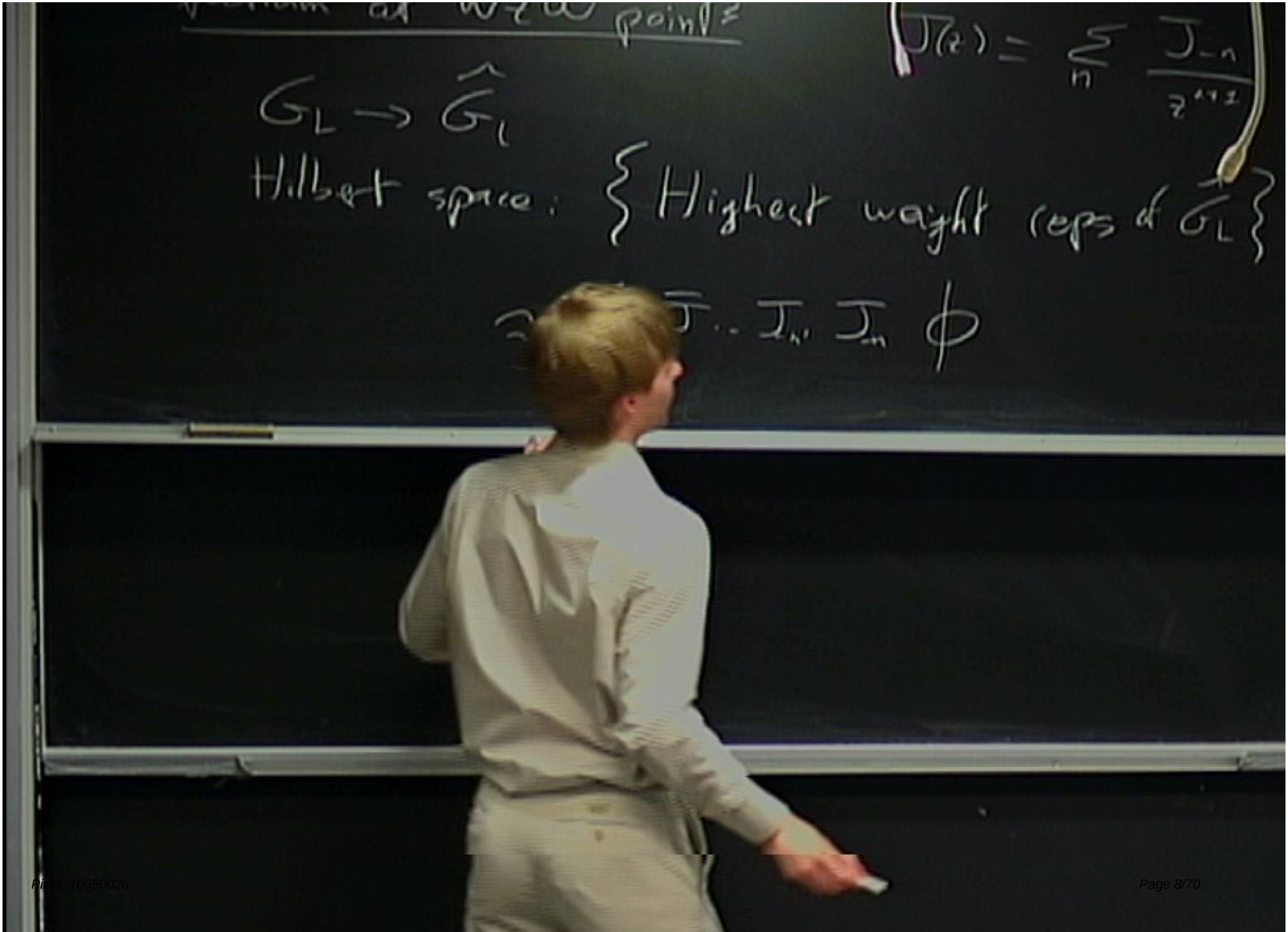

Current algebra in conformal
σ-models on supergroups - part 2
left current :
J = c ∂g g^{-1}
J̄ = c̄ ∂̄g g^{-1}
c = - (1+kf²)/(2f²)
c̄ = - (1-kf²)/(2f²)
J^a(z) J^b(w) = - c²/(c+c̄) κ^{ab}/(z-w)² + i f^{ab}_c c(c+2c̄)/(c+c̄)² J^c(w)/(z-w) + i f^{ab}_c c²/(c+c̄)² J̄^c(w) (z̄-w̄)/(z-w)² + ...
J^a(z) J̄^b(w) = ...
J̄^a(z) J̄^b(w) = ...

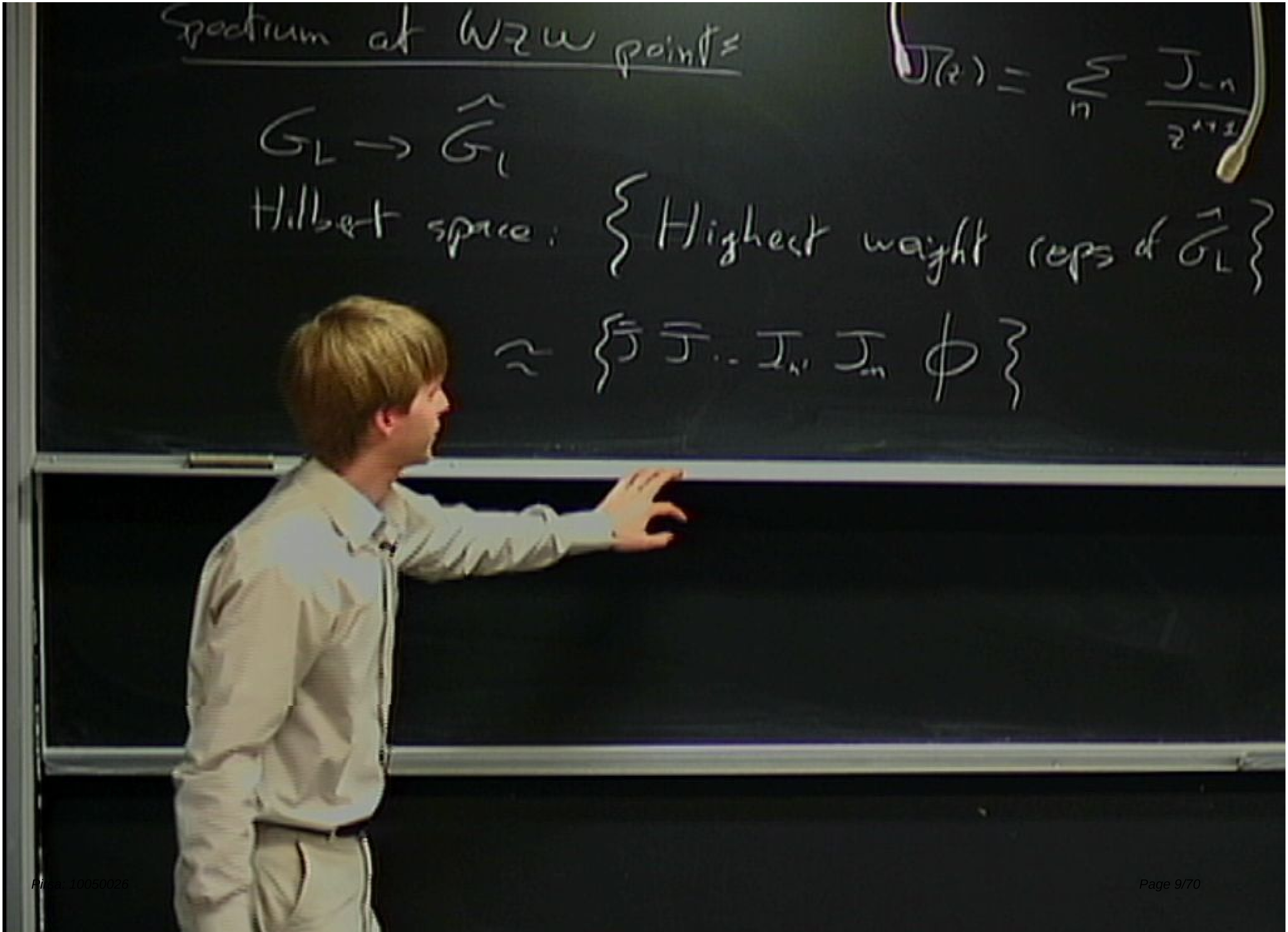

Introduction
Spectrum at WZW points
$G_L \to \hat{G}_L$
$J(z) = \sum_n \frac{J_{-n}}{z^{n+1}}$

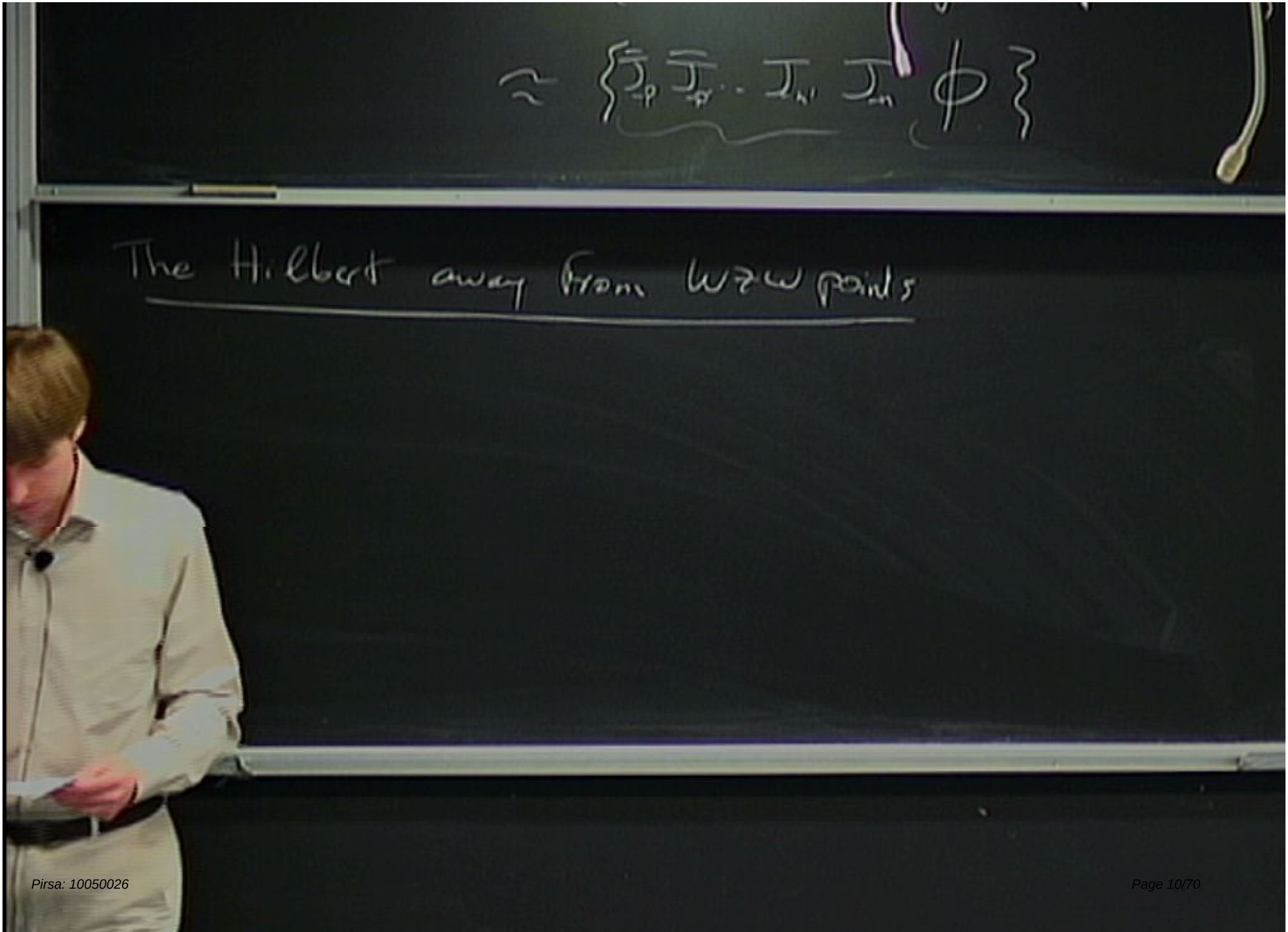
$G_L \to \hat{G}_L$

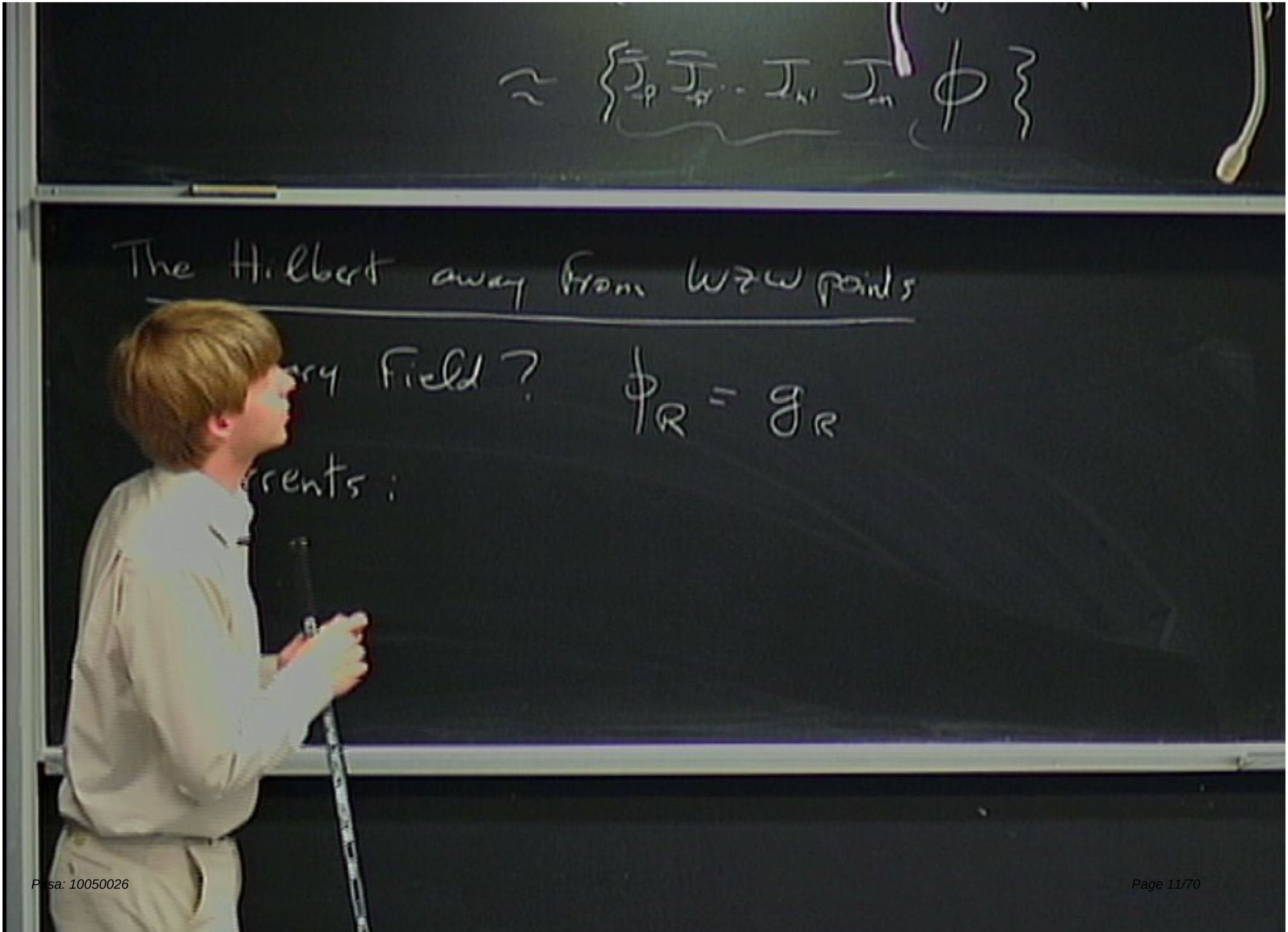
Hilbert space: $\{$ Highest weight reps of $\hat{G}$ $\}$

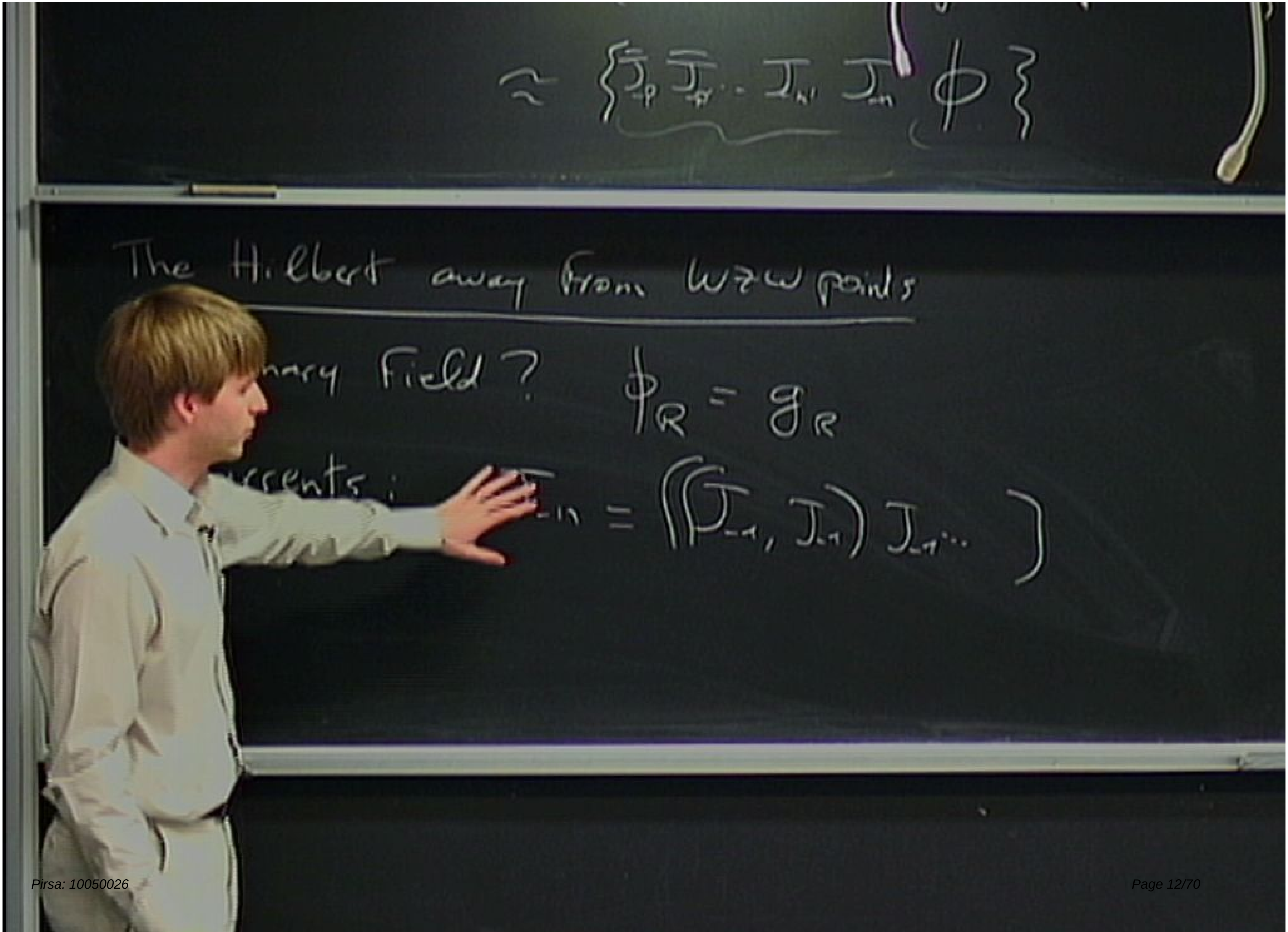

J(z) = Σ_n J_{-n} / z^{n+1}
G_L → Ĝ_L
Hilbert space: { Highest weight reps of G_L }

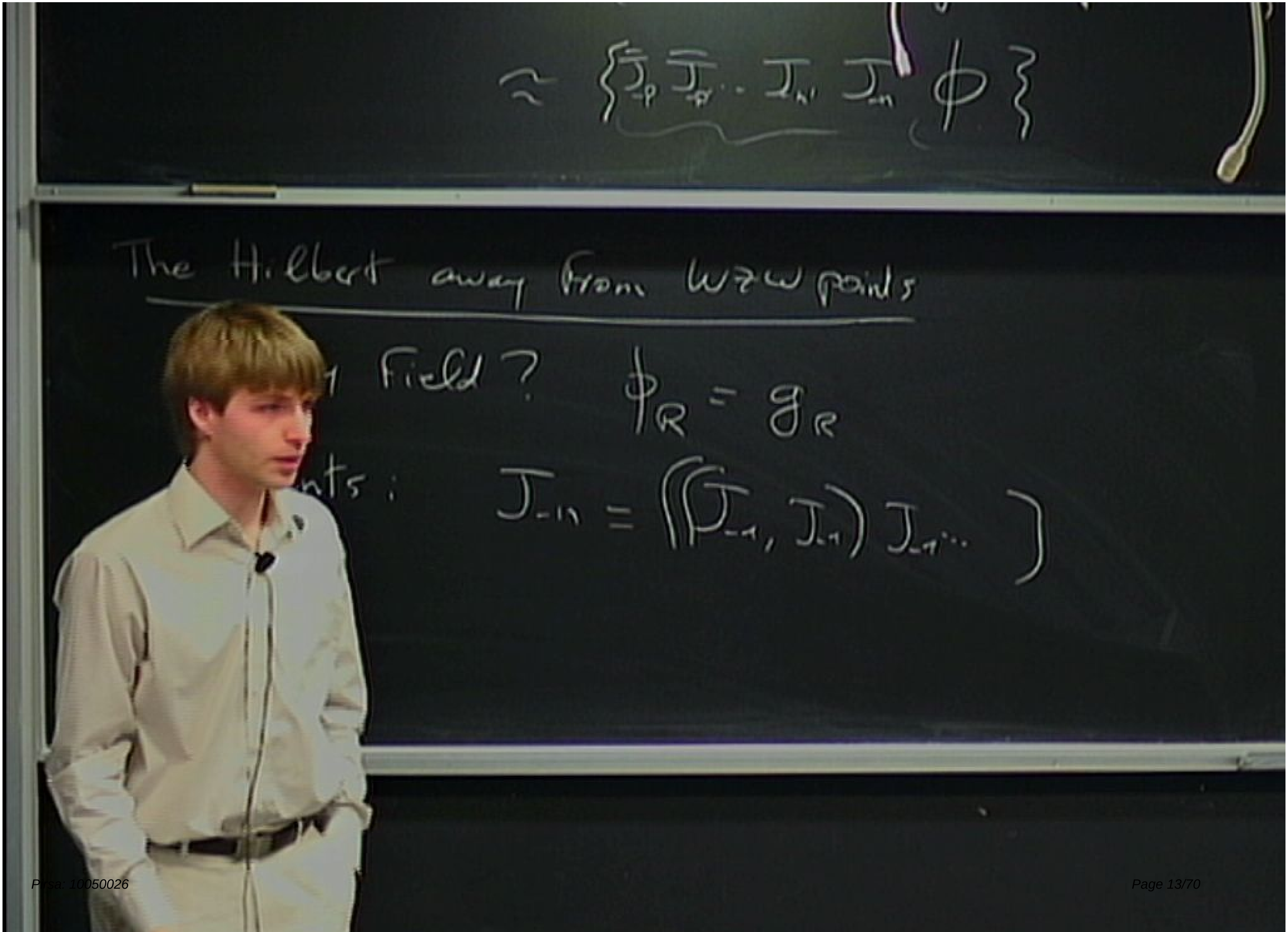

points
J(z) = Σ_n J_{-n} / z^{n+1}
G_L → Ĝ_l
Hilbert space: { Highest weight reps of Ĝ_L }
J.. J_{n'} J_{-n} φ

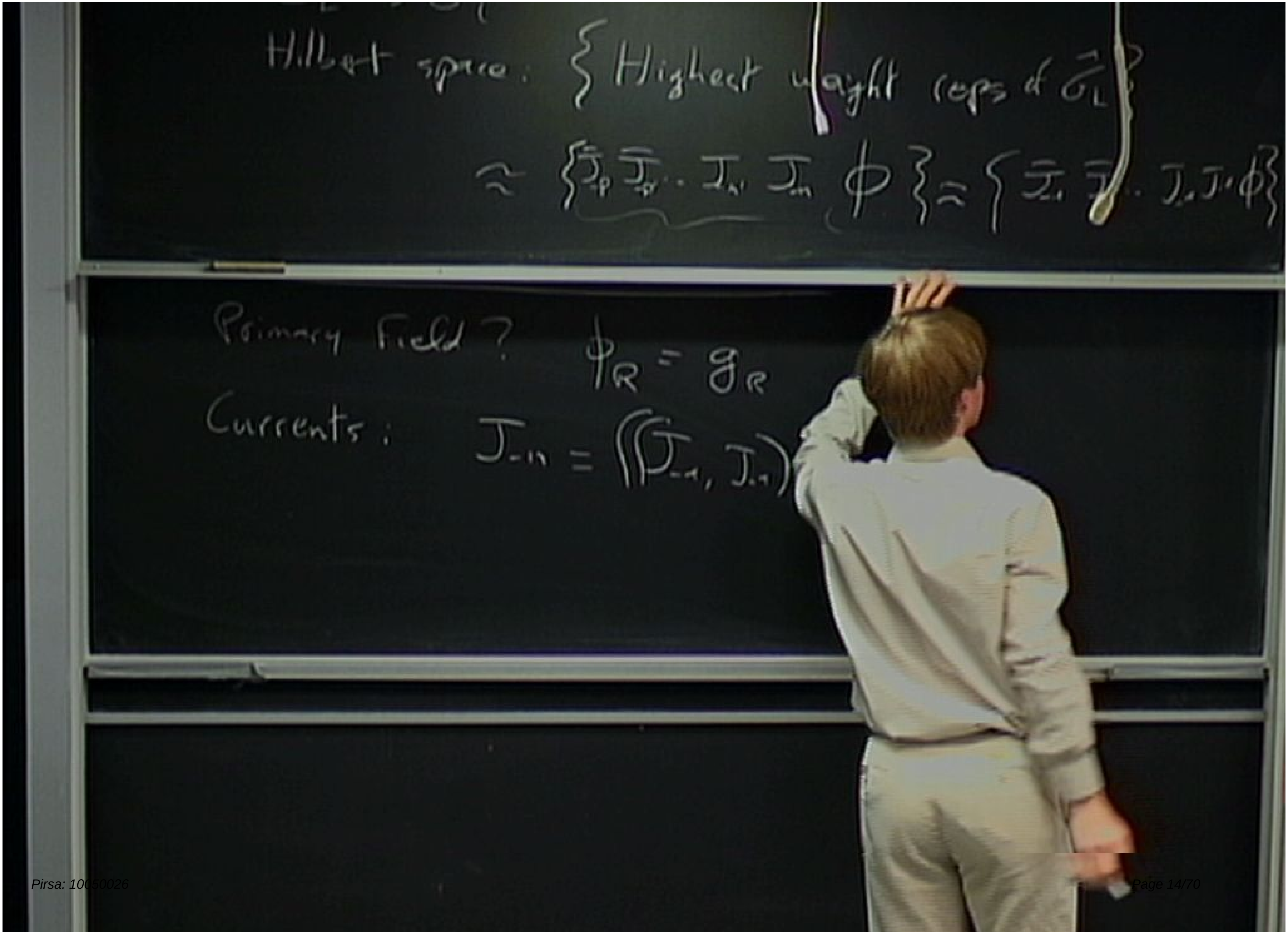

Spectrum at WZW points
G_L → Ĝ_L
Hilbert space: { Highest weight reps of Ĝ_L }
≃ { J J .. J_n' J_n φ }
J(z) = Σ_n J_-n / z^(n+1)

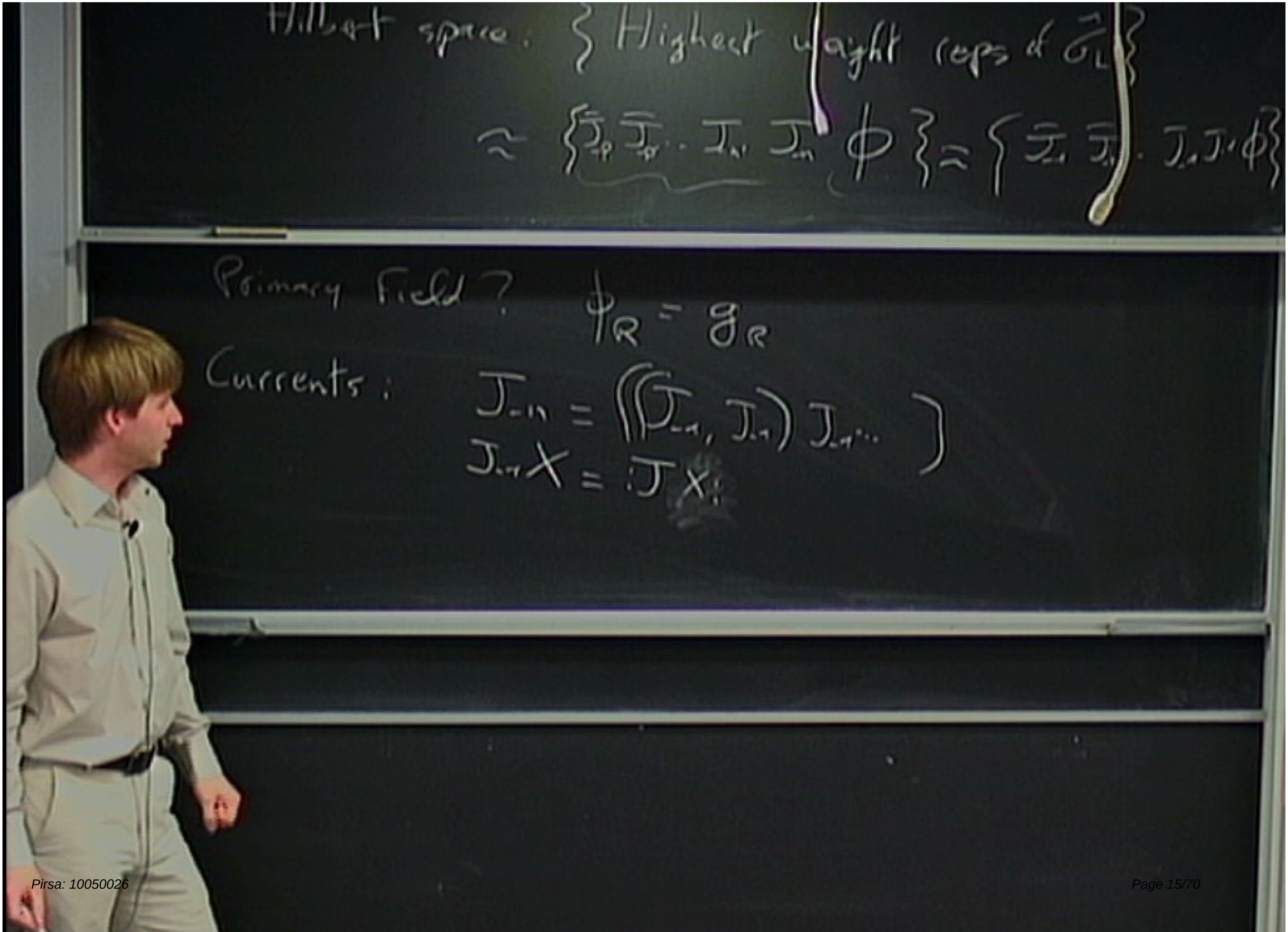

The Hilbert away from WZW points

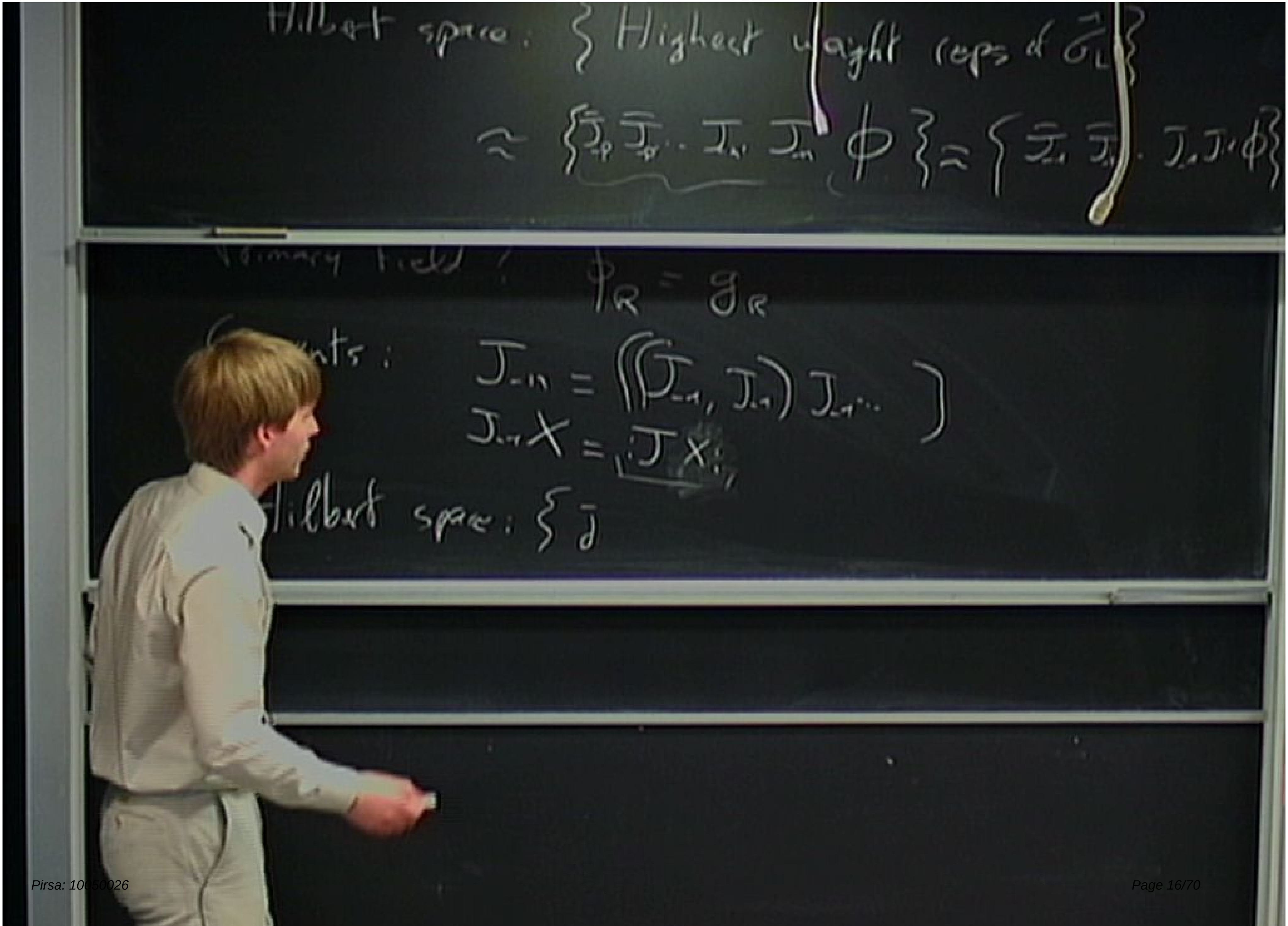

The Hilbert away from WZW points
Field ?
$\phi_R = g_R$
rents :


$\simeq \{ J_{-p} J_{-p'} \cdots J_{-n'} J_{-n} \phi \}$
The Hilbert away from WZW points
mary Field ?
$\phi_R = g_R$
rrents :
$J_{-n} = ((J_{-1}, J_{-1}) J_{-1} \cdots )$


The Hilbert away from WZW points
Field ?
φ_R = g_R
J_{-n} = ((J_{-1}, J_{-1}) J_{-1} ... ]


Hilbert space: { Highest weight reps of $\vec{g}_L$ }
$\simeq \{ J_{-p} J_{-q} \cdots J_{-n'} J_{-n} \phi \} \simeq \{ J_{-1} J_{-1} \cdots J_{-1} J_{-1} \phi \}$
Primary Field ? $\phi_R = g_R$
Currents : $J_{-n} = ((J_{-1}, J_{-1})$


Hilbert space: { Highest weight reps of $\vec{G}_L$ }
Primary Field? $\phi_R = g_R$
Currents: $J_{-n} = ((J_{-1}, J_{n}) J_{-1} \ldots )$
$J_{-1} X = :J X:$


Hilbert space: { Highest weight reps of
Currents:
Hilbert space: {

Hilbert space: $\{$ Highest weight reps of $\vec{\mathcal{G}}_L \}$

$$\simeq \{ \bar{J}_{-p} \bar{J}_{-q} \cdots J_{-n} J_{-m} \phi \} \simeq \{ \bar{J}_{-1} \bar{J}_{-1} \cdots J_{-1} J_{-1} \phi \}$$

Primary field: $\phi_R = g_R$

Currents: $J_{-n} = ((J_{-1}, J_{-1}) J_{-1} \cdots )$

$J_{-1} X = :J X:$

Hilbert space: $\{ :J \cdots :J :J \cdots :J \phi :: \}$

Hilbert space: $\{$ Highest weight reps of $\vec{\mathcal{J}}_L \}$

$$\simeq \{ \underbrace{J_{-p} J_{-q} \cdots J_{-n}} J_{-m} \phi \} \simeq \{ J_{-n} J_{-1} \cdots J_{-1} J_{-1} \phi \}$$

Primary field? $\phi_R = g_R$

Currents: $J_{-n} = ((J_{-1}, J_{-1}) J_{-1} \cdots ]$

$J_{-1} X = \underline{:J X:}$

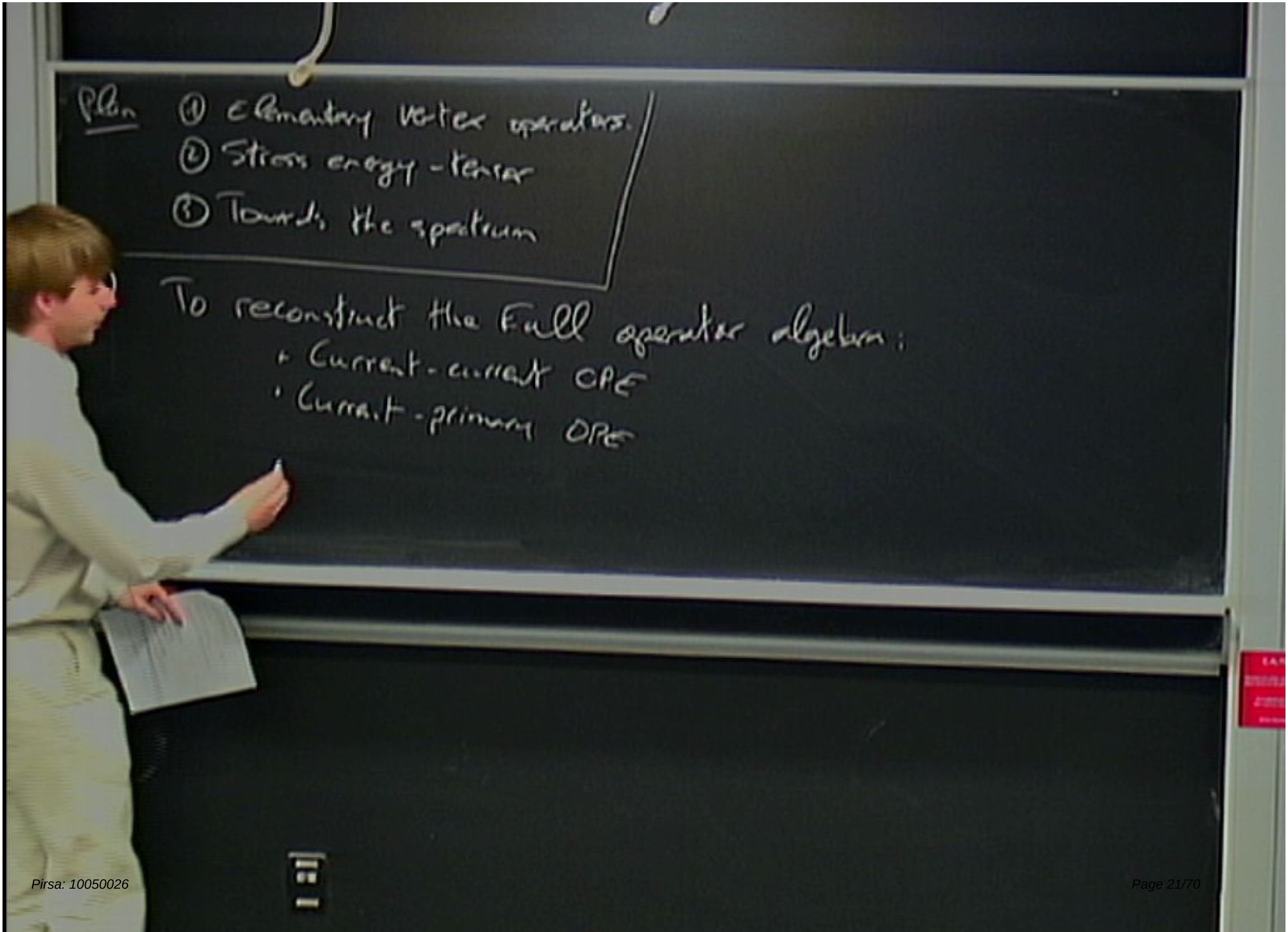
Hilbert space: $\{ :J \cdots :J :J \cdots :J \phi :: \}$

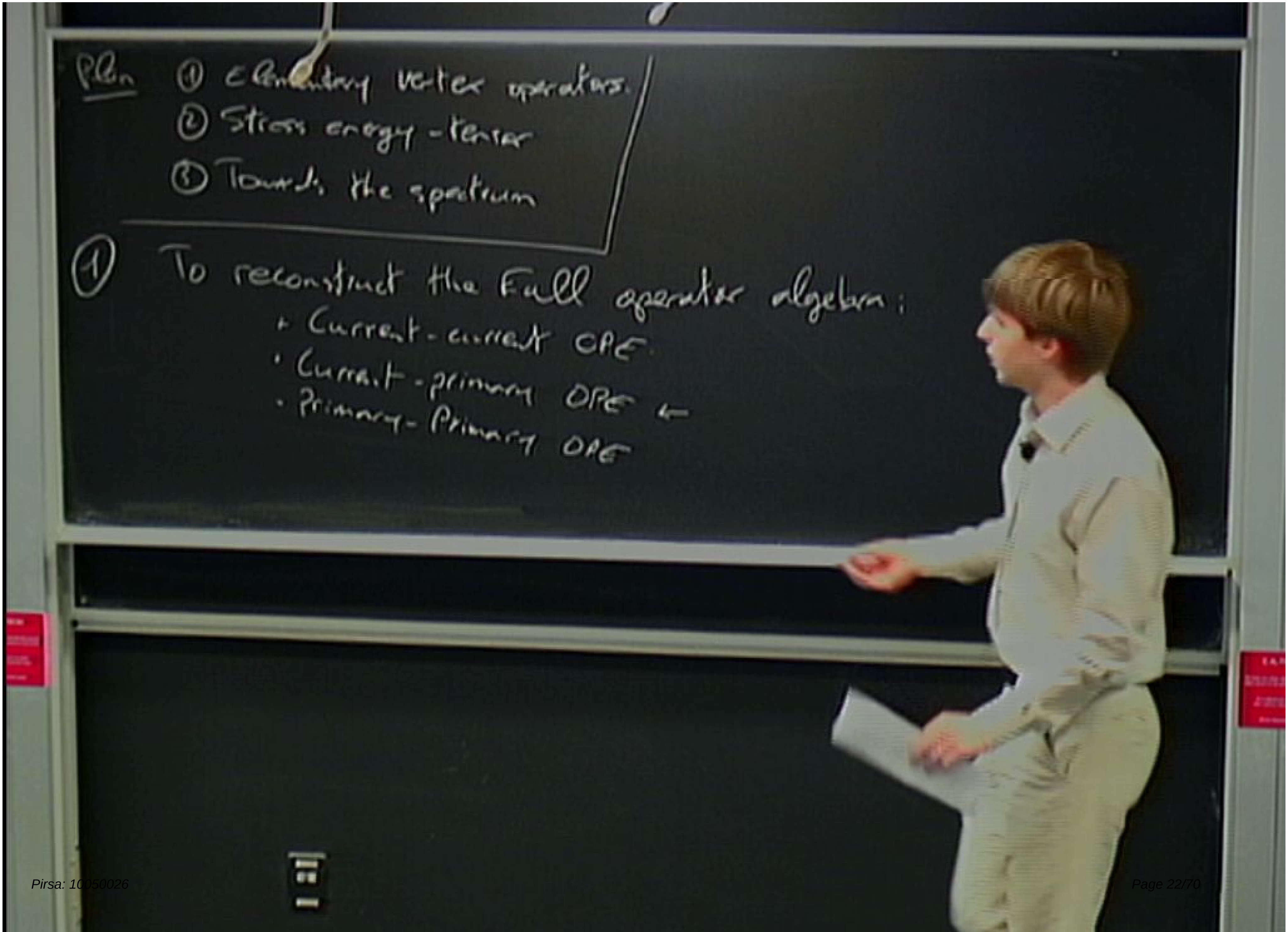
Plan

1. Elementary vertex operators
2. Stress energy-tensor
3.

Plan

1. Elementary vertex operators
2. Stress energy-tensor
3. Towards the spectrum


Plan
① Elementary vertex operators.
② Stress energy - tensor
③ Towards the spectrum
To reconstruct the full operator algebra:
• Current - current OPE
• Current - primary OPE

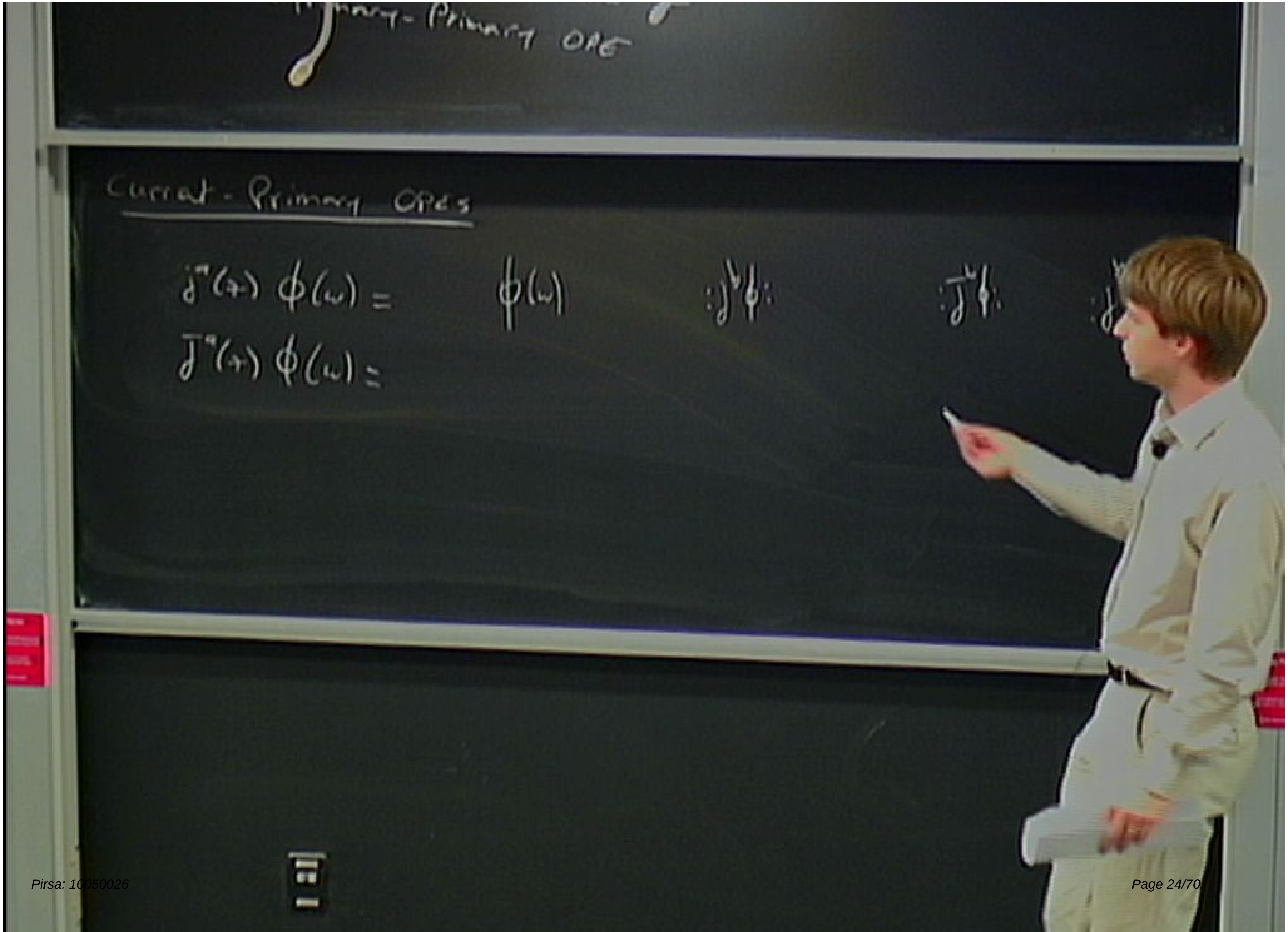

Plan
① Elementary vertex operators.
② Stress energy-tensor
③ Towards the spectrum
① To reconstruct the full operator algebra:
• Current-current OPE
• Current-primary OPE
• Primary-Primary OPE

Primary - Primary OPE

Current - Primary OPEs

$$J^a(z)\,\phi(w) =$$

$$\bar{J}^a(\bar{z})\,\phi(w) =$$

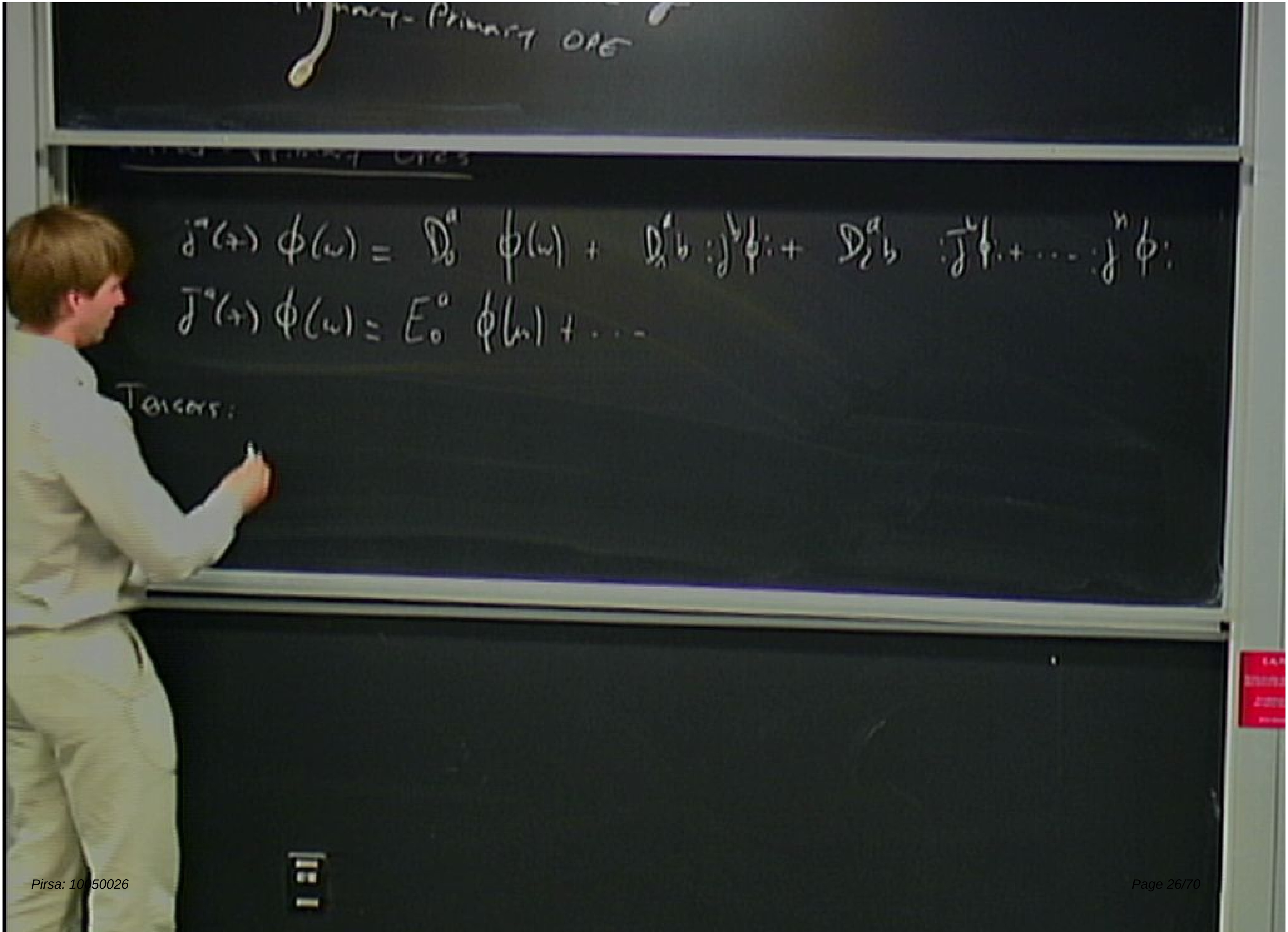

Primary - Primary OPE
Current - Primary OPEs
$J^a(z)\,\phi(w) =$
$\phi(w)$
$:J^b\phi:$
$:\bar{J}^b\phi:$
$\bar{J}^a(z)\,\phi(w) =$

Primary - Primary OPE

## Current - Primary OPEs

$$J^a(z)\,\phi(w) = \qquad \phi(w) \qquad :J^b\phi: \qquad :\bar{J}^b\phi: \qquad :J^n\phi:$$

$$\bar{J}^a(z)\,\phi(w) =$$


Primary-Primary OPE
Tensors:

Primary - Primary OPE

Primary - Primary OPEs

$$J^a(z)\,\phi(w) = D^a_0\,\phi(w) + D^a_{1b}\,:J^b\phi: + D^a_{2b}\,:J^bJ^b\phi: + \cdots :J^n\phi:$$

$$\bar{J}^a(\bar{z})\,\phi(w) = E^a_0\,\phi(w) + \cdots$$

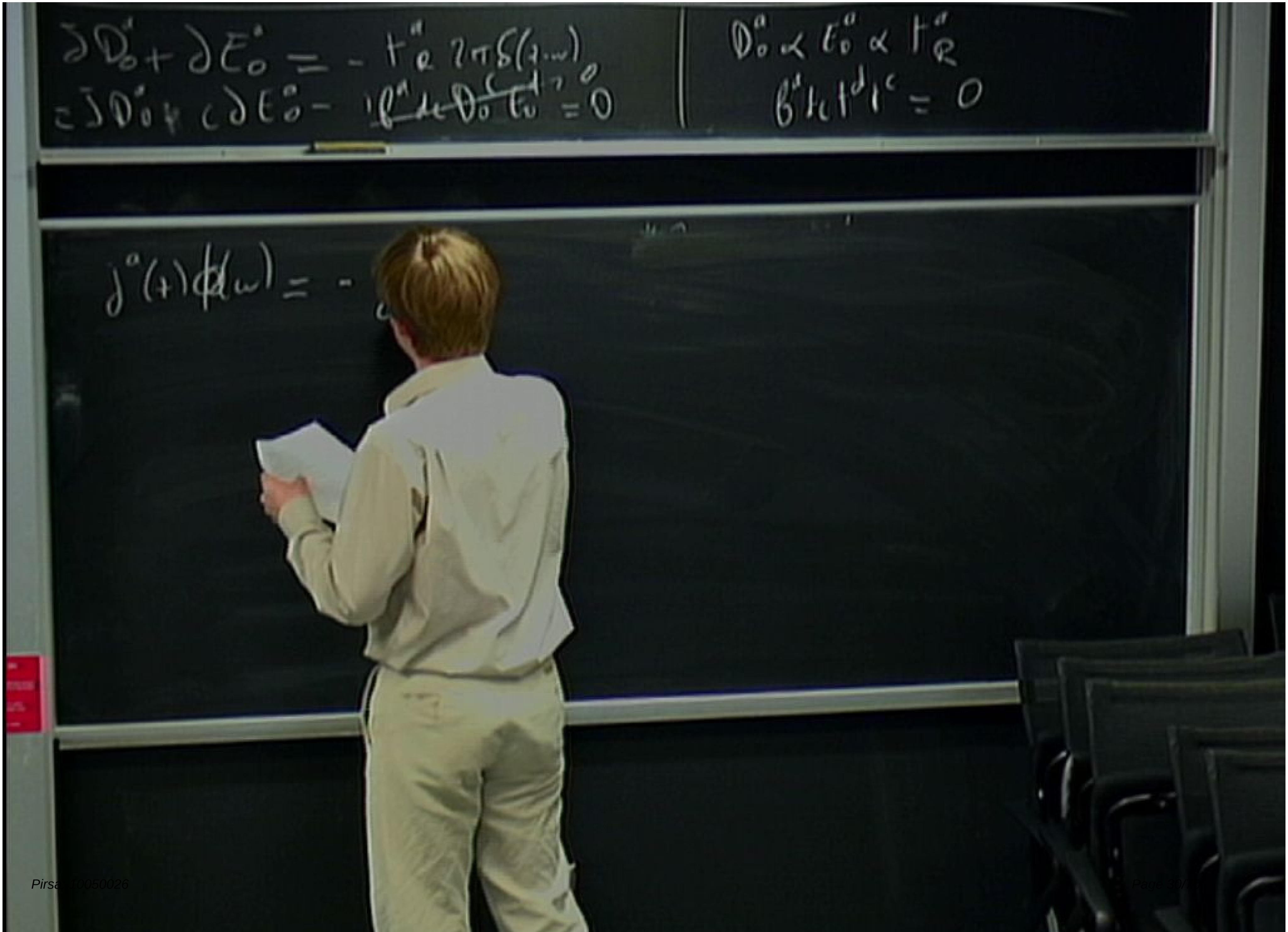
Tensors: $k^{ab}$, $f^{abc}$, $t^a_R$

Current conservation $\left(\partial j^a(z) + \bar{\partial} \bar{j}^a(\bar{z})\right) \phi(w, \bar{w}) =$

$$\delta g(z) = \epsilon_a(z)\, t^a g(z)$$

$$\Rightarrow \delta\phi(w) = \epsilon_a(z)\, t^a_R\, \phi(w)$$

<u>Maurer-Cartan</u>: $\left(\bar{c}\,\partial j^a(z) - c\,\bar{\partial} \bar{j}^a(z) - i f^a{}_{cd}\, :j^c \bar{j}^d:(z)\right)\phi(w) = 0$



$$\bar\partial D^a_0 + \partial E^a_0 = -t^a_R\, 2\pi\delta(z-w)$$

$$= \bar\partial D^a_0 + c\,\partial E^a_0 - \cancel{i f^a{}_{bc} D^b_0 E^c_0} = 0$$

$$D^a_0 \propto E^a_0 \propto t^a_R$$

$$f^a{}_{bc}\, t^b t^c = 0$$

$$j^a(z)\phi(w) = -\frac{c}{c+\bar c}\, t^a_R\, \frac{\phi(w)}{z-w} + i f^a{}_{bc}\, t^c_R\, \frac{\bar c}{(c+\bar c)^2} :j^b\phi:(w) \log|z-w|^2$$

$$+\, i f^a{}_{bc}\, t^c_R\, \frac{c}{(c+\bar c)^2} :\bar j^b\phi:(w)\, \frac{\bar z-\bar w}{z-w}$$

$$+ :j^a\phi:(w) + \ldots$$

$$\bar\partial D_0^a + \partial E_0^a = -t_R^a\, 2\pi\,\delta(z-w)$$

$$= \bar\partial D_0^a + c\,\partial E_0^a - i f^a{}_{bc} D_0^b E_0^c = 0$$

$$D_0^a \propto E_0^a \propto t_R^a$$

$$f^a{}_{bc}\, t^d t^c = 0$$

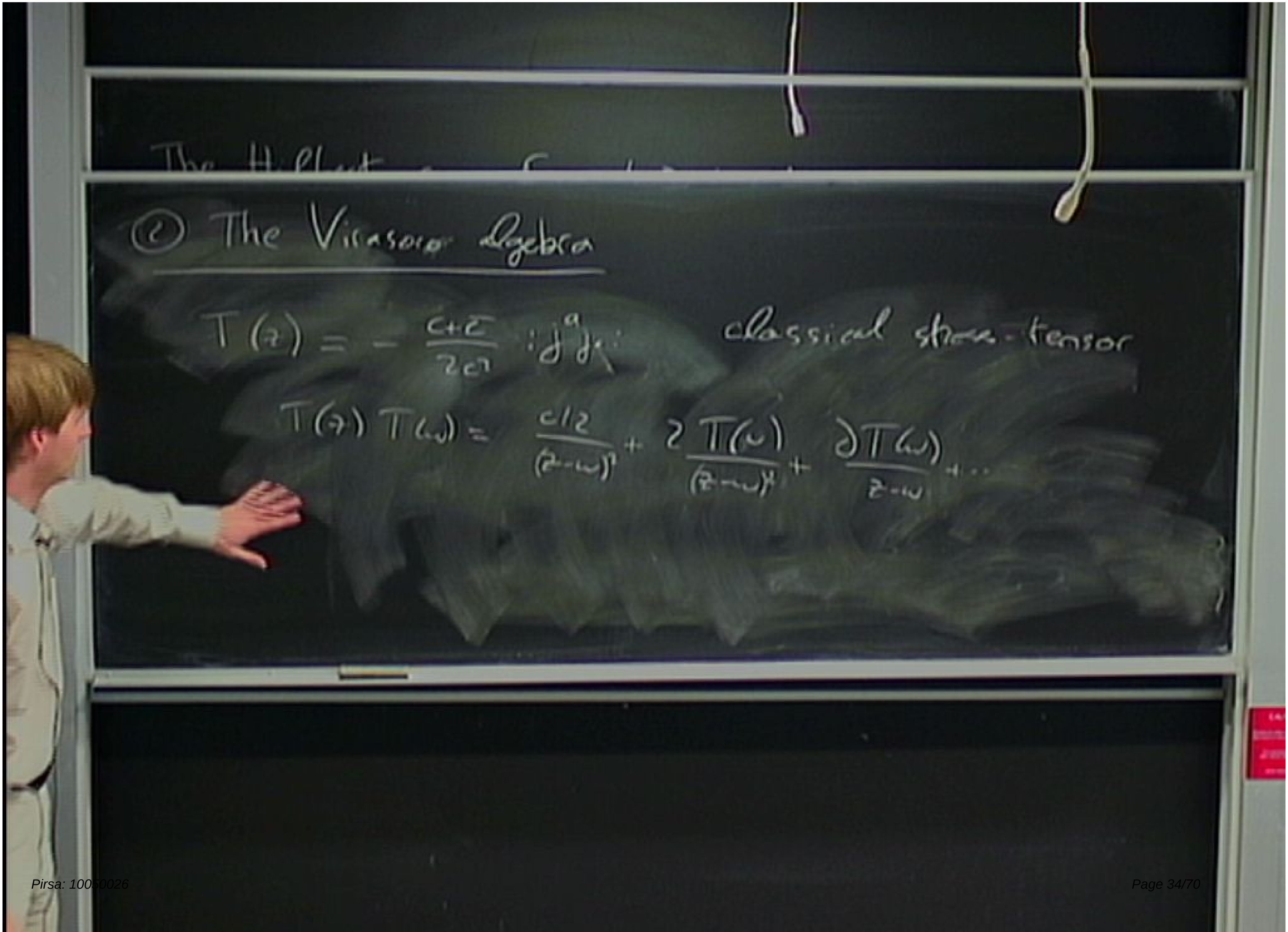
$$j^a(z)\phi(w) = -\frac{c}{c+\bar c}\, t_R^a\, \frac{\phi(w)}{z-w} + i f^a{}_{bc}\, t_R^c\, \frac{\bar c}{(c+\bar c)^2} :j^b\phi:(w)\, \log|z-w|^2$$

$$+ i f^a{}_{bc}\, t_R^c\, \frac{c}{(c+\bar c)^2} :\bar j^b\phi:(w)\, \frac{\bar z-\bar w}{z-w}$$

$$+ :j^a\phi:(w) + \ldots$$

The Hilbert away from WZW points

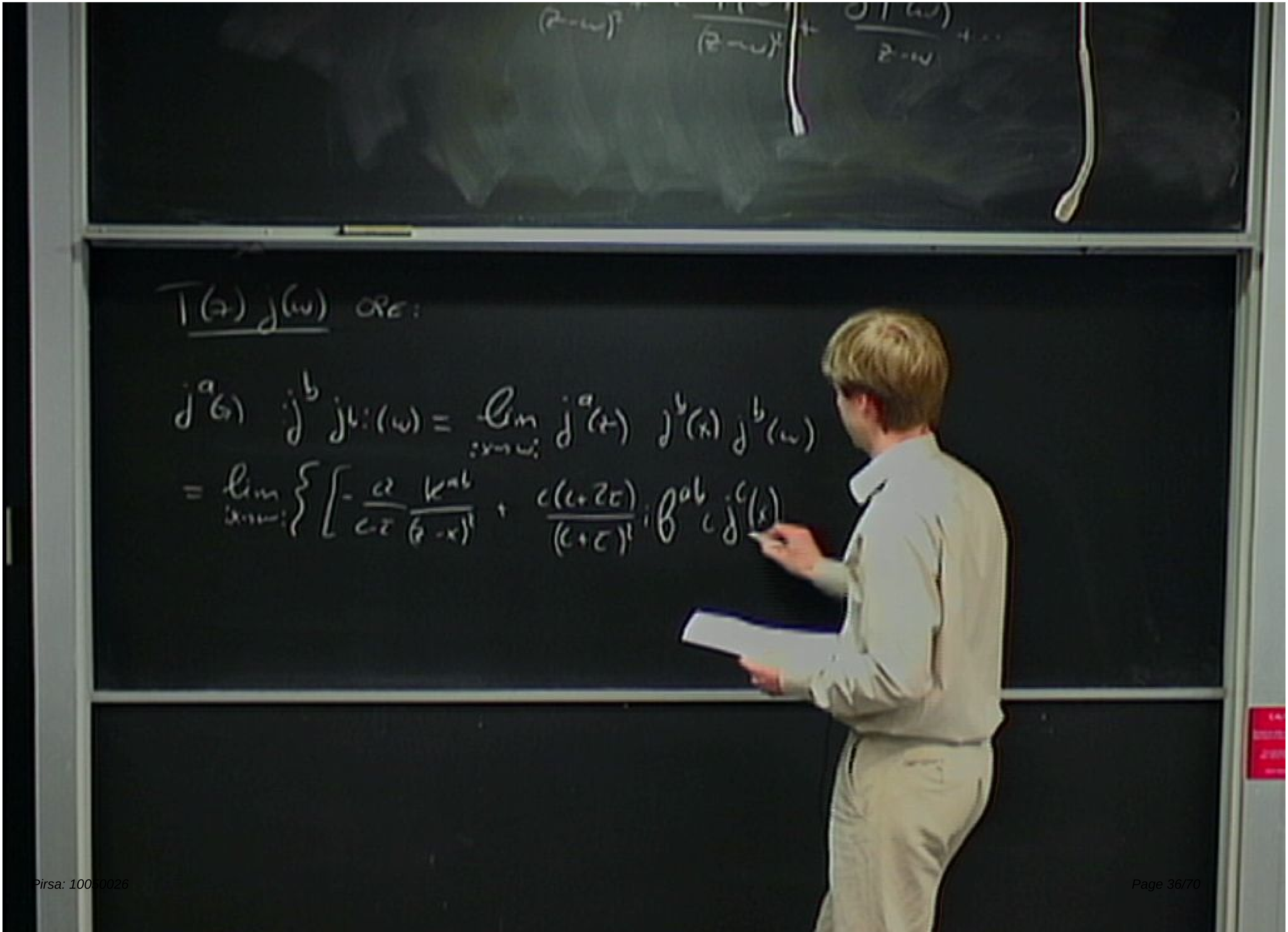
(2) The Virasoro algebra

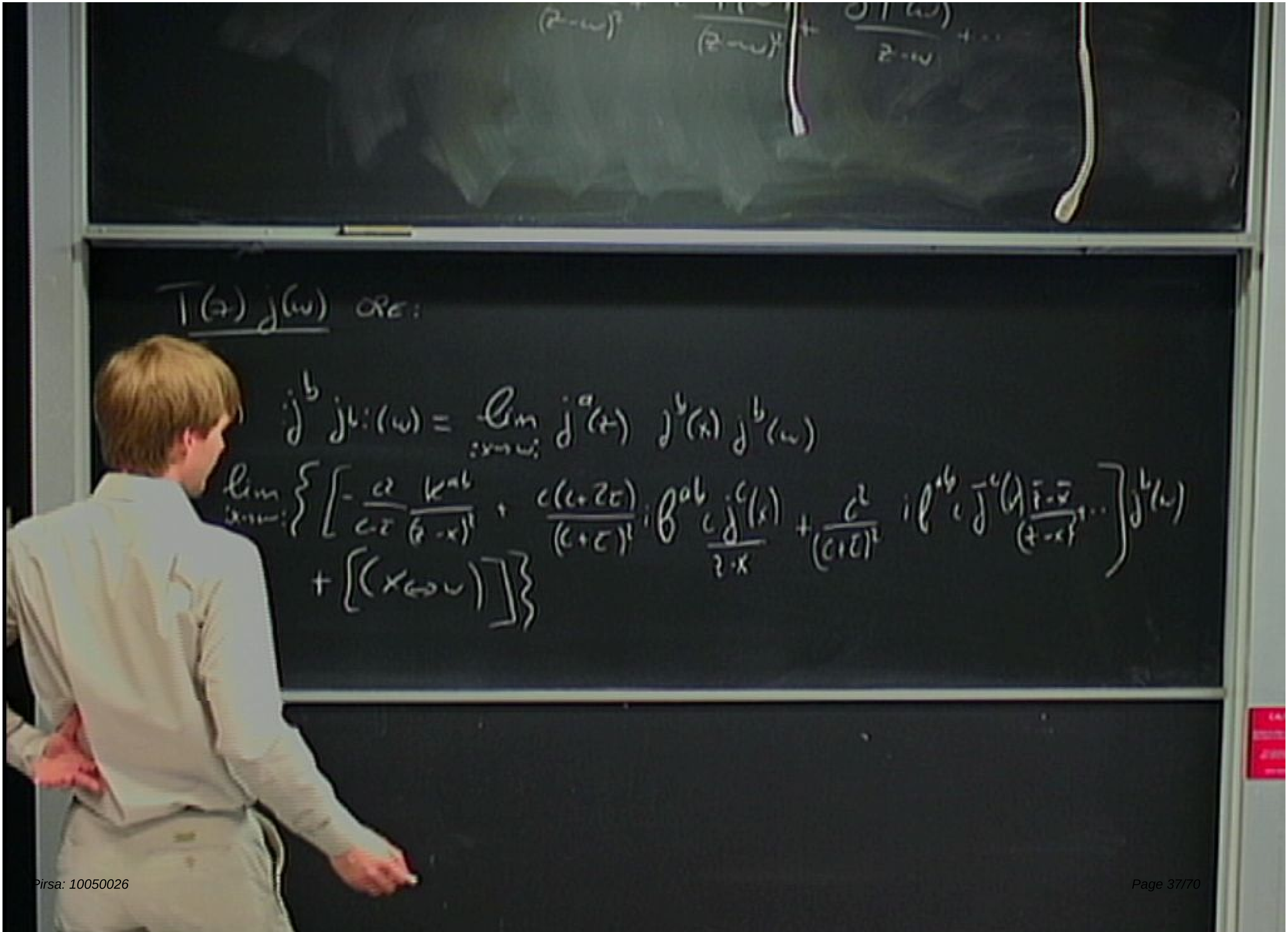

② The Virasoro Algebra
classical stress-tensor
T(z) T(w) = c/2 / (z-w)^4 + 2 T(w) / (z-w)^2 + ∂T(w) / (z-w) + ...

$$\frac{\ldots}{(z-w)^3} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$T(z)\, J(w)$ OPE:

$$J^a(z) \quad :J^b J^b:(w) = \lim_{y \to w} J^a(z)\; J^b(y)\, J^b(w)$$


T(z) j(w) OPE:
j^a(z) :j^b j^b:(w) = lim_{x→w} j^a(z) j^b(x) j^b(w)
= lim_{x→w} { [ - c/(c+τ) k^{ab}/(z-x)^2 + c(c+2τ)/((c+τ)^2) i f^{ab}_c j^c(x)


T(z) J(w) OPE:

$$\ldots (z-w)^4 \ldots \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

$T(z)\, j(w)$ OPE:

$$j^a(z)\, :j^b j^b:(w) = \lim_{x \to w} j^a(z)\, j^b(x)\, j^b(w)$$

$$= \lim_{x\to w} \Big\{ \Big[ -\frac{c^2}{c+\tilde{c}} \frac{k^{ab}}{(z-x)^2} + \frac{c(c+2\tilde{c})}{(c+\tilde{c})^2}\, i f^{abc} \frac{j^c(x)}{z-x} + \frac{c^2}{(c+\tilde{c})^2}\, i f^{abc}\, \bar{j}^c(\bar{x}) \frac{\bar{z}-\bar{x}}{(z-x)^2} + \dots \Big] j^b(w)$$

$$+ \big[ (x \leftrightarrow w) \big] \Big\}$$

$= -$

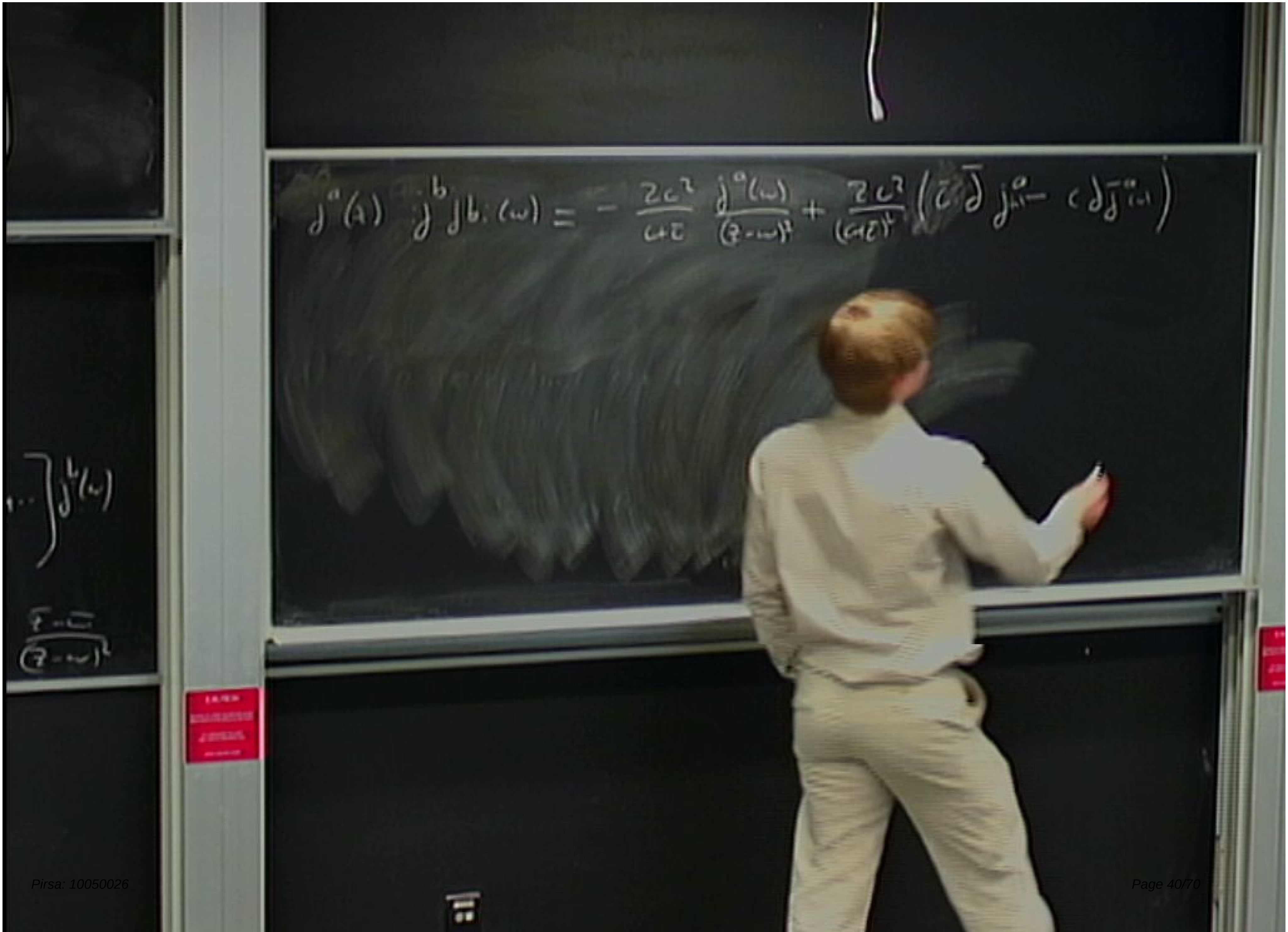
$$\frac{\quad}{(z-w)^2} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

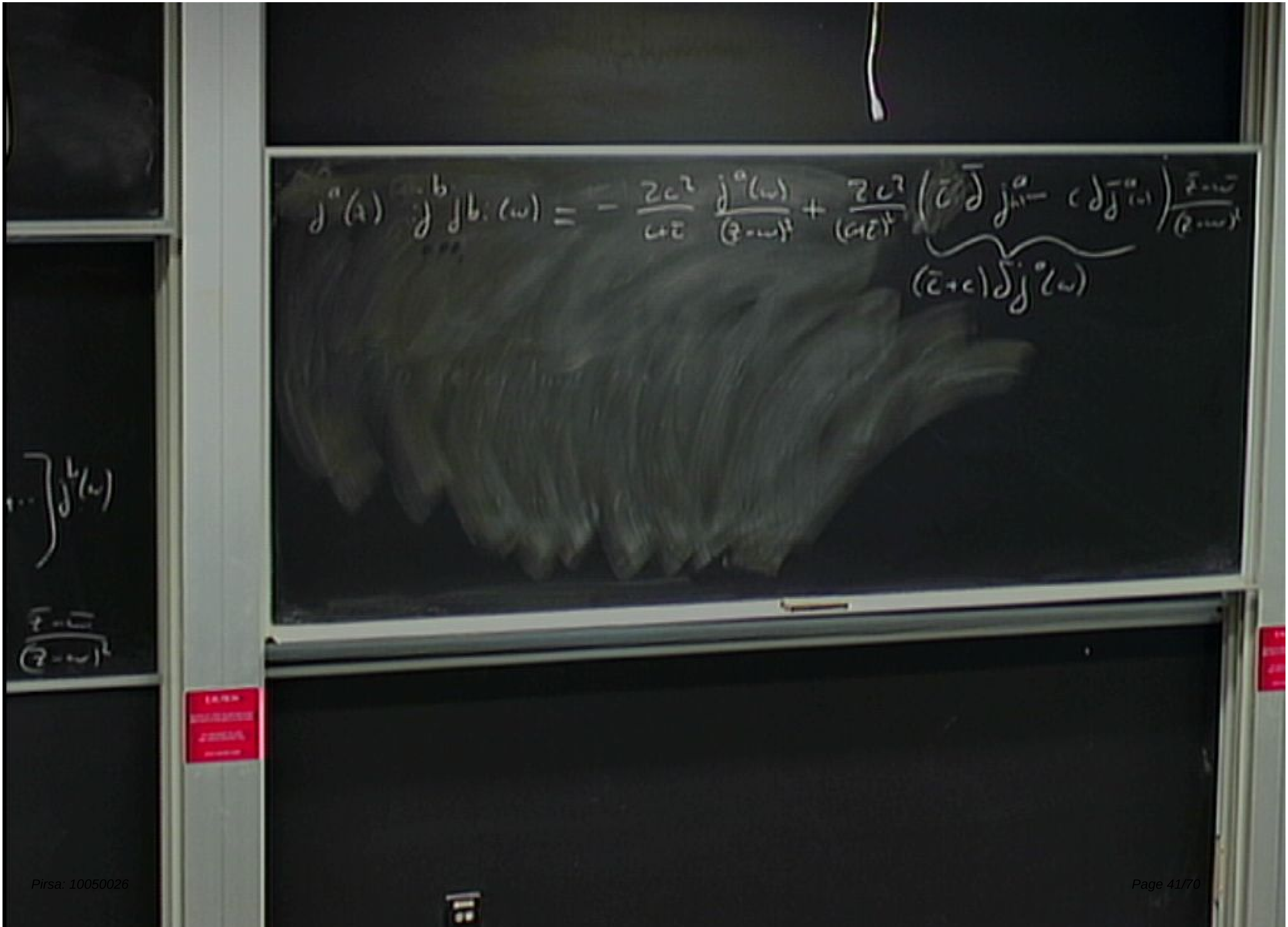
$T(z)\, j(w)$ OPE:

$$j^a(z)\; :j^b j^b:(w) = \lim_{:x\to w:} j^a(z)\; j^b(x)\, j^b(w)$$

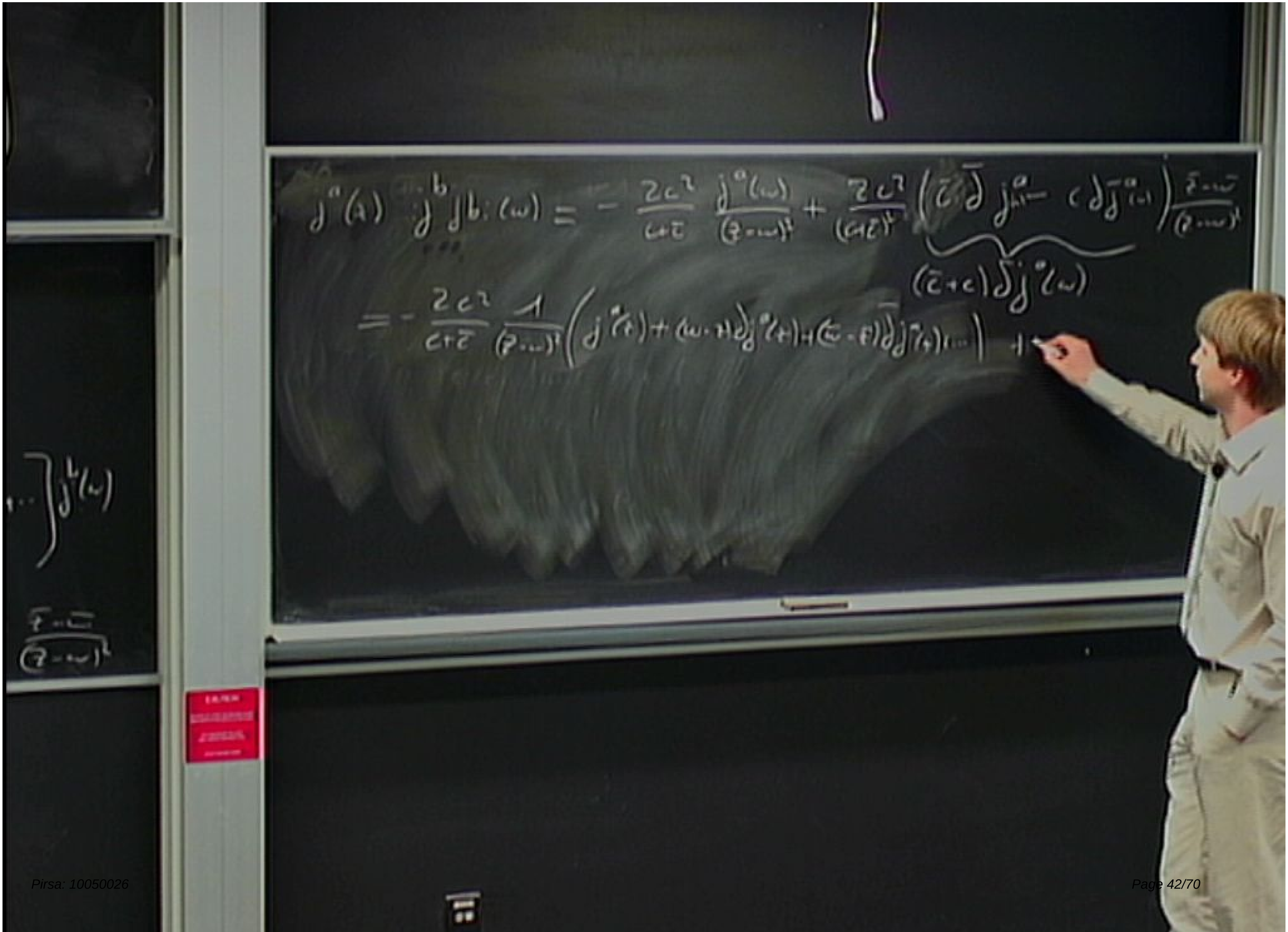
$$= \lim_{:x\to w:}\Big\{\Big[-\frac{c^2}{c+\tilde c}\frac{k^{ab}}{(z-x)^2} + \frac{c(c+2\tilde c)}{(c+\tilde c)^2}\, i f^{ab}{}_c \frac{j^c(x)}{z-x} + \frac{c^2}{(c+\tilde c)^2}\, i f^{ab}{}_c\, \bar j^c(x)\frac{\bar z-\bar x}{(z-x)^2}+\dots\Big] j^b(w)$$
$$+ \Big[(x\leftrightarrow w)\Big]\Big\}$$

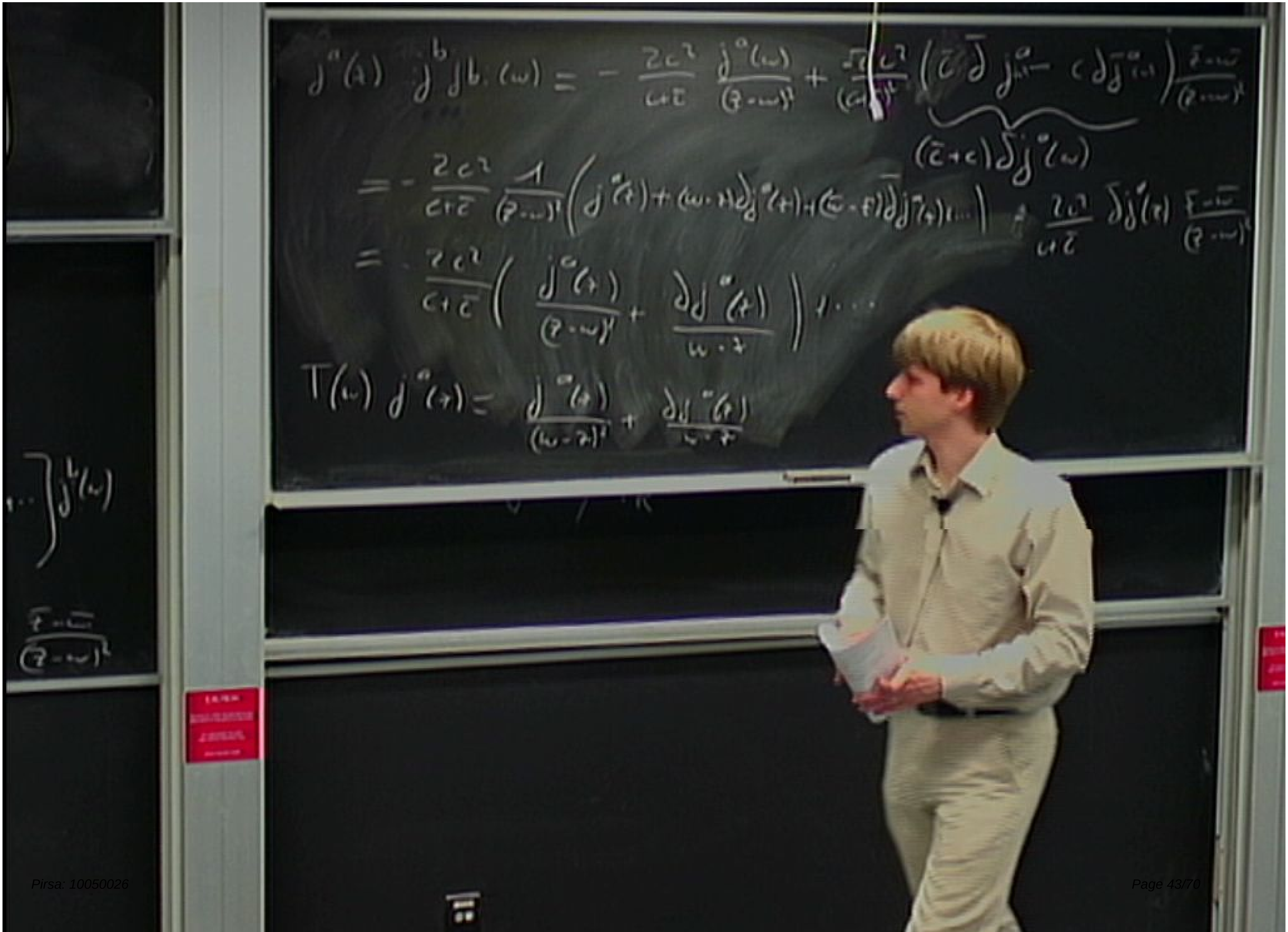
$$= -\frac{2c^2}{c+\tilde c}\frac{j^a(w)}{(z-w)^2} + \frac{2c(c+2\tilde c)}{(c+\tilde c)^2}\, i f^a{}_{bc} \frac{:j^c j^b:(w)}{z-w} + \frac{2c^2}{(c+\tilde c)^2}\, i f^a{}_{bc} :\bar j^c j^b:(w)$$





$j^a(z)\, j^b j^b{:}(w) = -\frac{2c^2}{c+\bar c}\frac{j^a(w)}{(z-w)^2} + \frac{2c^2}{(c+\bar c)^2}\big(\underbrace{\bar c\,\bar\partial j^a - c\,\partial \bar j^a(w)}_{(\bar c + c)\partial j^a(w)}\big)\frac{\bar z-\bar w}{(z-w)^2}$


$j^a(z)\, :j^b j^b:(w) = -\frac{2c^2}{c+\bar c}\frac{j^a(w)}{(z-w)^2} + \frac{2c^2}{(c+\bar c)^2}\left(\bar c\,\bar\partial j^a(w) - c\,\partial j^a(w)\right)\frac{\bar z-\bar w}{(z-w)^2}$
$(\bar c + c)\,\partial j^a(w)$
$= -\frac{2c^2}{c+\bar c}\frac{1}{(z-w)^2}\left(j^a(z) + (w-z)\partial j^a(z) + (\bar w-\bar z)\bar\partial j^a(z) + \dots\right) +$
$\dots]\, j^b(w)$
$\frac{\bar z-\bar w}{(z-w)^2}$


$T(w)\, j^a(z) = \frac{j^a(z)}{(w-z)^2} + \frac{\partial j^a(z)}{w-z}$

$$\frac{\quad}{c+\bar{c}}\left(\frac{j^a(z)}{(z-w)^2}+\frac{\partial j^a(z)}{w-z}\right)+\cdots$$

$$T(w)\, j^a(z) = \frac{j^a(z)}{(w-z)^2} + \frac{\partial j^a(z)}{w-z}$$

Virasoro

$j^a$ : Primary of dim 1

$\ldots\Big]\, j^b(w)$

$$\frac{\bar{z}-\bar{w}}{(z-w)^2}$$

$T(z)\, T(w)$ OPE

$$T(w)\, J^a(z) = \frac{J^a(z)}{(w-z)^2} + \frac{\partial J^a(z)}{w-z}$$

Virasoro

$J^a$ : Primary of dim 1

$T(z)\,T(w)$ OPE

$T(z)$

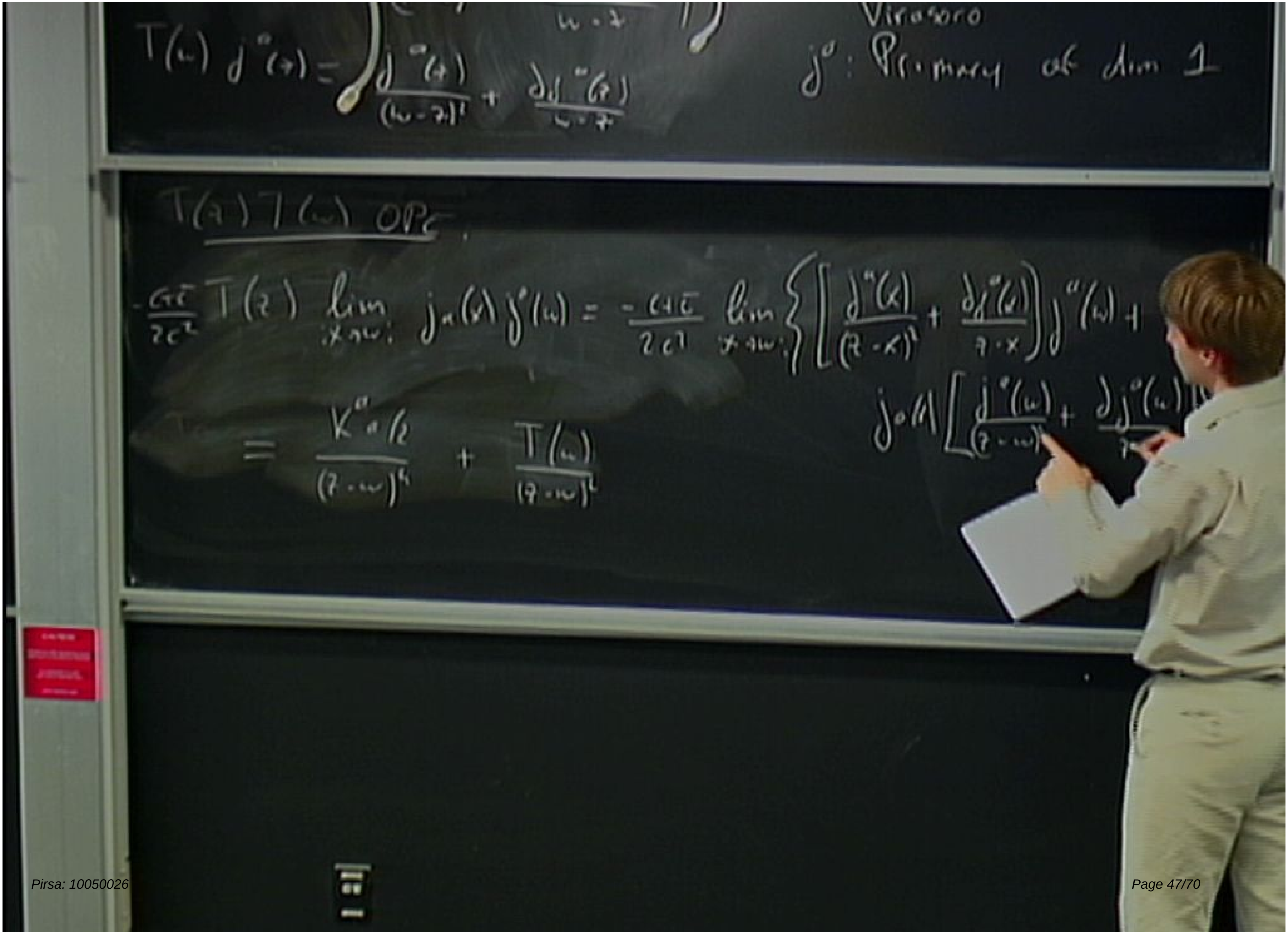
Virasoro

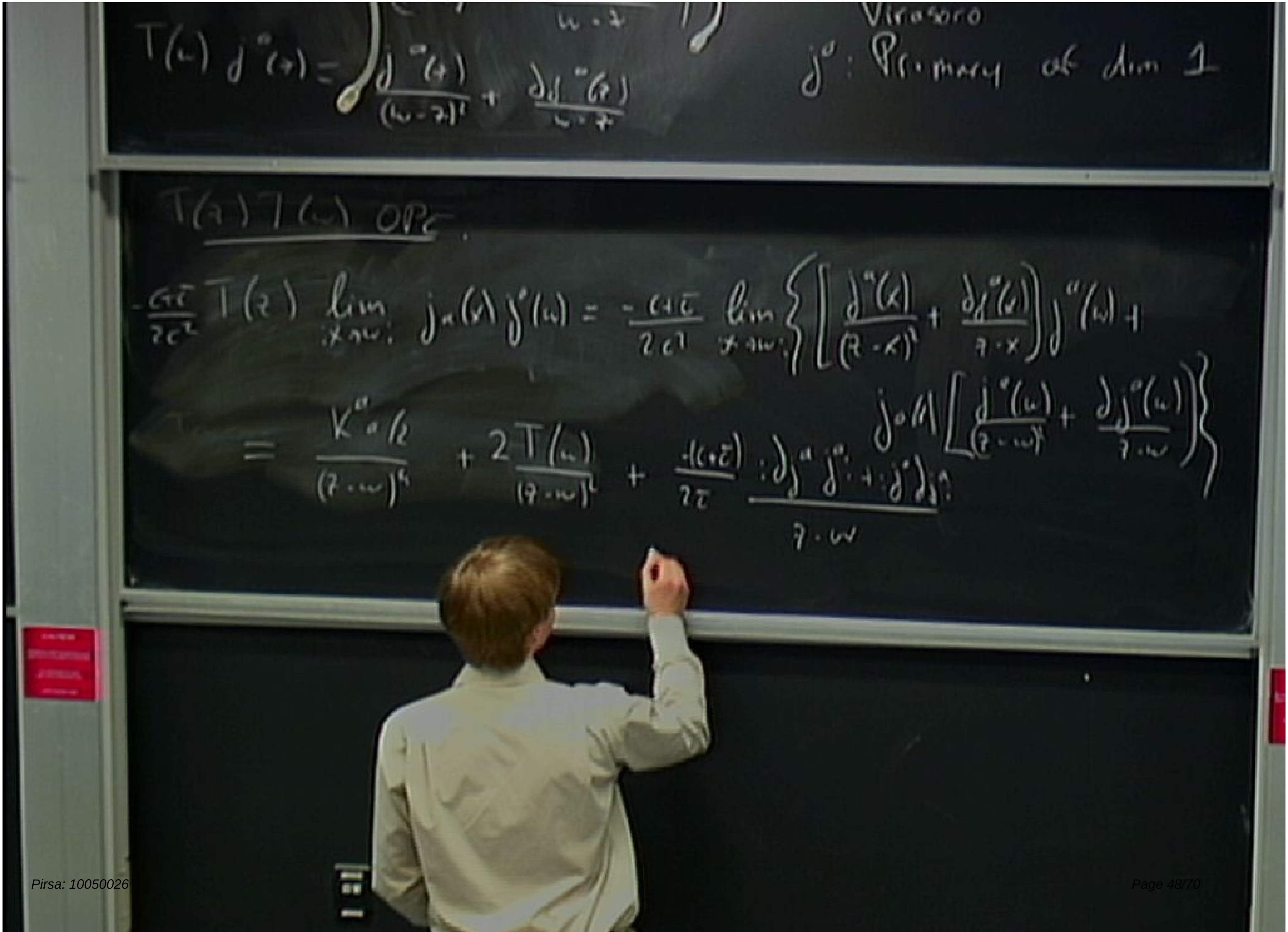
$$T(w)\, j^a(z) = \frac{j^a(z)}{(w-z)^2} + \frac{\partial j^a(z)}{w-z}$$

$j^a$ : Primary of dim 1

T(z) T(w) OPE

$$-\frac{c+\bar{c}}{2c^2}\, T(z) \lim_{x\to w} j_a(x)\, j^a(w) = -\frac{c+\bar{c}}{2c^2} \lim_{x\to w} \left\{ \left[ \frac{j^a(x)}{(z-x)^2} + \frac{\partial j^a(x)}{z-x} \right] j^a(w) + \right.$$


Virasoro
j^a : Primary of dim 1
T(z) T(w) OPE


Virasoro
j^a : Primary of dim 1
T(z) T(w) OPE

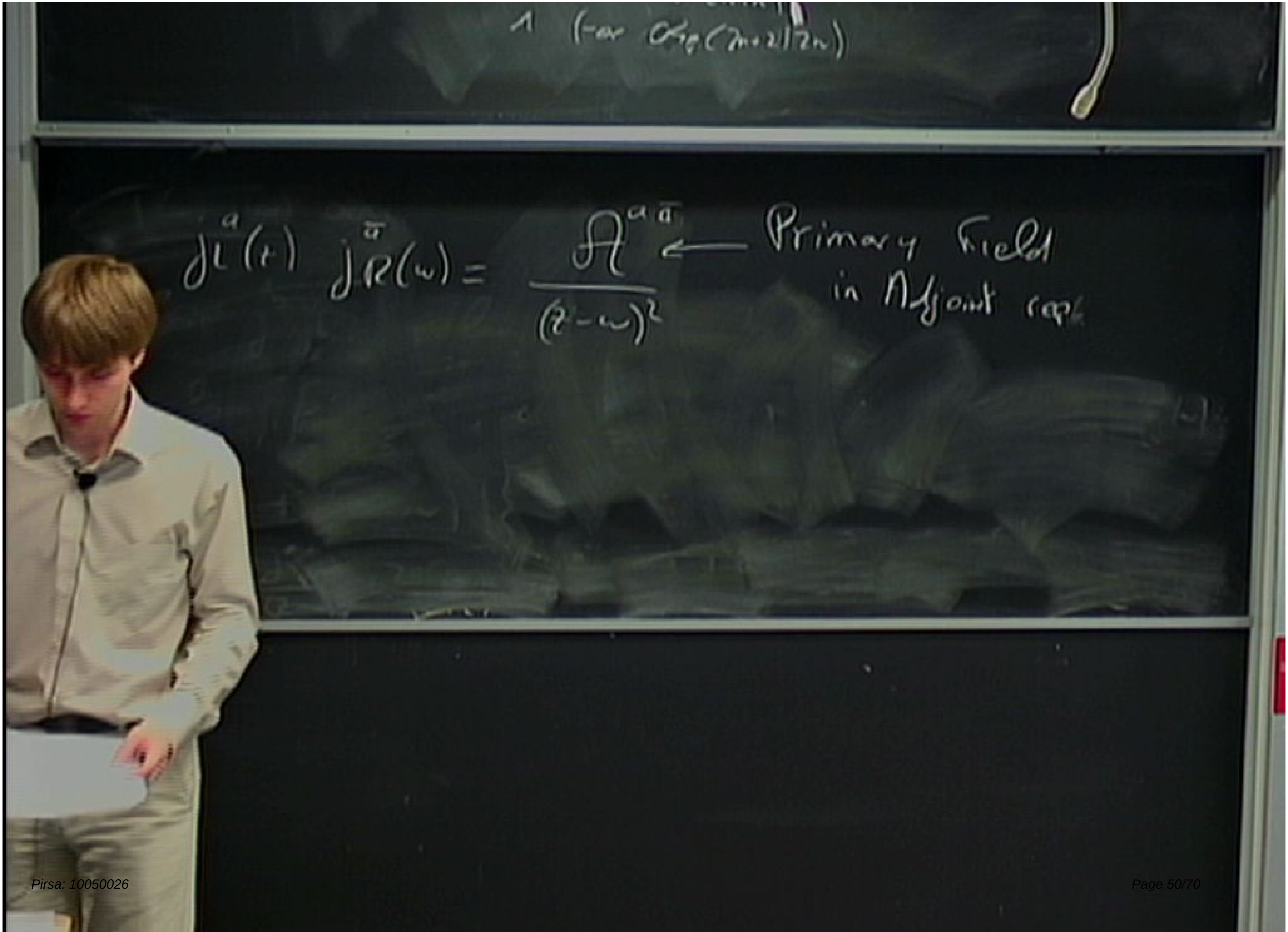
$$\ldots)\, j^a(z) = \oint \frac{j^a(z)}{(w-z)^2} + \frac{\partial j^a(z)}{w-z}$$

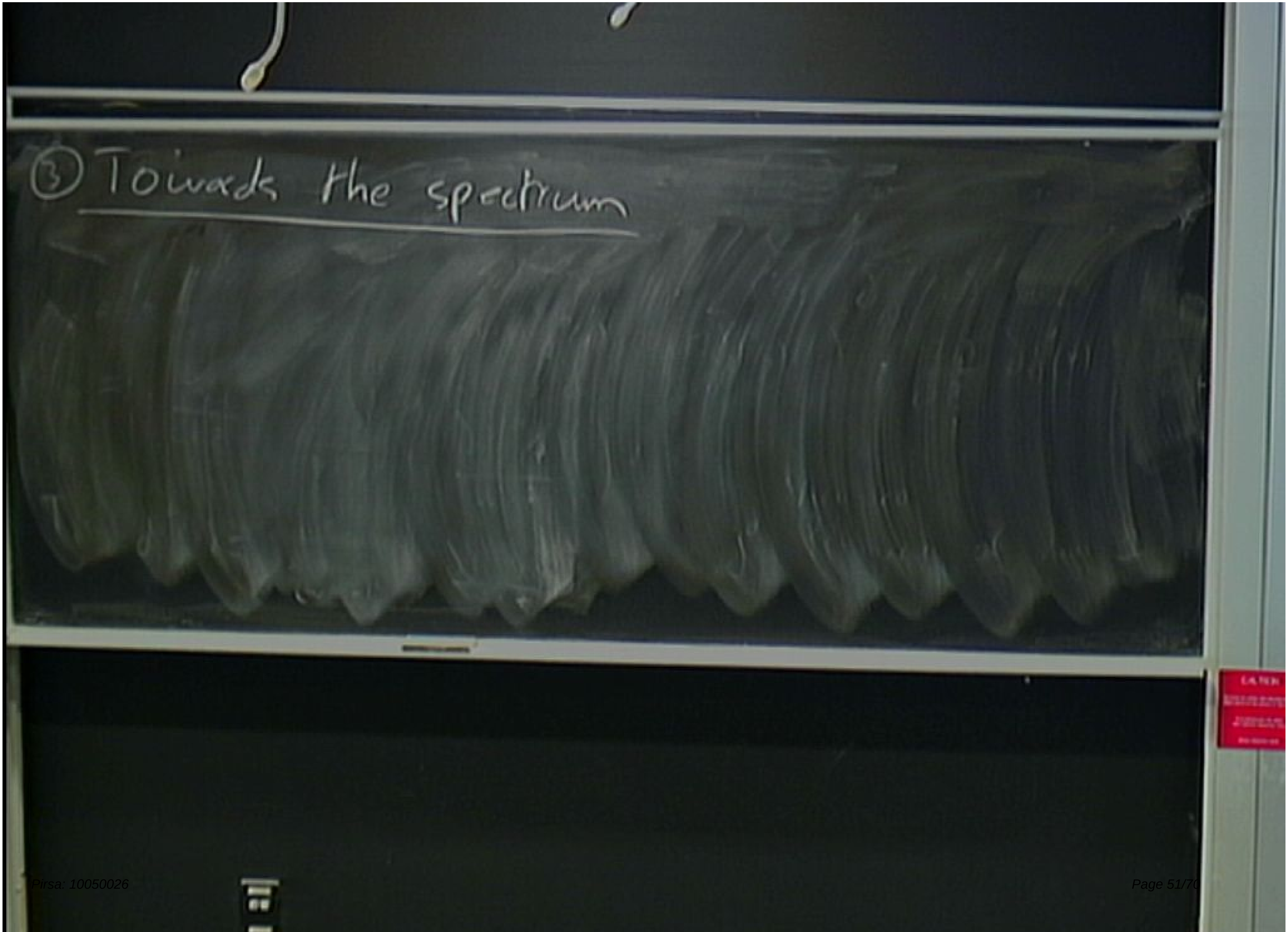
$j^a$ : primary of dim 1

## $T(z)T(w)$ OPE

$$= T(z) \lim_{x\to w} :j_a(x) j^a(w): = \frac{-(c+\tilde{c})}{2c^2} \lim_{x\to w} \left\{ \left[ \frac{j^a(x)}{(z-x)^2} + \frac{\partial j^a(x)}{z-x} \right] j^a(w) + j_a(x) \left[ \frac{j^a(w)}{(z-w)^2} + \frac{\partial j^a(w)}{z-w} \right] \right\}$$

$$= \frac{k^a{}_a/2}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \underbrace{\frac{-(c+\tilde{c})}{2\tilde{c}} \frac{:\partial j^a j^a: + :j^a \partial j_a:}{z-w}}_{\frac{\partial T}{z-w}}$$


for Osp(2n+2|2n)
$j_L^a(z)\, j_R^{\bar a}(w) = \frac{A^{a\bar a}}{(z-w)^2}$
Primary field
in Adjoint rep


③ Towards the spectrum

## ③ Towards the spectrum

$$T(z)\, X(w) = \sum_n \frac{L_n X(w)}{(z-w)^{2+n}}$$

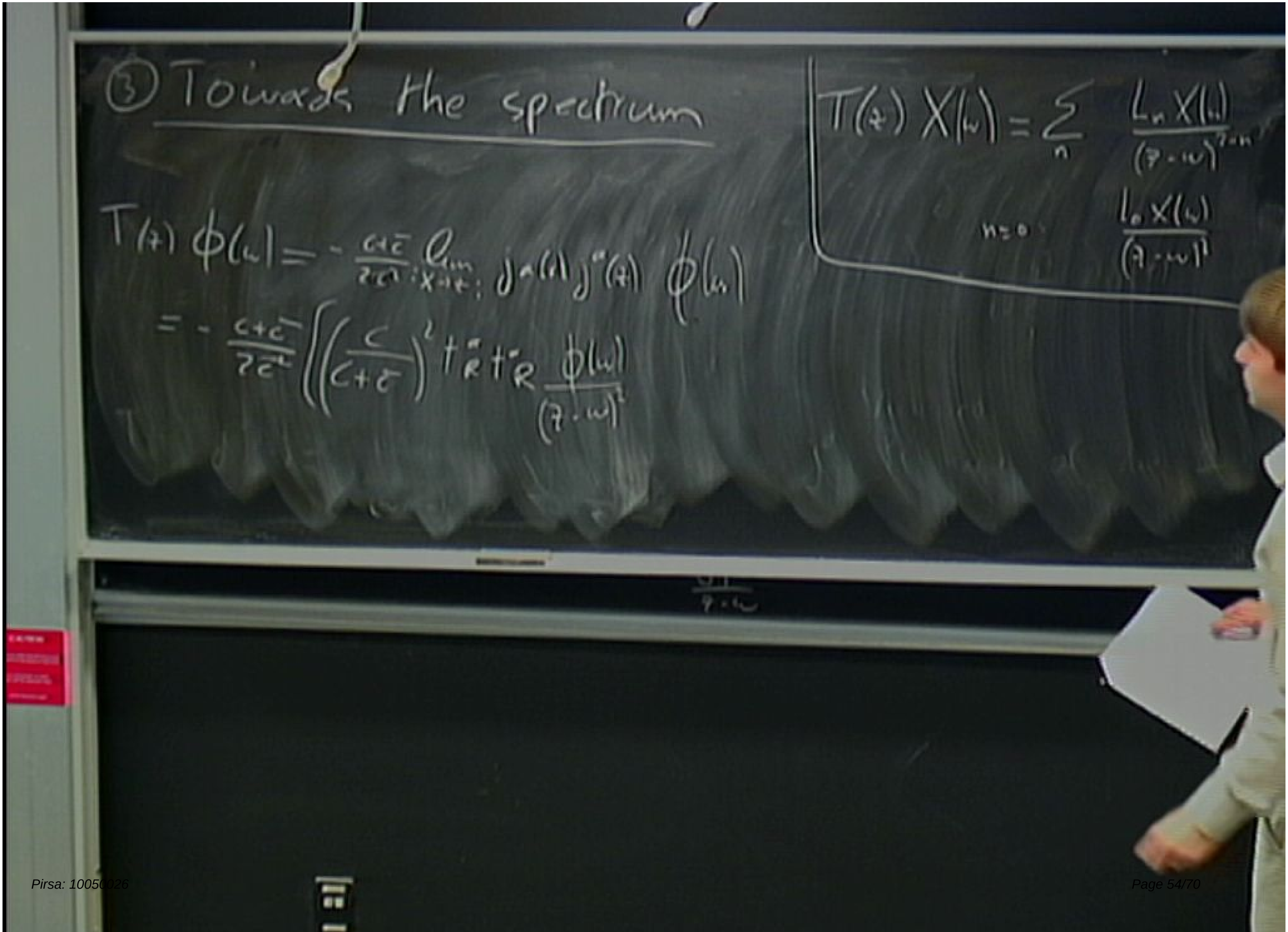
$n=0$: $\dfrac{L_0 X(w)}{(z-w)^2}$

## 3 Towards the spectrum

$$T(z)\,\phi(w) = -\,[\text{illegible}]\lim_{x\to z} :j^\mu(x)\, j^\mu(z):\ \phi(w)$$

$$T(z)\,X(w) = \sum_n \frac{L_n X(w)}{(z-w)^{2+n}}$$

$n=0$: $$\frac{L_0\, X(w)}{(z-w)^2}$$


③ Towards the spectrum
$T(z)\,X(w) = \sum_n \frac{L_n X(w)}{(z-w)^{2+n}}$
$n=0: \quad \frac{L_0 X(w)}{(z-w)^2}$
$T(z)\,\phi(w) = -\frac{c+\bar{c}}{2c^2} \lim :X+\bar{X}: j^a(z) j^a(z)\ \phi(w)$
$= -\frac{c+\bar{c}}{2c^2} \left( \left(\frac{c}{c+\bar{c}}\right)^2 t^a_R t^a_R \right) \frac{\phi(w)}{(z-w)^2}$

## ③ Towards the spectrum

$$T(z)\,\phi(w) = -\frac{c+\bar c}{2c\bar c}\lim_{x\to z} :j^a(x)\,j^a(z):\,\phi(w)$$

$$= -\frac{c+\bar c}{2c\bar c}\left(\left(\frac{c}{c+\bar c}\right)^2 t^a_R t^a_R \frac{\phi(w)}{(z-w)^2} + 2\,\frac{c}{c+\bar c}\,\frac{t^a_R :j^a\psi:(w)}{z-w}\right)$$

$$T(z)\,X(w) = \sum_n \frac{L_n X(w)}{(z-w)^{2+n}}$$

$n=0$: $$\frac{L_0 X(w)}{(z-w)^2}$$

$$= \frac{1}{2}\beta^2 + t^a_R \frac{\phi(w)}{(z-w)^2} + \frac{1}{2} t^a_{R\,ij} \partial^a \frac{\phi_i(w)}{z-w} + \ldots$$

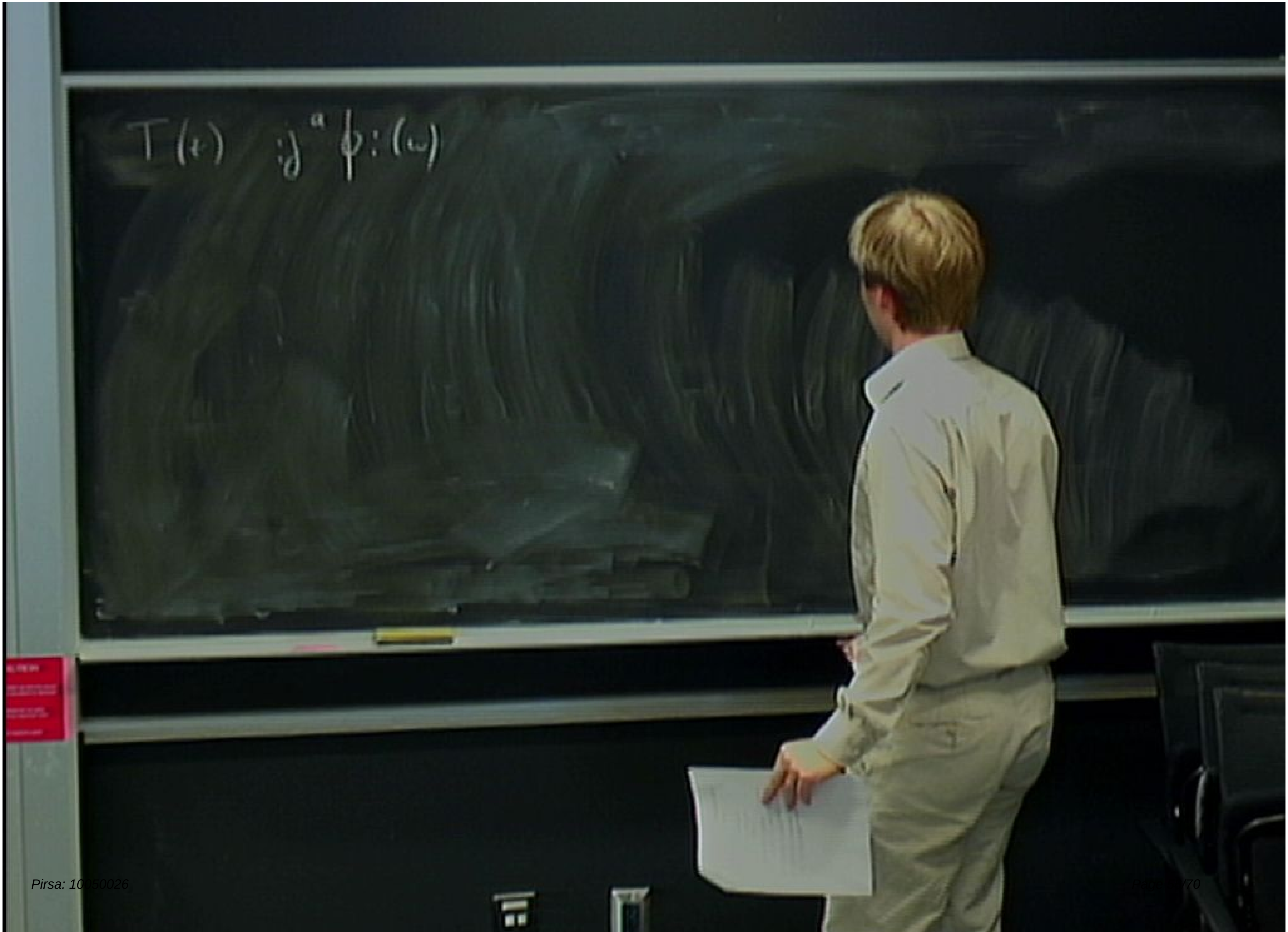
$L_{n>0}\,\phi = 0 \Rightarrow \phi$ Virasoro primaries

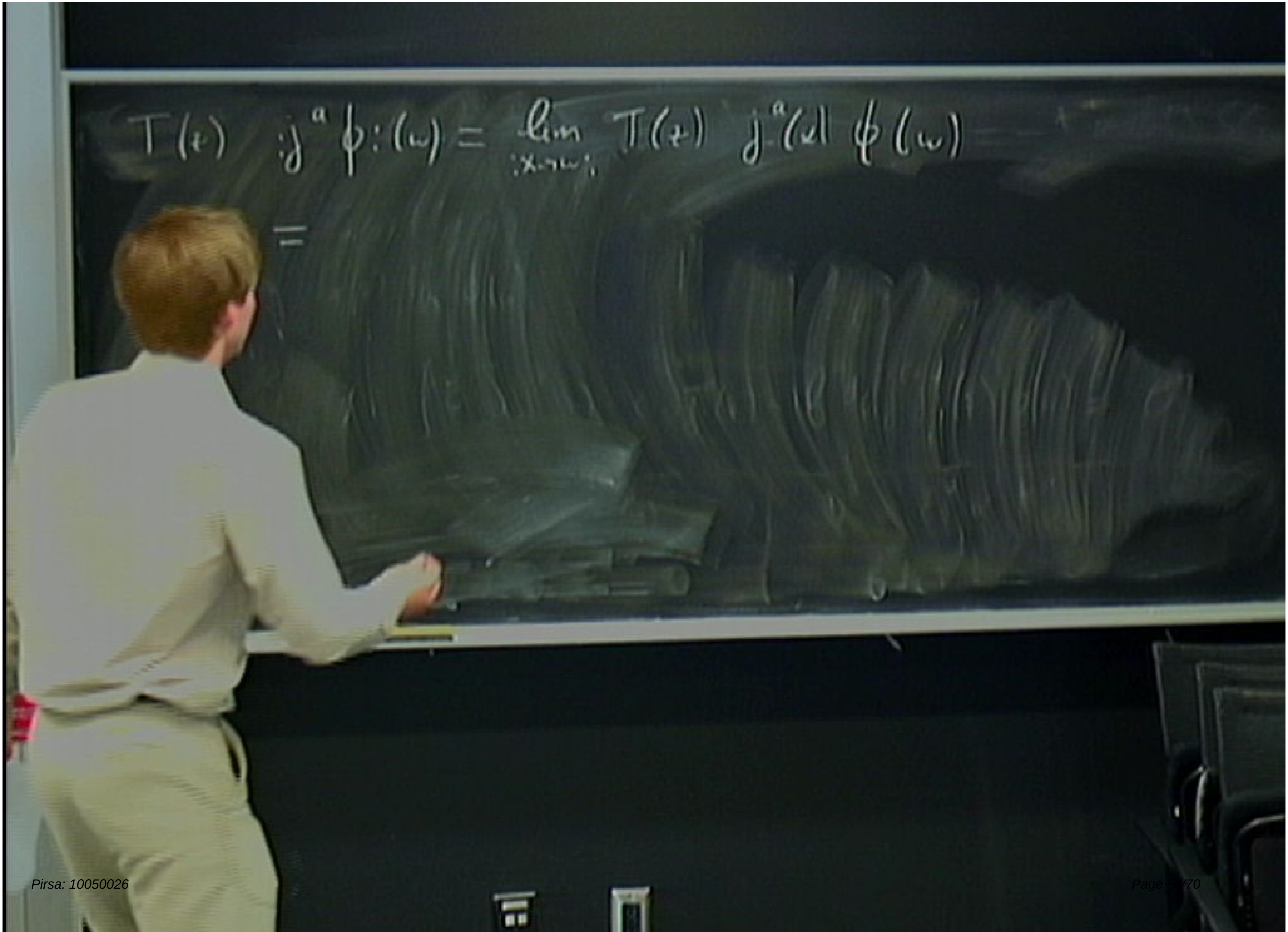
$$\Delta_\phi = \frac{\beta^2}{2} c^{(2)}_R \qquad \partial\phi = \frac{1}{2} t^a_{R\,ij} J^a \phi_i$$

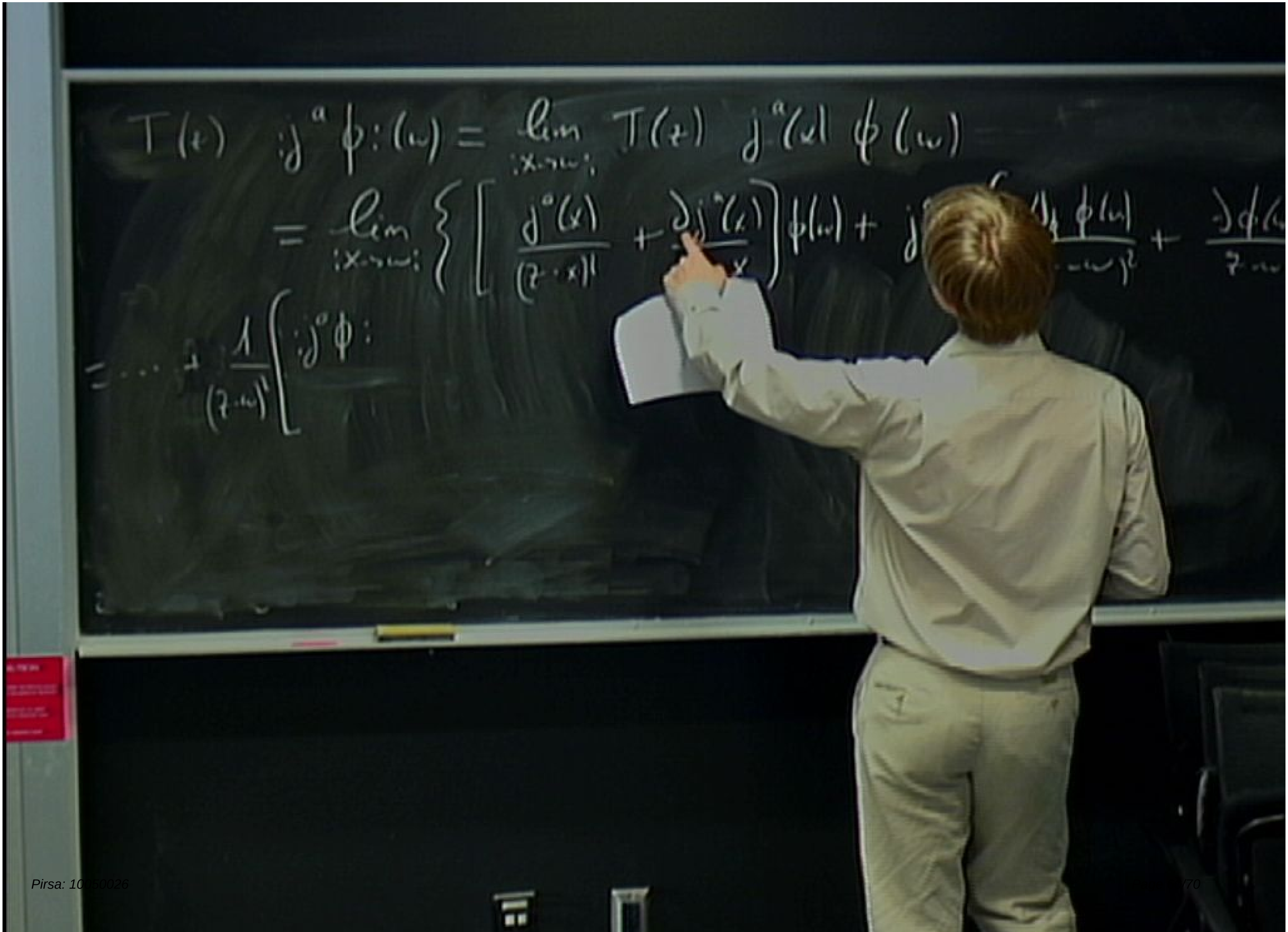
$$= \frac{1}{2}\beta^2 t^a_R t^a_R \frac{\phi(w)}{(z-w)^2} + \frac{1}{c} t^a_R j^a \frac{\phi_i(w)}{z-w} + \dots$$

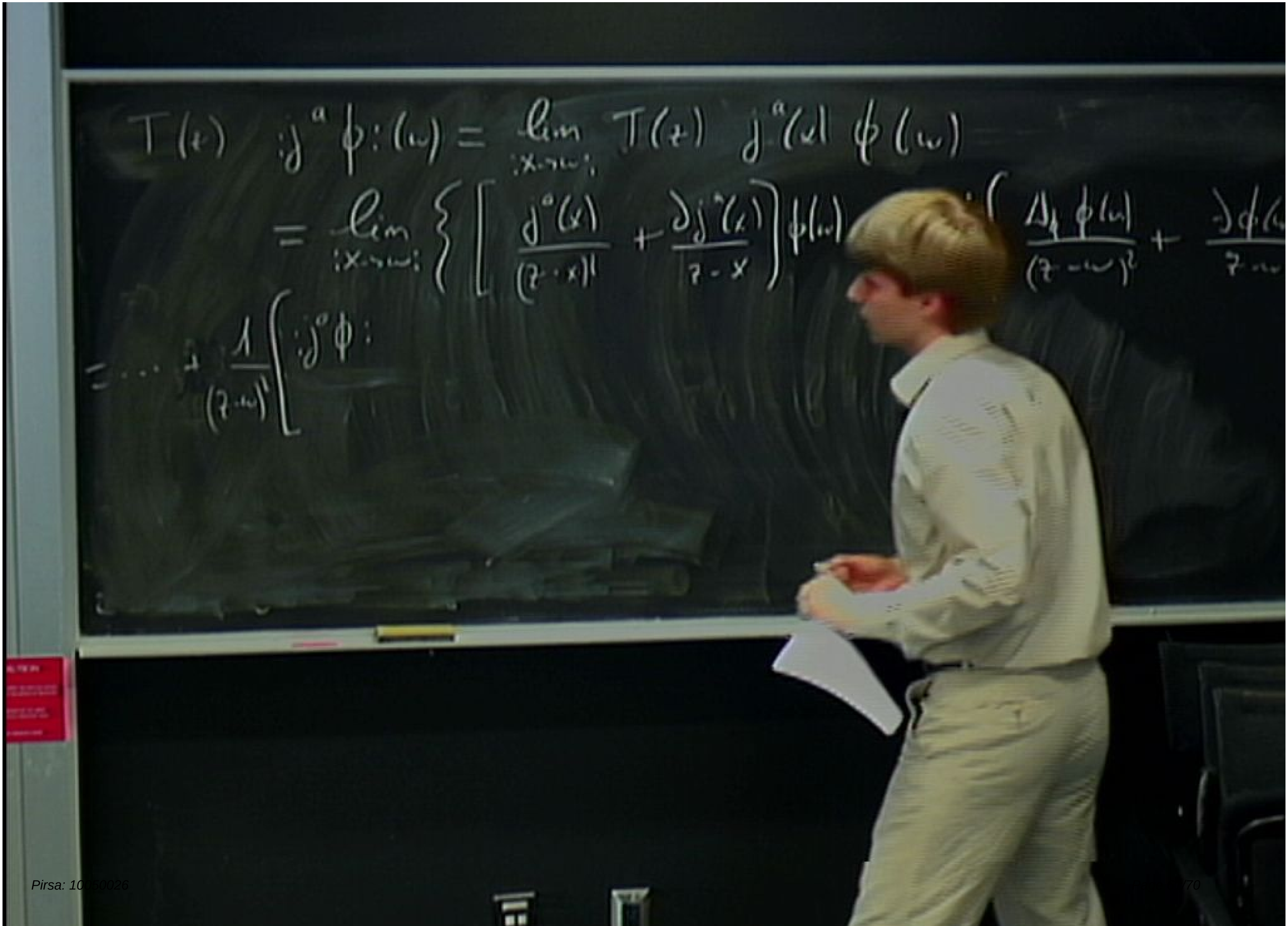
$L_{n>0}\,\phi = 0 \Rightarrow \phi$ Virasoro primaries

$$\Delta_\phi = \frac{\beta^2}{2} c^{(2)}_R \qquad \partial\phi = \frac{1}{c} t^a_R\, j^a \phi$$

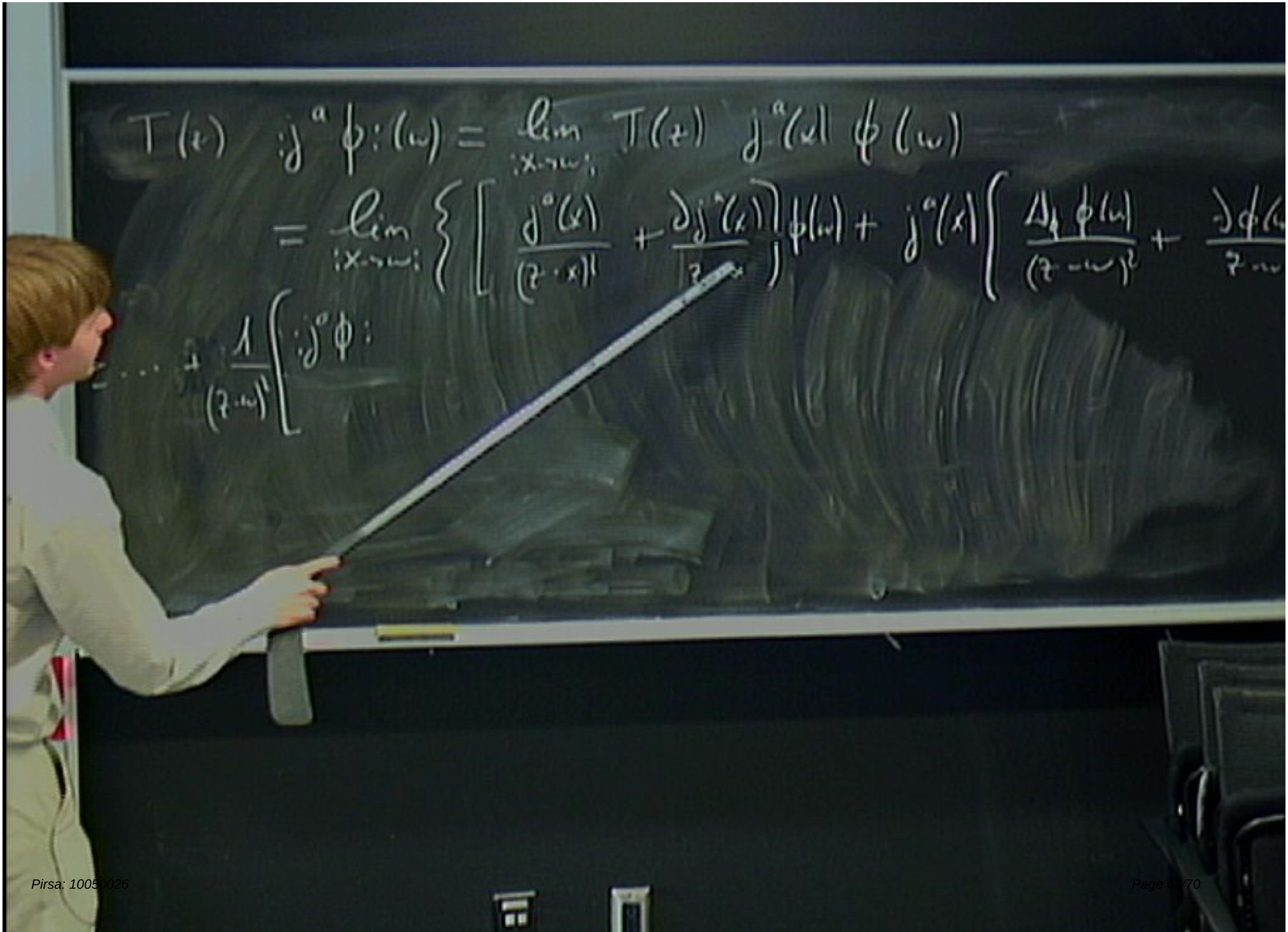

$T(z) \quad :J^a \phi:(w)$

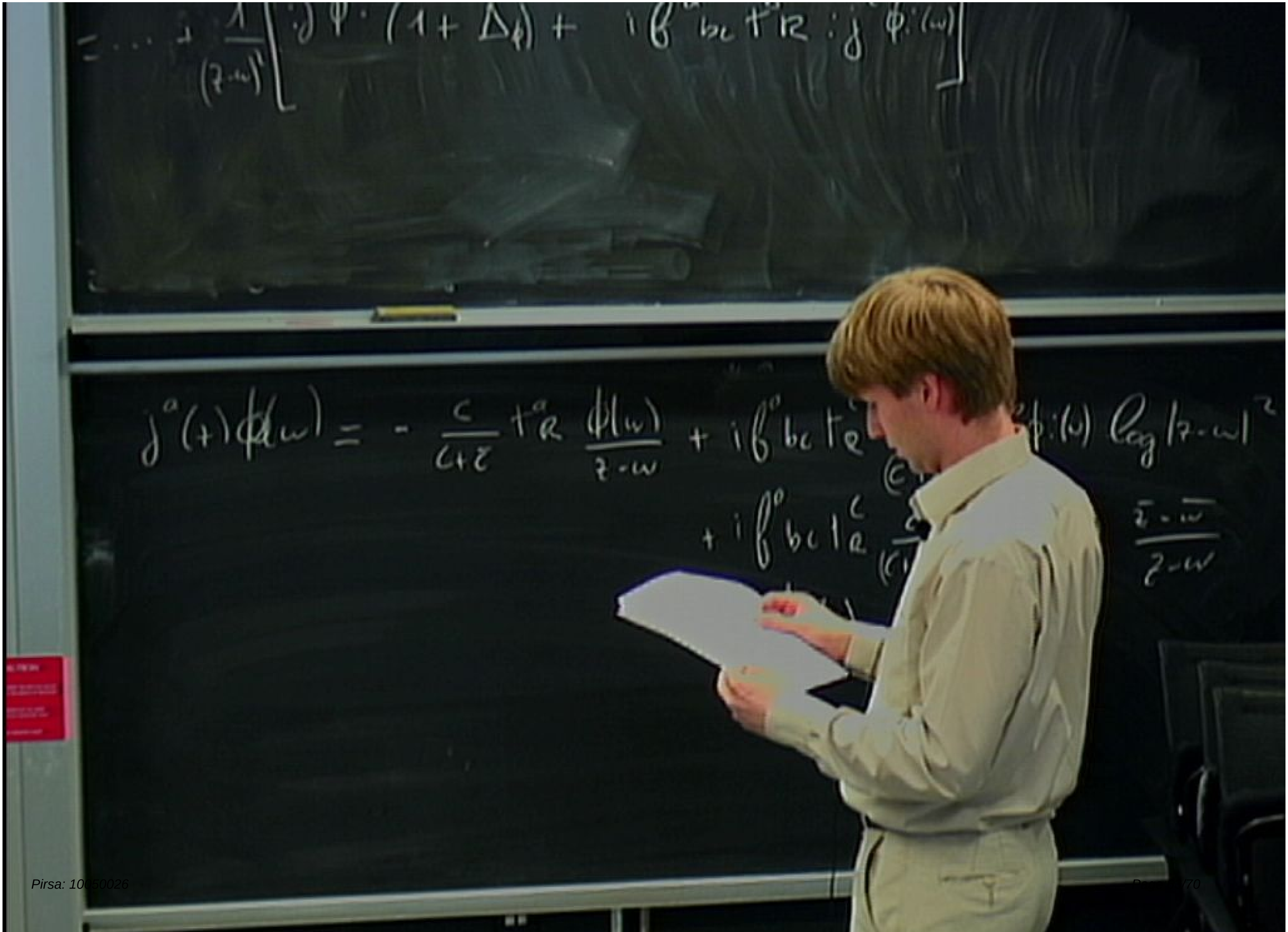

$T(z) \; :j^a \phi:(w) = \lim_{x \to w} T(z) \; j^a(x) \; \phi(w)$
$=$


$T(z)\ :j^a\phi:(w) = \lim_{:x\to w:} T(z)\ j^a(x)\ \phi(w)$



$T(z) \; :j^a \phi:(w) = \lim_{:x \to w:} T(z) \; j^a(x) \; \phi(w)$
$= \lim_{:x \to w:} \Big\{ \Big[ \frac{j^a(x)}{(z-x)^2} + \frac{\partial j^a(x)}{z-x} \Big] \phi(w)$
$\Big( \frac{\Delta_\phi \phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w}$
$= \cdots + \frac{1}{(z-w)^2} \Big[ :j^a \phi:$


$T(z)\ :j^a\phi:(w) = \lim_{:x\to w:} T(z)\ j^a(x)\ \phi(w)$
$= \lim_{:x\to w:} \Big\{ \Big[ \frac{j^a(x)}{(z-x)^2} + \frac{\partial j^a(x)}{z-x} \Big] \phi(w) + j^a(x) \Big( \frac{\Delta_\phi\, \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}$
$+ \frac{1}{(z-w)^2} \Big[ :j^a\phi:$

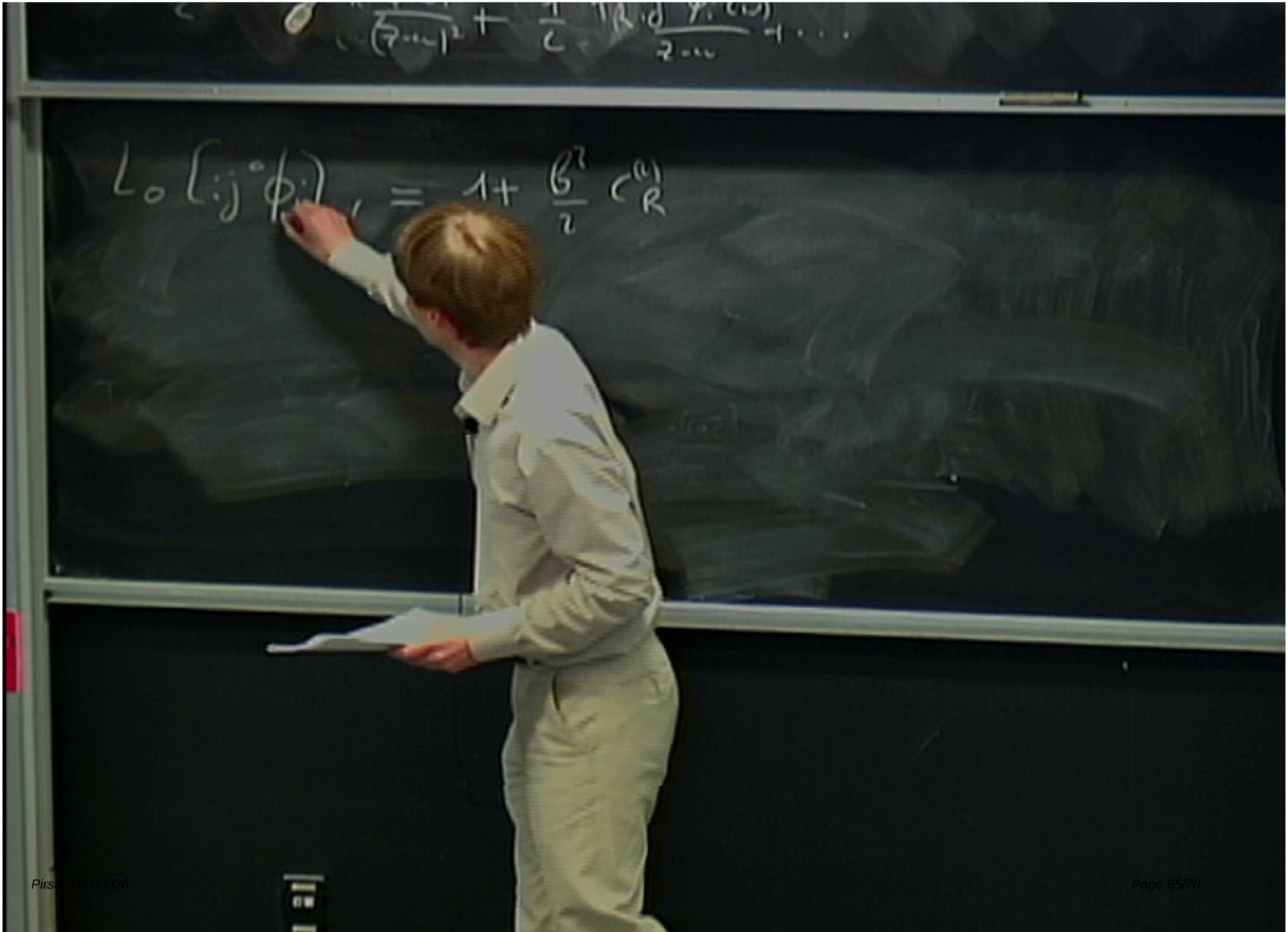


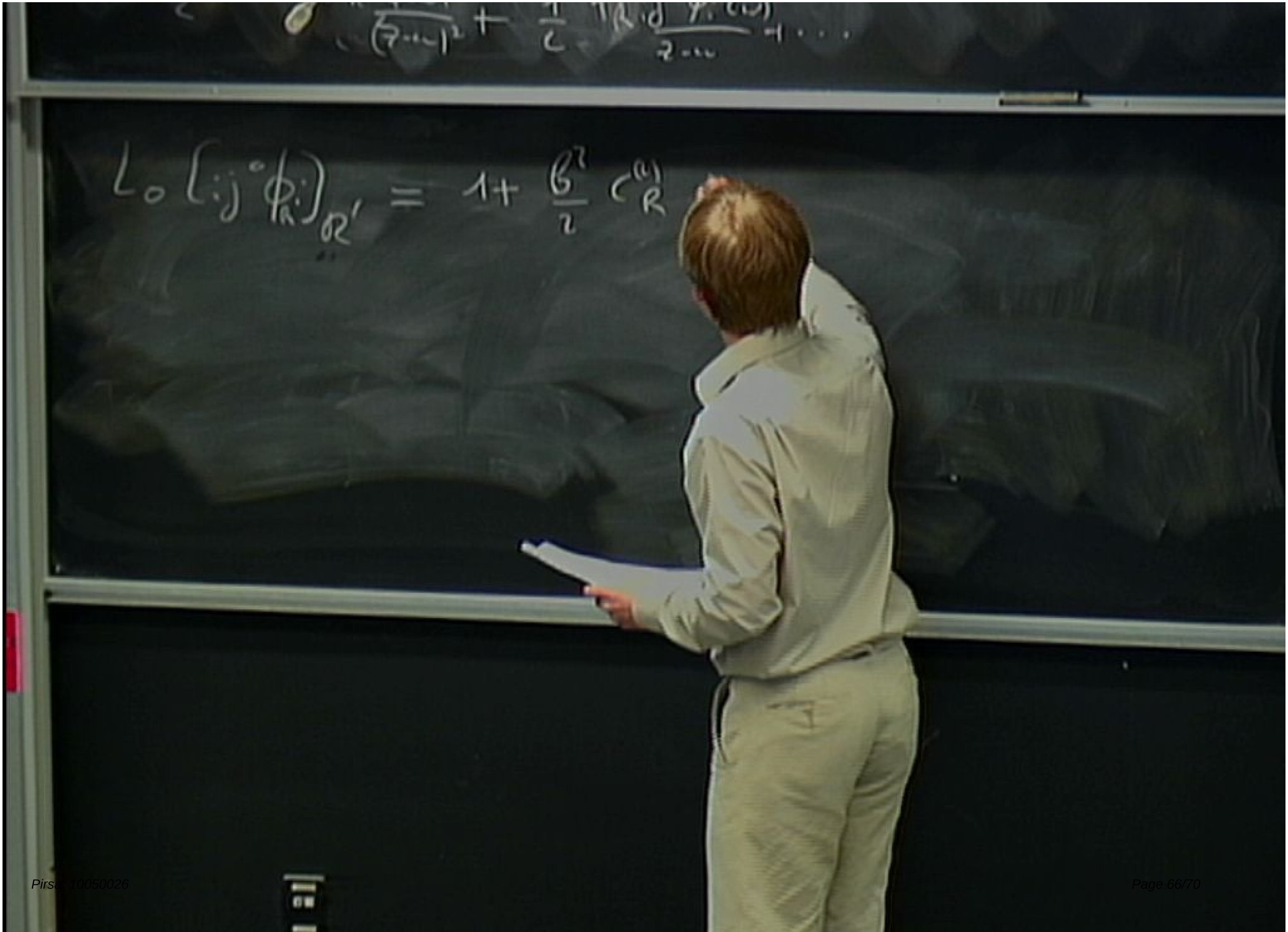
$$i f^{a}{}_{bc}\, t^{b}_{R} = t^{b}_{Adj} \otimes t^{b}_{R} = \frac{1}{2}\left( C^{(2)}_{Adj\otimes R} - C^{(2)}_{R} \right)$$

$\partial^{a}\phi$ transforms $\underline{Adj \otimes R}$

$$t^{a}_{Adj\otimes R} = t^{a}_{Adj} \otimes Id + Id \otimes t^{a}_{R}$$

$$C^{(2)}_{Adj\otimes R} = \cancel{C^{(2)}_{Adj}}^{\,0} + C^{(2)}_{R} + 2\, t^{a}_{Adj} \otimes t^{a}_{R}$$





$$L_0 [:j^\circ \phi_R:]_{R'} = 1 + \frac{b^2}{2} C^{(2)}_R + \frac{b^2}{2}(1-kb^2)\left(C^{(2)}_{R'} - C^{(2)}_R\right)$$

$$\frac{1}{2}\ \frac{\ldots}{(2\ldots)^2} + \frac{1}{2}\ \frac{\ldots}{2\ldots} + \ldots$$

$$L_0 \left[:j^0 \phi_R:\right]_{R'} = 1 + \frac{b^2}{2} c^{(2)}_R + \frac{b^2}{2}(1 - kb^2)\left(c^{(2)}_{R'} - c^{(2)}_R\right)$$

$$= 1 + c^{(2)}_R\left(\frac{b^2}{2} kb^2\right) + c^{(2)}_{R'} \frac{b^2}{2}\left(1 - kb^2\right)$$

$$L_0 [:j^0 \phi_R:]_{R'} = 1 + \frac{b^2}{2} c_R^{(2)} + \frac{b^2}{2}(1-kb^2)\left(c_{R'}^{(2)} - c_R^{(2)}\right)$$

$$= 1 + c_R^{(2)}\left(\frac{b^2}{2} k b^2\right) + c_{R'}^{(2)} \frac{b^2}{2}\left(1-kb^2\right)$$

$$L_0 \left[ {}_{j} \phi_{R} \right]_{R'} = 1 + \frac{b^2}{2} c^{(2)}_{R} + \frac{b^2}{2}(1 - k b^2)\left( c^{(2)}_{R'} - c^{(2)}_{R} \right)$$

$$= 1 + c^{(2)}_{R}\left( \frac{b^2}{2} k b^2 \right) + c^{(2)}_{R'} \frac{b^2}{2}\left( 1 - k b^2 \right)$$